Numerieke methoden

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Richting	Fysica
Jaar	2BFYS

Bespreking

Bij het examen kijkt de prof je examen na tijdens het mondeling en duidt op eventueel domme foutjes of ontbrekende verklaringen.

Puntenverdeling

10% van de punten staat op je doorgestuurde code uit de practica. 30% staat op je antwoorden op de vragen van de testjes die bij de practica horen. Het project staat op 10% en het examen staat op 50%. Je moet er zowel op het examen als de permanente evaluatie door zijn. Voor de permanente evaluatie heb je geen herkansing tijdens de herexamens, dus het is belangrijk dat je hier tijdens het jaar zeker je best voor doet.

Examenvragen

Academiejaar 2019-2020 1ste zit

Prof. Milorad Milosevic
Vraag 1
Give the general definition of a floating-point representation (5p), and how many different values can be represented in the given representation (5p). Write decimal number 39.7 in 64-bit binary form (5p).

Vraag 2
Derive and explain Newton method for solving nonlinear equations (10p). Tip: link to the fixed-point problem (think about convergence)

Vraag 3
Explain and formulate the Jacobi and the Gauss-Seidel iterative methods in the matrix form, for solving a system of linear equations Ax=b (7p). If A is a diagonally-dominant matrix, how does convergence of above methods depend on the initial conditions (3p)?

Vraag 4
The precision of the Simpson rule is 3, although the method is based on interpolation by quadratic polynomials. Demonstrate how that is achieved (10p). Derive the expression for the error of the Simpson rule (5p).

Vraag 5
Demonstrate (based on condition number) that direct solving of normal equations is not a stable algorithm for the least-squares problem (5p). Explain the alternative to solve the least-squares problem (5p).

Vraag 6
One possible method to solve the wave-equation is the "staggered leap-frog" algorithm. In one dimension, the finite differencing gives:
an+1j−2anj+an−1jτ2=c2anj+1−2anj+anj−1h2

where anj=a(jh,nτ), with h the grid step and τ the time step. Perform Von Neumann stability analysis and determine the condition of stability (15p). **Bonus**: Demonstrate how the stability analysis is performed in case of a given boundary condition a00=α(10p).

Vraag 7
Explain the idea behind Adams multistep methods for solving ODEs (5p). Name two advantages and two disadvantages for such methods, compared to Runge-Kutta methods of same order (10p).

Vraag 8
What is the shifted power method (3p)? Show mathematically how it works (7p).

Academiejaar 2018-2019 1ste zit

Prof. Milorad Milosevic
Vraag 1
Give the general definition of a floating-point representation and show how many different values can be represented in the given representation. (5p) Convert decimal number 217.2 to 64-bit binary form. (5p) **Bonus:** How about −217.2? (5p)

Vraag 2
Derive and explain Newton's method for solving nonlineair equations. (10p) How is it related to the fixed-point problem? (5p)

Vraag 3
For
A=(4111),b=(33),x0=(00),

one can apply the iterative Jacobi method to solve the system Ax=b, starting from x0. Is it possible to use the residual to determine when to stop the iterative Jacobi procedure? Demonstrate and explain. (15p)

Vraag 4
Derive the expression for Romberg integration using the Simpson rule. (10p) Compare the error of the formula to that of the Simpson rule. (5p)

Vraag 5
The Richardson method to solve the diffusion equation is given as:
Tn+1i−Tn−1i2τ=κTni+1+Tni−1−2Tnih2

, where Tni=T(ih,nτ), with h the grid step and τ the time step. Perform Von Neumann stability analysis and determine the condition of stability. (15p) What is the main shortcoming of the Von Neumann stability analysis? (2p) Show how that shortcoming can be addressed. (3p)

Vraag 6

What is a Runge-Kutta method and why is it better than a conceptually simpler higher-order Taylor expansion? (3p) What determines the order of the Runge-Kutta method? (3p) Which order should we choose as optimal? (2p) Are coefficients in the Runge-Kutta method of order n uniquely defined and why? (2p) Are multistep methods better and why? (5p)

