Graph theory

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February 4, 2025

Contents

Introduction

These are my lecture notes on the course Graph theory in the year 2023/24. The lecturer that year was izr. prof. PhD Csilla Bujtás.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

Common notation

Unless otherwise specified, we use the following notation:

G	Graph G with vertices V and edges E
$n(G)$	Number of vertices in G, $n(G) = \|V\|$ (if not ambiguous also just n)
$m(G)$	Number of edges in G, $m(G) = \|E\|$ (if not ambiguous also just m)
$G[S]$	Subgraph of G with vertices in S
$G[E']$	Subgraph of G with edges in E', $G[E'] = (\bigcup E', E')$
$\delta(G)$	Minimal degree in G, $\delta(G) = \min\limits_{v \in V} \deg(v)$
$\Delta(G)$	Maximal degree in G, $\Delta(G) = \max\limits_{v \in V} \deg(v)$
$N(S)$	Neighbouring vertices of $S \subseteq V$

1 Matchings

1.1 Independence, matchings and covers

Definition 1.1.1. A set $S \subseteq V$ is *independent* if $G[S]$ contains no edges. We denote the maximal cardinality of an independent set, the independence number, by $\alpha(G)$.

Definition 1.1.2. A set $T \subseteq V$ is a *vertex cover* if it contains a vertex of each edge. We denote the minimal cardinality of a vertex cover, the vertex cover number, by $\beta(G)$.

Proposition 1.1.3. The equality in $\alpha(G) + \beta(G) = n$ holds.

Proof. The complement of an independent set is a vertex cover and vice-versa. □

Definition 1.1.4. A set $M \subseteq E$ is a *matching* if no two of its edges contain the same vertex. We denote the maximal cardinality of a matching, the matching number, by $\alpha'(G)$.

Definition 1.1.5. A set $C \subseteq E$ is an *edge cover* if $\bigcup C = V$. If $\delta(G) \geq 1$, we denote the minimal cardinality of an edge cover, the edge cover number, by $\beta'(G)$.

Proposition 1.1.6. We have $\alpha'(G) \leq \beta(G)$.

Proof. We must choose a vertex from each edge of a matching to get a vertex cover. □

Proposition 1.1.7. We have $\alpha(G) \leq \beta'(G)$.

Proof. Every edge of an edge cover contains at most one vertex of an independent set. □

Proposition 1.1.8. We have $\alpha'(G) \leq \frac{n}{2} \leq \beta'(G)$.

Proof. The proof is obvious and need not be mentioned. □

Theorem 1.1.9 (Gallai)**.** If $\delta(G) \geq 1$, then $\alpha'(G) + \beta'(G) = n$.

Proof. Take a maximum matching M on G and let S be its vertex set. We can construct an edge cover from M by adding an edge for each missing vertex, resulting in x new vertices. Then

$$\alpha'(G) + \beta'(G) \leq |M| + x \leq 2 \cdot |M| + |S^{\mathsf{c}}| = n.$$

Now let C be a minimum edge cover. Note that each of its edges covers a vertex that is not covered by any other edge in C. That is, the graph $G[S]$ is a forest of k stars. To construct a matching, we can choose an arbitrary edge of each star, which gives

$$\alpha'(G) + \beta'(G) \geq k + (n - k) = n.$$

□

Definition 1.1.10. Let M be a matching. A path is an *M-alternating path* if its edges alternate between M and M^{c}.

Definition 1.1.11. An M-alternating path is called *M-augmenting* if its ends are not covered by M.

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Proposition 1.1.12. Maximum matchings do not contain M-augmenting paths.

Proof. We can construct a larger matching $M' = M \oplus P$, where P is an M-augmenting path. □

Theorem 1.1.13 (König)**.** Let G be a bipartite graph. Then $\alpha'(G) = \beta(G)$. If M is a matching in G that contains no M-augmenting path, then it is a maximum matching.

Proof. Let M be a matching such that no M-augmenting path exists in G, and let A and B be the parts of G. Denote $X = A \setminus V(M)$ and $Y = B \setminus V(M)$. Now let A_1 and B_1 be the set of vertices in A and B respectively that can be reached via an M-alternating path from X. Furthermore, let $A_2 = A \setminus (A_1 \cup X)$ and $B_2 = B \setminus (B_1 \cup Y)$. Then $A_2 \cup B_1$ is a vertex cover, as there are no edges in the pairs (X, Y), (X, B_2), (A_1, B_2) and (A_1, Y). We constructed a vertex cover of the same cardinality as M, hence M must be a maximum matching and $\alpha'(G) = \beta(G)$. □

Corollary 1.1.13.1. If G is a bipartite graph, then $\alpha(G) = \beta'(G)$.

Proof. We have
$$\alpha(G) = n - \beta(G) = n - \alpha'(G) = \beta'(G).$$
□

Theorem 1.1.14 (Hall)**.** Let G be a bipartite graph with parts A and B. Then the equality $\alpha'(G) = |A|$ holds if and only if $|S| \leq |N(S)|$ for all $S \subseteq A$.

Proof. The first implication is evident. Suppose now that $\alpha'(G) \neq |A|$ and take a maximum matching M in G. Using the notation from König's theorem, let $S = A_1 \cup X$. Then $N(S) = B_1$, therefore
$$|N(S)| = |B_1| = |A_1| < |S|\,.$$
□

Definition 1.1.15. A matching M is *perfect* if it covers all vertices.

Corollary 1.1.15.1. In a bipartite graph G a perfect matching exists if and only if $|A| = |B|$ and Hall's condition holds.

Definition 1.1.16. Let $S \subseteq A$ in a bipartite graph. The *deficiency* of S is defined as
$$\operatorname{def}(S) = |S| - |N(S)|\,.$$

Theorem 1.1.17. In a bipartite graph G, we have
$$\alpha'(G) = |A| - \max_{S \subseteq A} (\operatorname{def}(S))\,.$$

Theorem 1.1.18. If G is a regular bipartite graph, it has a perfect matching.

Proof. The proof is obvious and need not be mentioned. □

Theorem 1.1.19. Suppose M is a matching in G. Then there exists an M-augmenting path in G if and only if M is not a maximum matching.

Proof. One implication is precisely proposition 1.1.12. Suppose now that M is a non-maximum matching. That is, there exists a matching M' with $|M'| > |M|$. Consider $G' = G[M \oplus M']$. The maximal degree in G' is clearly at most 2, hence every component is either a path or a cycle. As we have no odd cycles, by $|M'| > |M|$ there exists an odd-length path in G' with both extreme edges are in M', which that is an M-augmenting path. $\square$

Remark 1.1.19.1. Maximum matchings can be found in polynomial time.

