Fiber bundles

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November 5, 2024

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Introduction

These are my lecture notes on the course Fiber bundles in the year 2023/24. The lecturer that year was doc. dr. Riccardo Ugolini.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Manifolds

1.1 Smooth manifolds

Definition 1.1.1. A space M is *paracompact* if for every open covering $\{U_j \mid j \in I\}$ there exists a locally finite refinement. That is, every point $p \in M$ has a neighbourhood that intersects only finitely many sets of the refinement.

Remark 1.1.1.1. Instead of second-countable, we require manifolds to be paracompact.

Definition 1.1.2. The *Grassmanian* is the space

$$G_{k,n}(\mathbb{C}) = \left\{V \le \mathbb{C}^n \mid \dim V = k\right\}.$$

Proposition 1.1.3. The Grassmanian is a complex manifold of dimension $k(n-k)$.

Proof. Let $M_{k,n}(\mathbb{C})$ be the set of $k \times n$ matrices of maximal rank. This is an open subspace of $\mathbb{C}^{k\times n}$, therefore it is a manifold of dimension $k \cdot n$. We can then define a map $\pi \colon M_{k,n}(\mathbb{C}) \to G_{k,n}(\mathbb{C})$ mapping each matrix to the span of its rows. Note that $\pi(A) = \pi(B)$ if and only if there exists some $g \in \mathrm{GL}_k(\mathbb{C})$ with $gA = B$. That is,

$$G_{k,n}(\mathbb{C}) = M_{k,n}(\mathbb{C}) \big/ \mathrm{GL}_k(\mathbb{C}) \,.$$

For $A \in M_{k,n}(\mathbb{C})$, denote by $A_1, \dots, A_\ell$ its $k \times k$-minors. As $\operatorname{rank} A = k$, there exists an index j such that $\det A_j \ne 0$. Let

$$U_j = \left\{\pi(A) \mid \det A_j \ne 0\right\}.$$

By the above observation, this is an open covering of $G_{k,n}(\mathbb{C})$. Define $\varphi_j \colon U_j \to \mathbb{C}^{k(n-k)}$ as follows – choose a matrix A such that $A_j \ne 0$. Then define

$$\varphi_j(\pi(A)) = A_j^{-1} \cdot B_j,$$

where B_j is A without the columns of A_j. This is clearly a homomorphism. It is easy to see that the transition maps are holomorphic. □

Definition 1.1.4. A smooth manifold is *orientable* if there exists an atlas such that the jacobians of its transition maps have positive determinants.

Proposition 1.1.5. Let M be a complex manifold of complex dimension n. Then M is also a smooth orientable manifold of real dimension $2n$.

Proof. We can clearly see that M is a smooth manifold of real dimension $2n$. Let $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ be a complex atlas for M. For $\alpha, \beta \in I$, take $F = \varphi_\alpha \circ \varphi_\beta^{-1}$ and write $F(z) = u(x,y) + iv(x,y)$, where $z = x + iy$ for $x, y \in \mathbb{R}^n$. Then

$$dF = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{bmatrix},$$

therefore

$$\det(dF) = \left(\det \frac{\partial u}{\partial x}\right)^2 + \left(\det \frac{\partial u}{\partial y}\right)^2 > 0.$$ □

1.2 Structure sheaf, tangent bundle, and differentials

Definition 1.2.1. Let M be a smooth manifold and $p \in M$. Consider[1]

$$G_p = \{(U, f) \mid p \in U \wedge f \in \mathcal{C}^\infty(U) \wedge U \text{ is open}\}.$$

We define an equivalence relation on G_p by $(U, f) \sim (V, g)$ if there exists an open set $W \subseteq U \cap V$ containing p such that $f|_W = g|_W$. The set $G_p/\sim$ is the space of *germs* of functions at p, denoted by $\mathcal{C}^\infty_{M,p}$. The equivalence class of (U, f) is denoted by f_p.

Remark 1.2.1.1. The sets $\mathcal{C}^\infty_{M,p}$ and $\mathcal{O}_{M,p}$ are commutative rings with a neutral element. The latter is also an integral domain.

Remark 1.2.1.2. The map $\mathcal{C}^\infty_{M,p} \to \mathbb{R}$, given by $f_p \mapsto f(p)$, is well defined and the set $\mathcal{M}_{M,p} = \{f_p \mid f(p) = 0\}$ is a maximal ideal.

Definition 1.2.2. Let $U \subseteq M$ be an open set. Define

$$\mathcal{C}^\infty_M(U) = \{f \colon U \to \mathbb{R} \mid f \text{ is smooth}\}.$$

The operator $\mathcal{C}^\infty_M$ is called the *structure sheaf* of M.

Definition 1.2.3. A *derivation* v on $\mathcal{C}^\infty_{M,p}$ is a linear operator $v \colon \mathcal{C}^\infty_{M,p} \to \mathbb{R}$ such that

$$v(f_p g_p) = f(p)v(g_p) + g(p)v(f_p)$$

for all $f_p, g_p \in \mathcal{C}^\infty_{M,p}$. The set of all derivations on $\mathcal{C}^\infty_{M,p}$ is denoted by T_pM.

Remark 1.2.3.1. Let (U, φ) be a local chart around p such that $\varphi(p) = 0$. Then $\varphi^* \colon \mathcal{C}^\infty_{\mathbb{R}^n,0} \to \mathcal{C}^\infty_{M,p}$, given by $\varphi^*(f) = f \circ \varphi$, is a ring isomorphism.

Remark 1.2.3.2. The map $d\varphi_p \colon \mathrm{T}_p M \to \mathrm{T}_0 \mathbb{R}^n$, given by

$$d\varphi_p(v)(f) = v(f \circ \varphi),$$

is an isomorphism of vector spaces.

Definition 1.2.4. We define

$$\frac{\partial}{\partial x_j^\alpha}(p)(f) = \left.\frac{\partial \left(f \circ \varphi_\alpha^{-1}\right)}{\partial x_j}\right|_0$$

for $f \in \mathcal{C}^\infty_{M,p}$.