Vraag 7

Describe the inverse power method mathematically. How can it be used to find **all** the eigenvalues of a given matrix? (10p)

Academiejaar 2016-2017 1ste zit

1. Give the general definition of a floating-point representation (5p), and how many different values are represented in the given system (5p). Write the decimal number 0.1789112 in a floating point representation with basis 2, precision 3 and exponent series [-1,1] (5p).
2. What is a fixed-point problem (3p)? Show this employed in the Newton method (3p)? What determines if Newton method will converge? (2p)? Give the upperbound for the error of the Newton method (2p).
3. Explain LU decomposition as a direct method to solve a system of linear equations Ax = b (7p). For which type of problems is this numerical method particularly advantageous? Calculate determinant of A using LU composition. (3p) How can the solution of LU method be conveniently improved (5p)?
4. Discuss the sensitivity of the solution of a least-squares problem to the error in the data, using condition number (5p). What are orthogonal transformations and why are they useful for the least-squares problem (5p)?
5. A possible method to solve the advection equation is the "leap-frog" algorithm
   an+1j=<n−1j−cτh(anj+1−anj−1)

   Determine the stability condition of this algorithm using the von Neumann analysis (10p). What is the main shortcomming of the con Neumann stability analysis (2p)? How to adress that shortcomming when considering stability of an iterative algorithm (3p)?
6. Derive the error estimate for the Simpson rule (5p). Derive the expression for Romberg integration using the Simpson rule.
7. What is the advantage and what the disadvantage of a multistep method compared to a Runge-Kutta method of the same order (5p)? Derive the iterative expression for the 3rd order Adams-Bashforth scheme (10p).

Academiejaar 2015-2016 1ste zit

1. Give the general definition of a floating-point representation (5p), and how many different values can be represented in the given system (5p). Write the decimal number 29.1 in 64-bit form (5p). **Bonus:** how about -29.1? (5p)
2. Derive and explain Newton method for solving non-linear equations

3. Explain and formulate the jacobi and Gauss-seidel iterative methods in the matrix form, for solving a system of linear equations Ax=b (7p). If A is a diagonally dominant matrix, how does convergence of above methods depend on the initial conditions (3p)?
4. The precision of the simpson rule is 3, although the method is based on interpolation by quadratix polynomials. Demonstrate how that is achieved (10p). Derive expression for the error of the simpson rule (5p).
5. Demonstrate (based on the condition number) that direct solving of normal equations is not a stable algorithm for the least squares problem (5p). Explain the alternative to solve the least-squares problem (5p)
6. One possible method to solve the wave-equation is the "staggered leap-frog" algorithm. In one dimension, the finite differencing gives an+1j−2anj+an−1jτ2=c2anj+1−2anj+anj−1h2 where anj=a(jh,nτ), with h the grid step and τ the time step. Perform Von Neumann stability analysis and determine the condition of the stability (10p). What is the main shortcoming of Von Neumann stability analysis (2p)? How to adress that shortcomming when considering stability of an iterative algorithm (3p)?
7. Explain the idea behind Adams multistep for solving ODEs (5p). Name two advantages and two disadvantages of such methods, compared to Runga-Kutta methods of the same order (10p).
8. What is the shifted power method (3p)? Show mathematically how it works (7p)

