Exercise 2.2.5 b)

Give DFA's accepting the following languages over the alphabet $\{0, 1\}$: The set of all strings whose tenth symbol from the right end is a 1.

$A = (Q, \Sigma, \delta, q_0, F)$, $\Sigma = \{0, 1\}$

$Q = \{a_1 a_2 \ldots a_n \mid 1 \leq n \leq 10, a_i \in \Sigma\} \cup \{q_0\}$

$\delta(q_0, 0) = 0$, $\delta(q_0, 1) = 1$

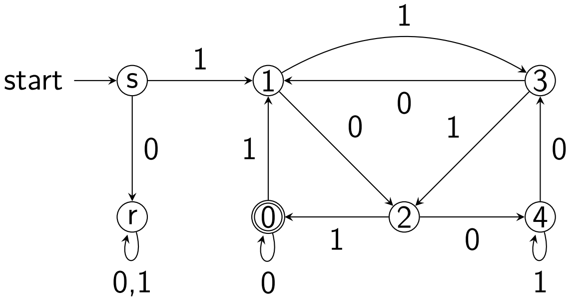
$$\delta(a_1 a_2 \ldots a_n, b) = \begin{cases} a_1 a_2 \ldots a_n b & \text{if } n < 10 \\ a_2 \ldots a_n b & \text{if } n = 10 \end{cases}, \; b \in \Sigma$$

$F = 1(0 + 1)^9$

Exercise 2.2.6 a)

The set of all strings beginning with a 1 that, when interpreted as binary integer, is a multiple of 5. for example, strings 101(5), 1010(10), and 1111(15) are in the language; 0, 100(4) and 111(7) are not.

	0	1
→s	r	q1
*q0	q0	q1
q1	q2	q3
q2	q4	q0
q3	q1	q2
q4	q3	q4
r	r	r



Exercise 2.2.6 b)

The set of all strings that, when interpreted *in reverse* as a binary integer, is divisible by 5. Examples of string in the language are 0, 10011(25), 1001100(25), and 0101(10).

Solutions: $A = (Q, \Sigma = \{0,1\}, \delta, q_0, F)$

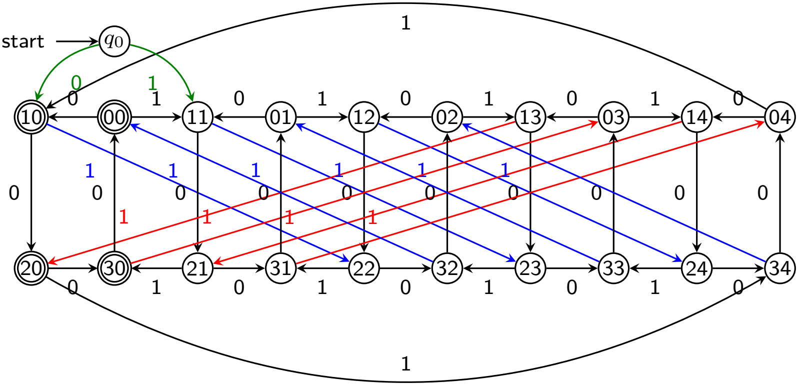
$$Q = \left\{ (x,y) \mid \begin{array}{ll} x \in \{0,1,2,3\}, & \text{len}(w) \mod 4 \\ y \in \{0,1,2,3,4\}, & \text{bin}(\overleftarrow{w}) \mod 5 \end{array} \right\} \cup \{q_0\}$$

$$f(x) \stackrel{def}{=} \left\{ \begin{array}{r|cccc} x & 0 & 1 & 2 & 3 \\ f(x) & 1 & 2 & 4 & 3 \end{array} \right\}$$

$$\delta \left\{ \begin{array}{lcl} \delta((x,y),0) & = & ((x+1) \mod 4, y) \\ \delta((x,y),1) & = & ((x+1) \mod 4, (y+f(x)) \mod 5) \\ \delta(q_0,0) & = & (1,0) \\ \delta(q_0,1) & = & (1,1) \end{array} \right.$$

$$F = \{(x,0) \mid x \in \{0,1,2,3\}\}$$

Exercise 2.2.6 b)



Exercise 2.2.7

Let A be a DFA and q a particular state of A, such that $\delta(q,a)=q$ for all input symbols a. Show by induction on the length of the input that for all input strings w, $\hat{\delta}(q,w)=q$.

