Number theory

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Contents

Introduction

These are my lecture notes on the course Number theory in the year 2023/24. The lecturer that year was gost. izr. prof. dr. rer. nat. Daniel Smertnig.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

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1 Distribution of prime numbers

1.1 Riemann zeta function

Definition 1.1.1. The *prime counting function* is defined as

$$\pi(x) = \left|\{p \in \mathbb{P} \mid p \leq x\}\right|.$$

Definition 1.1.2. Let $(a_n)_n \subseteq \mathbb{C}$ be a sequence. The infinite product

$$\prod_{n=1}^{\infty} a_n$$

converges *absolutely* if it converges normally as a product of constant functions.

Theorem 1.1.3. Let $\sigma > 1$ be a real number. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq \sigma$, we have

$$\sum_{n=1}^{\infty} = \prod_{p \in \mathbb{P}} (1 - p^{-s})^{-1},$$

with both the product and sum converging uniformly and absolutely.[1]

Proof. Note that

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{\mathrm{Re}(s)}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}}$$

is convergent, hence the given series converges as well. To prove the convergence of the product, first note that

$$\prod_{p \in \mathbb{P}} (1 - p^{-s})^{-1} = \prod_{p \in \mathbb{P}} \left(\sum_{k=0}^{\infty} p^{-sk}\right).$$

As

$$\sum_{p \in \mathbb{P}} \left|\sum_{k=1}^{\infty} p^{-sk}\right| \leq \sum_{p \in \mathbb{P}} \left(\sum_{k=1}^{\infty} (p^k)^{-\sigma}\right) \leq \sum_{n=1}^{\infty} n^{-\sigma}$$

converges normally, so does the product. To prove equality, we can bound

$$\left|\prod_{\substack{p \in \mathbb{P} \\ p \leq x}} \frac{1}{1 - p^{-s}} - \sum_{n=1}^{x} \frac{1}{n^s}\right| \leq \sum_{n=x+1}^{\infty} \left|\frac{1}{n^s}\right| \leq \sum_{n=x+1}^{\infty} \frac{1}{n^{\sigma}},$$

which converges to 0 as $x \to \infty$. □

Definition 1.1.4. The *Riemann zeta function* is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\mathrm{Re}(s) > 1$.

Lemma 1.1.5. If $\mathrm{Re}(s) > 1$, then $\zeta(s) \neq 0$.

[1] See Complex analysis, section 3 for definition and properties of convergence for products.

Proof. No term in the infinite product is equal to 0. □

Proposition 1.1.6. The function $\zeta(s)-\frac{1}{s-1}$ has a holomorphic continuation to $\mathrm{Re}(s) > 0$.

Proof. We can write

$$\begin{aligned}\zeta(s) - \frac{1}{s-1} &= \sum_{n=1}^{\infty} n^{-s} - \int_1^{\infty} x^{-s}\,dx \\ &= \sum_{n=1}^{\infty} \left(n^{-s} - \int_n^{n+1} x^{-s}\,dx\right) \\ &= \sum_{n=1}^{\infty} \int_n^{n+1} \left(n^{-s} - x^{-s}\right)\,dx\end{aligned}$$

as long as $\mathrm{Re}(s) > 1$. Now, for $n \leq x \leq n+1$, we can bound

$$\left|n^{-s} - x^{-s}\right| = \left|\int_n^x su^{-s-1}\,du\right| \leq \frac{|s|}{n^{\mathrm{Re}(s)+1}}.$$

Let $L \subseteq \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ be a compact set. As

$$\left|\sum_{n=1}^{\infty} \int_n^{n+1} \left(n^{-s} - x^{-s}\right)\,dx\right| \leq \sum_{n=1}^{\infty} \frac{|s|}{n^{\mathrm{Re}(s)+1}} \leq \|\mathrm{id}\|_L \cdot \sum_{n=1}^{\infty} \frac{1}{n^{\sigma+1}}$$

for all $s \in L$, where $\sigma = \min_L |z|$, the series converges uniformly on compact sets. □

Remark 1.1.6.1. The ζ function can be analytically extended to $\mathbb{C} \setminus \{1\}$ by

$$\zeta(1-s) = 2 \cdot (2\pi)^{-s} \cos\left(\frac{\pi}{2}s\right)\Gamma(s)\zeta(s).$$

It has a simple pole with residue 1 at 1.

Lemma 1.1.7. The equation $\overline{\zeta(\overline{s})} = \zeta(s)$ holds for all $s \in \mathbb{C} \setminus \{1\}$.

