Commutatieve algebra

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Commutatieve algebra	
Richting	Wiskunde
Jaar	3BWIS

Bespreking

In dit vak leer je enkele eigenschappen over ringen, modules en algebra's. De nadruk ligt vooral op de ringen en op priemidealen in ringen en hoe deze veranderen in een integrale uitbreiding en in quotiënten.

Professor Becher verwacht niet dat de studenten de bewijzen uit de cursus kunnen reproduceren, maar dat ze deze kunnen gebruiken in oefeningen. Deze oefeningen blijven echter wel abstract! Je kan best zo veel mogelijk tegenvoorbeelden zoeken bij elke stelling in de cursus waar veronderstellingen worden gemaakt. Het maken van de vragen uit de tuyaux kunnen ook een grote hulp zijn bij het studeren, daar je deze als extra oefeningen op examenniveau kunt bekijken.

Het punten systeem bij het vak is zo dat er punten zijn op medewerking tijdens de oefeningen, op een tussentijdse test en op het examen. Indien je slaagt op het examen ben je er zeker door, anders kunnen de test en je medewerking je er nog door helpen!

Augustus 2018

1. Definieer volgende zaken:
    - Het radicaal van een ring.
    - Het radicaal van een ideaal.
    - De dimensie van een ring.
    - Een noetherse van een ring.
2. Formuleer Hilbert's Nullstellensatz (sterke versie). Gebruik dit om het radicaalvan $(x2y3)\subset C[x,y](x2y3)\subset C[x,y]$ te vinden.

3. Stel R=C[s]/(s5−2s3)R=C[s]/(s5−2s3).
    - R is isomorf met 1 van volgende ringen, welke
      C5,C

      C5,C

      x C[x]/(x2),C[x]/(x3)C[x]/(x2),C[x]/(x3) x CC x C,C[x]/(x4)C,C[x]/(x4) x CC Je mag voor het vervolg werken met de gekozen ring.
    - Is R noethers?
    - Is R artins?
    - Wat is de Krull-dimensie van R?
    - Wat is het radicaal van R? Het Jacobson radicaal van R?
    - Met welke ring is R/Jac(R) isomorf?

Januari 2018

Theorie

1. Toon aan:
    - (Nakayama lemma) AA ring, MM eindig voortgebracht AA-moduul. Als Jac(A)M=M⇒M={0}Jac(A)M=M⇒M={0}
    - In de les: R lokale ring, MM eindig voortgebracht RR - moduul, {m}=Max(R){m}=Max(R), k=R/mk=R/m. Als dimkM/mM=m<∞dimkM/mM=m<∞, dan MM voortgebracht door mm elementen. Vraag : Is MM noodzakelijk vrij? Toon aan of geef tegenvoorbeeld.
2. Toon aan: als RR noethers, p∈Spec(R),ht(p)=m⇒pp∈Spec(R),ht(p)=m⇒p minimaal over ideaal voortgebracht door mm elementen.
3. (Mondeling) R=C[s](s4−s2)R=C[s](s4−s2)
    - RR is isomorf met 1 van de volgende ringen, welke? C4C4 , C×C[x]/(x3)C×C[x]/(x3) , (C[x]/(x2))2(C[x]/(x2))2 , C2×C[x]/(x2)C2×C[x]/(x2)
    - RR noethers/artins?
    - dim(R)=dim(R)=?
    - Rad(R)=Rad(R)=?, Jac(R)=Jac(R)=?
    - R/(Jac(R))=R/(Jac(R))=?
4. (Extra vraag) RR noethers domein, K=Frac(R)K=Frac(R). RR DVR ⇔∀x∈K∖{0}:x∈R⇔∀x∈K∖{0}:x∈R of x−1∈Rx−1∈R

Oefeningen

1. True or false
    - A local ring in Noetherian.
    - For an Artinian ring AA, every Noetherian module is also Artinian.
    - Every finitely generated module over a PID is either torsion or torsion free.
    - If dim(A)=dim(B)dim(A)=dim(B) for a ring extension A⊂BA⊂B, then B/AB/A is integral.

