CSC418 F2015 A1

December 2, 2015

1 Question 1

We are given the following parametric form for the "wiggly-ellipse":

$$\begin{aligned} x(t) &= 4\cos(2\pi t) + \frac{1}{16}\cos(32\pi t) \\ y(t) &= 2\sin(2\pi t) + \frac{1}{16}\sin(32\pi t) \end{aligned}$$

1.1 Tangent Vector

The tangent vector of the "wiggly-ellipse" is given by the two temporal derivatives of the parametric form. That is, the tangent vector

$$T(t) = \begin{bmatrix} T_x(t) \\ T_y(t) \end{bmatrix}$$

where T_x, T_y given given by

$$\begin{aligned} T_x(t) &= \frac{\partial x}{\partial t} = -8\pi\sin(2\pi t) - 2\pi\sin(32\pi t) \\ T_y(t) &= \frac{\partial y}{\partial t} = 4\pi\cos(2\pi t) + 2\pi\cos(32\pi t) \end{aligned}$$

1.2 Normal Vector

In 2D a normal vector is given by a $\frac{\pi}{2}$ rotation of a tangent vector. This is therefore

$$N(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} T(t) = \begin{bmatrix} N_x(t) \\ N_y(t) \end{bmatrix}$$

for N_x, N_y given by

$$\begin{aligned} N_x(t) &= -4\pi\cos(2\pi t) - 2\pi\cos(32\pi t) \\ N_y(t) &= -8\pi\sin(2\pi t) - 2\pi\sin(32\pi t) \end{aligned}$$

1.3 Symmetry

1.3.1 X-axis

The curve is symmetric over the x-axis if for every t, there exists a $\hat{t}$ such that

$$\begin{aligned} x(t) &= x(\hat{t}) \\ y(t) &= -y(\hat{t}). \end{aligned}$$

We see that setting $\hat{t} = -t$, and given that $\cos(-t) = \cos(t), \sin(-t) = -\sin(t)$

$$\begin{aligned}
x(\hat{t}) &= x(-t) = 4\cos(2\pi(-t)) + \frac{1}{16}\cos(32\pi(-t)) \\
&= 4\cos(2\pi t) + \frac{1}{16}\cos(32\pi t) \\
&= x(t) \\
y(\hat{t}) &= y(-t) = 2\sin(2\pi(-t)) + \frac{1}{16}\sin(32\pi(-t)) \\
&= -2\sin(2\pi t) - \frac{1}{16}\sin(32\pi t) \\
&= -(2\sin(2\pi t) + \frac{1}{16}\sin(32\pi t)) \\
&= -y(t)
\end{aligned}$$

1.3.2 Y-axis

In this case we see a counter-example for symmetry over the y-axis which is defined by the existence of a $\hat{t}$ such that

$$\begin{aligned}
x(t) &= -x(\hat{t}) \\
y(t) &= y(\hat{t}).
\end{aligned}$$

The easiest counterexample is given by $t = 0, \hat{t} = 0.5$, where we see that

$$y(0) = 2\sin(2\pi 0) + \frac{1}{16}\sin(32\pi 0) = 0 = 2\sin(2\pi 0.5) + \frac{1}{16}\sin(32\pi 0.5) = y(0.5).$$

and

$$x(0) = 4\cos(2\pi 0) + \frac{1}{16}\cos(32\pi 0) = \frac{65}{16} \neq -\frac{63}{16} = 4\cos(2\pi 0.5) + \frac{1}{16}\cos(32\pi 0.5). = x(.5)$$

Though no marks were removed for only saying the above, the following is theoretically required to prove the lack of symmetry. This is although we have above found a point where $y(t) = y(\hat{t}), x(t) \neq -x(\hat{t})$ we have to prove that there is no other $\hat{t}$ such that $y(t) = y(\hat{t}), x(t) = -x(\hat{t})$. This fact can be shown by proving that although $x(0) = \frac{65}{16}$, there is no value of t such taht $x(t) = -\frac{65}{16}$.

We know that in fact $x(t)$ never takes on the value $-\frac{65}{16}$ because of hte following analysis on the extrema of $x(t)$, given by $T_x(t) = 0$.

$$\begin{aligned}
T_x(t) &= 0 \\
-8\pi\sin(2\pi t) - 2\pi\sin(32\pi t) &= 0 \\
8\pi\sin(2\pi t) &= -2\pi\sin(32\pi t) \\
4\sin(2\pi t) &= -\sin(32\pi t) \\
4\sin(2\pi t) &= -\sin(16\pi t)\cos(16\pi t) \\
&\cdots \\
4\sin(2\pi t) &= -\sin(2\pi t)\cos(2\pi t)\cos(4\pi t)\cos(8\pi t)\cos(16\pi t) \\
4v &= -v\cos(2\pi t)\sin(4\pi t)\cos(8\pi t)\cos(16\pi t)
\end{aligned}$$

where the final equation is only equal to 0 if $v = 0$ (as a product of cosines will never be 4). The intermediary steps are multiple applications of the double-angle formula. Therefore $\sin(2\pi t) = 0$ so this function finds extrema at $t = 0, 0.5$, which we have computed to take on the values $\frac{65}{16}, -\frac{63}{16}$ respectively. This implices that

$$x(t) \in [-\frac{63}{16}, \frac{65}{16}].$$

2 Tori

$$p(\lambda) = p_0 + \lambda d$$

for point p_0 and vector d in 2D. Furthermore the implicit formula for the circle is

$$f(q) = \|q - p_1\|^2 - r^2$$

for point p_1 and radius r. A donut is defined by two radii r_1, r_2 such that $r_1 < r_2$.

