Commutative algebra

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Introduction

These are my lecture notes on the course Commutative algebra in the year 2024/25. The lecturer that year was gost. izr. prof. dr. rer. nat. Daniel Smertnig.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labelled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Rings and modules

1.1 Rings and ring homomorphisms

Definition 1.1.1. Unless stated otherwise, rings always have a unit and are commutative.

Definition 1.1.2. Let A be a ring. The set A^{\bullet} denotes the set of non-zero-divisors.

Definition 1.1.3. A ring A is a *domain* if 0 is the only zero-divisor of A.

Definition 1.1.4. Let $A \subseteq B$ be rings and $S \subseteq B$ a subset. The ring

$$A[S] = \bigcap_{\substack{A \subseteq A' \subseteq B \\ S \subseteq A'}} A'$$

is the subring of B obtained by adjoining S to A.

Definition 1.1.5. The set Spec(A) denotes the prime ideals of A.

Definition 1.1.6. The radical of an ideal I is defined as

$$\sqrt{I} = \{ a \in A \mid \exists n \in \mathbb{N} \colon a^n \in I \} .$$

Proposition 1.1.7. The radical of an ideal is again an ideal.

Proof. It suffices to show that for any $a, b \in \sqrt{I}$ their sum is also in \sqrt{I} . Suppose that $a^n, b^m \in I$. Then

$$(a+b)^{n+m-1} = \sum_{k=0}^{n+m-1} {m+n-1 \choose k} a^k b^{n+m-1-k}$$
$$= b^m \sum_{k=0}^{n-1} {m+n-1 \choose k} a^k b^{n-1-k} + a^n \sum_{k=n}^{n+m-1} {m+n-1 \choose k} a^{k-n} b^{n+m-1-k} \in I. \square$$

Definition 1.1.8. The *nilradical* of A is the set $\mathcal{N}(A) = \sqrt{(0)}$.

Definition 1.1.9. The *Jacobson radical* $\mathcal{J}(A)$ is the intersection of all maximal ideals in A

Lemma 1.1.10. The nilradical is contained in the Jacobson radical.

Proof. Let $a \in \mathcal{N}(A)$ and suppose that $a^n = 0$. For any maximal ideal M, we know that $a^n \in M$. Since M is prime, we deduce $a \in M$.

Lemma 1.1.11. We have

$$\mathcal{J}(A) = \left\{ a \in A \mid \forall b \in A \colon 1 - ba \in A^{\times} \right\}.$$

Proof. Let $a \in \mathcal{J}(A)$ and $b \in A$. Note that $1 - ab \notin M$ for any maximal ideal M, since $ab \in M$. As 1 - ab is not contained in any maximal ideal, it follows that (1 - ab) = A, hence 1 - ab is invertible.

Suppose now that $1 - ab \in A^{\times}$ for all $b \in A$. Let M be a maximal ideal and suppose $a \notin M$. Then (M, a) = A. In particular, we can write 1 = m + xa with $m \in M$ and $x \in A$. Rearranging, m = 1 - xa, which is a contradiction, as 1 - xa is invertible. \square

Lemma 1.1.12. The following statements hold:

- i) Let $I \triangleleft A$ and $P_1, \ldots, P_n \in \text{Spec}(A)$. If $I \subseteq P_1 \cup \cdots \cup P_n$, there exists some k such that $I \subseteq P_k$.
- ii) Let $I_1, \ldots, I_n \triangleleft A$ and $P \in \operatorname{Spec}(A)$. If $I_1 \cap \cdots \cap I_n \subseteq P$, then there exists some k such that $I_k \subseteq P$.

Proof.

i) We induct on n, noting that the statement trivially holds for n = 1.