2. Give examples of the following
   - A ring with $0\neq N(A)\neq J(A)$.
   - A ring with infinitely many maximal ideals but only one minimal prime ideal.
   - A ring of dimension 0 with exactly 3 maximal ideals.
   - A ring extension $A\subset B$ and a B-module M that is finitely generated as B-module, but not as A-module.
3. Let A be a ring. Show that A as an A-module has finite length $l(A)<+\infty$ if and only if A is Noetherian and $\dim(A)=0$.
4. Let $n\in \mathbb{Z}$. Show that the ring $\mathbb{Z}/n\mathbb{Z}$ is reduced if and only if n is square-free. Calculate $\dim(\mathbb{Z}/n\mathbb{Z})$ for these n.
5. Let B/A be integral.
   - Show that if $y\in A$ has an inverse $y^{-1}\in B$ that $y^{-1}\in A$.
   - Show that $J(B)\cap A\subset J(A)$.
   - Show that for any maximal ideal $M\in \mathrm{Max}(B)$, the intersection $M\cap A\in \mathrm{Max}(A)$.
   - Conclude that $J(A)=J(B)\cap A$.
   - Give an example where the equality fram (d) does not hold for an extension of rings that is not integral.
6. Assume that A is Noetherian and reduced. Let S be the set of all non-zero divisors in A. Show the following:
   - S is multiplicatively closed.
   - $\dim(A_S)=0$.
   - A_S is isomorphic to a finite product of fields.
7. Let $\omega=e^{\frac{2\pi i}{5}}$ be a primitive 5-th root of unity. Determine the Krull dimension of the following ring $A=\mathbb{Z}[\omega+\omega^3][X,Y,Z,W]/(XYZ-W^2)$

November 2016

Test 1

A is a commutative ring.

1. Prove or give a counterexample:
   - If A is a field, then $\dim(A)=0$.
   - If A is factorial, then $\dim(A)\leq 1$.
   - Let A be a domain, then $\mathrm{Jac}(A)=\mathrm{Nil}(A)$.
2. Let $f\in \mathbb{Q}[X]$. Then show that the following are equivalent:
   - $\mathbb{Q}[X]/(f)$ is reduced.
   - $\mathbb{Q}[X]/(f)$ is isomorphic to a finite product of fields.
   - f has only simple roots in $\mathbb{C}$.
3. Every simple A-module is isomorhpic to A/m for an $m\in \mathrm{Max}(A)$.
4. Let A be a domain, but not a field. Show that the field of fractions of A is not finitely generated as an A-module. (Hint: First look at the case where A is local.)

Test 2

Let A be a commutative ring and K a field.

1. Prove or give a counterexample
    - $K[X]$ is integrally closed.
    - If B/A is a ring extension with $\dim(A)=\dim(B)$, then it is an integral extension.
2. Prove that if A is a local ring, then any two simple A-modules are isomorphic.
3. Assume $A\subseteq B\subseteq K$ (A and B subrings). Let B be finitely generated as an A-module, then
    - A en B have the same integral closure in K
    - $\mathrm{Jac}(B)\cap A=\mathrm{Jac}(A)$
4. Consider the ring $\mathbb{C}[X,Y]/(Y^2-X^3-X^2)$. Show that
    - It is a domain
    - It has Krull-dimension 1

Januari 2016

A a commutative ring.

