Commutative algebra

Luka Horjak (luka1.horjak@gmail.com)

February 27, 2025

Contents

Introduction

These are my lecture notes on the course Commutative algebra in the year 2024/25. The lecturer that year was gost. izr. prof. dr. rer. nat. Daniel Smertnig.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labelled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Rings and modules

1.1 Rings and ring homomorphisms

Definition 1.1.1. Unless stated otherwise, rings always have a unit and are commutative.

Definition 1.1.2. Let A be a ring. The set $A^\bullet$ denotes the set of non-zero-divisors.

Definition 1.1.3. A ring A is a *domain* if 0 is the only zero-divisor of A.

Definition 1.1.4. Let $A \subseteq B$ be rings and $S \subseteq B$ a subset. The ring

$$A[S] = \bigcap_{\substack{A \subseteq A' \subseteq B \\ S \subseteq A'}} A'$$

is the subring of B obtained by *adjoining* S to A.

Definition 1.1.5. The set $\operatorname{Spec}(A)$ denotes the prime ideals of A.

Definition 1.1.6. The *radical* of an ideal I is defined as

$$\sqrt{I} = \{a \in A \mid \exists n \in \mathbb{N} \colon a^n \in I\}.$$

Proposition 1.1.7. The radical of an ideal is again an ideal.

Proof. It suffices to show that for any $a, b \in \sqrt{I}$ their sum is also in $\sqrt{I}$. Suppose that $a^n, b^m \in I$. Then

$$\begin{aligned}(a+b)^{n+m-1} &= \sum_{k=0}^{n+m-1} \binom{m+n-1}{k} a^k b^{n+m-1-k} \\ &= b^m \sum_{k=0}^{n-1} \binom{m+n-1}{k} a^k b^{n-1-k} + a^n \sum_{k=n}^{n+m-1} \binom{m+n-1}{k} a^{k-n} b^{n+m-1-k} \in I. \square\end{aligned}$$

Definition 1.1.8. The *nilradical* of A is the set $\mathcal{N}(A) = \sqrt{(0)}$.

Definition 1.1.9. The *Jacobson radical* $\mathcal{J}(A)$ is the intersection of all maximal ideals in A.

Lemma 1.1.10. The nilradical is contained in the Jacobson radical.

Proof. Let $a \in \mathcal{N}(A)$ and suppose that $a^n = 0$. For any maximal ideal M, we know that $a^n \in M$. Since M is prime, we deduce $a \in M$. $\square$

Lemma 1.1.11. We have

$$\mathcal{J}(A) = \left\{a \in A \;\middle|\; \forall b \in A \colon 1 - ba \in A^\times\right\}.$$

Proof. Let $a \in \mathcal{J}(A)$ and $b \in A$. Note that $1 - ab \notin M$ for any maximal ideal M, since $ab \in M$. As $1 - ab$ is not contained in any maximal ideal, it follows that $(1 - ab) = A$, hence $1 - ab$ is invertible.

Suppose now that $1 - ab \in A^\times$ for all $b \in A$. Let M be a maximal ideal and suppose $a \notin M$. Then $(M, a) = A$. In particular, we can write $1 = m + xa$ with $m \in M$ and $x \in A$. Rearranging, $m = 1 - xa$, which is a contradiction, as $1 - xa$ is invertible. $\square$

February 17, 2025

Lemma 1.1.12. The following statements hold:

i) Let $I \triangleleft A$ and $P_1, \dots, P_n \in \operatorname{Spec}(A)$. If $I \subseteq P_1 \cup \dots \cup P_n$, there exists some k such that $I \subseteq P_k$.

ii) Let $I_1, \dots, I_n \triangleleft A$ and $P \in \operatorname{Spec}(A)$. If $I_1 \cap \dots \cap I_n \subseteq P$, then there exists some k such that $I_k \subseteq P$.

Proof.

i) We induct on n, noting that the statement trivially holds for $n = 1$.