首先, 对于 $|w|=0$ 的 w, 显然成立.

假设当 $|w|=n$ 时成立, 则当 $|w|=n+1$ 时, 不妨设 $w=xa$, 有

$$
\begin{aligned}
\hat{\delta}(q,w) &= \hat{\delta}(q,xa) \\
&= \delta(\hat{\delta}(q,x),a) \\
&= \delta(q,a) \\
&= q
\end{aligned}
$$

Exercise 2.2.8

Let A be a DFA and a a particular input symbol of A, such that for all states q of A we have $\delta(q, a) = q$.

a) Show by induction on n that for all $n \geq 0$, $\hat{\delta}(q, a^n) = q$, where a^n is the string consisting of n a's.

归纳基础 $\hat{\delta}(q, a^0) = \hat{\delta}(q, \varepsilon) = q$, 归纳递推
$\hat{\delta}(q, a^{n+1}) = \hat{\delta}(q, a^n a) = \delta(\hat{\delta}(q, a^n), a) = \delta(q, a) = q$

b) Show that either $\{a\}^* \subseteq L(A)$ or $\{a\}^* \cap L(A) = \emptyset$.

Exercise 2.2.9

Let $A = (Q, \Sigma, \delta, q_0, \{q_f\})$ be a DFA, and suppose that for all a in Σ we have $\delta(q_0, a) = \delta(q_f, a)$

a) Show that for all $w \neq \varepsilon$ we have $\hat{\delta}(q_0, w) = \hat{\delta}(q_f, w)$.

b) Show that if x is a nonempty string in $L(A)$, then for all $k > 0$, x^k (i.e. x written k times) is also in $L(A)$.

Exercise 2.2.9

Let $A = (Q, \Sigma, \delta, q_0, \{q_f\})$ be a DFA, and suppose that for all a in Σ we have $\delta(q_0, a) = \delta(q_f, a)$

a) Show that for all $w \neq \varepsilon$ we have $\hat{\delta}(q_0, w) = \hat{\delta}(q_f, w)$.

当 $|w| = 1$ 时显然成立, 假设 $|w| = n$ 时成立, 当 $|w| = n + 1$ 时 $w = xa$, 有

$$\begin{aligned}\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, xa) = \delta(\hat{\delta}(q_0, x), a) \\ &= \delta(\hat{\delta}(q_f, x), a) = \hat{\delta}(q_f, xa) \\ &= \hat{\delta}(q_f, w)\end{aligned}$$

b) Show that if x is a nonempty string in $L(A)$, then for all $k > 0$, x^k (i.e. x written k times) is also in $L(A)$.

Exercise 2.2.9

Let $A = (Q, \Sigma, \delta, q_0, \{q_f\})$ be a DFA, and suppose that for all a in Σ we have $\delta(q_0, a) = \delta(q_f, a)$

a) Show that for all $w \neq \varepsilon$ we have $\hat{\delta}(q_0, w) = \hat{\delta}(q_f, w)$.

b) Show that if x is a nonempty string in $L(A)$, then for all $k > 0$, x^k (i.e. x written k times) is also in $L(A)$.

Exercise 2.2.9

Let $A = (Q, \Sigma, \delta, q_0, \{q_f\})$ be a DFA, and suppose that for all a in Σ we have $\delta(q_0, a) = \delta(q_f, a)$

a) Show that for all $w \neq \varepsilon$ we have $\hat{\delta}(q_0, w) = \hat{\delta}(q_f, w)$.

b) Show that if x is a nonempty string in $L(A)$, then for all $k > 0$, x^k (i.e. x written k times) is also in $L(A)$.

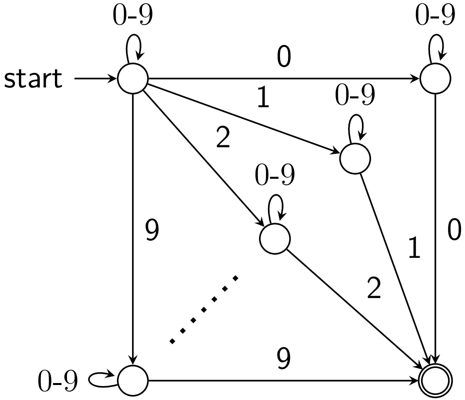
如果 $x \in L(A)$, 则有 $\hat{\delta}(q_0, x) = q_f$, 即 $k = 1$ 成立；假设 $k = n - 1$ 时, $x^k \in L(A)$ 成立, 那么当 $k = n$ 时

$$\hat{\delta}(q_0, x^n) = \hat{\delta}(\hat{\delta}(q_0, x^{n-1}), x) = \hat{\delta}(q_f, x) = \hat{\delta}(q_0, x) = q_f$$