1. Prove or give a counterexample
    - There exists a domain whose nilradical is not finitely generated.
    - Every submodule of a finitely generated module is finitely generated.
    - There exists a commutative ring with exactly four prime ideals, two maximal and two minimal.
    - A has finitie length as an A-module if and only if A is noetherian and $\dim(A)=0$.
2. Let A be a domain and K its field of fractions and I an A-submodule of K. Show that:
    - If I is finitely generated then $\lambda I\subseteq A$ for some $\lambda\in A\setminus\{0\}$.
    - If A noetherian and $\lambda I\subseteq A$ for some $\lambda\in A\setminus\{0\}$, then I finitely generated.
    - If A noetherian, J a nonzero ideal of A and $x\in K$ such that $xJ\subseteq J$, then x is integral. (Hint: Apply the second result to $I=A[x]$.)
3. Let A be the integral closure of $\mathbb{Z}$ in $\mathbb{C}$. Determine $\dim(A)$ and show that A is neither noetherian nor factorial. (Hint: Look at $\sqrt[2m]{2}$ for $m\geq 1$.)
4. Assume K field and $f,g\in K[X,Y]$ where f is irreducible and does not divide g. Show that:
    - $K[X,Y]/(f,g)$ is artinian.
    - Only finitely many prime ideals of $K[X,Y]$ contain both f and g.
    - There are only finitely many points $(a,b)\in K\times K$ with $f(a,b)=g(a,b)=0$.

5. Let AA be noetherian and reduced and SS the set of all non-zero divisors in AA. Show
    - SS is multiplicatively closed.
    - dim(AS)=0dim(AS)=0
    - ASAS is isomorphic to a finite product of fields.
6. Consider A=R[X,Y]/(X2−Y3)A=R[X,Y]/(X2−Y3)
    - AA a domain but no field.
    - AA isomorphic to the subring R[Ti,Tj]R[Ti,Tj] of R[T]R[T] for certain i,j∈Ni,j∈N.
    - AA not integrally closed.
    - AA has trancendence degree 1 over RR.

Januari 2015

Let AA always be a commutative ring.

1. **Exercise 1.** Suppose AA is local. Show the following:
    - x∈A×x∈A× or 1−x∈A×1−x∈A×
    - x2=x⇔x=0x2=x⇔x=0 or x=1x=1.
2. **Exercise 2.** Suppose for every x∈Ax∈A there exists n∈Nn∈N such that xn=xxn=x.
    - Prove that every prime ideal is maximal.
    - Give an example of such a ring that is not Artinian.
3. **Exercise 3.** Let MM be an AA-module and UU be a submodule of MM. Decide which of the following statements hold in general or give a counter-example.
    - If UU and M/UM/U are finitely generated, then MM is finitely generated.
    - If MM is finitely generated, then UU is finitely generated.
    - If MM is finitely generated, then M/UM/U is finitely generated.
4. **Exercise 4.** Let n∈Nn∈N
    - Suppose f:M→Nf:M→N is a homomorphism of modules where NN is finitely generated and that im(f)+Jac(A)∗N=Nim(f)+Jac(A)∗N=N. Show that ff is surjective.
    - Suppose A=C[T1,…,Tn]A=C[T1,…,Tn] and m∈Max(A)m∈Max(A). Show that there is a unique automorphism ϕ:A→Aϕ:A→A such that f(m)=(X1,…,Xn)f(m)=(X1,…,Xn)
5. **Exercise 5.**
    - Suppose A→A→ is an integral extension. Show that A∩B×=A×A∩B×=A×
    - Suppose that if B∖AB∖A is multiplicatively closed, then A is integrally closed in B.
    - Give an example of a proper extension where B∖AB∖A is multiplicatively closed.
6. **Exercise 6.** Determine the Krull dimension of the following rings and give a sequence of prime ideals of maximum length.
    - Z[X1,…,Xn]Z[X1,…,Xn]
    - R[X,Y,Z]/(X2+Y2+Z2)R[X,Y,Z]/(X2+Y2+Z2)