Lemma 1.2.5. The set

$$\left\{\frac{\partial}{\partial x_j}(p) \;\middle|\; j \le n\right\}$$

is a basis of T_pM.

Proof. We can assume $M = \mathbb{R}^n$ and $p = 0$. Take $f \in \mathcal{C}^\infty_{\mathbb{R}^n,0}$ and $v \in \mathrm{T}_0\mathbb{R}^n$. Then

$$v(f) = v\left(f(0) + \sum_{j=1}^n c_j x_j + O\left(|x|^2\right)\right) = \sum_{j=1}^n c_j v(x_j),$$

[1] Replace $\mathcal{C}^\infty$ with $\mathcal{O}$ for complex manifolds.

since $v\left(O\left(|x|^2\right)\right) = 0$. Hence

$$v = \sum_{j=1}^{n} v(x_j) \frac{\partial}{\partial x_j}(p)$$

as $c_j = \frac{\partial f}{\partial x_j}(0)$. It follows that the above set is in fact a generator of T_pM. As they are clearly linearly independent, they indeed form a basis. □

Definition 1.2.6. Let M and N be manifolds and $f \colon M \to N$ be a smooth map. For $p \in M$, define $df_p \colon \mathrm{T}_pM \to \mathrm{T}_{f(p)}N$ via

$$df_p(v)(h) = v(h \circ f)$$

for $v \in \mathrm{T}_pM$ and $h \in \mathcal{C}^\infty_{N,f(p)}$.

Proposition 1.2.7. Let $p \in M$ and $f \colon M \to N$ be a smooth map. Let (U, φ) be a local chart around p and (V, ψ) be a local chart around $f(p)$. Then the matrix of the linear map df_p with respect to the standard bases is given by the jacobian matrix of $\psi \circ f \circ \varphi^{-1}$ in $\varphi(p)$.

Definition 1.2.8. Let M be a smooth manifold of dimension n. A topological subspace $N \subseteq M$ is a *regular submanifold* of codimension k if for every point $p \in N$ there exists a local chart (U, φ) near $p \in M$ such that

$$\varphi(N \cap U) = \{x \in \varphi(U) \subseteq \mathbb{R}^n \mid \forall j \leq k \colon x_j = 0\}.$$

Proposition 1.2.9. If N is a regular submanifold, then for every point $p \in N$ there exists a neighbourhood U of p and a smooth map $F \colon U \to \mathbb{R}^k$ such that $N \cap U = \{q \in U \mid f(q) = 0\}$ and dF_q is surjective for all $q \in N \cap U$.

Proof. Take F to be the projection on the first k coordinates. □

Theorem 1.2.10 (Rank)**.** Let M_1 and M_2 be smooth manifolds of dimensions n and m. Let $F \colon M_1 \to M_2$ be a smooth map. Assume that $\operatorname{rank}(df_p) = k \in \mathbb{N}$ for all $p \in M_1$. Then for all $a \in F(M_1)$ we have that $F^{-1}(a)$ is a regular submanifold of dimension k in M_1.

Proposition 1.2.11. Let $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ and define

$$H = \left\{ [z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \;\middle|\; \sum_{j=0}^{n} a_j z_j = 0 \right\}.$$

Then H is a submanifold.

1.3 Coverings and group actions

Proposition 1.3.1. Let M be a smooth manifold with covering $\left(\widetilde{M}, \pi\right)$. Then $\widetilde{M}$ has a unique smooth structure such that π is smooth.

Definition 1.3.2. A group G is a *Lie group* if it is a topological space admitting a smooth structure such that the operation $G \times G \to G$, given by $(g, h) \mapsto gh^{-1}$, is smooth.

Definition 1.3.3. For a smooth manifold M denote

$$\mathrm{Aut}(M) = \{f \colon M \to M \mid f \text{ is a diffeomorphism}\}.$$

A group G *acts* on M via $\varphi \colon G \to \mathrm{Aut}(M)$ if φ is a homomorphism. The action is *faithful* or *effective* if φ is injective.

Definition 1.3.4. Given an action $\varphi \colon G \to \mathrm{Aut}(M)$, we define the *orbit* of $x \in M$ as

$$Gx = \{\varphi(g)(x) \mid g \in G\}.$$

Define an equivalence relation on M as $x \sim y \iff Gx = Gy$. Denote $M/G = M/\sim$.

Remark 1.3.4.1. For a smooth manifold M, $M/\mathrm{Aut}(M)$ is a point.

Definition 1.3.5. A *complex torus* is the quotient $\mathbb{C}^n/G$, where

$$G = \left\{ z \mapsto z + \sum_{j=1}^{2n} \lambda_j w_j \;\middle|\; \forall j \colon \lambda_j \in \mathbb{Z} \right\}$$

for $\mathbb{R}$-linearly independent vectors w_j.

2 Fiber bundles

2.1 Basic definitions

Definition 2.1.1. Let E and M be smooth manifolds and $\pi\colon E \to M$ be a smooth map. The triple (E, M, π) is a *submersion* if $d\pi_p\colon \mathrm{T}_pE \to \mathrm{T}_{\pi(p)}M$ is surjective for all $p \in E$.

Definition 2.1.2. A *fiber bundle* is a quadruple (E, M, F, π), where E, M and F are smooth manifolds and $\pi\colon E \to M$ is a surjective submersion such that every point $x \in M$ has a neighbourhood $U \subseteq M$ and a diffeomorphism $\varphi\colon \pi^{-1}(U) \to U \times F$ with $\pi = \pi_1 \circ \varphi$, where $\pi_1\colon U \times F \to U$ is the projection.

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\pi_1"] \\
& U
\end{tikzcd}$$

Definition 2.1.3. Let G be a Lie group. A fiber bundle (E, M, F, π) is a *bundle with structure group* G if the following conditions hold:

i) The group G acts effectively on F.

ii) There exists an open covering $\{U_\alpha \mid \alpha \in I\}$ of M such that for all $\alpha \in I$ there exists diffeomorphism $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$, such that if $U_\alpha \cap U_\beta \neq \emptyset$ there exists a smooth map $g_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to G$ satisfying

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x, f) = (x, g_{\alpha,\beta}(x)f)$$

for all $x \in U_\alpha \cap U_\beta$ and $f \in F$.