Academiejaar 2014-2015 1ste zit

Groep 1

1. Discuss all different factors that determine sensitivity of the solution of th given set of linear equations (10p)
2. What is a fixed-point problem (3p)? When does a fixed-point iteration result in quadratic convergence (5p)? Show how this is employed in the Newton method for solving nonlinear equations (7p).
3. The error of a polynomial interpolation of a function f(x) in the interval [a,b] using polynomial Pn−1(x) of order n-1 trough n points xi∈[a,b], satisfies that for every x∈[a,b] there exists c∈[a,b] for which f(x)−Pn−1(x)=(x−x1)(x−x2)…(x−xn−1)(x−xn)n!f(n)(c) Use this to show that the precision of a trapezoidal rule is 1 (10p)
4. Explain and formulate the Jacobi and the Gauss-Seidel iterative methods in the matrix form, for solving a system of linear equations Ax=b (7p). If A is a diagonally-dominant matrix, how does convergence of above methods depend on the initial conditions (3p)?
5. What is QR factorization (2p)? Show how it helps solve the least-squares problem (10p). What are other methods to facilitate the least-squares solution (3p)?
6. One method do solve advection equation ∂a∂t=−c∂a∂x is the method of LAX=an+1j=12(anj+1+anj−1)−cτ2h(anj+1+anj−1) where tau is the time step, and h the distance between grid points. Determine the stability condition of this algorithm using the von Neumann analysis (10p). What is the main shortcomming of the con Neumann stability analysis (2p)? How to adress that shortcomming when considering stability of an iterative algorithm (3p)?
7. Determine the coefficients a1,a2,p1,p2 for the following Rune-Kutta method for y′=f(t,y(t))
yn+1=yn+h[a1f(tn,yn)+a2f(tn+p1h,yn+p2hf(tn,yn))]

   (10p). What is the global error of this Runge-Kutta method (5p)?

8. Explain in detail the power method (4p), inverse power method (3p) and shifted power method (3p)

Groep 2

1. Give the general definition of a floating-point representation (5p), and how many different values are represented in the given system (5p).
2. What is a fixed point problem (3p)? When does a fixed-point iteration result in quadratic convergence (5p)? Show how this is employed in the Newton method for solving nonlinear equations (7p)
3. Explain and formulate the Jacobi and the Gauss-Seidel iterative methods in the matrix form, for solving a system of linear equations Ax=b (7p). If A is a diagonally-dominant matrix, how does convergence of above methods depend on the initial conditions (3p)?
4. Discuss the sensitivity of the solution of a least-squares problem to the error in the data, using condition number (5p). How is a pseudo-inverse matrix defined (5p)? is it wise to use it directly to solve the least-squares problem and why (5p)?
5. A possible method to solve the advection equation is the "leap-frog" algorithm
   an+1j=<n−1j−cτh(anj+1−anj−1)

   Determine the stability condition of this algorithm using the von Neumann analysis (10p). What is the main shortcomming of the con Neumann stability analysis (2p)? How to adress that shortcomming when considering stability of an iterative algorithm (3p)?
6. Name ETn the error of the composite trapezoidal rule Tn with n intervals
   ETn=I−Tn

   , where I is the exact value of the integral. Show that, for large n, ETnET2n≈4 (5p). Use this to find the improved approximation for I (i.e. the Romberg integration formula) (3p), and show that obtained expression matches the rule of Simpson (2p).
7. Determine the coefficients a1,a2,p1,p2 for the following Rune-Kutta method for y′=f(t,y(t))
   yn+1=yn+h[a1f(tn,yn)+a2f(tn+p1h,yn+p2hf(tn,yn))]

   (10p). What is the global error of this Runge-Kutta method (5p)?
8. Which eigenvalues of a given matrix are determined by the power method (5p)? What is the shifted power method (5p)?

Academiejaar 2013-2014 2^{de} zit

1. Consider the quadrature formula
   ∫−11f(x)|x|dx≈w1f(−1)+w2f(x2).

   a. Determine the constants w1, w2 and x2 so that this formula has maximal precision.
   b. What is the precision of this formula?
   c. Describe in general how adaptive quadrature works.
2. Consider the implicit Euler method for solving the equation dx/dt=f(t,x), x(t0)=x0:
   xn+1=xnhf(tn+1,xn+1).

   Show that
   a. this method has a local error of O(h2);
   b. this method is A-stable (by considering the stability for dx/dt=λx).

3. Discuss which different factors determine the sensitivity of the solution of a system of linear equations.
4. The Richardson method to solve the diffusion equation is given as:
   $\frac{T^{n+1}_i - T^{n-1}_i}{2\tau} = \kappa \frac{T^n_{i+1} + T^n_{i-1} - 2T^n_i}{h^2}$.