Test December 2014

1. **Exercise 1.** Give examples of the following:
    - A finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{Q}(\sqrt{-1})$ that is not integral over $\mathbb{Z}$.
    - A finitely generated commutative $\mathbb{Z}$-algebra that is not finitely generated as a $\mathbb{Z}$-module.
    - A domain D with fraction field $K=D[x]$ for some $x\in K\setminus D$
2. **Exercise 2.** Let A be a commutative ring. Compare the following properties of an A-module M. (Decide by proofs or counter-examples which of the following properties and which implications between them do hold.)
    - P1 M has a minimal set of generators.
    - P2 M contains a maximal A-linearly independent subset.
    - P3 M is free.
3. **Exercise 3.** Let $A=\mathbb{R}[X,Y]/(1+X^2+Y^2)$. Show the following:
    - A is a domain.
    - A is an integral extention of a PID.
    - A has Krull dimension 1.
4. **Exercise 4.** Let A be a domain such that the set of ideals of A is totally ordered by inclusion. Show the following:
    - A is integrally closed.
    - Every finitely generated ideal of A is principal.
    - If there exists $\pi\in A$ such that $A\pi\in\operatorname{Max}(A)$ and $\bigcap_{n\in\mathbb{N}}A\pi^n=\{0\}$ then every ideal of A is of the form $A\pi^n$ for some $n\in\mathbb{N}$.

Test November 2014

1. Exercise 1. Decide (by proof or counter-example) which of the following statements are true for every finite commutative ring A:
    - $\operatorname{Jac}(A)=\operatorname{Nil}(A)$
    - $\operatorname{Nil}(A)=\{0\}$
    - Every finitely generated A-module is noetherian.
    - Every ring homomorphism $A\to\mathbb{Z}$ is surjective.
2. Exercise 2. Let $A=\mathbb{R}[T]$ and let $\mathfrak{b}$ be a nonzero ideal of A. Show the following:
    - The Krull dimension of $A/\mathfrak{b}$ is zero.
    - $A/\sqrt{\mathfrak{b}}\approx\mathbb{R}^m\times\mathbb{C}^n$ for some $m,n\in\mathbb{N}$.
3. Exercise 3. Let $n\in\mathbb{N}$, let A be a commutative ring and $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ minimal prime ideals of A. Consider the multiplicative set $S=A\setminus\bigcup_{i=1}^n\mathfrak{p}_i$. Show the following:
    - Every element of A_S is either a zero-divisor or a unit.
    - A_S has precisely n prime ideals.
    - For $a\in\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_n$ there exists $b\in S$ and $r\in\mathbb{N}$ with $a^rb=0$.

4. Exercise 4. Give examples of the following:
    - A local domain that is not a field and where 1+1+1=01+1+1=0.
    - A commutative ring containing a field and a non-zero nilpotent element.
    - A ZZ-module that is neither free nor finitely generated.
    - A commutative ring AA with an AA-module that is finitely generated but not noetherian.
5. Exercise 5. Let AA be a local commutative ring whose maximal ideal mm is finitely generated. Show that mm is a principal ideal if and only if for every i∈Ni∈N there exist x∈Ax∈A such that mi=mi+1+Axmi=mi+1+Ax. (Met de machten word het ideaalproduct bedoeld.)

Test November 2013

1. Let *n* be a positive integer and p1,...,pn∈Spec(A)p1,...,pn∈Spec(A) such that pi⊄pjpi⊄pj, whenever i≠ji≠j. Show that S=A∖(p1∪...∪pn)S=A∖(p1∪...∪pn) is a multiplicative set and that the ring ASAS has exactly *n* maximal ideals.
2. Assume that A is a pricipal ideal domain. Show the following:
    - Every nonzero prime ideal of A is maximal.
    - If aa is a nonzero radical ideal of A, then A/aA/a is isomorphic to a finite product of fields.
3. Let M be a finitely generated A-module. Show that M contains a maximal proper submodule. Give a counterexample to this statement where M is a non-finitely generated module over some ring A.
4. Give examples for the following:
    - An extention of commutative rings that is not integral.
    - A local domain with a nonzero prime ideal that is not maximal.
    - A module that is not free.
    - A cummutative ring that is not a domain and for which the nilradical coincides with the Jacobson radical.
5. Assume that A is a local ring and its maximal ideal m is finitely generated. Explain that m/m2m/m2 is a vector space over the field k=A/mk=A/m. Show that the minimal number of generators of m is equal to dimkm/m2dimkm/m2.