The set $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ is called the *trivializing atlas*, while $\{g_{\alpha,\beta} \mid \alpha, \beta \in I\}$ are the *(local) transition functions*.

Remark 2.1.3.1. Transition functions satisfy $g_{\alpha,\alpha}(x) = \mathrm{id}_F$ for all $x \in U_\alpha$. Furthermore, if $x \in U_\alpha \cap U_\beta$, then

$$g_{\beta,\alpha}(x) \cdot g_{\alpha,\beta}(x) = \mathrm{id}_F\,.$$

Finally, if $x \in U_\alpha \cap U_\beta \cap U_\gamma$, then

$$g_{\alpha,\beta}(x) \cdot g_{\beta,\gamma}(x) \cdot g_{\gamma,\alpha}(x) = \mathrm{id}_F\,.$$

These are called *cocycle conditions*.

Definition 2.1.4. Let M and M' be manifolds with fiber bundles $\pi_E\colon E \to M$ and $\pi_{E'}\colon E' \to M$. A *morphism of bundles* is a pair (f, φ) of maps $f\colon M \to M'$ and $\varphi\colon E \to E'$ with $\pi_{E'} \circ \varphi = f \circ \pi_E$.

$$\begin{tikzcd}
E \arrow[r, "\varphi"] \arrow[d, "\pi_E"'] & E' \arrow[d, "\pi_{E'}"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}$$

If f and φ are diffeomorphisms, the bundles are *equivalent*.

Definition 2.1.5. Let $U \subseteq M$ be an open set. The bundle E is *trivial over* U if the bundle $E|_U = \pi^{-1}(U)$ is equivalent to $U \times F$.

Proposition 2.1.6. Let M be a manifold, E a bundle over M with fiber F, and G its structure group. Let $\left\{\left(U_\alpha, \varphi_\alpha^E\right) \;\middle|\; \alpha \in I\right\}$ be a trivializing atlas for E with transition functions $\{g_{\alpha,\beta} \mid \alpha, \beta \in I\}$. Let E' be another bundle over the manifold M with trivializing atlas $\left\{\left(U_\alpha, \varphi_\alpha^{E'}\right) \;\middle|\; \alpha \in I\right\}$, fiber F', structure group G' and transition functions $\{h_{\alpha,\beta} \mid \alpha, \beta \in I\}$. If a smooth map $\psi \colon E \to E'$ induces an equivalence (id, ψ) of bundles, then for all $\alpha \in I$, posing

$$\psi_\alpha = \varphi_\alpha^{E'} \circ \psi \circ \left(\varphi_\alpha^E\right)^{-1} = (\psi'_\alpha, \psi''_\alpha) : U_\alpha \times F \to U_\alpha \times F',$$

we have $\psi'_\alpha = \mathrm{id}$, $\psi_\alpha(x, \cdot) \colon \{x\} \times F \to \{x\} \times F$ is a diffeomorphism for all x, and if $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\psi''_\beta(x, t) = h_{\beta,\alpha}(x) \cdot \psi''_\alpha\left(x, g_{\alpha,\beta}(x)t\right).$$

Vice-versa, suppose that there exists a family $\{\psi_\alpha = (\mathrm{id}, \psi''_\alpha) \mid \alpha \in I\}$ of smooth maps $\psi_\alpha \colon U_\alpha \times F \to U_\alpha \times F'$ such that, for every $x \in U_\alpha$, $\psi''_\alpha(x, \cdot) \colon \{x\} \times F \to \{x\} \times F$ is a diffeomorphism satisfying the above equation. Then there exists a bundle equivalence (id, ψ) such that $\psi_\alpha = \varphi_\alpha^{E'} \circ \psi \circ \left(\varphi_\alpha^E\right)^{-1}$ for all $\alpha \in I$.

Proof. Let $(x, t) \in U_\alpha \times F$ and consider ψ_α. We see that

$$\begin{aligned}
\psi_\beta &= \varphi_\beta^{E'} \circ \psi \circ \left(\varphi_\beta^E\right)^{-1} \\
&= \varphi_\beta^{E'} \circ \left(\varphi_\alpha^{E'}\right)^{-1} \circ \varphi_\alpha^{E'} \circ \psi \circ \left(\varphi_\alpha^E\right)^{-1} \circ \varphi_\alpha^E \circ \left(\varphi_\beta^E\right)^{-1} \\
&= (\mathrm{id}, h_{\beta,\alpha}) \circ \psi_\alpha \circ (\mathrm{id}, g_{\alpha,\beta}),
\end{aligned}$$

as required.

Now define $\psi \colon E \to E'$ as

$$\psi(e) = \left(\varphi_\alpha^{E'}\right)^{-1} \circ \psi_\alpha \circ \varphi_\alpha^E(e)$$

for $e \in \pi^{-1}(U_\alpha)$. It can be verified that ψ is well defined and is an equivalence of fibrations. □

Theorem 2.1.7. Let M and F be manifolds and G a Lie group acting effectively on F. Let $\{U_\alpha \mid \alpha \in I\}$ be an open cover of M with maps $g_{\alpha,\beta} \colon U_\alpha \cap U_\beta \to G$ whenever $U_\alpha \cap U_\beta \neq \emptyset$ satisfying the cocycle condition. Then there exists a unique[2] bundle E with base M, fiber F, structure group G and transition functions $\{g_{\alpha,\beta} \mid \alpha, \beta \in I\}$.

Proof. Define

$$E = \left. \bigsqcup_{\alpha \in I} U_\alpha \times F \middle/ \sim \right.,$$

where $(x, f) \sim (y, f')$ if $x = y$ and there exist $\alpha, \beta \in I$ such that $x \in U_\alpha \cap U_\beta$ and $f = g_{\alpha,\beta}(x) f'$. By the cocycle condition, this is in fact an equivalence relation. We define $\pi([x, f]) = x$, which is well defined and continuous. The local trivializations

[2] Up to equivalence.