Theorem 1.1.20 (Tutte)**.** Denote by $\sigma(G)$ the number of odd components in G. A graph G has a perfect matching if and only if the inequality

$$|S| \geq \sigma(G[V \setminus S])$$

holds for every $S \subseteq V$.

Proof. Suppose G has a perfect matching. Then every odd component of $G[V \setminus S]$ is matched to a distinct vertex in S, hence Tutte's condition holds.

Now suppose that Tutte's condition holds for G. Note that this implies that $2 \mid n$, as we can take $S = \emptyset$. Furthermore, suppose that G is a maximal counterexample, that is, adding any edge to G produces a graph that either breaks Tutte's condition or contains a perfect matching. We can check that the former is actually impossible, as adding edges can only decrease the number $\sigma(G[V \setminus S])$.

Denote $U = \{x \in V \mid \deg(x) = n - 1\}$. Clearly, $G[U]$ is a complete graph, and hence $U \neq V$. We consider two cases:

i) Every component H of $G[U^\mathsf{c}]$ induces a complete graph. In this case, just take a maximum matching of each component and match the last remaining vertex in odd components with vertices in U. This can clearly be done by Tutte's condition.

ii) Some component H of $G[U^\mathsf{c}]$ is not complete. Take $x, y \in H$ with $d(x, y) = 2$, and let $xz, yz \in E$. As $z \notin U$, there exists some vertex $w \in V$ such that $zw \notin E$. Consider the graphs G_1 and G_2 that we get by adding edges xy and zw to G, respectively. By our assumption they have perfect matchings M_1 and M_2. Clearly they contain xy and zw respectively.

   Now consider $M_1 \oplus M_2$. As every vertex has degree 0 or 2, the graph $G' = G[M_1 \oplus M_2]$ splits into isolated vertices and cycles. Clearly, the cycles have even length. If xy and zw belong to different cycles, we can just switch the edges of M_1 in the cycle containing xy, which produces a perfect matching in G.

   Now suppose that the same cycle contains both xy and zw. We choose the edge xz or yz, such that the cycle splits into two even components. We can clearly produce a perfect matching in both components. By adding the edges of M_1 from every other component, we have in fact constructed a perfect matching. $\square$

Theorem 1.1.21 (Berge-Tutte formula)**.** The maximum matching leaves exactly

$$\max_{S \subseteq V} (\sigma(G[S^{\mathsf{c}}]) - |S|)$$

vertices uncovered.

Definition 1.1.22. A *factor* of a graph is a spanning subgraph. A k-factor is a k-regular spanning subgraph.

Remark 1.1.22.1. A 1-factor is just a perfect matching.

Theorem 1.1.23 (Peterson)**.** Every bridgeless cubic[1] graph has a perfect matching.

Proof. We will prove that Tutte's condition holds for every set $S \subseteq V$. Denote by $E(S, S^{\mathsf{c}})$ the edges between S and S^{c}. Clearly, $|E(S, S^{\mathsf{c}})| \leq 3|S|$. By the handshake lemma, we can see that every odd component H of $G[S^{\mathsf{c}}]$ is connected to S by an odd number of edges. As the graph is bridgeless, we can infer that $|E(V(H), S)| \geq 3$. Therefore

$$3|S| \geq |E(S, S^{\mathsf{c}})| \geq 3\sigma(G[S^{\mathsf{c}}]). \qquad \square$$

Theorem 1.1.24. If G is a cubic graph with at most one bridge, then G has a perfect matching.

Proof. Repeating the proof of Peterson's theorem, we find that

$$3|S| \geq |E(S, S^{\mathsf{c}})| \geq 3\sigma(G[S^{\mathsf{c}}]) - 2. \qquad \square$$

Theorem 1.1.25. If G is a cubic graph and all cut edges lie on the same path, then G has a perfect matching.

Theorem 1.1.26. If G is a k-regular graph and k is even, then G splits into 2-factors.

Proof. It suffices to find one 2-factor and proceed by induction. It is clearly enough to consider connected graphs. By Euler's theorem there exists an Eulerian circuit C in G, which induces a directed graph. Define a bipartite graph F_G by taking $A = \{a_i \mid i \leq n\}$, $B = \{b_i \mid i \leq n\}$, and take $a_i b_j$ as an edge in F_G if $v_i v_j \in E\left(\vec{G}\right)$, where v_k are vertices in G. This is a regular bipartite graph. Its perfect matching coincides with a 2-factor of G. $\square$

[1] 3-regular.

2 Connectivity

2.1 Connectivity number

Definition 2.1.1. The *connectivity number* $\kappa(G)$ is the minimum number of vertices such that we get either a disconnected graph or one vertex upon removing them. We say that G is *k-connected* if $\kappa(G) \geq k$.

Remark 2.1.1.1. We see that $\kappa(G) \leq \delta(G)$.

Remark 2.1.1.2. As an independent set is always disconnected (or just one vertex), we see that

$$\kappa(G) \leq n - \alpha(G) = \beta(G).$$

Theorem 2.1.2. The minimal number of edges in a k-connected graph of order n is $\left\lceil \frac{nk}{2} \right\rceil$.

Proof. We see that $k \leq \kappa(G) \leq \delta(G)$, hence

$$m(G) = \frac{1}{2} \sum_{v \in V} \deg(v) \geq \frac{nk}{2}.$$

It remains to show that the bound $\left\lceil \frac{nk}{2} \right\rceil$ is achievable. We consider the following graphs:

i) If k is even, take $H_{n,k} = C_n^{\frac{k}{2}}$.[2]

ii) If k is odd and n is even, take $H_{n,k}$ to be $C_n^{\frac{k-1}{2}}$ with additional edges between every pair of diametrically opposite vertices.

iii) If both n and k are odd, take $H_{n,k}$ to be $C_n^{\frac{k-1}{2}}$ with additional edges between v_i and $v_{i+\frac{n-1}{2}}$ for $i \leq \frac{n+1}{2}$.