Exercise 2.3.4

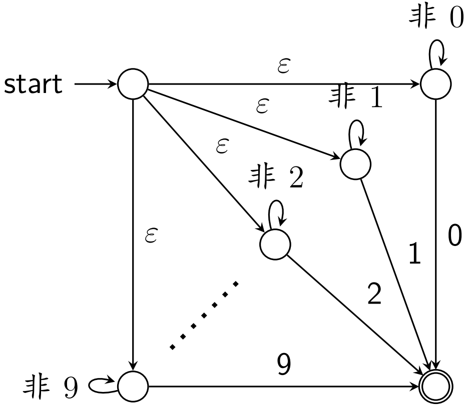
Give NFA, try to take advantage of nondeterminism as much as possible.

a) The set of strings over alphabet $\{0, 1, \cdots, 9\}$ such that the final digit has appear before.



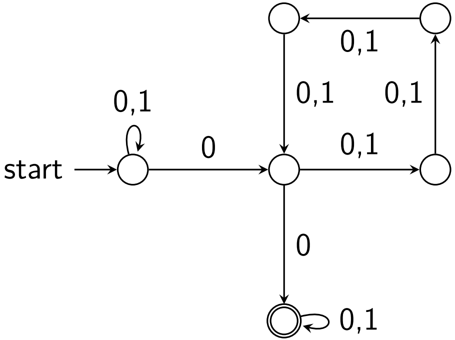
Exercise 2.3.4

b) The set of strings over alphabet $\{0, 1, \cdots, 9\}$ such that the final digit has *not* appeared before.



Exercise 2.3.4

c) The set of strings of 0's and 1's such that there are two 0's separated by a number of positions that is a mutiple of 4. (Note that 0 is an allowable multiple of 4.)



Exercise 3.1.1

The set of strings over alphabet $\{a, b, c\}$ containing at least one a and at least one b.

$$(\mathbf{a}+\mathbf{b}+\mathbf{c})^*(\mathbf{a}(\mathbf{a}+\mathbf{b}+\mathbf{c})^*\mathbf{b}+\mathbf{b}(\mathbf{a}+\mathbf{b}+\mathbf{c})^*\mathbf{a})(\mathbf{a}+\mathbf{b}+\mathbf{c})^*$$

The set of strings of 0's and 1's whose tenth symbol from the right end is 1.

$$(\mathbf{0}+\mathbf{1})^*\mathbf{1}(\mathbf{0}+\mathbf{1})^{\mathbf{9}}$$

The set of strings of 0's and 1's with at most one pair of consecutive 1's.

$$(\mathbf{0}+\mathbf{10})^*(\boldsymbol{\varepsilon}+\mathbf{1}+\mathbf{11})(\mathbf{0}+\mathbf{01})^*$$

Exercise 3.1.2

Write regular expressions for the following languages:

a) The set of all strings of 0's and 1's such that every pair of adjacent 0's appears before any pair of adjacent 1's.
$(\mathbf{0}+\mathbf{10})^*(\mathbf{01}+\mathbf{1})^*(\varepsilon+\mathbf{0})$

b) The set of strings of 0's and 1's whose number of 0's is divisible by five.
$(\mathbf{01}^*\mathbf{01}^*\mathbf{01}^*\mathbf{01}^*\mathbf{0}+\mathbf{1})^*$

Exercise 3.1.3

a) The set of all strings of 0's and 1's not containing 101 as a substring.
$0^*(1+000^*)^*0^*$ or $(0+\varepsilon)(1+000^*)^*(0+\varepsilon)$ or
$(0+\varepsilon)(1+00+000)^*(0+\varepsilon)$

b) The set of all strings with an equal number of 0's and 1's, such that no prefix has two more 0's than 1's, nor two more 1's than 0's.
$(01+10)^*$

c) The set of all strings of 0's and 1's whose number of 0's is divisible by five and whose number of 1's is even.

Exercise 3.1.4

Give English descriptions of the languages of the following regular expressions:

a) $(1+\varepsilon)(00^*1)^*0^*$

没有连续的 1

b) $(0^*1^*)^*000(0+1)^*$

有 3 个连续 0 的串

c) $(0+10)^*1^*$

任何连续 1 以后没有 0

Exercise 4.1.2

Prove that the following are not regular languages.

d) The set of strings of 0's and 1's whose length is a perfect square.