$\varphi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$ are given by $\left[\widetilde{x}, \widetilde{f}\right] \mapsto (x, f)$, where (x, f) is the unique representative of $\left[\widetilde{x}, \widetilde{f}\right]$ in $U_\alpha \times F$. We see that φ_α is bijective, its inverse being the quotient map ρ. It follows that it is a homeomorphism.

Up to refining $\{U_\alpha \mid \alpha \in I\}$, we can assume that (U_α, ψ_α) are local charts for M. Let $\{(W_j, \theta_j) \mid j \in J\}$ be another atlas for F. Then $\{[U_\alpha \times W_j] \mid \alpha \in I \wedge j \in J\}$ is an open cover of E. Define $\widetilde{\varphi}_{\alpha,j} \colon \psi_\alpha(U_\alpha) \times \theta_j(W_j)$ as $[x, f] \mapsto (\psi_\alpha(x), \theta_j(f))$. We need to check that transition maps are smooth. Let $(p, t) \in \psi_\beta(U_\alpha \cap U_\beta) \times \theta_k(W_j \cap W_k)$. Then,

$$\begin{aligned}
\widetilde{\varphi}_{\alpha,j} \circ \left(\widetilde{\varphi}_{\beta,k}\right)^{-1}(p, t) &= \widetilde{\varphi}_{\alpha,j}\left(\left[\psi_\beta^{-1}(p), \theta_k^{-1}(t)\right]\right) \\
&= \widetilde{\varphi}_{\alpha,j}\left(\left[\psi_\beta^{-1}(p), g_{\alpha,\beta}(\psi_\beta^{-1}(p)) \cdot \theta_k^{-1}(t)\right]\right) \\
&= \left(\psi_\alpha \circ \psi_\beta^{-1}(p), \theta_j\left(g_{\alpha,\beta}\left(\psi_\beta^{-1}(p)\right)\theta_k^{-1}(t)\right)\right),
\end{aligned}$$

which is smooth, hence E is a smooth manifold. We can check that E has the given transition functions. $\square$

2.2 Vector bundles and principal bundles

Definition 2.2.1. A bundle (E, M, π) with fiber $\mathbb{R}^k$ and structure group $\mathrm{GL}_k(\mathbb{R})$ is a *vector bundle* of rank k, if there exists a trivializing atlas $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ for E such that for every point $x \in U_\alpha$, the map $\varphi_\alpha|_{E_x} : E_x \to \{x\} \times \mathbb{R}^k$ is a vector space isomorphism.

Remark 2.2.1.1. Let M be a manifold and $f \colon \mathbb{R} \to \mathbb{R}$ a non-linear diffeomorphism. Define $E = M \times \mathbb{R}$ with cover $\{M\}$ that trivializes E with the map $\varphi \colon E \to M \times \mathbb{R}$, $\varphi(x, v) \mapsto (x, f(v))$. This is a bundle with fiber $\mathbb{R}$ and structure group $\mathrm{GL}_1(\mathbb{R})$, but not a vector bundle.

Definition 2.2.2. Let M and M' be manifolds with vector bundles E and E'. A bundle morphism (f, ρ) is a *vector bundle morphism* if the map $\varphi_x = \varphi|_{E_x} : E_x \to E'_{f(x)}$ is a linear map for all $x \in M$.

Lemma 2.2.3. Let M be a manifold and E a bundle over M with fiber $\mathbb{R}^k$ and structure group $\mathrm{GL}_k(\mathbb{R})$. Then there exists a vector space bundle E' on M that is equivalent to E as a bundle.

Definition 2.2.4. A Lie group G *acts on the right* on a manifold F if $R \colon G \to \mathrm{Diff}(F)$ is such that $R(e) = \mathrm{id}$, $R\left(g^{-1}\right) = R(g)^{-1}$ and $R(gh) = R(h)R(g)$.

Remark 2.2.4.1. If L is an action, then $R(g) = L\left(g^{-1}\right)$ is a right action.

Example 2.2.4.2. Let E be a bundle with fiber $F \cong G$ and structure group G. Then G acts on F as follows: Let $\theta \colon F \to G$ be a diffeomorphism. Define

$$R_g(f) = \theta^{-1}(\theta(f)g)$$

for $g \in G$, $f \in F$. Then $R_g = R(g) \in \mathrm{Diff}(F)$. We write $R_g(f) = fg$ when the choice for θ is clear.

Definition 2.2.5. Let G be a Lie group. A bundle (P, M, π) with fiber G and structure group G is a *principal bundle* if there exists a trivializing atlas $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ for P such that $\varphi_\alpha|_{P_x} : P_x \to \{x\} \times G$ is G-equivariant for every $x \in U_\alpha$. That is, if $\varphi_\alpha(v) = (x, \varphi''_\alpha(x, v))$ for $v \in P_x$, then $\varphi''_\alpha(x, vg) = \varphi''_\alpha(x, v)g$ for all $g \in G$.

Remark 2.2.5.1. This action on the right commutes with the action we have from the fact that P is a fiber bundle with structure group G.

Definition 2.2.6. Let M and M' be manifolds with principal bundles P and P' with groups G and G'. Let $\rho \colon G \to G'$ be a Lie group morphism. A bundle morphism (f, φ) is a *principal bundle ρ-morphism* if $\varphi_x(pg) = \varphi_x(p)\rho(g)$ holds for all $x \in M$, $p \in P_x$ and $g \in G$.

Lemma 2.2.7. Let M be a manifold with bundle P with fiber G and structure group G. Then there exists a principal bundle P' on M which is equivalent to P as a bundle.

Definition 2.2.8. Let G be a Lie group with Lie subgroup H and let $f \colon H \hookrightarrow G$ be an immersion. Let P and P' be principal bundles on a manifold M with group G and H respectively. Then P' is a *reduction* of P if there exists a ρ-morphism of principal bundles (id, h), where with $h \colon P' \to P$ is injective.

Proposition 2.2.9. Let P be a principal bundle over M with group G. Let $H \leq G$ be a Lie subgroup. Then we can reduce G to H if and only if there exists a trivializing atlas of P with transition functions in H.