It is clear that $m(H_{n,k}) = \left\lceil \frac{nk}{2} \right\rceil$. Next, we prove that each of these graphs is k-connected. Consider the graph $H_{n,k}$ with $k-1$ vertices removed.

i) Note that we can always go from one vertex to the next one left in the cycle, unless we removed $\frac{k}{2}$ consecutive vertices. But that can only happen once in the whole cycle, meaning we can just take the other way around.

ii) We can again try to go to the next vertex in the cycle. To have two breaks in the cycle, all $k-1$ removed vertices must be in the breaks. But the two components are still connected by a diameter.

iii) Same as the previous case. □

[2] Here G^k is the graph with the same vertices as G, and $xy \in V(G^k)$ if and only if $d(x,y) \leq k$ in G.

2.2 Edge connectivity

Definition 2.2.1. A set $F \subseteq E$ is a *disconnecting set* if $G \setminus F$ is disconnected.

Definition 2.2.2. Let $A \subseteq V$. The set of edges $E(A, A^{\mathsf{c}})$ is called an *edge cut*.

Remark 2.2.2.1. Every nontrivial edge cut is a disconnecting set. Every minimal disconnecting set is an edge cut.

Definition 2.2.3. The *edge connectivity number* of G is the minimum number of edges in a disconnecting set in an edge cut. We denote it by $\kappa'(G)$.

Definition 2.2.4. A graph G is *k-edge-connected* if the removal of less than k edges results in a connected graph. Equivalently, $k \leq \kappa'(G)$.

Theorem 2.2.5. Let G be a simple graph with $n \geq 2$ with $n \geq 2$. Then

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Proof. The second inequality results from the edge cut with $A = \{v\}$, where v is a vertex of minimal degree.

Let $F \subseteq E(G)$ be an edge cut in G with minimal cardinality, that is $|F| = \kappa'(G)$. We consider two cases:

i) If F forms a complete bipartite graph, then

$$\kappa'(G) = |A| \cdot |A^{\mathsf{c}}| = |A| \cdot (n - |A|) \geq n - 1 \geq \kappa(G).$$

ii) If F does not form a complete bipartite graph, consider vertices $x \in A$ and $y \in A^{\mathsf{c}}$ with $xy \notin E$. For each edge in F, choose an endpoint that is different from x and y. They clearly form a vertex cut of cardinality at most $|F|$, hence $\kappa(G) \leq \kappa'(G)$. □

Corollary 2.2.5.1. The minimal number of edges in a k-edge-connected graph on n vertices is $\left\lceil \frac{kn}{2} \right\rceil$ when $n > k \geq 2$.

Proof. Note that $k \leq \kappa'(G) \leq \delta(G)$. By the handshake lemma, we find that $m(G) \geq \frac{nk}{2}$. As $H_{n,k}$ is k-connected, it is also k-edge-connected. □

2.3 2-connected graphs

Theorem 2.3.1 (Whitney)**.** If G is a 2-connected graph, then for every distinct vertices $u, v \in G$ there exist two internally disjoint uv-paths.

Proof. Suppose the statement is false. Take a counterexample with minimal $k = d(u, v)$. Note that $k \geq 2$, as otherwise G is not 2-edge-connected, and hence is not 2-connected.

Let w be a vertex with $d(u, w) = k - 1$ and $d(v, w) = 1$. Note that there exists a uv-path P not containing w since G is 2-connected. Now consider two uw-paths, which exist by minimality. If v is in this cycle, we trivially get two uv-paths. Otherwise, we get three uv-paths. To get two disjoint paths, travel along P until the first intersection with one of the other paths, then switch to that one. □

Lemma 2.3.2 (Expansion)**.** Let G be a k-connected graph. If we construct a graph by adding a new vertex and connecting it to at least k vertices of G, the resulting graph is again k-connected.

Proof. The proof is obvious and need not be mentioned. □

Theorem 2.3.3. If G is a graph with $n \geq 3$, the following statements are equivalent:

i) The graph G is 2-connected.

ii) The graph G is connected with no cut-vertex.

iii) For every vertices u and v there exist at least two internally vertex-disjoint paths between them.

iv) There exists a cycle through any two vertices.

v) There exists a cycle through any two edges and $\delta(G) \geq 1$.

Proof. Using Whitney's theorem, we see that the first four statements are clearly equivalent. Suppose now that G is 2-connected consider two distinct edges e and f. Expand G by adding new vertices w and w', where w is connected to the vertices of e and w' is connected to the vertices of f. By the expansion lemma, there exists a cycle through w and w', which induces the sought cycle in G. If $e = f$, take another edge e'. By the above argument, there exists a cycle through e and e', which is the required cycle.

Suppose now that the last condition holds. In particular, G has no isolated edges. For any vertices $u, v \in G$, we can therefore take distinct edges $e, f \in E$ with $u \in e$ and $v \in f$. Since any cycle through e and f is also a cycle through u and v, G is 2-connected. □

Lemma 2.3.4 (Subdivision)**.** Let G' be a graph from G that is obtained by subdividing an edge with a vertex. Then G' is 2-connected if and only if G is 2-connected.

Proof. For any two edges in G', take the corresponding edges in G (instead of taking subdivisions, take the whole edge). Cycles in G' containing these two edges correspond precisely with cycles in G containing the corresponding edges. □

2.4 Ear decomposition of graphs

Definition 2.4.1. In a graph G, a path P is an *open ear* if all internal vertices of P are of degree 2, while the endpoints have degree at least 3 in G.

Definition 2.4.2. An *open ear decomposition* of G is a sequence $P_0, P_1, \ldots, P_k$, where P_0 is a cycle in G and P_i is an ear for $i > 0$ in the graph

$$G_i = \bigcup_{j=0}^{i} P_j$$

and $G_k = G$. Furthermore, we require that P_i be edge-disjoint.[3]

Theorem 2.4.3. A graph G is 2-connected if and only if it admits an ear decomposition.

Proof. Suppose that G has an ear decomposition. By induction, we can prove that G_i is 2-connected, as we can apply the expansion and subdivision[4] lemmas.

Now suppose that G is 2-connected. Set P_0 to be an arbitrary cycle in G. If G_i is not an induced graph of G, let P_{i+1} be a missing edge. Otherwise, choose a vertex u not in G_i. Take two edges, one in G_i and one with vertex u. These lie in a cycle, which includes an ear containing u, which is our P_{i+1}. As we cover some edges on each step, the process is finite. □

Proposition 2.4.4. A graph G is 2-edge-connected if and only if it is connected and every edge of G lies in a cycle.

Proof. The proof is obvious and need not be mentioned. □

Definition 2.4.5. In a graph G, a cycle P is a *closed ear* if all but one vertex of P are of degree 2, while the last one has degree at least 4 in G.