取 $w = 0^{N^2}$

e) The set of strings of 0's and 1's that are of the form ww, that is some string repeated.

取 $w = 0^N 1 0^N 1$

h) The set of strings of the form $w1^n$, where w is a string of 0's and 1's of length n.

Exercise 4.1.3 b)

The set of strings of the form 0^i1^j such that the greatest common divisor of i and j is 1.

证明: 设 $L = \{0^i1^j \mid \gcd(i,j) = 1\}$, 则
$L' = \overline{L} \cap \mathbf{0}^*\mathbf{1}^* = \{0^i1^j \mid \gcd(i,j) > 1\}$.

由泵引理证明 L' 非正则. 使用比 N 大的素数 P 取
$w = 0^P1^P$, 则 $0^{P-m}1^P \notin L'$, 因为 $\gcd(P-m, P) = 1$.

Exercise 4.2.2

If L is a language, and a is a symbol, then L/a, the quotient of L and a, is the set of strings w such that wa is in L. For example, if $L = \{a, aab, baa\}$, then $L/a = \{\varepsilon, ba\}$. Prove that if L is regular, so is L/a. Hint: Start with a DFA for L and consider the set of accepting states.

证明: 设识别 L 的 DFA $M = (Q, \Sigma, \delta, q_0, F)$, 即 $L = L(M)$.
构造 DFA $M' = (Q, \Sigma, \delta, q_0, F')$, 其接受状态

$$F' = \{q \mid \delta(q, a) \in F\},$$

其中 $q \in Q, a \in \Sigma$. 只需证明 $L(M') = L/a$.

$\because \forall w \in L(M')$, 有 $\delta(q_0, w) \in F'$, 即 $\delta(\delta(q_0, w), a) \in F$
$\therefore w \in L/a$;

又 $\because \forall w \in L/a$ 有 $wa \in L$ 即 $\delta(q_0, wa) \in F$ 即
$\delta(\delta(q_0, w), a) \in F$ 即 $\delta(q_0, w) \in F'$ $\therefore w \in L(M')$. $\square$

Exercise 4.2.6 a)

Show that the regular languages are closed under the following operations:

min(L) = $\{w \mid w$ is in L, but no proper prefix of w is in L $\}$.

$$\delta'(q,a) = \begin{cases} \delta(q,a) & \text{if } q \notin F \\ \varnothing & \text{if } q \in F \end{cases} \tag{1}$$

证明 $L(M') = \min(L)$

1° $\forall w \in L(M')$ 存在转移序列 $q_0 q_1 \cdots q_n \in F$ 使 M' 接受 w 其中 $q_i \notin F, 0 \le i \le n-1$ $\therefore w \in \min(L)$

2° $\forall w \in \min(L)$ 有 $w \in L$, 如果 M 接受 w 的状态序列为 $q_0 q_1 \cdots q_n \in F$ 则显然 $q_i \notin F, 0 \le i \le n-1$(因为否则, w 有 L 可接受的前缀) $\therefore w \in L(M')$

Exercise 4.2.6 a)

min(L) $= \{w \mid w$ is in L, but no proper prefix of w is in $L\}$.

用封闭性证明

$$\min(L) = L - L\Sigma^+$$

Exercise 4.2.6 b)

max$(L) = \{\ w \mid w$ is in L and for no x other than ε is wx in $L\}$

由 $M = (Q, \Sigma, \delta, q_0, F)$ 构造 $M' = (Q, \Sigma, \delta, q_0, F')$ 其中

$$F' = \{f \mid f \in F, \forall x \in \Sigma^+, \hat{\delta}(f, x) \notin F\}$$

则 $L(M') = \max(L)$

Exercise 4.2.6 b)

max(L) = $\{\ w \mid w$ is in L and for no x other than ε is wx in $L\}$

利用封闭性. 如果 $\Sigma = \{a, b, \cdots\}$, 设 $\Gamma = \{a, \hat{a}, b, \hat{b}, \cdots\}$, 定义同态 h $(\Gamma \to \Sigma^*)$ 和 g $(\Gamma \to \Sigma^*)$:

$$\begin{array}{ll} h(a) = a & g(a) = a \\ h(\hat{a}) = a & g(\hat{a}) = \varepsilon \\ h(b) = b & g(b) = b \\ h(\hat{b}) = b & g(\hat{b}) = \varepsilon \end{array}$$