Proof. Suppose that $\{h_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to H \mid \alpha, \beta \in I\}$ are transition functions. They define a principal bundle P' on M with structure group H. Recall that

$$P' = \bigsqcup_{\alpha \in I} U_\alpha \times H \Big/ \sim \quad \text{and} \quad P = \bigsqcup_{\alpha \in I} U_\alpha \times G \Big/ \sim .$$

Note that there is a well defined map $U_\alpha \times H \hookrightarrow U_\alpha \times G$ given by $(x, h) \mapsto (x, h)$.

Let $[x, g] = [x, g_{\alpha,\beta}(x)g] \in P'$ for $g \in H$. The above map hence induces a well defined map on the quotient, therefore we get an injective morphism of principal bundles.

Now assume that $h\colon P' \to P$ is a reduction. Let $\{U_\alpha \mid \alpha \in I\}$ be a trivializing atlas for P and P', and denote by $\varphi'_\alpha\colon\ P'|_{U_\alpha} \to U_\alpha \times H$ the local trivialization for P'. We see that

$$\varphi'_\alpha(p') = (x, \widetilde{\varphi}'_\alpha(x, p')) = (x, \widetilde{\varphi}'_\alpha(x, e)p') .$$

Let $p \in P_x$ for $x \in U_\alpha$. Then there exists some $g \in G$ and $p' \in P'_x$ such that $p = h(p')g$ and if $p = h(p'_1)g_1$, then

$$gg_1^{-1} = h(p')^{-1}h(p'_1) = h\left((p')^{-1}p'_1\right) \in H,$$

hence $p' = p'_1\,(g_1g^{-1})$. We obtain that

$$\widetilde{\varphi}'_\alpha(x, p')g = \widetilde{\varphi}'_\alpha\left(x, p'_1g_1g^{-1}\right) = \widetilde{\varphi}'_\alpha\,(x, p'_1)\,g_1.$$

We can now define $\psi_\alpha\colon\ P|_{U_\alpha} \to U_\alpha \times G$ by $p \mapsto (x, \widetilde{\varphi}'_\alpha(x, p')g)$, where $p = h(p')$.

Consider the transition functions $\varphi'_{\alpha,\beta}$ for P'. Observe that

$$\varphi'_{\alpha,\beta}(x) = \widetilde{\varphi}'_\alpha(x, e)\left(\widetilde{\varphi}'_\beta(x, e)\right)^{-1}$$

For $x \in U_\alpha \cap U_\beta$ and $t \in G$ we can now write

$$\psi_\alpha \circ \psi_\beta^{-1}(x, t) = \psi_\alpha(p)$$

for $p = h(p')g \in P_x$, where $p' \in P'_x$ and $g \in G$. This is now further equal to

$$\begin{aligned}
\psi_\alpha(p) &= (x, \widetilde{\varphi}'_\alpha(x, p')g) \\
&= \left(x, \widetilde{\varphi}'_\alpha(x, p')\left(\widetilde{\varphi}'_\beta(x, p')\right)^{-1} t\right) \\
&= \left(x, \widetilde{\varphi}'_\alpha(x, e)p'\,(p')^{-1}\,\widetilde{\varphi}'_\beta(x, e)^{-1}t\right) \\
&= \left(x, \varphi'_{\alpha,\beta}(x)t\right) .
\end{aligned}$$

□

Example 2.2.9.1. Let E be a vector bundle over a manifold M. Then its transition maps $g_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R})$ give rise to a principal bundle $P(E)$ with fiber $\mathrm{GL}_n(\mathbb{R})$. Vice-versa is also true – principal $\mathrm{GL}_n(\mathbb{R})$ bundles have unique associated vector bundles.

Definition 2.2.10. A *complex vector bundle of complex rank k* is a a vector bundle over a manifold M with fiber $\mathbb{C}^k$ and structure group $\mathrm{GL}_k(\mathbb{C})$. If M is a complex manifold and $\pi\colon E \to M$ is holomorphic, then E is a *holomorphic bundle*.

Proposition 2.2.11. Let L and L' be holomorphic fiber bundles of rank 1 in a complex manifold M. Denote by $\{g_{\alpha,\beta} \mid \alpha, \beta \in I\}$ and $\left\{g'_{\alpha,\beta} \;\middle|\; \alpha, \beta \in I\right\}$ their respective transition functions. A holomorphic vector bundle isomorphism $f \colon L \to L'$ exists if and only if there exist holomorphic functions $f_\alpha \colon U_\alpha \to \mathbb{C}^*$ such that

$$\left.\frac{f_\beta}{f_\alpha}\right|_{U_\alpha \cap U_\beta} = \frac{g_{\alpha,\beta}}{g'_{\alpha,\beta}}$$

whenever $U_\alpha \cap U_\beta \neq \emptyset$.

Definition 2.2.12. The *tangent bundle* of a manifold M is defined as $\mathrm{T}M = \bigsqcup_{p \in M} \mathrm{T}_p M$.

Proposition 2.2.13. The tangent bundle is a vector bundle of dimension $n = \dim M$. If $\varphi_{\alpha,\beta}$ are the transition maps of the manifold, then $g_{\alpha,\beta}(p)$ is the matrix of $(d\varphi_{\alpha,\beta})_p$ in the standard basis of $\mathbb{R}^n$.

Definition 2.2.14. Let E be a vector bundle of rank r on a manifold M. For $p \in M$ let $F(E)_p$ denote the ordered bases of E_p. Then $F(E) = \bigsqcup_{p \in M} F(E)_p$ with the trivial projection is the *frame bundle*.

Proposition 2.2.15. The frame bundle is a principal bundle with fiber $\mathrm{GL}_r(\mathbb{R})$. It is equivalent to $P(E)$ as a principal bundle.

Definition 2.2.16. When $E = \mathrm{T}M$, we write $FM = F(\mathrm{T}M)$.

Example 2.2.16.1. The map $S^{2n+1} \to \mathbb{C}\mathrm{P}^n$, given by $(z_0, \ldots, z_n) \mapsto [z_0 : \cdots : z_n]$,[3] is a principal bundle with fiber S^1.