The area of the donut is given by $\pi(r_2^2 - r_1^2)$.

There can be between 0 and 4 intersections between the line and the boundary of the donut:

- 0)Line misses
- 1)Line tangent to outer circle
- 2)Line goes through outer circle but misses inner circle
- 3)Line goes through outer circle and tangnet ot inner circle
- 4)Line goes through both circles

We will describe a procedure for the intersections with each circle, For the number of intersections and the intersection points we simply add the number of intersections or union the set of intersections.

The intersection between a circle and a line can be determined by plugging the parametric representation into the implicit function:

$$\begin{aligned}
f(p(\lambda)) &= \|p(\lambda) - p_1\|^2 - r^2 \\
&= \|p_0 + \lambda d - p_1\|^2 - r^2 \\
&= \|\lambda d + (p_0 - p_1)\|^2 - r^2 \\
&= \lambda^2\|d\|^2 + 2\lambda d \cdot (p_0 - p_1) + \|p_0 - p_1\|^2 - r^2
\end{aligned}$$

so if we see have a quadratic equation in terms of λ where the zeros of this function correspond to intersections.

Let

$$\begin{aligned}
A &= \|d\|^2 \\
B &= 2\lambda d \cdot (p_0 - p_1) \\
C &= \|p_0 - p_1\|^2 - r^2 \\
D &= B^2 - 4AC.
\end{aligned}$$

Then

- $D > 0 \Rightarrow 2$ intersections
- $D = 0 \Rightarrow 1$ intersections
- $D < 0 \Rightarrow 0$ intersections.

Let $\hat{\lambda}$ be a zero of the above quadratic equation (the number of valid $\hat{\lambda}$ of course depends on the sign of D). Then the intersection point is $p(\hat{\lambda}) = p_0 + \hat{\lambda} d$.

Thus if we do the procedure we just described for each circle we can find the number of intersection points and intersections.

If the line and donut are both transformed by a non-uniform scale S around its origin we would see that the modified implicit equation is

$$\bar{f}(q) = f(S^{-1}(q))$$

and the parametric equation changes by

$$\bar{p}(\lambda) = Sp(\lambda)$$

so clearly

$$\bar{f}(\bar{p}(\lambda)) = f(p(\lambda))$$

and so the number of intersections would not change. However, the intersection points sould be scaled by S as they come from $\bar{p} = Sp$.

If one were to only apply the transformation to the donut the set of intersections and their points can change arbitrarily. For instance, consider

$$p(\lambda) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $p_1 = 0$, $r_1 = 1$, $r_2 = 2$. In the original configuration we have one intersection, but if we scale uniformly by 2 we have 3 intersections. Furthermore if we scale by 3 we get 4 intesections and scale by .5 we get 0 intersections.

3 Transformations

3.1 Translation and uniform scaling

This is false by counterexample. Consider translation

$$T(x) = x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and uniform scaling

$$S(x) = 2x$$

$$T(S(x)) = 2x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$S(T(x)) = 2x + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

so if $x = 0$ we clearly see that the results are different.

3.2 Translation and non-uniform scaling

This is false by counterexample.

Consider translation

$$T(x) = x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and uniform scaling

$$S(x) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$T(S(x)) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$S(T(x)) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

so if $x = 0$ we clearly see that the results are different.

3.3 Scaling and Rotation with same fixed points

Let R and S be a rotation and scaling around the same point, which we will say is the origin. We pick the following scaling and rotations:

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$SR \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

$$RS \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

So these do not commute.

3.4 Scaling and rotation with different centers

Let R be a rotation around a point away from the origin and S a scaling around the origin. For instance, in homogeneous coordinates let

$$R = C\hat{R}C^{-1}$$

for

$$\hat{R} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$RS = \begin{bmatrix} 0 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$SR = \begin{bmatrix} 0 & -1 & -2 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so they don't commute.

3.4.1 Translation and shear

Let translation T be

$$T(x) = x + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and shear S be

$$S(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x$$

then

$$T(S(x)) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S(T(x)) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so if $x = 0$ the results are clearly not the same and so they don't commute.

4 Triangles

A triangle can be seen as the intersection of three half-planes, where each each half-plane can be defined implicitly by two points A, B on the boundary and one on the interior I:

$$f_I^{AB}(p) = (p - A) \cdot \left[(I - A) - \frac{(I - A) \cdot (B - A)}{\|B - A\|^2}(B - A) \right]$$

Then the function

$$g_{v_0,v_1,v_2}(p) = \min(f_{v_0}^{v_1 v_2}(p), f_{v_1}^{v_0 v_2}(p), f_{v_2}^{v_1 v_0}(p))$$

is an implicit function for the triangle. In this context $g(q) > 0$ implies the point is on the interior and $g(q) = 0$ implies q is on the boundary.

Given a square with indices

$$\begin{matrix} A & - & B \\ | & & | \\ C & - & D \end{matrix}$$

we can triangulate it with the two triangles ABC, BDC.

In general, given a series of in-order vertices $v_1, \ldots, v_n$ it is triangulated by the union of triangles

$$\bigcup_2^{n-1} T(v_1, v_i, v_{i+1}).$$

This procedure will not work for concave polygons, such as

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

A convex polygon is the intersection of a series of half-planes, so

$$g_{v_1,\ldots,v_N}(p) = \min_1^N (f_{v_{n-1}}^{v_n v_{n+1}}(p))$$

where $v_0 = v_N, v_{N+1} = v_1$ is an implicit function for the convex polygon and once again $g > 0$ is the interiorand $g = 0$ is the boundary.