Riemann surfaces

Luka Horjak (luka1.horjak@gmail.com)

February 27, 2025

Contents

Introduction **3**

1 Riemann surfaces **4**

1.1 Definition and holomorphic maps 4

1.2 Homotopy and the fundamental group 7

Introduction

These are my lecture notes on the course Riemann surfaces in the year 2024/25. The lecturer that year was viš. znan. sod. dr. Rafael Benedikt Andrist.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Riemann surfaces

1.1 Definition and holomorphic maps

February 20, 2025

Definition 1.1.1. A *surface* is a manifold of complex dimension 1.

Definition 1.1.2. A *Riemann surface* is a connected complex surface.

Definition 1.1.3. The *Riemann sphere* is defined as $\widehat{\mathbb{C}} = \mathbb{CP}^1$ with the usual complex structure.[1]

Definition 1.1.4. A *complex torus* is given by a quotient

$$T = \mathbb{C} \big/ a\mathbb{Z} \oplus b\mathbb{Z}\,,$$

where $a, b \in \mathbb{C}$ are $\mathbb{R}$-linearly independent. The parallelogram bounded by 0, a, b and $a+b$ is called the *fundamental domain* of T.

Theorem 1.1.5 (Identity)**.** Let X and Y be Riemann surfaces and $f, g \colon X \to Y$ be holomorphic maps. If the set $A = \{x \in X \mid f(x) = g(x)\}$ has an accumulation point, then $f = g$ on X.

Proof. We prove that the set of accumulation points is open. Take an accumulation point $a \in X$. Note that, by continuity, $f(a) = g(a)$. Consider charts $\varphi \colon U \to V$ on X and $\psi \colon W \to Z$ on Y such that $a \in U$, $f(a) \in W$ and $f(U) \subseteq W$. Applying the identity theorem for holomorphic functions on the function $\psi \circ f \circ \varphi^{-1}$, we find that f and g agree on U, which is a neighbourhood of a. All such points are accumulation points of A.

Note that this means that the set of accumulation points of A is both open and closed. As X is connected and this set is non-empty, $A = X$. By continuity, $f = g$ on $A = X$. □

Theorem 1.1.6 (Riemann's removable singularity theorem)**.** Let X be a Riemann surface, $U \subseteq X$ an open set and $a \in U$. Suppose that $f \colon U \setminus \{a\} \to \mathbb{C}$ is a holomorphic function that is bounded on $U \setminus \{a\}$. Then f can be extended uniquely to a holomorphic function $\tilde{f} \colon U \to \mathbb{C}$ with $\tilde{f}\big|_{U \setminus \{a\}} = f$.

Proof. First note that we can shrink U down to obtain a chart $\varphi \colon U \to \mathbb{C}$, then apply Riemann's removable singularity theorem to the function $f \circ \varphi^{-1}$ in the point $\varphi(a)$ and define $\tilde{f}(a) = \left(\widetilde{f \circ \varphi^{-1}}\right)(\varphi(a))$. As it's a composition of holomorphic functions, is is itself holomorphic. By continuity, the extension is unique. □

February 27, 2025

Definition 1.1.7. Let X be a Riemann surface. A *meromorphic* function f on X is a function $f \colon X \setminus A \to \mathbb{C}$ such that $f|_{X \setminus A}$ is holomorphic, A is a closed set of isolated points, and

$$\lim_{\substack{z \to a \\ z \in X \setminus A}} |f(z)| = \infty$$

for all $a \in A$. We denote the set of meromorphic functions on X by $\mathcal{M}(X)$.

Remark 1.1.7.1. The set $\mathcal{M}(X)$ is a field.

[1] Also denoted by P^1.

Theorem 1.1.8. Let X be a Riemann surface and $f \in \mathcal{M}(X)$. For each pole p of f we set $f(p) = \infty \in \widehat{\mathbb{C}}$. Then f is a holomorphic map from X to $\widehat{\mathbb{C}}$. Conversely, every holomorphic map $f \colon X \to \widehat{\mathbb{C}}$ that is not identically ∞ defines a meromorphic function on X.

Proof. Note that f is clearly continuous. It therefore suffices to show that f is holomorphic at every pole. Recall that a chart around ∞ is given by $\varphi \colon z \mapsto \frac{1}{z}$. Let U be a neighbourhood of p that contains no other pole, and define $g \colon U \setminus \{p\} \to \mathbb{C}$ by $g(z) = \left(f \circ \varphi^{-1}(z)\right)^{-1}$. Using Riemann's removable singularity theorem, this map has a unique holomorphic extension with $g(p) = 0$ by continuity. But that means that the proposed extension of f is indeed holomorphic at p.

Suppose $f \colon X \to \widehat{\mathbb{C}}$ is holomorphic. Define $A = \{z \in X \mid f(z) = \infty\}$. Then $f|_{X \setminus A}$ is clearly a meromorphic function. □

Theorem 1.1.9. Let X and Y be Riemann surfaces and $f \colon X \to Y$ a holomorphic map. For any point $p \in X$ there exist charts $\varphi \colon U \to V$ and $\psi \colon Z \to W$ such that $p \in U$, $f(p) \in Z$, $\varphi(p) = 0 = \psi(f(p))$, $f(U) \subseteq Z$ and

$$\psi \circ f \circ \varphi^{-1}(z) = z^k$$

for some integer $k \in \mathbb{N}_0$. This integer is determined uniquely.