   Suppose the statement doesn't hold for n. By the induction hypothesis we can find

$$a_i \in I \Big\backslash \bigcup_{j \neq i} P_j$$

   for any i. Then $a_i \in P_i$. Consider the element

$$a = \sum_{i=1}^{n} \prod_{j \neq i} a_j.$$

   Note that all but one of the above terms are an element of P_i. But then a is not an element of any P_i, which is a contradiction.

ii) Suppose the contrary and let $a_j \in I_j \setminus P$ for all j. But then

$$\prod_{j=1}^{n} a_j \subseteq \prod_{j=1}^{n} I_j \subseteq \bigcap_{j=1}^{n} I_j \subseteq P,$$

   which is a contradiction. $\square$

Remark 1.1.12.1. The first statement is called *prime avoidance*.

Proposition 1.1.13. Let $f\colon A \to B$ be a ring homomorphism. If $I \triangleleft B$, then $f^{-1}(I) \triangleleft A$. Furthermore, if $P \in \operatorname{Spec}(B)$, then $f^{-1}(P) \in \operatorname{Spec}(A)$.

Proposition 1.1.14 (Universal property)**.** Let $I \triangleleft A$ and $\pi\colon A \to A/I$ be the canonical epimorphism. For every ring homomorphism $f\colon B$ with $I \subseteq \ker(f)$, there exists a unique ring homomorphism $\hat{f}\colon A/I \to B$ such that $f = \hat{f} \circ \pi$.

$$\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[dr, "f"'] & A/I \arrow[d, dashed, "\hat{f}"] \\
& B
\end{tikzcd}$$

Corollary 1.1.14.1. If $f\colon A \to B$ is a ring homomorphism, then $A/\ker f \cong f(A)$.

Theorem 1.1.15 (Isomorphism theorems)**.** The following statements hold:

i) Let $I \triangleleft A$. There is a bijective correspondence

$$\{J \triangleleft A \mid I \subseteq J\} \leftrightarrow \left\{\overline{J} \triangleleft A/I\right\},$$

   given by $J \mapsto J/I$ and $\overline{J} \mapsto \pi^{-1}\left(\overline{J}\right)$.

ii) If $I, J \triangleleft A$ with $I \subseteq J$, then

$$A \big/ J \cong {}^{A/I} \big/ {}_{J/I} .$$

iii) Let $B \subseteq A$ be a subring and $I \triangleleft A$. Then $I \cap B \triangleleft B$ and

$$B + I \big/ I \cong B \big/ B \cap I .$$

Theorem 1.1.16 (Chinese remainder theorem)**.** If $I_1, \ldots, I_n \triangleleft A$ are pairwise comaximal, then

$$A \big/ I_1 \cap \cdots \cap I_n \cong \prod_{k=1}^{n} A \big/ I_k .$$

February 20, 2025

1.2 Modules

Definition 1.2.1. Let M be an A-module and $E \subseteq M$. The *A-module generated by E* is denoted by
$$\langle E \rangle_A = \left\{ \sum_{k=1}^{n} a_k m_k \;\middle|\; a_k \in A \wedge m_k \in E \right\}.$$

Proposition 1.2.2. Let M be an A-module and $I \triangleleft A$. Then M/IM is an A/I-module via the natural product.

Remark 1.2.2.1. Categorically, A/I-modules are equivalent to A-modules M with $IM = 0$.

Theorem 1.2.3 (Universal property)**.** Let $N \leq M$ be A-modules and $\pi \colon M \to M/N$ be the canonical epimorphism. If $f \colon M \to X$ is an A-module homomorphism with $N \subseteq \ker f$, then there exists a unique homomorphism $\hat{f} \colon M/N \to X$ such that $f = \hat{f} \circ \pi$.

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "f"'] & M/N \arrow[d, dashed, "\hat{f}"] \\ & X \end{tikzcd}$$

Theorem 1.2.4 (Isomorphism theorems)**.** The following statements hold:

i) We have $f(M) \cong M/\ker(f)$.

ii) If $N \leq M$, then submodules $N \leq X \leq M$ are in bijective correspondence with submodules of M/N.

iii) If $N \leq X \leq M$, then
$$M/X \cong {}^{M/N}\big/_{X/N}.$$

iv) If $N, N' \leq M$, then
$$N + N'/N \cong N'/N \cap N'.$$

Theorem 1.2.5 (Universal property)**.** If $(f_i \colon M_i \to X)_{i \in I}$ is a family of A-module homomorphisms, then there exists a unique homomorphism
$$\hat{f} \colon \bigoplus_{i \in I} M_i \to X$$
such that $f_i = \hat{f} \circ \varepsilon_i$ for all $i \in I$.