那么

$$\max(L) = L - g(h^{-1}(L) \cap (a+b)^*(\hat{a}+\hat{b})^+)$$

Exercise 4.2.6 c)

init(L)={ w | for some x, wx is in L }

用同样的同态 h 和 g, 则

$$\mathrm{init}(L) = g(h^{-1}(L) \cap (a+b)^*(\hat{a}+\hat{b})^*)$$

Exercise 4.2.6 c)

init(L)={ w | for some x, wx is in L }

由 $M = (Q, \Sigma, \delta, q_0, F)$ 构造 $M' = (Q, \Sigma, \delta, q_0, Q - Q')$ 其中

$Q' = \{ q \mid q \in Q$, 没有从 q 到终态的路径 $\}$.

$$q \in Q - Q' \;\Leftrightarrow\; \exists x,\; \hat{\delta}(q, x) \in F$$

$$\forall w \in \Sigma^*, \hat{\delta}(q_0, w) \in Q - Q' \Leftrightarrow \exists x,\; \hat{\delta}(\hat{\delta}(q_0, w), x) \in F$$

即 $L(M') = \mathrm{init}(L)$.

Exercise 5.1.3

Show that every regular laugnage is a context-free laugnage.
Hint: Construct a CFG by induction on the number of operators in the regular expression.

证明：对正则表达式 R 中运算符的个数 n 进行归纳.

当 $n = 0$ 时, R 只能是 $\boldsymbol{\varepsilon}$, $\varnothing$ 或 $\mathbf{a}$ $(a \in \Sigma)$, 可以构造仅有一条产生式的文法 $S \to \varepsilon$, $S \to \varnothing$ 或 $S \to a$ 得到.

1) 若 $R = R_1 + R_2$, 则 R_1 和 R_2 中运算符都不超过 m, 所以都存在文法 G_1 和 G_2, 分别开始于 S_1 和 S_2, 只需构造新产生式和开始符号 $S \to S_1|S_2$, 连同 G_1 和 G_2 的产生式, 构成 R 的文法;

2) 若 $R = R_1R_2$, 则同理构造 $S \to S_1S_2$ 即可; 3) 若 $R = R_1^*$, 则构造 $S \to SS_1 \mid \varepsilon$ 即可.

且每种构造, 文法的语言与该表达式的语言等价. $\square$

Exercise 5.1.5

Let $T = \{0, 1, (,), +, *, \varnothing, e\}$. We may think of T as the set of symbols used by regular expressions over the alphabet $\{0, 1\}$; the only difference is that we use e for symbol ε, to avoid potential confusion in what follows. Your task is to design a CFG with set of terminals T that generates exactly the regular expressions with alphabet $\{0, 1\}$.

解: $S \to S + S \mid SS \mid S^* \mid (S) \mid 0 \mid 1 \mid \varnothing \mid e$.

Exercise 5.2.2

Suppose that G is a CFG without any productions that have ε as the right side. If w is in $L(G)$, the length of w is n, and w has a derivation of m steps, show that w has a parse tree with $n+m$ nodes.

证明:

1. 派生 w 的每一步推导都对应语法树的一个内节点, 所以 w 语法树中共有 m 个内节点;

2. 每个 w 的终结符都构成一个叶节点, 所以至少有 n 个叶节点, 而由于 G 中没有空产生式, 因此不会有标记为 ε 的叶节点, 所以只能有 n 个叶节点. 所以 w 的语法树有 $n+m$ 个节点. $\square$

Exercise 5.2.3

Suppose all is as in Exercise 5.2.2, but G may have some productions with ε as the right side. Show that a parse tree for a string w other than ε may have as many as $n + 2m - 1$ nodes, but no more.

证明:

1. 派生 w 的每一步推导都对应语法树的一个内节点, 所以 w 语法树中共有 m 个内节点.

2. 每个 w 的终结符都构成一个叶节点, 所以至少有 n 个叶节点.

3. 推导过程中, 每次空产生式的应用, 都会增加一个标记 ε 的叶节点, 但显然不能全部的 m 步都使用空产生式, 所以最多增加 $m - 1$ 个 ε 叶节点. 因此 w 的语法树有最多 $m + n + m - 1 = n + 2m - 1$ 个节点. $\square$