Suppose the statement doesn't hold for n. By the induction hypothesis we can find

$$a_i \in I \setminus \bigcup_{j \neq i} P_j$$

for any i. Then $a_i \in P_i$. Consider the element

$$a = \sum_{i=1}^{n} \prod_{j \neq i} a_j.$$

Note that all but one of the above terms are an element of P_i . But then a is not an element of any P_i , which is a contradiction.

ii) Suppose the contrary and let $a_j \in I_j \setminus P$ for all j. But then

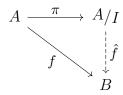
$$\prod_{j=1}^{n} a_j \subseteq \prod_{j=1}^{n} I_j \subseteq \bigcap_{j=1}^{n} I_j \subseteq P,$$

which is a contradiction.

Remark 1.1.12.1. The first statement is called *prime avoidance*.

Proposition 1.1.13. Let $f: A \to B$ be a ring homomorphism. If $I \triangleleft B$, then $f^{-1}(I) \triangleleft A$. Furthermore, if $P \in \text{Spec}(B)$, then $f^{-1}(P) \in \text{Spec}(A)$.

Proposition 1.1.14 (Universal property). Let $I \triangleleft A$ and $\pi \colon A \to A/I$ be the canonical epimorphism. For every ring homomorphism $f \colon B$ with $I \subseteq \ker(f)$, there exists a unique ring homomorphism $\hat{f} \colon A/I \to B$ such that $f = \hat{f} \circ \pi$.



Corollary 1.1.14.1. If $f: A \to B$ is a ring homomorphism, then $A/\ker f \cong f(A)$.

Theorem 1.1.15 (Isomorphism theorems). The following statements hold:

i) Let $I \triangleleft A$. There is a bijective correspondence

$$\{J \triangleleft A \mid I \subseteq J\} \leftrightarrow \{\overline{J} \triangleleft A/I\},$$

given by $J \mapsto J/I$ and $\overline{J} \mapsto \pi^{-1}(\overline{J})$.

ii) If $I, J \triangleleft A$ with $I \subseteq J$, then

$$A/J \cong A/I/J/I$$
.

iii) Let $B\subseteq A$ be a subring and $I\triangleleft A.$ Then $I\cap B\triangleleft B$ and

$$B+I/I \cong B/B \cap I$$
.

Theorem 1.1.16 (Chinese remainder theorem). If $I_1, \ldots, I_n \triangleleft A$ are pairwise comaximal, then

$$A/I_1 \cap \cdots \cap I_n \cong \prod_{k=1}^n A/I_k$$
.

1.2 Modules

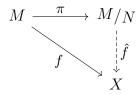
Definition 1.2.1. Let M be an A-module and $E \subseteq M$. The A-module generated by E is denoted by

$$\langle E \rangle_A = \left\{ \sum_{k=1}^n a_k m_k \mid a_k \in A \land m_k \in E \right\}.$$

Proposition 1.2.2. Let M be an A-module and $I \triangleleft A$. Then M/IM is an A/I-module via the natural product.

Remark 1.2.2.1. Categorically, A/I-modules are equivalent to A-modules M with IM = 0.

Theorem 1.2.3 (Universal property). Let $N \leq M$ be A-modules and $\pi: M \to M/N$ be the canonical epimorphism. If $f: M \to X$ is an A-module homomorphism with $N \subseteq \ker f$, then there exists a unique homomorphism $\hat{f}: M/N \to X$ such that $f = \hat{f} \circ \pi$.



Theorem 1.2.4 (Isomorphism theorems). The following statements hold:

- i) We have $f(M) \cong M/\ker(f)$.
- ii) If $N \leq M$, then submodules $N \leq X \leq M$ are in bijective correspondence with submodules of M/N.
- iii) If $N \leq X \leq M$, then

$$M/X \cong M/N/X/N$$
.

iv) If $N, N' \leq M$, then

$$N + N'/N \cong N'/N \cap N'$$
.

Theorem 1.2.5 (Universal property). If $(f_i: M_i \to X)_{i \in I}$ is a family of A-module homomorphisms, then there exists a unique homomorphism

$$\hat{f}: \bigoplus_{i \in I} M_i \to X$$

such that $f_i = \hat{f} \circ \varepsilon_i$ for all $i \in I$.

