Trace ideals and their applications

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Introduction

These are my lecture notes on the course Izbrana poglavja iz analize: Trace Ideals and Their Applications in the year 2023/24. The lecturer that year was prof. dr. Oleksiy Kostenko.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Operators on Hilbert spaces

1.1 Matrices and bounded operators

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Definition 1.1.1. Let V be a finite-dimensional vector space over a field K. A *linear operator* A in V is a linear map $A\colon V \to V$.

Definition 1.1.2. Let $A\colon V \to V$ be a linear operator. A closed subspace $U \leq V$ is an *invariant subspace* for A if $A(U) \subseteq U$. The set of all invariant subspaces of A is denoted by $\mathrm{Lat}(A)$.

Remark 1.1.2.1. An operator A is invariant for U if we can write

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

in the decomposition $V = U \oplus W$.

Remark 1.1.2.2. If $U \in \mathrm{Lat}(A)$, then $p_{A|_U} \mid p_A$.

Definition 1.1.3. Let $q \in \mathbb{N}$ be the minimal integer such that[1]

$$\ker(A - \lambda)^q = \ker(A - \lambda)^{q+1}.$$

The subspace $N_\lambda = \ker(A - \lambda)^q$ is called the *root subspace* of A.

Definition 1.1.4. A subspace $U \leq V$ is a *cyclic subspace* for $A\colon V \to V$ if

$$U = \mathrm{span}\left\{A^n x \mid 0 \leq n \leq q\right\}$$

for some $x \in V$.

Definition 1.1.5. Let X be a Banach space. A linear operator $A\colon X \to X$ is *bounded* if the set

$$\left\{ \frac{\|Ax\|}{\|x\|} \;\middle|\; x \in X \setminus \{0\} \right\}$$

is bounded.

[1] Such a q exists as V is finite-dimensional.

1.2 Compact operators on Banach spaces

Definition 1.2.1. Let X and Y be Banach spaces. A linear operator $T\colon X \to Y$ is *compact* if T maps bounded sets in X into pre-compact sets in Y. Equivalently, the set $T(B_X)$ is pre-compact.

Proposition 1.2.2. Let $k \in \mathcal{C}([0,1]^2)$ be a continuous function. Then the integral operator

$$(Kf)(x) = \int_0^1 k(x,y)f(y)\,dy$$

is a compact operator on $(\mathcal{C}([0,1]), \|\cdot\|_2)$ and $(\mathcal{C}([0,1]), \|\cdot\|_\infty)$.

Proof. Introduction to functional analysis, proposition 5.4.9. □

Definition 1.2.3. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is *weakly convergent* with limit x if

$$\lim_{n\to\infty} f(x_n) = f(x)$$

for all functionals $f \in X^*$.

Definition 1.2.4. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is *normally convergent* with limit x if

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Theorem 1.2.5. A compact operator maps weakly convergent sequences into normal convergent sequences.

Proof. Let $T\colon X \to Y$ be a compact operator and let $(x_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence with limit x. By the uniform boundedness principle,[2] the sequence $(\|x_n\|)_{n\in\mathbb{N}}$ is bounded. But then for every functional $f \in Y^*$ it holds that

$$f(Tx_n) - f(Tx) = (T^*f)(x_n - x),$$

hence $(Tx_n)_{n\in\mathbb{N}}$ is weakly convergent. Suppose that it is not normally convergent – that is, there exists some $\varepsilon > 0$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|Tx_{n_k} - Tx\| \geq \varepsilon$$

holds for all k. As T is compact, this subsequence has an accumulation point. The only possible accumulation point is clearly Tx. □

Remark 1.2.5.1. If T is a bounded operator on a reflexive Banach space X, the converse holds as well.

Theorem 1.2.6. If $(T_n)_{n\in\mathbb{N}}$ is a sequence of compact operator with bounded limit T, then T is compact.

Proof. Introduction to functional analysis, theorem 5.4.4. □

Theorem 1.2.7. Every compact operator on a Hilbert space is a normal limit of finite rank operators.

Proof. Introduction to functional analysis, theorem 5.4.10. □

[2] Introduction to functional analysis, theorem 3.3.5.

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