Fields and Galois Theory

Algebra III: Fields

Richting	Wiskunde
Jaar	2BWIS

Examenvragen

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1. (4pts) Let F be a field and R a domain, prove the following:
    1. If $F \subset R$ and $[R:F] < \infty \Rightarrow R$ is a field.
    2. Let $F \subset K$ be an extension of fields. For $\alpha \in K$, let $F[\alpha]$ be the F-algebra generated by α. Show that $[F[\alpha]:F] < \infty \Leftrightarrow \alpha$ is algebraic over F.
2. (4pts) Let $\mathbb{F}_q$ be a finite field with $q = p^m$, p prime, elements.
    1. Show that the Frobenius map $Fr: \mathbb{F}_q \to \mathbb{F}_q : x \mapsto x^p$ is an isomorphism and compute its order.
    2. Find an irreducible polynomial $f \in \mathbb{F}_2[X]$ sucht that $\mathbb{F}_2/(f) \cong \mathbb{F}_{2^3}$.
    3. Use the previous item to find a generator of the multiplicative group $\mathbb{F}_{2^3}$.
3. (7pts) Consider $\mathbb{Q} \subset \mathbb{Q}(\sqrt{3}, \sqrt{2})$.
    1. Show that $\mathbb{Q}(\sqrt{3}, \sqrt{2})$ is Galois.
    2. Compute the Galois group.
    3. Use the fundamental theorem of Galois Theory to describe all the intermediate field extensions $\mathbb{Q} \subset K \subset \mathbb{Q}(\sqrt{3}, \sqrt{2})$. For each K describe the corresponding subgroup $H \subset G$ for which $K = \mathbb{Q}(\sqrt{3}, \sqrt{2})^H$.
    4. $\forall g \in G$ write down $g(\sqrt{2} + \sqrt{3})$. Use this to compute the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$. Conclude that $\mathbb{Q}(\sqrt{3}, \sqrt{2}) = \mathbb{Q}(\sqrt{3} + \sqrt{2})$.
4. (3pts) Show that the field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{1 + \sqrt{2}})$ is not Galois.
5. (2pts) Let K be an algebraic field extension of F. Define $m_\alpha : K \to K : x \mapsto \alpha \cdot x$ for all $\alpha \in K$. Consider the F-linear trace map of K over F: $Tr_{K/F} : K \to F : \alpha \mapsto Tr(m_\alpha)$.
    1. Describe the trace map $Tr_{\mathbb{C}/\mathbb{R}}$.
    2. Show that if $x \in F$, then $Tr_{K/F}(x) = ([K:F])x$.
    3. Let $\alpha \in K$ be an element with minimal polynomial $f(X) = X^d + a_{d-1}X^{d-1} + \ldots + a_0 \in F[X]$. Show that $Tr_{K/F}(\alpha) = a_{d-1} \cdot \frac{[K:F]}{d}$. You may use the fact that if $F \subset L \subset K$, then $Tr_{K/F} = Tr_{L/F} \circ Tr_{K/L}$.

Categorieën:

- Wiskunde
- 2BWIS