Proof. Let $\tilde{\varphi} \colon U \to V$ be a chart on X with $p \in U$ such that $\tilde{\varphi}(p) = 0$. Furthermore, let $\psi \colon Z \to W$ be a chart on Y with $f(U) \subseteq Z$ and $\psi(f(p)) = 0$. Define $g = \psi \circ f \circ \tilde{\varphi}^{-1}$. Then $g(z) = z^k \cdot h(z)$, where $k \geq 1$, $h(0) \neq 0$ and h is a holomorphic function. Locally, h has a k-th root, hence

$$g(z) = \left(z \cdot \sqrt[k]{h(z)}\right)^k = w(z)^k.$$

Taking $\varphi = w \circ \tilde{\varphi}$, we get the sought charts on small enough domains. As k is equal to the number of preimages of points distinct from p, it is unique. □

Definition 1.1.10. Such integer k is called the *multiplicity* of f in p.

Corollary 1.1.10.1. Every non-constant holomorphic map $f \colon X \to Y$ is open.

Proof. In the charts from the above theorem, disks around φp map to disks. □

Corollary 1.1.10.2. Let X and Y be Riemann surfaces and $f \colon X \to Y$ a bijective holomorphic map. Then $f^{-1} \colon Y \to X$ is holomorphic.

Proof. As f is not constant, it is open, hence f^{-1} is continuous. In local coordinates, f is of the form $z \mapsto z^k$. As f is bijective, $k = 1$, hence the inverse is locally $z \mapsto z$, which is holomorphic. □

Corollary 1.1.10.3 (Maximum principle)**.** Let X be a Riemann surface and $f \colon X \to \mathbb{C}$ a non-constant holomorphic function. Then $|f|$ does not attain its maximum.

Proof. The map f is open. □

Theorem 1.1.11. Let X and Y be Riemann surfaces and $f\colon X \to Y$ a non-constant holomorphic map. If X is compact, then Y is compact and f is surjective.

Proof. Note that $f(X)$ is an open and closed subset of Y, which is connected. $\square$

Corollary 1.1.11.1. If $f\colon X \to \mathbb{C}$ is a holomorphic function for a compact Riemann surface X, then f is constant.

Proof. The proof is obvious and need not be mentioned. $\square$

Theorem 1.1.12 (Liouville)**.** Every bounded holomorphic function on complex numbers is constant.

Proof. We can extend f to a function $\widehat{\mathbb{C}} \to \mathbb{C}$ by Riemann's removable singularity theorem. Applying the above theorem, we get that f is constant as $\mathbb{C}$ is not compact. $\square$

Theorem 1.1.13 (Fundamental theorem of algebra)**.** Every non-constant polynomial with complex coefficients has a complex root.

Proof. The polynomial can be extended to a holomorphic map $p\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, which is surjective by theorem 1.1.11. As $p(\infty) = \infty$, the set $p^{-1}(0)$ contains a complex number. $\square$

Theorem 1.1.14. Every holomorphic function $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is either a rational function or $f \equiv \infty$.

Proof. If $f \not\equiv \infty$, the set $A = \{z \in \mathbb{C} \mid f(z) = \infty\}$ is finite – otherwise, it'd have an accumulation point. Let $A = \{a_i \mid i \leq n\}$. If needed, replace f by $\frac{1}{f}$ so that $\infty \notin A$, and repeat the argument. Now consider the function

$$g = f - \sum_{i=1}^{n} \sum_{k=1}^{N_i} d_{i,k} \cdot \frac{1}{(z - a_i)^k},$$

where $d_{i,k}$ are obtained from principal parts of Laurent series around a_i. As $g\colon \widehat{\mathbb{C}} \to \mathbb{C}$ is a holomorphic function, it is constant, hence f is a rational function. $\square$

1.2 Homotopy and the fundamental group

Definition 1.2.1. Denote the *homotopic* relation by $\sim$.

Proposition 1.2.2. Not all paths in $\mathbb{C}^*$ are homotopic.

Proof. Consider $\gamma(t) = e^{2\pi it}$ and $\delta(t) = e^{-2\pi it}$. Recall that

$$\frac{1}{2\pi i} \cdot \int\limits_{\gamma(t)} \frac{1}{z} \, dz = 1$$

and

$$\frac{1}{2\pi i} \cdot \int\limits_{\delta(t)} \frac{1}{z} \, dz = -1.$$

As integrals are a homotopy invariant, γ and δ are not homotopic. $\square$

Index

C
complex torus, 4

F
fundamental domain, 4
fundamental theorem of algebra, 6

H
homotopic, 7

I
identity theorem, 4

L
Liouville's theorem, 6

M
maximum principle, 5
meromorphic funcion, 4
multiplicity, 5

R
Riemann
  sphere, 4
  surface, 4
Riemann's removable singularity theorem, 4

S
surface, 4