Definition 2.4.6. A *closed ear decomposition* of G is a sequence $P_0, P_1, \ldots, P_k$, where P_0 is a cycle in G and P_i is an open or closed ear for $i > 0$ in the graph

$$G_i = \bigcup_{j=0}^{i} P_j$$

and $G_k = G$. Furthermore, we require that P_i be edge-disjoint.

Theorem 2.4.7. A graph G is 2-edge-connected if and only if it has a closed ear decomposition.

Proof. Analogous as theorem 2.4.3. □

Definition 2.4.8. A directed graph $\vec{G}$ is *strongly connected* if for every $u, v \in V\left(\vec{G}\right)$ there exists a directed path from u to v. A *strong orientation* of a graph G is a directed graph $\vec{G}$ which is strongly connected.

Theorem 2.4.9 (Robbin)**.** A graph G has a strong orientation if and only if it is 2-edge connected.

[3] Not stated in the lectures, but removes the edge case where $P_k = P_{k+1}$, which sounds annoying.
[4] Possibly the converse!

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Proof. If G has a strong orientation, it is clearly connected and every edge lies in a cycle. Now suppose that G is 2-edge connected. Let $P_0, P_1, \ldots, P_k$ be a closed ear decomposition of G. Direct the edges of the cycle in a cycle and along each ear in a path. It is clear that the resulting orientation is strong. $\square$

2.5 Minimal cuts

Definition 2.5.1. Let $x, y \in V$ be non-adjacent vertices in G. A set $S \subseteq V$ is an *x, y-cut* if x and y belong to different components of $G \setminus S$. We denote the minimum size of an x, y-cut by $\kappa_G(x, y)$.

Definition 2.5.2. For $x, y \in V$, we denote by $\lambda_G(x, y)$ the maximal number of pairwise internally vertex-disjoint x, y-path.

Theorem 2.5.3 (Menger)**.** Suppose that x and y are non-adjacent vertices in G. Then $\kappa_G(x, y) = \lambda_G(x, y)$.

Proof. For convenience, we denote the above numbers by κ and λ respectively. Clearly $\kappa \geq \lambda$, as we need to select at least one vertex from each disjoint x, y-paths to disconnect them.

To prove the reverse inequality, we induct on n. For $n = 2$, we clearly have $\kappa = \lambda = 0$.

Suppose now that $n \geq 3$ and consider two cases:

i) There exists a minimum x, y-cut S such that $S \neq N(x)$ and $S \neq N(y)$. Let V_x denote the set of vertices that can be reached from x by a path with no internal vertices from S, and define V_y analogously. By the minimality of S, we find that $S = V_x \cap V_y$.

   Let G_x be the graph obtained from $G[V_1]$ by adding a vertex y' that is adjacent to precisely the vertices in S. Note that, as $S \neq N(y)$, the number of vertices decreased, and that S is a minimum x, y'-cut in G_x. It follows that $\kappa = \kappa_G(x, y) = \kappa_{G_x}(x, y') = \lambda_{G_x}(x, y')$ by the induction hypothesis. Analogously, $\kappa = \lambda_{G_y}(x', y)$. By pairing up the x, y'-paths with x', y paths according to the visited vertex in S, we obtain κ internally vertex-disjoint x, y-paths, hence $\lambda \geq \kappa$.

ii) The only minimum x, y-cuts are $N(x)$ and/or $N(y)$. If x and y have a neighbour z in common, we can remove it from G and apply the induction hypothesis. Note that removing z reduced both the number of x, y-paths and the minimum size of an x, y-cut by 1, hence equality holds for G as well.

   Suppose then that $N(x)$ and $N(y)$ are disjoint. If $N(x) \cup N(y) \cup \{x, y\} = V$, we can construct a bipartite graph H with sets $N(x)$ and $N(y)$ (we disregard internal edges in both $N(x)$ and $N(y)$). The number of internally vertex-disjoint x, y-paths is clearly equal to the size of the maximum matching in H. Without loss of generality suppose that $N(x)$ is a minimum x, y-cut. Note that for every $A \subseteq N(x)$, we have $|N_H(A)| \geq |A|$, as otherwise we could obtain a smaller x, y-cut by replacing the vertices in A with those in $N_H(A)$. By Hall's theorem, there exists a perfect matching, hence $\lambda = \alpha'(H) = N(x) = \kappa$.

   Finally, if there exists a vertex $v \neq x, y$ with $v \notin N(x) \cup N(y)$, then v does not belong to any minimum x, y-cuts, therefore $\kappa_{G \setminus v}(x, y) = \kappa$. Applying the induction hypothesis, we can find κ internally vertex-disjoint x, y-paths in $G \setminus v$. Since these are also valid in G, we conclude $\lambda \geq \kappa$. $\square$

Definition 2.5.4. Let $x, y \in V$ be vertices in G. A set $R \subseteq E$ is an *x, y-edge cut* if x and y belong to different components of $G \setminus R$.

Definition 2.5.5. For $x, y \in V$, we denote by $\kappa'_G(x, y)$ the minimal cardinality of an x, y-edge cut in G, and by $\lambda'_G(x, y)$ the maximal number of edge-disjoint x, y-path.

Definition 2.5.6. The *line graph* $L(G)$ of a graph G has vertices representing the edges of G. Two vertices in $L(G)$ are connected if and only if they share a vertex in G.

Theorem 2.5.7 (Menger)**.** For every $x, y \in V$ we have $\kappa'_G(x, y) = \lambda'_G(x, y)$.

Proof. Define a new graph G' by adding vertices u and v to G, which are connected to x and y respectively, and consider its line graph. Note that any path between the new edges in $L(G')$ corresponds to a path between x and y in G. In particular, vertex-disjoint paths in $L(G')$ correspond to edge-disjoint paths in G. Hence

$$\lambda_{L(G')}(xu, yv) = \lambda'_G(x, y).$$

By Menger's theorem, we know that

$$\lambda_{L(G')}(xu, yv) = \kappa_{L(G')}(xu, yv).$$

Finally, by definition of a line graph, a vertex cut in $L(G')$ that separates xu and yv corresponds to an edge cut in G that separates x and y, hence

$$\kappa_{L(G')}(xu, yv) = \kappa_G(x, y).$$ □

Lemma 2.5.8. For each edge $e \in E$, we have

$$\kappa(G) - 1 \leq \kappa\left(G \setminus \{e\}\right) \leq \kappa(G).$$

Proof. The second inequality follows from the fact that each vertex cut in G is also a vertex cut in $G \setminus \{e\}$.