Example 2.2.16.2 (Homogeneous spaces)**.** Let G be a Lie group and $H \subseteq G$ be a Lie subgroup. Then $G \to G/H$ is a principal bundle with fiber H.

[3] The *Hopf fibration*.

2.3 Sections

Definition 2.3.1. Let M be a manifold with fiber bundle $\pi\colon E \to M$. Let $U \subseteq M$ be an open set. A *section* of E over U is a smooth map $s\colon U \to E$ such that $\pi \circ s = \mathrm{id}_U$. The set of sections of E over U is denoted by $\mathcal{C}^\infty(U, E)$. Sections with $U = M$ are called *global sections*.

Proposition 2.3.2. Let E be a fiber bundle over M with fiber F and structure group G. Let $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ be its trivializing atlas with transition maps $\{g_{\alpha,\beta} \mid \alpha, \beta \in I\}$. Let $U \subseteq M$ be an open set. If $s \in \mathcal{C}^\infty(U, E)$, denoting $s_\alpha = \pi_F \circ \varphi_\alpha \circ s$ we have

$$s_\alpha = g_{\alpha,\beta} s_\beta$$

for $U_\alpha \cap U_\beta \neq \emptyset$.

Vice-versa, if $\{s_\alpha \mid \alpha \in I\}$ are smooth functions satisfying the above equation, there exists a unique $s \in \mathcal{C}^\infty(U, E)$ with $s_\alpha = \pi_F \circ \varphi_\alpha \circ s$.

Proof. Note that $s(x) = \varphi_\alpha^{-1}(x, s_\alpha(x))$, therefore

$$(x, s_\alpha(x)) = \varphi_\alpha(s(x)) = \varphi_\alpha\left(\varphi_\beta^{-1}(x, s_\beta(x))\right) = (x, g_{\alpha,\beta}(x) s_\beta(x)).$$

If this equation holds, we can define $s(x) = \varphi_\alpha^{-1}(x, s_\alpha(x))$. The same computation as above shows that this is in fact a well-defined section. □

Definition 2.3.3. The maps s_α are called the *local data* of s.

Definition 2.3.4. Let E be a vector bundle of rank k over M and $U \subseteq M$ an open subset. Then E is trivial over U if and only if there exist $s_1, \dots, s_k \in \mathcal{C}^\infty(U, E)$ such that $\{s_j(x) \mid j \leq k\} \subseteq E_x$ is a basis for every $x \in U$.

Proof. The proof is obvious and need not be mentioned. □

Proposition 2.3.5. A principal bundle P over M admits a global section if and only if P is equivalent to $M \times G$ as a principal bundle.

Proof. Trivial bundles clearly have sections. Now suppose that $s\colon M \to P$ is a section. Then for every $g \in P_x$ there exists a unique $h \in G$ with $s(x)h = g$. We can now define $\Phi\colon P \to M \times G$ as $g \mapsto (x, h)$. □

Corollary 2.3.5.1. A principal bundle $P \to M$ is equivalent to $M \times G$ if and only if its structure group can be reduced to $\{e\}$.

Proof. If a reduction exists, there exists an embedding $\Phi\colon M \times \{e\} \hookrightarrow P$. But then $x \mapsto \Phi(x, e)$ is a global section. □

Definition 2.3.6. The *alternator* $A \in \mathrm{End}\left(V^{\otimes r}\right)$ is defined as

$$A(v_1 \otimes \cdots \otimes v_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}.$$

Remark 2.3.6.1. It holds that $A^2 = A$.

Definition 2.3.7. The *external product* is defined as

$$\bigwedge^r V = A\left(V^{\otimes r}\right).$$

We write $v_1 \wedge \cdots \wedge v_r = A(v_1 \otimes \cdots \otimes v_r)$.

Definition 2.3.8. A *differential form* is a section of $\mathrm{T}^* M$.

2.4 Pull-backs, subbundles and quotient bundles

Definition 2.4.1. Let M and N be manifolds, $f\colon M \to N$ a smooth map and $\pi\colon E \to N$ a bundle with fiber F and structure group G. The *pullback* f^*E with fiber F and structure group G over M is the bundle

$$f^*E = \{(m, e) \in M \times E \mid f(m) = \pi(e)\}\,.$$

Proposition 2.4.2. If $\{g_{\alpha,\beta} \mid \alpha, \beta \in I\}$ are the transition functions for E, then the functions $\{g_{\alpha,\beta} \circ f \mid \alpha, \beta \in I\}$ are the transition functions of f^*E.

Proof. Let $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ be the local trivialization for E. Define the map

$$\psi_\alpha\colon\ (\pi')^{-1}\left(f^{-1}(U_\alpha)\right) \to f^{-1}(U_\alpha) \times F$$

by $\psi_\alpha(m, e) = (m, \pi_F \circ \varphi_\alpha(e))$. These are trivializations for f^*E. Note that

$$\pi_F \circ \varphi_\beta(e) = \pi_F\left(g_{\beta,\alpha}\left(\pi(e)\right) \cdot \varphi_\alpha(e)\right) = \pi_F\left(g_{\beta,\alpha}\left(f(m)\right)\right) \cdot \pi_F(\varphi_\alpha(e)),$$

as required. □

Proposition 2.4.3. Let $f\colon M \to N$ be a smooth map. Let $\pi\colon E \to N$ and $\rho\colon E' \to M$ be fiber bundles with fibers F and structure groups G. Suppose that $(f, g)\colon E' \to E$ is a fiber bundle equivalence. Then $E' \cong f^*E$.

Proof. Let $\Phi\colon E' \to E$ be defined as $\widetilde{e} \mapsto (\rho\left(\widetilde{e}\right), g\left(\widetilde{e}\right))$.

$$\begin{array}{ccc} E' & \xrightarrow{\ g\ } & E \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ M & \xrightarrow[\ f\ ]{} & N \end{array}$$

□

Definition 2.4.4. Let E and F' be vector bundles over M. If there exists a vector bundle morphism $i\colon F' \to E$ which is injective on the fibers, we call $F = i(F')$ a *subbundle* of E.