If $(g_i \colon X \to M_i)_{i \in I}$ is a family of A-module homomorphisms, then there exists a unique homomorphism
$$\hat{g} \colon X \to \prod_{i \in I} M_i$$
such that $g_i = \pi_i \circ \hat{g}$ for all $i \in I$.

Remark 1.2.5.1. Note that
$$\bigoplus_{i\in I} M_i \subseteq \prod_{i\in I} M_i.$$
If I is finite, then A-modules form an abelian category.

Definition 1.2.6. An A-module M is *free* if
$$M \cong \bigoplus_{i\in I} A$$
for some set I. A *basis* of M is a family $(m_i)_{i\in I}$ such that the map
$$\bigoplus_{i\in I} A \to M, \quad (a_i)_{i\in I} \mapsto \sum_{i\in I} a_i m_i$$
is an isomorphism.

Lemma 1.2.7. The following statements hold:

i) Every module is a quotient of a free module.
ii) A module M is finitely generated if and only if there exists an epimorphism $\varphi\colon A^k \to M$ for some integer k.
iii) A module M is finitely generated and free if and only if $M \cong A^k$ for some integer k.

Lemma 1.2.8 (Nakayama)**.** Let M be a finitely generated module over A.

i) If $J(A)M = M$, then $M = (0)$.
ii) If $N \leq M$ such that $M = N + J(A)M$, then $N = M$.

Proof.

i) Assume $M \neq 0$ and let $m_1, \ldots, m_r \in M$ be a minimal generating set. Note that, as $M \neq 0$, $r \geq 1$. By our assumptions, we can write
$$m_r = \sum_{i=1}^{r} a_i m_i,$$
where $a_i \in J(A)$. But as $1 - a_r$ is invertible, we can express m_r as a linear combination of the other elements, which is a contradiction.
ii) Note that
$$J(A)\, {M}\big/{N} = {J(A)M + N}\big/{N} = {M}\big/{N},$$
hence ${M}/{N} = 0$. $\square$

Definition 1.2.9. A ring A is *local* if $A \neq 0$ and A has a unique maximal ideal. We denote it by $(A, \mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal.

Remark 1.2.9.1. If $(A, \mathfrak{m})$ is local, then ${A}/{\mathfrak{m}}$ is a field and $\mathfrak{m} = J(A)$.

Corollary 1.2.9.2. Let $(A, \mathfrak{m})$ be a local ring and M a finitely generated module. If $x_1 + \mathfrak{m}M, \ldots, x_r + \mathfrak{m}M$ is a basis of the ${A}/{\mathfrak{m}}$-vector space ${M}/{\mathfrak{m}M}$, then $x_1, \ldots, x_r \in M$ generate M.

Proposition 1.2.10. If $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ is a short exact sequence, then the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "f"', "\cong"] & N \arrow[r, "g"] \arrow[d, "\mathrm{id}"'] & P \arrow[r] \arrow[d, "(\hat{g})^{-1}"', "\cong"] & 0 \\
0 \arrow[r] & \mathrm{im}(f) \arrow[r, hook] & N \arrow[r, two heads, "\pi"] & N/K \arrow[r] & 0
\end{tikzcd}$$

commutes and has exact rows, where $\hat{g}(n+K) = g(n)$.

Lemma 1.2.11. The following statements hold:

i) A sequence $0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} P$ is exact if and only if the sequence

$$0 \longrightarrow \mathrm{Hom}(X, N) \xrightarrow{f_*} \mathrm{Hom}(X, M) \xrightarrow{g_*} \mathrm{Hom}(X, P)$$

is exact for every A-module X.

ii) A sequence $N \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$ is exact if and only if the sequence

$$0 \longrightarrow \mathrm{Hom}(P, X) \xrightarrow{f^*} \mathrm{Hom}(M, X) \xrightarrow{g^*} \mathrm{Hom}(N, X)$$

is exact for every A-module X.

Definition 1.2.12. Let $M_1, \ldots, M_n$ and P be A-modules. A map $f\colon M_1 \times \cdots \times M_n \to P$ is A-*multilinear* if it is linear in every component.