Suppose that $\kappa\left(G \setminus \{e\}\right) < \kappa(G)$. Let S be a minimum vertex cut in $G' = G \setminus \{e\}$. If any of vertices x and y has degree at least two, we can add it to S to get a vertex cut in G. Otherwise, we find that $|S| = n - 2$, hence $S \cup \{x\}$ is a vertex cut in G. □

Theorem 2.5.9 (Menger)**.** In any graph G with at least two vertices, the following statements hold:

i) We have $\kappa'(G) = \min\limits_{x \neq y} \lambda'_G(x, y)$.

ii) We have $\kappa(G) = \min\limits_{x \neq y} \lambda_G(x, y)$.

Proof. The only non-trivial part is showing that we can take the minimum over all $x \neq y$ in ii), not just non-adjacent ones.[5] It suffices to show that for every adjacent vertices x and y we have $\lambda_G(x, y) \geq \kappa(G)$. Denote $G' = G \setminus \{xy\}$ and note that

$$\lambda_G(x, y) = \lambda_{G'}(x, y) + 1.$$

Applying Menger's theorem and the above lemma, we get

$$\lambda_G(x, y) = \kappa_{G'}(x, y) + 1 \geq \kappa(G).$$ □

[5] Note that the equality clearly holds for complete graphs.

3 Colourings

3.1 Vertex colourings

Definition 3.1.1. Let G be a simple graph. A *k-colouring* of G is a map $\varphi\colon V \to [k]$ such that for all $xy \in E$ we have $\varphi(x) \neq \varphi(y)$. The graph G is *k-colourable*[6] if it has a k-colouring.

Definition 3.1.2. The *chromatic number* $\chi(G)$ is the smallest integer k such that G is k-colourable.

Definition 3.1.3. Denote by $\omega(G)$ the order of the largest clique in G.

Proposition 3.1.4. In a graph G, the inequality

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1$$

holds.

Proof. The proof is obvious and need not be mentioned. □

Proposition 3.1.5. In a graph G, we have

$$\frac{n}{\alpha(G)} \leq \chi(G).$$

Proof. As every colour class is an independent set, it contains at most $\alpha(G)$ vertices. □

Theorem 3.1.6 (Welsh-Powell)**.** If $d_1 \geq d_2 \geq \cdots \geq d_n$ are the degrees of vertices in G, then

$$\chi(G) \leq 1 + \max_{i \leq n} \left(\min(d_i, i-1)\right).$$

Proof. Colour the vertices in sequence $v_1, v_2, \dots, v_n$, always using the smallest possible number at each step. □

Proposition 3.1.7. We have the following characterisations:

i) For a graph G, $\chi(G) = 1$ if and only if $E = \emptyset$.

ii) For a graph G, $\chi(G) = 2$ if and only if G is bipartite and $E \neq \emptyset$.

Proof. The proof is obvious and need not be mentioned. □

Theorem 3.1.8 (Brooks)**.** Suppose G is a connected graph that is neither a complete graph nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Proof. Denote $\Delta(G) = k$. Suppose first that G is not regular and let $\deg(v) \leq k-1$. Colour the vertices greedily in decreasing order of distance from v. As we coloured at most $k-1$ neighbours of a vertex in each step, it is clear we need at most k colours.

Now consider regular graphs. For $k \leq 2$, excluding the special cases, the inequality clearly holds. Now assume $k \geq 3$. We consider three cases:

[6] Also k-partite.

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i) We have $\kappa(G) = 1$, that is, there exists a cut vertex x that splits G into parts V_1 and V_2. Denote $G_i = G[V_i \cup \{x\}]$. Note that G_i is not k-regular, as $\deg_{G_i}(x) \leq k - 1$. We can colour both V_1 and V_2 with k colours. By joining them at x, we find a k-colouring of G.

ii) Suppose that $G \setminus \{x, y\}$ is disconnected. Again, denote $G_i = G[V_i \cup \{x, y\}]$ and note that G_i is not k-regular. As above, we colour both graphs and attempt to join the colourings. This is not possible only when every colouring of G_1 assigns the same colour to both vertices, while every colouring of G_2 assigns different colours to them, or vice-versa. In particular, $G_1 + xy$ is not k-colourable. We deduce that $\Delta(G_1 + xy) = k$, hence it is a k-regular graph. But then $\deg_{G_2}(x) = \deg_{G_2}(y) = 1$. As $k \geq 3$, we can colour x and y with the same colour in G_2, hence this case is not possible.

iii) Finally, consider $\kappa(G) \geq 3$. As $G \neq K_n$, we can find vertices x and y such that $d(x, y) = 2$. Let $z \in N(x) \cap N(y)$. Note that $G \setminus \{x, y\}$ is connected, hence there exists a path which contains neither from z to any other vertex. We proceed to colour vertices greedily. First, colour x and y with the same colour. Then proceed to colour the vertices in decreasing order of distance from z in $G \setminus x, y$. As we coloured at most $k - 1$ neighbours of a vertex in each step, we need at most k colours for every vertex. The exception is z, but two of its neighbours are already coloured with the same colour. □

Definition 3.1.9. A *Mycielski construction* of a graph G with $V = \{v_i \mid i \leq n\}$ is a graph $M(G)$ with

$$V(M(G)) = V \cup \{u_i \mid i \leq n\} \cup \{z\}$$

and

$$E(M(G)) = E \cup \{u_i v_j \mid v_i v_j \in E\} \cup \{z u_i \mid i \leq n\}.$$

Theorem 3.1.10. If G is a graph with at least one edge, then $\chi(M(G)) = \chi(G) + 1$ and $\omega(M(G)) = \omega(G)$.

Proof. Let $\chi(G) = k$. Note first that $M(G)$ is $(k+1)$-colourable. Indeed, we can copy a k-colouring of G to both v_i and u_i, then colour z with a new colour. It is clear that this satisfies the conditions of a colouring.

Suppose that $\chi(M(G)) \leq k$ and consider a k-colouring φ of $M(G)$ where $\varphi(z) = k$. Denote $S = \{v_i \mid \varphi(v_i) = k\}$. We can define a new colouring on G as

$$\psi(v_i) = \begin{cases} \varphi(u_i), & v_i \in S, \\ \varphi(v_i), & v_i \notin S. \end{cases}$$

It is easy to see that this is a $(k-1)$-colouring of G, as $\varphi(u_i) \neq k$ for all i. This is a contradiction as $\chi(G) = k$, hence $\chi(M(G)) = k + 1$.