Proposition 2.4.5. Let M be a manifold with a vector bundle $\pi\colon E \to M$ of rank k and let $F = i(F')$ be a subbundle of rank $\ell \le k$. Then i is an embedding and $F \subseteq E$ is a submanifold. Moreover, $\pi|_F : F \to M$ is a vector bundle and there exists a trivializing atlas $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ of E such that $\{(U_\alpha, \pi_\ell \circ \varphi_\alpha) \mid \alpha \in I\}$ is a trivializing atlas for F, where π_ℓ is the projection on the first ℓ coordinates.

Example 2.4.5.1. Let $S \subseteq M$ be a submanifold and $i\colon S \hookrightarrow M$ the inclusion map. Then $di\colon \mathrm{T}S \to \mathrm{T}M$ is a fiber-wise injection. In particular, $\mathrm{T}S \subseteq \mathrm{T}M|_S$ is a subbundle over S.

Definition 2.4.6. Let M be a manifold with a vector bundle $E \to M$ of rank k. Let $F \subseteq E$ be a subbundle of rank ℓ. Define a relation $e \sim e' \iff e, e' \in E_x \wedge e - e' \in F_x$. We denote the set of equivalence classes by ${E}/{F}$.

Proposition 2.4.7. The set ${}^{E}\!/_{F}$ is a vector bundle such that $\rho\colon E \to {}^{E}\!/_{F}$, given by $e \mapsto [e]$, is a morphism, and $\left({}^{E}\!/_{F}\right)_x = {}^{E_x}\!/_{F_x}$ for all $x \in M$. Furthermore, if $Q \to M$ is a vector bundle with fibers ${}^{E_x}\!/_{F_x}$ and there exists a morphism $\rho'\colon E \to Q$ with $v \mapsto [v]$, then $Q \cong {}^{E}\!/_{F}$.

Proof. Let $\pi'\colon {}^{E}\!/_{F} \mapsto M$ be given by $[e] \mapsto \pi(e)$, which is clearly well-defined. Let $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ be a trivializing atlas, adapted to F. Define $P'\colon U_\alpha \times \mathbb{R}^k \to U_\alpha \times \mathbb{R}^{k-\ell}$ as the projection onto the last $k - \ell$ coordinates. Note that $e \sim e'$ if and only if we have $P'(\varphi_\alpha(e)) = P'(\varphi_\alpha(e))$.

Define a map $\psi_\alpha\colon {}^{E}\!/_{F}|_{U_\alpha} \to U_\alpha \times \mathbb{R}^{k-\ell}$ by $\psi_\alpha([e]) = (\pi'(e), P' \circ \varphi_\alpha(e))$. This is clearly well defined and bijective. The quotient topology on ${}^{E}\!/_{F}$ is the same as the one induced by $\{\psi_\alpha \mid \alpha \in I\}$. The transition maps are given by the block-matrix structure of subbundles transition maps. $\square$

Definition 2.4.8. The bundle ${}^{E}\!/_{F}$ is called the *quotient bundle*.

Definition 2.4.9. Let $S \subseteq M$ be a submanifold. The *normal bundle* to S in M is the quotient bundle $\mathrm{N}S = {}^{\mathrm{T}M|_S}\!\big/_{\mathrm{T}S}$.

2.5 Cartier divisors

Definition 2.5.1. Let M be a complex manifold. An *effective Cartier divisor* D on M is given by an open cover $\{U_\alpha \mid \alpha \in I\}$ and holomorphic functions $f_\alpha \colon U \to \mathbb{C}$, not identically zero, such that the quotients $f_{\alpha,\beta} = \frac{f_\alpha}{f_\beta}$ are holomorphic functions on $U_\alpha \cap U_\beta$ without zeros.

Remark 2.5.1.1. The functions $\{f_{\alpha,\beta} \mid \alpha, \beta \in I\}$ satisfy the cocycle conditions, hence they induce a line bundle $\mathcal{O}(D)$. Vice-versa, given a line bundle and a non-zero global holomorphic section, its local data define a divisor.

2.6 Kernel, image, and exact sequences

Proposition 2.6.1. Let $\varphi\colon E \to F$ be a morphism of vector bundles over a manifold M. Then $\ker\varphi$ and $\operatorname{im}\varphi$ are vector subbundles if and only if $\operatorname{rank}\varphi_x$ is constant for $x \in M$.

Proof. Let $k = \operatorname{rank}\varphi_x$, $m = \operatorname{rank} E$ and $l = \operatorname{rank} F$. Let $U \subseteq M$ be a trivializing open set for E and F.

Locally, the map $\varphi\colon U \times \mathbb{R}^m \to U \times \mathbb{R}^l$ is given by $\varphi(x, a) = (x, A(x)a)$ for a map $A\colon U \to \mathbb{R}^{l\times m}$. Let $x_0 \in U$ and assume that the first k rows of $A(x_0)$ are independent. Replace U by a smaller set such that the first k rows of $A(x)$ are independent for all $x \in U$. Denote the first k rows of $A(x)$ by $B(x)$. Then

$$\ker\varphi = \{(x, a) \in U \times \mathbb{R}^m \mid B(x)a = 0\}.$$

Note that the map $F\colon U \times \mathbb{R}^m \to \mathbb{R}^k$ has maximal rank, hence $F^{-1}(0) = \ker\varphi \cap E|_U$ is a submanifold.

We can also suppose that the $k \times k$ minor $B'(x)$ of $B(x)$, given by the first k columns, is non-singular – that is, $\det B'(x) \neq 0$. Let B'' be the remainder of the matrix. Then $B(x)a = 0$ if and only if $a' = -B'(x)^{-1}B''(x)a''$.

We found $(m - k)$ independent solutions depending smoothly on $x \in U$ and pointwise generating $\ker\varphi$. Let $v_j(x) \in E_x$ be the image of $a_j(x)$ under the trivialization. We can complete $\{v_j(x) \mid j \le m - k\}$ to a basis of sections of $E|_U$. This basis trivializes $E|_U$ as

$$\sum_{j=1}^{m} b_j v_j(x) \mapsto (x, b_1, \dots, b_m),$$

with

$$\ker\varphi = \{(x, b_1, \dots, b_m) \in U \times \mathbb{R}^m \mid \forall j > m - k\colon b_j = 0\}$$

locally. Thus,

$$\sum_{j=1}^{m-k} b_j v_j(x) \mapsto (x, b_1, \dots, b_{m-k}),$$

are local trivializations for $\ker\varphi$.

