Assignment 1

You can make use of the following fact regarding the spectral decomposition of symmetric matrix. Let $\mathbf{A}$ be a $p \times p$ symmetric matrix, then we have:

1) Suppose that $\lambda_1, \lambda_2, \cdots, \lambda_p$ are the eigenvalues of $\mathbf{A}$. Let $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_p$ be the associated normalized and mutually perpendicular eigenvectors ($\mathbf{e}_i'\mathbf{e}_i = 1, \mathbf{e}_i'\mathbf{e}_j = 0$ for $1 \leq i, j \leq p, i \neq j$.) Then $\mathbf{A}$ can be represented as

$$\mathbf{A} = \sum_{i=1}^{p} \lambda_i \mathbf{e}_i \mathbf{e}_i'.$$

2) If a $p \times p$ symmetric matrix can be represented as:

$$\mathbf{A} = \sum_{i=1}^{p} \lambda_i \mathbf{e}_i \mathbf{e}_i',$$

where $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_p$ are a sequence of normalized and mutually perpendicular eigenvectors ($\mathbf{e}_i'\mathbf{e}_i = 1, \mathbf{e}_i'\mathbf{e}_j = 0$ for $1 \leq i, j \leq p, i \neq j$.) Then $\lambda_1, \lambda_2, \cdots, \lambda_p$ are the eigenvalues of $\mathbf{A}$ and $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_p$ are the associated eigenvectors.

Problem 1. (Eigenvalue and Orthogonal Matrix)

Let $\mathbf{A}$ be a $p \times p$ symmetric matrix with spectral decomposition:

$$\mathbf{A} = \sum_{i=1}^{p} \lambda_i \mathbf{e}_i \mathbf{e}_i'.$$

Let $\mathbf{Q}$ be a $p \times p$ orthogonal matrix and define:

$$\mathbf{B} = \mathbf{Q}\mathbf{A}\mathbf{Q}'.$$

Answer the following questions:

1) Show that, the sequence of vectors:

$$\mathbf{Q}\mathbf{e}_1, \mathbf{Q}\mathbf{e}_2, \cdots, \mathbf{Q}\mathbf{e}_p,$$

are mutually perpendicular and the length of each vector equals 1.

2) Show that, for any $1 \leq i \leq p$.

$$\mathbf{B}\mathbf{Q}\mathbf{e}_i = \lambda_i \mathbf{Q}\mathbf{e}_i,$$

and write down the spectral decomposition of $\mathbf{B}$.

Problem 2. (Approximate a Symmetric Matrix)

1) Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times k$ matrices. Denote the entry on the ith row and jth column of $\mathbf{A}$ as a_{ij}, and the entry on the ith row and jth column of $\mathbf{B}$ as b_{ij} ($1 \leq i \leq m, 1 \leq j \leq k$.). We can then evaluate the difference between these two matrices by calculating the sum of square

differences between corresponded entries. Show that, this difference can be represented as following:

$$\sum_{i=1}^{m}\sum_{j=1}^{k}(a_{ij}-b_{ij})^2 = tr[(\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})']$$

2) Now let us focus on a a $p \times p$ symmetric matrix $\mathbf{A}$ with spectral decomposition:

$$\mathbf{A} = \sum_{i=1}^{p} \lambda_i \mathbf{e}_i \mathbf{e}_i'.$$

We can use a lower rank matrix $\mathbf{B}$ to approximate $\mathbf{A}$, in particular, let us define

$$\mathbf{B} = \sum_{i=1}^{q} \lambda_i \mathbf{e}_i \mathbf{e}_i',$$

for $1 \leq q < p$. Show that:

$$tr[(\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})'] = \sum_{i=q+1}^{p} \lambda_i^2.$$

That is, the error of approximation equals to the sum of the squares of the eigenvalues not included in $\mathbf{B}$.

Problem 3. (Equicorrelation Matrix) Let ρ be a constant and $-\frac{1}{n-1} < \rho < 1$. Define the $p \times p$ equicorrelation matrix as:

$$\mathbf{R} = (1-\rho)\mathbf{I}_p + \rho \mathbf{1}_p \mathbf{1}_p',$$

where $\mathbf{I}_p$ is $p \times p$ identity matrix and $\mathbf{1}_p = (1, 1, \cdots, 1)'$. Answer the following question:

1)Write down the detail expression of $\mathbf{R}$ for $p = 3$. Why is R called equicorrelation matrix?

2) Define

$$\mathbf{E} = \frac{1}{1-\rho}[\mathbf{I}_p - \frac{\rho}{1+(p-1)\rho}\mathbf{1}_p\mathbf{1}_p'],$$

Verify that $\mathbf{E}$ is the inverse of $\mathbf{R}$ by showing that $\mathbf{ER} = \mathbf{I}_p$.

3) Since all the eigenvalues of $\mathbf{I}_p$ equal 1, the spectral decomposition of $\mathbf{I}_p$ is:

$$\mathbf{I}_p = \sum_{i=1}^{p} \mathbf{e}_i \mathbf{e}_i'.$$

In particular, we can choose the first eigenvector $\mathbf{e}_1 = \frac{1}{\sqrt{p}}\mathbf{1}_p$.

Use the above fact to find out the spectral decomposition of $\mathbf{R}$ and figure out its eigenvalues.

4) Utilize the results you obtained in 3), show that, the determinant of $\mathbf{R}$ is:

$$|\mathbf{R}| = (1-\rho)^{p-1}[1+(p-1)\rho].$$

Problem 4. (Properties of Multivariate Normal Distribution) Answer the following questions on multivariate normal distribution.

1) Let $\mathbf{X} \sim N_p(\boldsymbol{\mu}_X, \boldsymbol{\Sigma}_X)$ and define $\mathbf{Y} = \boldsymbol{\alpha} + \mathbf{A}\mathbf{X} + \mathbf{Z}$. Suppose that $\boldsymbol{\alpha}$ is a $q \times 1$ constant vector, $\mathbf{A}$ is a $q \times p$ constant matrix, $\mathbf{Z} \sim N_q(\mathbf{0}, \boldsymbol{\Sigma})$ and $\mathbf{Z} \perp \mathbf{X}$. The the vector $(\mathbf{X}', \mathbf{Y}')'$ still follow multivariate normal distribution. Find out its mean vector and covariance matrix.

2) Let $\mathbf{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then the two linear transformations $\mathbf{AX}$ and $\mathbf{BX}$ are independent if $\mathbf{A}\boldsymbol{\Sigma}\mathbf{B}' = 0$. Here $\mathbf{A}, \mathbf{B}$ represent two constant matrices.

Problem 5. Property of Wishart Distribution For $p \times p$ random matrix $\mathbf{A} \sim W_p(\Sigma, m)$, show that, if $\mathbf{C}$ is a $q \times p$ matrix, then

$$\mathbf{CAC}' \sim W_q(\mathbf{C}\Sigma\mathbf{C}', m).$$