As G is a subgraph of $M(G)$, we obviously have $\omega(G) \leq \omega(M(G))$. If z is in a clique of $M(G)$, then it has order at most 2. Otherwise, if it contains a vertex u_i, it contains neither v_i nor any other vertex u_j. By replacing u_i with v_i, we preserve the clique. For every clique in $M(G)$, we found a corresponding clique in G of the same size, hence $\omega(M(G)) \leq \omega(G)$. □

Theorem 3.1.11. If G is a graph with $\chi(G) = k$, then $m(G) \geq \binom{k}{2}$.

Proof. There is at least one edge between each pair of colours. $\square$

3.2 Turan's theorem and chordal graphs

Definition 3.2.1. A graph G is a *complete k-partite graph* if all pairs of vertices from different colour classes are connected. We denote it by $K_{n_1,\dots,n_k}$, where n_i are sizes of the partite classes.

Definition 3.2.2. The *Turan graph* $T_{n,k}$ is the complete k-partite graph on n vertices such that each of the partite classes is of size $\left\lfloor \frac{n}{k} \right\rfloor$ of $\left\lceil \frac{n}{k} \right\rceil$.

Theorem 3.2.3 (Turan)**.** If G is a graph of order n with $\omega(G) \leq r$, then

$$m \leq m(T_{n,r}).$$

Proof. We induct on r. If $r = 1$, the inequality clearly holds. Now suppose $r \geq 2$ and denote $\Delta(G) = k$. In particular, let $\deg(v) = k$.

Now let $G' = G[N(v)]$. We can see that $\omega(G') \leq r - 1$. By the induction hypothesis, there are at most $m(T_{k,r-1})$ edges in G'.

Construct another graph H as follows – add $n - k$ vertices to $T_{k,r-1}$ and connect each of these vertices to each vertex in $T_{k,r-1}$. Note that $n(H) = n$ and $\omega(H) = r$. Observe that

$$m \leq \sum_{v \notin G'} \deg(v) + m(G') \leq (n-k) \cdot k + m(T_{k,r-1}) = m(H).$$

Furthermore, H is a complete r-partite graph, hence

$$E(H) = \sum_{i \neq j} |V_i| \cdot |V_j| = \frac{1}{2} \cdot \left(n^2 - \sum_{i=1}^{r} |V_i|^2 \right).$$

By Karamata's inequality, this is greatest when H is a Turan graph.[7] □

Remark 3.2.3.1. The bound is sharp with equality if and only if $G \cong T_{n,r}$.

Corollary 3.2.3.2. If G is a graph of order n with $\chi(G) = r$, then

$$m \leq m(T_{n,r}).$$

The equality holds if and only if $G \cong T_{n,r}$.

Proof. The proof is obvious and need not be mentioned. □

Definition 3.2.4. Denote by $\operatorname{ex}(n, F)$ the maximum number of edges in a graph G with $n(G) = n$ such that G does not contain F as a subgraph.

Definition 3.2.5. A graph G is a *chordal graph* if it has no induced subgraph that is isomorphic to a cycle C_k with $k \geq 4$.

Definition 3.2.6. A vertex v is a *simplicial vertex* in G if $N(v)$ is a clique.

Definition 3.2.7. A *simplicial elimination ordering* in G is an order $v_1, \dots, v_n$ of vertices such that $N(v_i) \cap \{v_j \mid j \geq i\}$ induces a clique.

[7] For non-Turan graphs, the number of edges increases upon moving a vertex from a class V_i into a class V_j if $|V_i| \geq |V_j| + 2$.

Theorem 3.2.8 (Voloshin's lemma)**.** If G is a chordal graph, then for every $x \in V(G)$ there exists a simplicial vertex among the ones farthest from x.

Proof. We induct on n. For $n = 1$, the statement trivially holds. For $n \geq 2$, consider an arbitrary vertex x. If x is a universal vertex in G, apply the induction hypothesis to $G \setminus x$. Otherwise, let T be the set of vertices farthest from x. Denote by H a component of $G[T]$ and let $S = N(H) \setminus H$. Finally, let Q be the component of $G \setminus S$ containing x.

We claim that S induces a clique. Let $u, v \in S$ be distinct. Each vertex in S clearly has neighbours both in H and in Q. Since both H and Q induce connected subgraphs, we can find u, v-paths with internal vertices in H and one with internal vertices in Q. Consider shortest such paths. These paths form a cycle of order $k \geq 4$, hence it also contains a chord. But by the above conditions, the only possible chord is uv. Applying this argument to all possible pairs of vertices in S, we conclude that S is a clique.

We now apply the induction hypothesis to $G[S \cup H]$. If this graph is a clique, then every vertex in H is simplicial. Otherwise, take a vertex $u \in S$ such that $H \not\subseteq N(u)$. Then the induction hypothesis supplies us with a simplicial vertex in H. Since any simplicial vertex in $G[S \cup H]$ is also simplicial in G, we found a simplicial vertex in T. $\square$

Theorem 3.2.9. A graph G is chordal if and only if there exists a simplicial elimination ordering of the vertices in G.

Proof. Suppose first that G is chordal. Applying Voloshin's lemma, we find that G has a simplicial vertex v_1. We can then apply Voloshin's lemma to $G \setminus v_1$. Repeating this process n times, we get a simplicial elimination ordering $v_1, v_2, \ldots, v_n$.

Suppose now that G is not chordal – that is, it contains an induced cycle C_k with $k \geq 4$. Then no vertex in this cycle can be the first one to appear in a simplicial elimination ordering, therefore it cannot exist. $\square$

Theorem 3.2.10. If a graph G is chordal, then $\chi(G) = \omega(G)$.

Proof. Recall that $\chi(G) \geq \omega(G)$, hence we need only prove the reverse inequality. Let $v_1, v_2, \ldots, v_n$ be a simplicial elimination ordering of G. Consider the greedy colouring of vertices in reverse order. It is clear that we only need $\omega(G)$ colours, as we coloured at most $\omega(G) - 1$ vertices of the considered vertex in each step. $\square$

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3.3 Perfect graphs and chromatic index

Definition 3.3.1. A graph G is *perfect* if $\chi(H) = \omega(H)$ holds for every induced subgraph H of G.

Theorem 3.3.2. Every chordal graph is perfect.