Januari 2013

1. Exercise 1. Give an example (with a short explanation) for each of the following:
    - A cummutative ring with three prime ideals that form a chain.
    - A cummutative ring with exactly five prime ideals.
    - A cummutative domain with exactly three maximal ideals.
    - A commutative noetherian ring that is not artinian.
    - A commutative ring that is not noetherian.
2. Exercise 2. Let e∈Ae∈A such that e2=ee2=e. Consider the multiplicative set S={1,e}S={1,e}, and the corresponding ring of fractions ASAS. Show that the natural homomorphism A→ASA→AS is surjective and determine its kernel. Show further that A≈AS×BA≈AS×B for a cummutative ring BB.

3. Exercise 3. Assume that AA is artinian. Show the following:
    - AA has only finitely many prime ideals.
    - Jac(A)∗n=0Jac(A)∗n=0 for some n∈Nn∈N.
    - AA is isomorphic to a finite product of atinian local rings.
4. Exercise 4. Let pp be a minimal prime ideal of AA. Prove that every element of pp is a zero divisor in AA. (Hint: for x∈px∈p, show that x1∈Apx1∈Ap is nilpotent.)
5. Exercise 5. Let AA be a finitely generated commutative KK-algebra over a field KK. Show that AA is artinian if and only if AA is finite dimentional as a KK-vector space. (Hint: what if A is a polynomail ring, or an integral extension thereof?)
6. Exercise 6. Let GG be a finite subgroup of the group of ring automorphisms of AA (i.e. ring isomorphisms A→AA→A) and consider the subring C={a∈A|g(a)=a∀g∈G}C={a∈A|g(a)=a∀g∈G}. Show the following:
    - The ring extention C→AC→A is integral. (Hint: for a∈Aa∈A consider the polynomial Πg∈G(T−g(a)).)Πg∈G(T−g(a)).)
    - For n∈Nn∈N, there exist prime ideals B0,…,BnB0,…,Bn in A with B0⊂…⊂BnB0⊂…⊂Bn if and only if there exists prime ideals p0,…,pnp0,…,pn in C with p0⊂…⊂pnp0⊂…⊂pn. (Opmerking, dit zijn strikte inclusies, ze zijn niet gelijk)

Januari 2014

Let *A* always be a commutative ring.

1. Show the following:
    - For any x∈Nil(A)x∈Nil(A) one has 1+x∈Ax1+x∈Ax.
    - For u∈Axu∈Ax and x∈Nil(A)x∈Nil(A) one has u+x∈Axu+x∈Ax.
2. Formulate *Nakayama's Lemma*. Use it to show the following: if *A* is local with maximal ideal mm and *M* is a finitely generated *A*-module, then there exists no proper submodule U⊊MU⊊M such that M=U+(m∗M)M=U+(m∗M).
3. Prove or disprove:
    - Every submodule of noetherian module is noetherian.
    - Every subring of noetherian ring is a noetherian ring.
    - The ring Z[15][X]Z[15][X] is noetherian.
4. Let A←BA←B be an integral extension of commutative rings. Prove or disprove:
    - Any radical ideal of *A* is of the form I∩AI∩A for a radical ideal II of BB.
    - If II is an ideal of BB such that I∩A∈Max(A)I∩A∈Max(A), then I∈Max(B)I∈Max(B).
    - Jac(A)=Jac(B)∩AJac(A)=Jac(B)∩A.
5. Assume that *A* is artinian. Show the following:
    - Every element of *A* is either a zero divisor or invertible.
    - There is a finite product of maximal ideals of *A* that is zero.
    - *A* is isomorphic to a finite product of local rings.

6. Determine the *Krull dimension* of the following rings:
    - Z[5–√]Z[5]
    - Z[X]Z[X]
    - C[X,Y]/(f)C[X,Y]/(f) where f$\in$C[X,Y]$\setminus$Cf$\in$C[X,Y]$\setminus$C.

Categorieën:

- Wiskunde
- 3BWIS