Let now $Q = E/\ker\varphi$. Then $\varphi\colon E \to F$ induces a map $\rho\colon Q \to F$ which is injective on the fibers, hence $\rho(Q) = \varphi(E) = \operatorname{im}\varphi$ is a subbundle. □

Definition 2.6.2. Let E, E' and E'' be vector bundles over a manifold M. Let $\alpha\colon E' \to E$ and $\beta\colon E \to E''$ be morphisms. If α is injective, β is surjective, and $\operatorname{im}\alpha = \ker\beta$, then

$$0 \longrightarrow E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \longrightarrow 0$$

is a *short exact sequence.*

Remark 2.6.2.1. In a short exact sequence, $E'' \cong E/E'$.

Proposition 2.6.3. Suppose E, E' and E'' form a short exact sequence with mophisms α and β as above, and let F be another vector bundle over M. Then the sequence

$$0 \longrightarrow E' \otimes F \xrightarrow{\alpha\otimes\mathrm{id}} E \otimes F \xrightarrow{\beta\otimes\mathrm{id}} E'' \otimes F \longrightarrow 0$$

is also exact. Similarly, the sequences

$$0 \longrightarrow \operatorname{Hom}(F, E') \xrightarrow{\alpha \circ \cdot} \operatorname{Hom}(F, E) \xrightarrow{\beta \circ \cdot} \operatorname{Hom}(F, E'') \otimes F \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Hom}(E', F) \xrightarrow{\cdot \circ \alpha} \operatorname{Hom}(E, F) \xrightarrow{\cdot \circ \beta} \operatorname{Hom}(E'', F) \otimes F \longrightarrow 0$$

are exact. Finally, if $f \colon N \to M$ is smooth, the sequence

$$0 \longrightarrow f^*(E') \longrightarrow f^*(E) \longrightarrow f^*(E'') \longrightarrow 0$$

is also exact.

Theorem 2.6.4. Suppose that E, E' and E'' form a short exact sequence as above. Then $\det E \cong \det E' \otimes \det E''$.

Proof. Take a trivialization adapted to $\alpha(E')$, that is

$$g^E_{\alpha,\beta} = \begin{bmatrix} g^{E'}_{\alpha,\beta} & k_{\alpha,\beta} \\ 0 & g^{E''}_{\alpha,\beta} \end{bmatrix}.$$

Then $\det g^E_{\alpha,\beta} = \det g^{E'}_{\alpha,\beta} \cdot \det g^{E''}_{\alpha,\beta}$. □

Definition 2.6.5. Let M be a manifold. Then $K_M = \det(\mathrm{T}^*M)$ is the *canonical bundle* of M.

Remark 2.6.5.1. Let L be a line bundle. Since $L \otimes L^* \cong \operatorname{Hom}(L, L)$, it admits a global section $x \mapsto \mathrm{id}_L$. Hence $L \otimes L^*$ is trivial and $E \otimes L \cong F$ if and only if $E \cong F \otimes L^*$ for any two vector bundles E and F.

Theorem 2.6.6. Let $S \subseteq M$ be a complex submanifold of codimension 1. Then

$$K_S = (K_M \otimes \mathcal{O}([S]))|_S \,.$$

Proof. We have a short exact sequence

$$0 \longrightarrow \mathrm{T}S \longrightarrow \mathrm{T}M|_S \longrightarrow \mathrm{N}S \longrightarrow 0$$

over S. Taking the dual and using the previous theorem, we find that

$$K_M|_S \cong K_S \otimes \mathcal{O}([S])^*|_S \,.$$

As $\mathcal{O}([S])$ is a line bundle, we are done by the above remark. □

2.7 Line bundles and the Picard group

For this section, let M denote a complex manifold of dimension n.

Definition 2.7.1. A holomorphic vector bundle of rank 1 on M is called a *line bundle*. The set of all line bundles (modulo equivalence) is the *Picard group* $\mathrm{Pic}(M)$.

Proposition 2.7.2. The set $\mathrm{Pic}(M)$ is an abelian group with $L^{-1} = L^*$ and neutral element $M \times \mathbb{C}$.

Proof. The proof is obvious and need not be mentioned. □

Definition 2.7.3. The *tautological bundle* on $\mathbb{C}\mathrm{P}^n$ is given by

$$\mathcal{O}(-1) = \left\{([p], v) \in \mathbb{C}\mathrm{P}^n \times \mathbb{C}^{n+1} \;\middle|\; v \in \mathbb{C} \times p\right\}.$$

Definition 2.7.4. Define $\mathcal{O}(1) = \mathcal{O}(-1)^*$ and $\mathcal{O}(k) = \mathcal{O}(1)^k$.

Proposition 2.7.5. Let $k \geq 0$. The space $\mathcal{O}\left(\mathbb{C}\mathrm{P}^n, \mathcal{O}(k)\right)$ is isomorphic to the space of homogeneous polynomials of degree k in $n+1$ variables. In particular, the bundles $\mathcal{O}(k)$ are pairwise distinct for $k \in \mathbb{Z}$.

Proof. Let $s \in \mathcal{O}\left(\mathbb{C}\mathrm{P}^n, \mathcal{O}(k)\right)$ and let $s_\alpha \colon U_\alpha \to \mathbb{C}$ be its local data. Observe that

$$s_\alpha([z]) = \left(\frac{z_\alpha}{z_\beta}\right)^{-k} s_\beta([z])$$

on $U_\alpha \cap U_\beta$. Thus $p(z) = z_\alpha^k s_\alpha$ is independent of α and holomorphic on $\mathbb{C}^{n+1}$. Furthermore, $p(\lambda z) = \lambda^k p(z)$, hence p is homogeneous of degree k. Since p is holomorphic on $\mathbb{C}^{n+1}$, we get $k \geq 0$. □

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