Definition 1.2.13. Let $M_1, \ldots, M_n$ be A-modules. The *tensor product* $M_1 \otimes \cdots \otimes M_n$ is the A-module together with the multilinear map

$$\otimes\colon \prod_{i=1}^{n} M_i \to \bigotimes_{i=1}^{n} M_i$$

defined by the following universal property: For every A-module P and every multilinear map $f\colon M_1 \times \cdots \times M_n \to P$ there exists a unique A-module homomorphism

$$\hat{f}\colon \bigotimes_{i=1}^{n} M_i \to P$$

such that $\hat{f} \circ \otimes = f$.

$$\begin{tikzcd}
\prod_{i=1}^{n} M_i \arrow[r, "\otimes"] \arrow[dr, "f"'] & \bigotimes_{i=1}^{n} M_i \arrow[d, dashed, "\hat{f}"] \\
& P
\end{tikzcd}$$

Remark 1.2.13.1. The tensor product is associative and commutative. It is functorial in each component.

Theorem 1.2.14 (Hom-$\otimes$ adjunction)**.** Let M be an A-module. Then $\cdot \otimes M$ is left-adjoint to $\operatorname{Hom}(M, \cdot)$. That is, for any A-modules M, N and P, there are A-isomorphisms $\operatorname{Hom}(N \otimes M, P) \to \operatorname{Hom}(N, \operatorname{Hom}(M, P))$, given by $f \mapsto (n \mapsto (m \mapsto f(n \otimes m)))$ with inverse $g \mapsto (n \otimes m \mapsto g(n)(m))$. These isomorphisms are natural transformations in N, M and P.

Corollary 1.2.14.1. The tensor product $M \otimes \cdot$ is right-exact. That is, for every exact sequence $N \xrightarrow{f} P \xrightarrow{g} Q \longrightarrow 0$ the sequence

$$M \otimes N \xrightarrow{\operatorname{id} \otimes f} M \otimes P \xrightarrow{\operatorname{id} \otimes g} M \otimes Q \longrightarrow 0$$

is also exact.

Proof. Applying lemma 1.2.11, we see that

$$0 \longrightarrow \operatorname{Hom}(Q, X) \xrightarrow{f^*} \operatorname{Hom}(P, X) \xrightarrow{g^*} \operatorname{Hom}(N, X)$$

is exact. Applying lemma 1.2.11 again, we see that

$$0 \longrightarrow \operatorname{Hom}(M, \operatorname{Hom}(Q, X)) \xrightarrow{(f^*)_*} \operatorname{Hom}(M, \operatorname{Hom}(P, X)) \xrightarrow{(g^*)_*} \operatorname{Hom}(M, \operatorname{Hom}(N, X))$$

is exact as well. Applying the previous theorem and lemma 1.2.11 again, we get the required sequence. □

Corollary 1.2.14.2. For any family $(N_i)_{i \in I}$ of A-modules we have

$$M \otimes \left(\bigoplus_{i \in I} N_i \right) \subseteq \bigoplus_{i \in I} (M \otimes N_i) .$$

Proof. We can construct the isomorphisms using the universal properties. □

Proposition 1.2.15. Let M be an A-module and $I \triangleleft A$. Then ${M}/{MI} \cong M \otimes_A {A}/{I}$.

Proof. As $0 \longrightarrow I \hookrightarrow A \longrightarrow {A}/{I} \longrightarrow 0$ is a short exact sequence, the sequence

$$I \otimes M \longrightarrow A \otimes M \longrightarrow {A}/{I} \otimes M \longrightarrow 0.$$

But as $A \otimes M \cong M$ under $\mu(a \otimes m) = am$ and $\mu|_{I \otimes M} = IM$, we get

$${M}\Big/{IM} \cong {A \otimes M}\Big/{I \otimes M} \cong {A}\Big/{I} \otimes M. \qquad \square$$

Proposition 1.2.16. If $A \neq 0$ and $A^{(I)} \cong A^{(J)}$, then $|I| = |J|$.[1]

[1] $A^{(I)} = \bigoplus_{i \in I} A.$

February 27, 2025

Proof. Let M be a maximal ideal in A, then $K = A/M$ is a field. Then

$$A^{(I)} \otimes A/M \cong \left(A \otimes A/M\right)^{(I)} \cong \left(A/M\right)^{(I)} \cong K^{(I)}$$

as A-modules and A/M-modules. Hence $K^{(I)} \cong K^{(J)}$, therefore $|I| = |J|$. □

Definition 1.2.17. If M is a finitely generated free module, its *rank* is the unique $n \in \mathbb{N}$ such that $M \cong A^n$.