If $(g_i: X \to M_i)_{i \in I}$ is a family of A-module homomorphisms, then there exists a unique homomorphism

$$\hat{g}\colon X\to\prod_{i\in I}M_i$$

such that $g_i = \pi_i \circ \hat{g}$ for all $i \in I$.

Remark 1.2.5.1. Note that

$$\bigoplus_{i\in I} M_i \subseteq \prod_{i\in I} M_i.$$

If I is finite, then A-modules form an abelian category.

Definition 1.2.6. An A-module M is free if

$$M \cong \bigoplus_{i \in I} A$$

for some set I. A basis of M is a family $(m_i)_{i\in I}$ such that the map

$$\bigoplus_{i \in I} A \to M, \quad (a_i)_{i \in I} \mapsto \sum_{i \in I} a_i m_i$$

is an isomorphism.

Lemma 1.2.7. The following statements hold:

- i) Every module is a quotient of a free module.
- ii) A module M is finitely generated if and only if there exists an epimorphism $\varphi \colon A^k \to M$ for some integer k.
- iii) A module M is finitely generated and free if and only if $M \cong A^k$ for some integer k.

Lemma 1.2.8 (Nakayama). Let M be a finitely generated module over A.

- i) If J(A)M = M, then M = (0).
- ii) If $N \leq M$ such that M = N + J(A)M, then N = M.

Proof.

i) Assume $M \neq 0$ and let $m_1, \ldots, m_r \in M$ be a minimal generating set. Note that, as $M \neq 0, r \geq 1$. By our assumptions, we can write

$$m_r = \sum_{i=1}^r a_i m_i,$$

where $a_i \in J(A)$. But as $1 - a_r$ is invertible, we can express m_r as a linear combination of the other elements, which is a contradiction.

ii) Note that

$$J(A) M/N = J(A)M + N/N = M/N,$$

hence M/N=0.

Definition 1.2.9. A ring A is *local* if $A \neq 0$ and A has a unique maximal ideal. We denote it by (A, \mathfrak{m}) , where \mathfrak{m} is the maximal ideal.

Remark 1.2.9.1. If (A, \mathfrak{m}) is local, then A/\mathfrak{m} is a field and $\mathfrak{m} = J(A)$.

Corollary 1.2.9.2. Let (A, \mathfrak{m}) be a local ring and M a finitely generated module. If $x_1 + \mathfrak{m}M, \ldots, x_r + \mathfrak{m}M$ is a basis of the A/\mathfrak{m} -vector space $M/\mathfrak{m}M$, then $x_1, \ldots, x_r \in M$ generate M.

Proposition 1.2.10. If $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is a short exact sequence, then the diagram

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

$$f \downarrow \cong \operatorname{id} \qquad (\hat{g})^{-1} \downarrow \cong$$

$$0 \longrightarrow \operatorname{im}(f) \hookrightarrow N \xrightarrow{\pi} N/K \longrightarrow 0$$

commutes and has exact rows, where $\hat{g}(n+K) = g(n)$.

Lemma 1.2.11. The following statements hold:

i) A sequence $0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} P$ is exact if and only if the sequence

$$0 \longrightarrow \operatorname{Hom}(X,N) \xrightarrow{f_*} \operatorname{Hom}(X,M) \xrightarrow{g_*} \operatorname{Hom}(X,P)$$

is exact for every A-module X.

ii) A sequence $N \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$ is exact if and only if the sequence

$$0 \longrightarrow \operatorname{Hom}(P,X) \stackrel{f^*}{\longrightarrow} \operatorname{Hom}(M,X) \stackrel{g^*}{\longrightarrow} \operatorname{Hom}(N,X)$$

is exact for every A-module X.

