Riemann surfaces

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Contents

Introduction

These are my lecture notes on the course Riemann surfaces in the year 2024/25. The lecturer that year was viš. znan. sod. dr. Rafael Benedikt Andrist.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labeled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Riemann surfaces

1.1 Definition and holomorphic maps

February 20, 2025

Definition 1.1.1. A *surface* is a manifold of complex dimension 1.

Definition 1.1.2. A *Riemann surface* is a connected complex surface.

Definition 1.1.3. The *Riemann sphere* is defined as $\widehat{\mathbb{C}} = \mathbb{C}\mathrm{P}^1$ with the usual complex structure.[1]

Definition 1.1.4. A *complex torus* is given by a quotient

$$T = \mathbb{C} \big/ a\mathbb{Z} \oplus b\mathbb{Z} \,,$$

where $a, b \in \mathbb{C}$ are $\mathbb{R}$-linearly independent. The parallelogram bounded by 0, a, b and $a + b$ is called the *fundamental domain* of T.

Theorem 1.1.5 (Identity)**.** Let X and Y be Riemann surfaces and $f, g \colon X \to Y$ be holomorphic maps. If the set $A = \{x \in X \mid f(x) = g(x)\}$ has an accumulation point, then $f = g$ on X.

Proof. We prove that the set of accumulation points is open. Take an accumulation point $a \in X$. Note that, by continuity, $f(a) = g(a)$. Consider charts $\varphi \colon U \to V$ on X and $\psi \colon W \to Z$ on Y such that $a \in U$, $f(a) \in W$ and $f(U) \subseteq W$. Applying the identity theorem for holomorphic functions on the function $\psi \circ f \circ \varphi^{-1}$, we find that f and g agree on U, which is a neighbourhood of a. All such points are accumulation points of A.

Note that this means that the set of accumulation points of A is both open and closed. As X is connected and this set is non-empty, $A = X$. By continuity, $f = g$ on $A = X$. $\square$

Theorem 1.1.6 (Riemann's removable singularity theorem)**.** Let X be a Riemann surface, $U \subseteq X$ an open set and $a \in U$. Suppose that $f \colon U \setminus \{a\} \to \mathbb{C}$ is a holomorphic function that is bounded on $U \setminus \{a\}$. Then f can be extended uniquely to a holomorphic function $\tilde{f} \colon U \to \mathbb{C}$ with $\tilde{f}\big|_{U \setminus \{a\}} = f$.

Proof. Shrink U down to obtain a chart $\varphi \colon U \to \mathbb{C}$, then apply Riemann's removable singularity theorem to the function $f \circ \varphi^{-1}$ in the point $\varphi(a)$ and define $\tilde{f}(a) = \left(f \circ \varphi^{-1}\right)(\varphi(a))$. As it's a composition of holomorphic functions, is is itself holomorphic. By continuity, the extension is unique. $\square$

[1] Also denoted by P^1.

Index