Definition 1.2.18. A module M is

- *projective* if $\operatorname{Hom}(M, \cdot)$ is exact.
- *injective* if $\operatorname{Hom}(\cdot, M)$ is exact.
- *flat* if $M \otimes \cdot$ is exact.

Theorem 1.2.19. The following statements are equivalent for an A-module P:

i) The module P is projective.

ii) For every epimorphism $g\colon M \to N$, the map $g_*\colon \operatorname{Hom}(P, M) \to \operatorname{Hom}(P, N)$ is an epimorphism.

iii) For every epimorphism $f\colon M \to N$ and homomoprhism $\varphi\colon P \to N$ there exists a homomorphism $\psi\colon P \to M$ with $f \circ \psi = \varphi$.

$$\begin{tikzcd} & P \arrow[dl, dash, "\psi"'] \arrow[d, "\varphi"] \\ M \arrow[r, two heads, "f"'] & N \end{tikzcd}$$

iv) Every epimorphism $g\colon M \to P$ splits.

v) There exists an A-module M such that $P \oplus M$ is free.

Proof. The first statement implies the second by definition.

Now assume that the second statement holds. The map $f_*\colon \operatorname{Hom}(P, M) \to \operatorname{Hom}(P, N)$ is therefore an epimorphism. By definition, we can construct a homomorphism $\psi\colon P \to M$ that maps to f.

Assume now that the diagram condition holds. Then there exists a homomorphism $s\colon P \to M$ such that $g \circ s = \mathrm{id}$.

Suppose every epimorphism $g\colon M \to P$ splits. In particular, this holds for an epimorphism $g\colon A^{(I)} \to P$. But then

$$0 \longrightarrow \ker g \longrightarrow A^{(I)} \xrightarrow{\ g\ } P \longrightarrow 0$$

is a short exact sequence. As it splits, $P \oplus \ker g \cong A^{(I)}$.

Finally, suppose that $P \oplus C \cong A^{(I)}$ is a free module. Take a short exact sequence

$$0 \longrightarrow M \xrightarrow{\ f\ } N \xrightarrow{\ g\ } Q \longrightarrow 0.$$

As $\mathrm{Hom}(P, \cdot)$ is left-exact, we only need to check that $g_*\colon \mathrm{Hom}(P, N) \to \mathrm{Hom}(P, Q)$ is surjective. Let $\varphi \in \mathrm{Hom}(P, Q)$. Let $\pi\colon A^{(I)} \to P$ be the canonical projection and $\varepsilon\colon P \hookrightarrow A^{(I)}$ the embedding. Then $\pi \circ \varepsilon = \mathrm{id}_P$.

For each basis vector $e_i \in A^{(I)}$, choose $n_i \in N$ such that $g(n_i) = \varphi \circ \pi(e_i)$, which is possible by surjectivity of g.

$$\begin{tikzcd}
 & A^{(I)} \arrow[d, shift left, "\pi"] \arrow[ddl, dashed, "\Psi"'] & \\
 & P \arrow[u, shift left, "\varepsilon"] \arrow[d, "\varphi"] \arrow[dl, dashed, "\psi"] & \\
N \arrow[r, two heads, "g"'] & Q \arrow[r] & 0
\end{tikzcd}$$

Construct a homomorphism $\Psi\colon \mathrm{Hom}(A^{(I)}, N)$ by $\Psi(e_i) = n_i$. Then $g \circ \Psi = \varphi \circ \pi$. Now define $\psi = \Psi \circ \varepsilon$. Then, for every $p \in P$, we have

$$g \circ \psi(p) = g \circ \Psi \circ \varepsilon(p) = \varphi \circ \pi \circ \varepsilon(p) = \varphi(p),$$

hence $g_*(\psi) = \varphi$. $\square$

Index