Proof. The proof is obvious and need not be mentioned. □

Theorem 3.3.3. Bipartite graphs are perfect.

Proof. All induced subgraphs of G are clearly bipartite as well, hence we only need to show that $\chi(G) = \omega(G)$, which is clear – they are both 1 if $E = \emptyset$, otherwise they are both equal to 2. □

Definition 3.3.4. An *edge colouring* of G is a function $c \colon E(G) \to \mathbb{N}$ such that every two distinct edges e and f with a vertex in common satisfy $c(e) \neq c(f)$. The *chromatic index* $\chi'(G)$ is the minimum number of colours needed for an edge colouring.

Theorem 3.3.5 (Vizing)**.** For every graph G we have $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Theorem 3.3.6. If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.

Proof. We consider two cases:

i) If G is regular, then we can find a perfect matching. We colour all the edges in this matching with one colour and remove them. We can repeat this process $\Delta(G)$ times to obtain the desired colouring.

ii) Suppose that G is not regular. First, add vertices to G in such a way that both its parts have the same size. Repeat the following process until the graph is regular: Choose a vertex in each part with degree less than $k = \Delta(G)$. Add the graph $K_{k,k}$ with one missing edge to G, then connect the chosen vertices to the endpoints of the missing edge. As
$$\sum_{v \in G} (k - \deg(v))$$
is a decreasing monovariant, the process eventually stops. We can thus colour the edges in the resulting graph with k colours, which induces a colouring of our initial graph. □

Theorem 3.3.7. If G is a bipartite graph, then its line graph is perfect.

Proof. Note that $\chi(L(G)) = \chi'(G)$. Since G is bipartite, we have
$$\chi'(G) = \Delta(G) = \omega(L(G)),$$
since there are no K_3 subgraphs in G. Thus $\chi(L(G)) = \omega(L(G))$. Note that every induced subgraph of a line graph is a linegraph of a subgraph of the original graph. In particular, any induced subgraph H of $L(G)$ is a line graph of a subgraph of G, hence it is a line graph of a bipartite graph as well. As above, $\chi(H) = \omega(H)$, hence $L(G)$ is perfect. □

Theorem 3.3.8 (Perfect graph theorem)**.** A graph G is perfect if and only if $\overline{G}$ is perfect.

Theorem 3.3.9 (Strong perfect graph theorem)**.** A graph G is perfect if and only if neither G not $\overline{G}$ has an induced odd cycle of length $k \geq 5$.

Definition 3.3.10. A graph G is (β, α')-*perfect* if every induced subgraph H of G satisfies $\beta(H) = \alpha'(H)$.

Theorem 3.3.11. A graph G is (β, α')-perfect if and only if G is bipartite.

Proof. Bipartite graphs are clearly (β, α')-perfect by König's theorem. Suppose that G is not bipartite and consider the shortest odd cycle in G. It must have no chords, hence the graph H induced by these vertices is an odd cycle, therefore

$$\beta(H) = \left\lceil \frac{n(H)}{2} \right\rceil \neq \left\lfloor \frac{n(H)}{2} \right\rfloor = \alpha'(H). \qquad \square$$

Theorem 3.3.12 (Gallai-Roy-Vitaver)**.** Let G be a simple graph. Then

$$\chi(G) = \min_{\vec{D} \text{ is an orientation}} \left(\max_{p \text{ is a path}} \ell(p) + 1 \right).$$

Proof. Let $\vec{D}$ be an orientation on G. Choose a maximal acyclic subgraph $\vec{D}'$. Let $d(v)$ be the length of the longest directed path in $\vec{D}'$ that ends in v and define $c(v) = d(v) + 1$. We claim that this is a proper colouring. If $\vec{uv} \in \vec{D}'$, then clearly $c(u) \neq c(v)$. If $\vec{uv} \in \vec{D} \setminus \vec{D}'$, then by adding $\vec{uv}$ to $\vec{D}'$ we'd get a cycle, hence there exists a path from v to u. As such, $c(u) \neq c(v)$. This shows that

$$\chi(G) \leq \max_{p \text{ is a path}} \ell(p) + 1.$$

It remains to prove that the inequality is reversed for a particular orientation $\vec{D}$. Let c be a colouring of G with $\chi(G)$ colours and orient edges in G such that $\vec{uv} \in \vec{D}$ if and only if $c(u) < c(v)$. Then clearly $\ell(p) + 1 \leq \chi(G)$ for all directed paths p. $\square$

4 Planar graphs

4.1 Definition and Euler's formula

Definition 4.1.1. Let G be a graph. The *drawing of G into a plane* is a function h on $V(G) \cup E(G)$ such that $h(v) \in \mathbb{R}^2$ for all $v \in V$ and $h(uv)$ is a continuous $h(u)h(v)$-curve for each $uv \in E$.

Definition 4.1.2. A *planar embedding*[8] of a graph G is a drawing where the curves corresponding to edges intersect only in the common end vertices.

Definition 4.1.3. A graph G is *planar* if it admits a planar embedding.

Theorem 4.1.4 (Jordan)**.** Every closed simple curve in the plane divides it into exactly two regions.

Definition 4.1.5. Let G be a plane graph. A *face* of G is a maximal region that contains no points from the image of the embedding function.

Definition 4.1.6. A *dual graph* of a plane graph G is a graph G^* with faces of G as vertices, in which two vertices are connected if and only if their corresponding faces have an edge in common.

Remark 4.1.6.1. A dual graph need not be simple, even if G is.

Definition 4.1.7. The *length* $\ell(F)$ of a face F in a plane graph G is the total length of walks along the boundary of F.

Theorem 4.1.8. Let G be a plane graph. The following statements are equivalent:

i) The graph G is bipartite.

ii) Every face of G has even length.

iii) The graph G^* is Eulerian.

Proof. If G is bipartite, then every face must clearly have even length. If G is not bipartite, let C be an odd cycle. Then

$$\sum_{F \text{ is inside } C} \ell(F) = \ell(C) + 2 \cdot \sum_{e \text{ is inside } C} 1 \equiv 1 \pmod 2,$$

therefore at least one face has odd length.

Now we'll prove that the last two statements are equivalent. First note that G^* is connected. As lengths of faces coincide with degrees in G^*, this is just Euler's theorem. □

Theorem 4.1.9. Let G is a plane graph and $D \subseteq E(G)$. The set D is a set of edges of a cycle if and only if the corresponding dual edge set D^* is a minimal edge cut.