Proof. The function $\overline{\zeta(\overline{s})}$ is holomorphic. As it coincides with $\zeta(s)$ for $s \geq 1$, the functions are equal. □

1.2 Prime number theorem

Proposition 1.2.1. The series
$$\sum_{p\in\mathbb{P}} \frac{\log(p)}{p^s}$$
converges uniformly and absolutely for $\mathrm{Re}(s) \geq \sigma > 1$.

Proof. We can bound
$$\sum_{p\in\mathbb{P}} \left|\frac{\log(p)}{p^s}\right| \leq \sum_{p\in\mathbb{P}} \frac{\log(p)}{p^\sigma} \leq \sum_{n=1}^{\infty} \frac{\log(p)}{n^\varepsilon} \cdot \frac{1}{n^{\sigma-\varepsilon}},$$
which clearly converges for $0 < \varepsilon < \sigma - 1$. □

Definition 1.2.2. We define functions
$$\theta(x) = \sum_{\substack{p\in\mathbb{P}\\ p\leq x}} \log(p)$$
and
$$\phi(s) = \sum_{p\in\mathbb{P}} \frac{\log(p)}{p^s}.$$

Remark 1.2.2.1. The function ϕ is holomorphic for $\mathrm{Re}(s) > 1$.

Proposition 1.2.3. The function ϕ has a meromorphic continuation to $\mathrm{Re}(s) > \frac{1}{2}$. It has simple poles at points $s = 1$ and zeros of $\zeta(s)$.

Proof. Calculate the logarithmic derivative of ζ as
$$\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\sum_{p\in\mathbb{P}} \frac{\left((1-p^{-s})^{-1}\right)'}{(1-p^{-s})^{-1}} \\
&= -\sum_{p\in\mathbb{P}} \frac{-(1-p^{-s})^{-2}\cdot p^{-s}\log(p)}{(1-p^{-s})^{-1}} \\
&= \sum_{p\in\mathbb{P}} \frac{\log(p)}{p^s-1} \\
&= \phi(s) + \sum_{p\in\mathbb{P}} \frac{\log(p)}{p^s(p^s-1)}.
\end{aligned}$$
Similarly as in the proof of proposition 1.2.1, we can show that the above series converges uniformly and absolutely for $\mathrm{Re}(s) > \frac{1}{2}$. □

Theorem 1.2.4. If $\mathrm{Re}(s) = 1$, then $\zeta(s) \neq 0$.

Proof. Let $\mu = \mathrm{ord}_{1+ib}\,\zeta \geq 0$. As $\zeta(\overline{z}) = \overline{\zeta(z)}$, we also have $\mu = \mathrm{ord}_{1-ib}\,\zeta$. $\theta = \mathrm{ord}_{1+2ib}\,\zeta = \mathrm{ord}_{1-2ib}\,\zeta$. As ϕ has a simple pole at 1, we have
$$\lim_{\varepsilon\to 0} \varepsilon\phi(1+\varepsilon) = 1.$$

Similarly,

$$\lim_{\varepsilon\to 0} \varepsilon\phi(1+\varepsilon \pm ib) = -\mu,$$

as the logarithmic derivative of ζ at b has residue $-\mu$, and

$$\lim_{\varepsilon\to 0} \varepsilon\phi(1+\varepsilon \pm 2ib) = -\theta.$$

Now compute

$$f(\varepsilon) = \sum_{r=-2}^{2} \binom{4}{2+r} \phi(1+\varepsilon+rib) = \sum_{p\in\mathbb{P}} \frac{\log(p)}{p^{1+\varepsilon}} \cdot \left(p^{\frac{ib}{2}} - p^{-\frac{ib}{2}}\right)^4 = \sum_{p\in\mathbb{P}} \frac{\log(p)}{p^{1+\varepsilon}} \cdot \left(2\operatorname{Re}\left(p^{\frac{ib}{2}}\right)\right)^4.$$

It follows that

$$0 \le \lim_{\varepsilon\to 0} \varepsilon \cdot f(\varepsilon) = 6 - 8\mu - 2\theta.$$

As $\theta \ge 0$, we have $\mu = 0$. □

Corollary 1.2.4.1. The function ϕ is holomorphic for $\operatorname{Re}(s) = 1$, except for a simple pole with residue 1 at 1. In particular, the function

$$g(z) = \frac{\phi(z+1)}{z+1} - \frac{1}{z}$$

is holomorphic for $\operatorname{Re}(z) \ge 0$.

Index