Definition 1.2.12. Let M_1, \ldots, M_n and P be A-modules. A map $f: M_1 \times \cdots \times M_n \to P$ is A-multilinear if it is linear in every component.

Definition 1.2.13. Let M_1, \ldots, M_n be A-modules. The tensor product $M_1 \otimes \cdots \otimes M_n$ is the A-module together with the multilinear map

$$\otimes : \prod_{i=1}^{n} M_i \to \bigotimes_{i=1}^{n} M_i$$

defined by the following universal property: For every A-module P and every multilinear map $f: M_1 \times \cdots \times M_n \to P$ there exists a unique A-module homomorphism

$$\hat{f} : \bigotimes_{i=1}^{n} M_i \to P$$

such that $\hat{f} \circ \otimes = f$.

$$\prod_{i=1}^{n} M_{i} \xrightarrow{\otimes} \bigotimes_{i=1}^{n} M_{i}$$

$$f \qquad \downarrow \hat{f}$$

$$P$$

Remark 1.2.13.1. The tensor product is associative and commutative. It is functorial in each component.

Theorem 1.2.14 (Hom- \otimes adjunction). Let M be an A-module. Then $\cdot \otimes M$ is left-adjoint to $\operatorname{Hom}(M,\cdot)$. That is, for any A-modules M, N and P, there are A-isomorphisms $\operatorname{Hom}(N\otimes M,P)\to\operatorname{Hom}(N,\operatorname{Hom}(M,P))$, given by $f\mapsto (n\mapsto (m\mapsto f(n\otimes m)))$ with inverse $g\mapsto (n\otimes m\mapsto g(n)(m))$. These isomorphisms are natural transformations in N, M and P.

Corollary 1.2.14.1. The tensor product $M \otimes \cdot$ is right-exact. That is, for every exact sequence $N \xrightarrow{f} P \xrightarrow{g} Q \longrightarrow 0$ the sequence

$$M \otimes N \xrightarrow{\operatorname{id} \otimes f} M \otimes P \xrightarrow{\operatorname{id} \otimes g} M \otimes Q \longrightarrow 0$$

is also exact.

Proof. Applying lemma 1.2.11, we see that

$$0 \longrightarrow \operatorname{Hom}(Q, X) \xrightarrow{f^*} \operatorname{Hom}(P, X) \xrightarrow{g^*} \operatorname{Hom}(N, X)$$

is exact. Applying lemma 1.2.11 again, we see that

$$0 \longrightarrow \operatorname{Hom}(M, \operatorname{Hom}(Q, X)) \stackrel{(f^*)_*}{\longrightarrow} \operatorname{Hom}(M, \operatorname{Hom}(P, X)) \stackrel{(g^*)_*}{\longrightarrow} \operatorname{Hom}(M, \operatorname{Hom}(N, X))$$

is exact as well. Applying the previous theorem and lemma 1.2.11 again, we get the required sequence. \Box

Corollary 1.2.14.2. For any family $(N_i)_{i \in I}$ of A-modules we have

$$M \otimes \left(\bigoplus_{i \in I} N_i\right) \subseteq \bigoplus_{i \in I} \left(M \otimes N_i\right).$$

Proof. We can construct the isomorphisms using the universal properties.

Proposition 1.2.15. Let M be an A-module and $I \triangleleft A$. Then $M/MI \cong M \otimes_A A/I$.

Proof. As $0 \to I \hookrightarrow A \to A/I \to 0$ is a short exact sequence, the sequence

$$I\otimes M \longrightarrow A\otimes M \longrightarrow A/I\otimes M \longrightarrow 0.$$

But as $A \otimes M \cong M$ under $\mu(a \otimes m) = am$ and $\mu|_{I \otimes M} = IM$, we get

$$M/_{IM} \cong A \otimes M/_{I \otimes M} \cong A/_{I} \otimes M.$$

Proposition 1.2.16. If $A \neq 0$ and $A^{(I)} \cong A^{(J)}$, then |I| = |J|.

$$^{1} A^{(I)} = \bigoplus_{i \in I} A.$$

Proof. Let M be a maximal ideal in A, then K = A/M is a field. Then

$$A^{(I)} \otimes A \big/ M \cong \left(A \otimes A \big/ M \right)^{(I)} \cong \left(A \big/ M \right)^{(I)} \cong K^{(I)}$$

as A-modules and A/M-modules. Hence $K^{(I)} \cong K^{(J)}$, therefore |I| = |J|.

