Commutative algebra

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Introduction

These are my lecture notes on the course Commutative algebra in the year 2024/25. The lecturer that year was gost. izr. prof. dr. rer. nat. Daniel Smertnig.

The notes are not perfect. I did not write down most of the examples that help with understanding the course material. I also did not formally prove every theorem and may have labelled some as trivial or only wrote down the main ideas.

I have most likely made some mistakes when writing these notes – feel free to correct them.

1 Rings and modules

1.1 Rings and ring homomorphisms

Definition 1.1.1. Unless stated otherwise, rings always have a unit and are commutative.

Definition 1.1.2. Let A be a ring. The set $A^\bullet$ denotes the set of non-zero-divisors.

Definition 1.1.3. A ring A is a *domain* if 0 is the only zero-divisor of A.

Definition 1.1.4. Let $A \subseteq B$ be rings and $S \subseteq B$ a subset. The ring

$$A[S] = \bigcap_{\substack{A \subseteq A' \subseteq B \\ S \subseteq A'}} A'$$

is the subring of B obtained by *adjoining* S to A.

Definition 1.1.5. The set $\operatorname{Spec}(A)$ denotes the prime ideals of A.

Definition 1.1.6. The *radical* of an ideal I is defined as

$$\sqrt{I} = \{a \in A \mid \exists n \in \mathbb{N} \colon a^n \in I\}.$$

Proposition 1.1.7. The radical of an ideal is again an ideal.

Proof. It suffices to show that for any $a, b \in \sqrt{I}$ their sum is also in $\sqrt{I}$. Suppose that $a^n, b^m \in I$. Then

$$\begin{aligned} (a+b)^{n+m-1} &= \sum_{k=0}^{n+m-1} \binom{m+n-1}{k} a^k b^{n+m-1-k} \\ &= b^m \sum_{k=0}^{n-1} \binom{m+n-1}{k} a^k b^{n-1-k} + a^n \sum_{k=n}^{n+m-1} \binom{m+n-1}{k} a^{k-n} b^{n+m-1-k} \in I. \end{aligned}$$ □

Definition 1.1.8. The *nilradical* of A is the set $\mathcal{N}(A) = \sqrt{(0)}$.

Definition 1.1.9. The *Jacobson radical* $\mathcal{J}(A)$ is the intersection of all maximal ideals in A.

Lemma 1.1.10. The nilradical is contained in the Jacobson radical.

Proof. Let $a \in \mathcal{N}(A)$ and suppose that $a^n = 0$. For any maximal ideal M, we know that $a^n \in M$. Since M is prime, we deduce $a \in M$. □

Lemma 1.1.11. We have

$$\mathcal{J}(A) = \left\{a \in A \;\middle|\; \forall b \in A \colon 1 - ba \in A^\times\right\}.$$

Proof. Let $a \in \mathcal{J}(A)$ and $b \in A$. Note that $1 - ab \notin M$ for any maximal ideal M, since $ab \in M$. As $1 - ab$ is not contained in any maximal ideal, it follows that $(1 - ab) = A$, hence $1 - ab$ is invertible.

Suppose now that $1 - ab \in A^\times$ for all $b \in A$. Let M be a maximal ideal and suppose $a \notin M$. Then $(M, a) = A$. In particular, we can write $1 = m + xa$ with $m \in M$ and $x \in A$. Rearranging, $m = 1 - xa$, which is a contradiction, as $1 - xa$ is invertible. □

Lemma 1.1.12. The following statements hold:

i) Let $I \lhd A$ and $P_1, \dots, P_n \in \operatorname{Spec}(A)$. If $I \subseteq P_1 \cup \dots \cup P_n$, there exists some k such that $I \subseteq P_k$.

ii) Let $I_1, \dots, I_n \lhd A$ and $P \in \operatorname{Spec}(A)$. If $I_1 \cap \dots \cap I_n \subseteq P$, then there exists some k such that $I_k \subseteq P$.

Proof.

i) We induct on n, noting that the statement trivially holds for $n = 1$.

Suppose the statement doesn't hold for n. By the induction hypothesis we can find

$$a_i \in I \Big\backslash \bigcup_{j \neq i} P_j$$

for any i. Then $a_i \in P_i$. Consider the element

$$a = \sum_{i=1}^{n} \prod_{j \neq i} a_j.$$

Note that all but one of the above terms are an element of P_i. But then a is not an element of any P_i, which is a contradiction.

ii) Suppose the contrary and let $a_j \in I_j \setminus P$ for all j. But then

$$\prod_{j=1}^{n} a_j \subseteq \prod_{j=1}^{n} I_j \subseteq \bigcap_{j=1}^{n} I_j \subseteq P,$$

which is a contradiction. $\square$

Remark 1.1.12.1. The first statement is called *prime avoidance*.