Proof. If D is a cycle, then D^* separates faces inside the cycle from the ones outside. It is clear that it is a minimal edge cut. If $E(C)$ is a proper subset of D, then D^* is not a minimal cut. If D does not contain a cycle, then D^* is not even an edge cut. □

[8] Also *plane graph*.

Definition 4.1.10. The planar graph G is *outerplanar* if there exists an embedding such that the outer face contains all vertices.

Remark 4.1.10.1. If G is outerplane and 2-edge-connected, then it is Hamiltonian.

Theorem 4.1.11. If G is a simple outerplanar graph then $\delta(G) \leq 2$.

Proof. The statement clearly holds for $n \leq 3$. For $n \geq 4$, we prove a stronger statement with induction – there exist two distinct non neighbouring vertices u and v with degrees at most 2. Since K_4 is not outerplanar, the statement holds for $n = 4$. For general n, consider two cases:

i) There is a cut vertex v. Let G_i be the graphs obtained by adding back v to the components of $G \setminus v_i$. As every graph G_i is outerplanar, they all contain at least one vertex of degree at most 2 that is distinct from v. Taking one from each from G_1 and G_2, we satisfy all conditions.

ii) There is no cut vertex in G. In particular, there is no cut edge in G, hence the outer face is a Hamiltonian cycle. If G is a cycle, the statement clearly holds. Otherwise, consider a chord xy. It splits the graph into two outerplanar graph, each containing a vertex of degree at most 2 that is distinct from x and y. Furthermore, there is clearly no edge between such two vertices. □

Theorem 4.1.12 (Euler's formula)**.** Let G be a plane graph with k components. Then

$$n + f - e = k + 1.$$

Remark 4.1.12.1. If G is a simple planar graph, then $e \leq 3n - 6$. Furthermore, if G is K_3-free, then $e \leq 2n - 4$.

Definition 4.1.13. A *subdivision* of G is a graph that is obtained by replacing some edges of G with internally vertex-disjoint paths.

Remark 4.1.13.1. A subdivision of G is planar if and only if G is planar.

Definition 4.1.14. A *Kuratowski graph* is a subdivision of K_5 or $K_{3,3}$.

Proposition 4.1.15. If G is a planar graph, it contains no Kuratowski graph.

Proof. Neither K_5 or $K_{3,3}$ are planar graphs. □

Lemma 4.1.16. Let G be a planar graph and $e \in E$. Then there exists an embedding of G such that e is on the boundary of the outer face.

Proof. Apply an inversion with center inside one of the neighbouring faces. □

Lemma 4.1.17. If G is a minimal non-planar graph, then G is 2-connected.

Proof. Note that G is clearly connected. Suppose that G has a cut-vertex v, and let $G_1, G_2, \dots$ denote the subgraphs containing v and a connected component of $G \setminus v$. By the minimality assumption, these are all planar. By the previous lemma, each has an embedding such that v is on the edge of the outer face. Using an appropriate transformation, we can connect all these embeddings into an embedding of G. □

Lemma 4.1.18. If $S = \{x, y\}$ is a minimum vertex-cut in G and G is non-planar, then $G \setminus S$ contains a component G_i such that the S-lobe H_i with the added edge xy is non-planar.

Proof. Otherwise, embed each such H_i into the plane such that xy is on the boundary of the outer face. Using suitable transformations we can attach such embeddings using x and y. This gives us an embedding of $G \cup xy$, which induces an embedding for G. □

Lemma 4.1.19. If G is a non-planar graph without Kuratowski subgraphs and has the minimum number of edges among such graphs, then G is 3-connected.

Proof. As G is minimal, it is 2-connected. Suppose that $S = \{x, y\}$ is a vertex cut. Using the notation from the previous lemma, there exists a graph H_i which is not planar. By the minimality of $m(G)$, it has a Kuratowski subgraph F, which must contain xy. But since there exists a path between x and y in $G \setminus F$, we can replace xy with this path to obtain a Kuratowski subgraph in G. □

Definition 4.1.20. A *contraction* $G \cdot e$ is the graph $G/_{xy}$.

Theorem 4.1.21. If G is a 3-connected graph with $n \geq 5$, then there exists an edge $e \in E$ such that $G \cdot e$ is 3-connected.

Proof. Suppose otherwise. Let S be a vertex cut of $G \cdot e$ with 2 vertices. If $w = [x] \notin S$, then S remains a a vertex cut in G, which is not possible. Hence the minimum vertex cut contains w. Thus there exists a vertex cut $S' = \{x, y, z\}$ in G.

We consider an edge $f = uv$ such that $G \setminus \{u, v, z\}$ has the largest possible component G_i.

Let z' be a vertex adjacent to z which is not in $S \cup G_i$. Then there exists a vertex z^* such that $\{z, z', z^*\}$ is a vertex cut.

Denote by H the subgraph induced by $G_i \cup \{u, v\}$. We consider three cases:

i) If $z^* \in V(H)$ and $H \setminus z^*$ is disconnected, then $\{z, z^*\}$ is a vertex cut in G, which is not possible.

ii) If $z^* \in V(H)$ and $H \setminus z^*$ is connected, then $\{z, z'\}$ is a vertex cut in G, which is again not possible.

iii) If $z^* \notin V(H)$, we get a contradiction with maximality. □

Lemma 4.1.22. If G contains no Kuratowski subgraph, then $G \cdot e$ contains no Kuratowski subgraph for any edge e.

Proof. Suppose otherwise. Then the Kuratowski subgraph F clearly contains w. We consider the following cases:

i) If $\deg(w) = 2$, then G clearly already contained a Kuratowski subgraph.

ii) If $\deg(w) \geq 3$ and at most one neighbour of w is not a neighbour of x, we can replace w with x to get a Kuratowski subgraph in G.

iii) In all other cases, we find that $\deg_F(w) \geq 4$, but as F is a Kuratowski subgraph, we deduce that $\deg_F(w) = 4$ and that F is a subdivision of K_5. It is easy to see that G then contains a subdivision of $K_{3,3}$. □

Definition 4.1.23. An embedding is *convex* if the boundary of every face is a convex polygon.

Theorem 4.1.24. If G is a 3-connected graph without Kuratowski subgraphs, there exists a convex embedding of G such that no three vertices are on a line.

Proof. We induct on n. The statement clearly holds for $n = 4$. □

Theorem 4.1.25 (Kuratowski)**.** A graph G is planar if and only if it contains no Kuratowski subgraph.

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