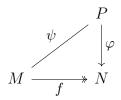
Definition 1.2.17. If M is a finitely generated free module, its rank is the unique $n \in \mathbb{N}$ such that $M \cong A^n$.

Definition 1.2.18. A module M is

- $projective if Hom(M, \cdot)$ is exact.
- injective if $Hom(\cdot, M)$ is exact.
- $flat \text{ if } M \otimes \cdot \text{ is exact.}$

Theorem 1.2.19. The following statements are equivalent for an A-module P:

- i) The module P is projective.
- ii) For every epimorphism $g: M \to N$, the map $g_*: \operatorname{Hom}(P, M) \to \operatorname{Hom}(P, N)$ is an epimorphism.
- iii) For every epimorphism $f: M \to N$ and homomorphism $\varphi: P \to N$ there exists a homomorphism $\psi: P \to M$ with $f \circ \psi = \varphi$.



- iv) Every epimorphism $g: M \to P$ splits.
- v) There exists an A-module M such that $P \oplus M$ is free.

Proof. The first statement implies the second by definition.

Now assume that the second statement holds. The map f_* : $\operatorname{Hom}(P, M) \to \operatorname{Hom}(P, N)$ is therefore an epimorphism. By definition, we can construct a homomorphism $\psi \colon P \to M$ that maps to f.

Assume now that the diagram condition holds. Then there exists a homomorphism $s \colon P \to M$ such that $q \circ s = \mathrm{id}$.

Suppose every epimorphism $g \colon M \to P$ splits. In particular, this holds for an epimorphism $g \colon A^{(I)} \to P$. But then

$$0 \longrightarrow \ker g \longrightarrow A^{(I)} \stackrel{g}{\longrightarrow} P \longrightarrow 0$$

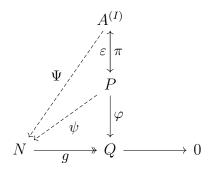
is a short exact sequence. As it splits, $P \oplus \ker g \cong A^{(I)}$.

Finally, suppose that $P \oplus C \cong A^{(I)}$ is a free module. Take a short exact sequence

$$0 \longrightarrow M \stackrel{f}{\longrightarrow} N \stackrel{g}{\longrightarrow} Q \longrightarrow 0.$$

As $\operatorname{Hom}(P,\cdot)$ is left-exact, we only need to check that $g_*\colon \operatorname{Hom}(P,N)\to \operatorname{Hom}(P,Q)$ is surjective. Let $\varphi\in \operatorname{Hom}(P,Q)$. Let $\pi\colon A^{(I)}\to P$ be the canonical projection and $\varepsilon\colon P\hookrightarrow A^{(I)}$ the embedding. Then $\pi\circ\varepsilon=\operatorname{id}_P$.

For each basis vector $e_i \in A^{(I)}$, choose $n_i \in N$ such that $g(n_i) = \varphi \circ \pi(e_i)$, which is possible by surjectivity of g.



Construct a homomorphism Ψ : Hom $(A^{(I)}, N)$ by $\Psi(e_i) = n_i$. Then $g \circ \Psi = \varphi \circ \pi$. Now define $\psi = \Psi \circ \varepsilon$. Then, for every $p \in P$, we have

$$g \circ \psi(p) = g \circ \Psi \circ \varepsilon(p) = \varphi \circ \pi \circ \varepsilon(p) = \varphi(p),$$

hence
$$g_*(\psi) = \varphi$$
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