Proposition 1.1.13. Let $f \colon A \to B$ be a ring homomorphism. If $I \lhd B$, then $f^{-1}(I) \lhd A$. Furthermore, if $P \in \operatorname{Spec}(B)$, then $f^{-1}(P) \in \operatorname{Spec}(A)$.

Proposition 1.1.14 (Universal property)**.** Let $I \lhd A$ and $\pi \colon A \to A/I$ be the canonical epimorphism. For every ring homomorphism $f \colon B$ with $I \subseteq \ker(f)$, there exists a unique ring homomorphism $\hat{f} \colon A/I \to B$ such that $f = \hat{f} \circ \pi$.

$$\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[dr, "f"'] & A/I \arrow[d, dashed, "\hat{f}"] \\
& B
\end{tikzcd}$$

Corollary 1.1.14.1. If $f \colon A \to B$ is a ring homomorphism, then $A/\ker f \cong f(A)$.

Theorem 1.1.15 (Isomorphism theorems)**.** The following statements hold:

i) Let $I \lhd A$. There is a bijective correspondence

$$\{J \lhd A \mid I \subseteq J\} \leftrightarrow \left\{\overline{J} \lhd A/I\right\},$$

given by $J \mapsto J/I$ and $\overline{J} \mapsto \pi^{-1}\left(\overline{J}\right)$.

ii) If $I, J \triangleleft A$ with $I \subseteq J$, then

$$A/J \cong {}^{A/I}\Big/_{J/I} .$$

iii) Let $B \subseteq A$ be a subring and $I \triangleleft A$. Then $I \cap B \triangleleft B$ and

$$B + I/I \cong B/B \cap I .$$

Theorem 1.1.16 (Chinese remainder theorem)**.** If $I_1, \ldots, I_n \triangleleft A$ are pairwise comaximal, then

$$A/I_1 \cap \cdots \cap I_n \cong \prod_{k=1}^{n} A/I_k .$$

1.2 Modules

Definition 1.2.1. Let M be an A-module and $E \subseteq M$. The *A-module generated by E* is denoted by

$$\langle E\rangle_A = \left\{ \sum_{k=1}^{n} a_k m_k \;\middle|\; a_k \in A \land m_k \in E \right\}.$$

Proposition 1.2.2. Let M be an A-module and $I \triangleleft A$. Then M/IM is an A/I-module via the natural product.

Remark 1.2.2.1. Categorically, A/I-modules are equivalent to A-modules M with $IM = 0$.

Theorem 1.2.3 (Universal property)**.** Let $N \leq M$ be A-modules and $\pi\colon M \to M/N$ be the canonical epimorphism. If $f\colon M \to X$ is an A-module homomorphism with $N \subseteq \ker f$, then there exists a unique homomorphism $\hat{f}\colon M/N \to X$ such that $f = \hat{f} \circ \pi$.

$$\begin{tikzcd}
M \arrow[r, "\pi"] \arrow[dr, "f"'] & M/N \arrow[d, dashed, "\hat{f}"] \\
& X
\end{tikzcd}$$

Theorem 1.2.4 (Isomorphism theorems)**.** The following statements hold:

i) We have $f(M) \cong M/\ker(f)$.

ii) If $N \leq M$, then submodules $N \leq X \leq M$ are in bijective correspondence with submodules of M/N.

iii) If $N \leq X \leq M$, then
$$M/X \cong {}^{M/N}\big/_{X/N}.$$

iv) If $N, N' \leq M$, then
$$N + N'/N \cong N'/N \cap N'.$$

Theorem 1.2.5 (Universal property)**.** If $(f_i\colon M_i \to X)_{i\in I}$ is a family of A-module homomorphisms, then there exists a unique homomorphism

$$\hat{f}\colon \bigoplus_{i\in I} M_i \to X$$

such that $f_i = \hat{f} \circ \varepsilon_i$ for all $i \in I$.

If $(g_i\colon X \to M_i)_{i\in I}$ is a family of A-module homomorphisms, then there exists a unique homomorphism

$$\hat{g}\colon X \to \prod_{i\in I} M_i$$

such that $g_i = \pi_i \circ \hat{g}$ for all $i \in I$.

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