   Perform VOn Neumann stability analysis, and prove that this algorithm is not stable.
5. Determine the coefficients a1, a2, p1, p2 of the following Runge-Kutta method for $y' = f(t, y(t))$:
   $y_{n+1} = y_n + h[a_1 f(t_n, y_n) + a_2 f(t_n + p_1 h, y_n + p_2 h f(t_n, y_n))]$.

   What is the global error of this Runge-Kutta method?

Academiejaar 2013-2014 1ste zit

Sinds dit jaar wordt het vak gegeven door prof. Milosevic.

1. For
   $A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, x_0 = b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

   one can apply iterative Jacobi method to solve the system $Ax=b$, starting from x_0. Is it possible to use residual to determine when to stop the iterative Jacobi procedure? Demonstrate and explain.

2. Discuss different factors which determine sensitivity of the least-squares method.
3. Derive the expression for Romberg integration using the Simpson rule.
4. Name the mai advantage and the main disadvantage of the Runge-Kutta method for ODEs. Determine the coefficients a_1, a_2, p_1, p_2 for the following Runge-Kutta method for $y' = f(t, y(t))$:
   $y_{n+1} = y_n + h[a_1 f(t_n, y_n) + a_2 f(t_n + p_1 h, y_n + p_2 h f(t_n, y_n))]$

   .

5. One possible method to solve the wave-equation is the "staggered leap-frog" algorithm. In one dimension, the finite differencing gives
   $\frac{a^{n+1}_j - 2a^n_j + a^{n-1}_j}{\tau^2} = c^2 \frac{a^n_{j+1} - 2a^n_j + a^n_{j-1}}{h^2}$

   ,

   where $a^n_j = a(jh, n\tau)$, with h the grid step and τ the time step. Perform Von Neumann stability analysis and determine the condition of stability in the 1D and the 2D case.

Academiejaar 2012-2013 1ste zit

Bart Partoens

1. Stel volgende kwadratuurformule
   $\int_{-1}^{1} |x| f(x) dx \approx A f(-1) + B f(x_2)$

   . Bepaal A, B, x_2 zodat deze formule een maximale precisie heeft. Wat is deze maximale precisie?

2. Leg uit welke verschillende factoren de gevoeligheid van het kleinste kwadraten probleem bepalen.
3. Toon aan dat de precisie van de regel van Simpson 3 is, ondanks het feit dat men gebruik maakt van een kwadratische interpolerende polynoom.
4. Beschouw volgende Runga-Kutta regel
   $$y_{n+1}=y_n+h[a_1f(t_n,y_n)+a_2f(t_n+p_1h,y_n+p_2hf(t_n,y_n))]$$
   - Bepaal de coefficienten a_1,a_2,p_1,p_2
   - Wat is de globale fout van deze Runga-Kutta methode?
5. $u^{n+1}_j-u^n_j=\frac{\Delta t}{h^2}[\alpha(u^n_{j-1}-2u^n_j+u^n_{j+1})+(1-\alpha)(u^{n+1}_{j-1}-2u^{n+1}_j+u^{n+1}_{j+1})]$ is een methode voor de diffusie vergelijking $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$. $u^n_j=u(jh,n\Delta t)$ met h de stepgrid, Δt de tijdstap, $\alpha\in[0,1]$. Bepaal de voorwaarde op α door middel van stabiliteitsanalyse zodat deze methode stabiel is.

Academiejaar 2011-2012 1^ste zit

Bart Partoens

1. Gegeven volgend recursief voorschrift
   $$x_{n+1}=2x_n-\pi x_n^2$$
   met $x_0=1/3$
   Bepaal de convergentiesnelheid z en de constante c van $\lim_{n\to\infty}\frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^z}=c$
2. Stel volgende kwadratuurformule
   $$\int_{-1}^{1}|x|f(x)dx\approx A[f(x_1)+f(x_2)]$$
   . Bepaal A,x_1,x_2 zodat deze formule een maximale precisie heeft. Wat is deze maximale precisie?
3. De Trapeziumregel.
   - De fout op een polynoom-benadering op $[a,b]$ waarbij de polynoom door n punten $(x_n,f(x_n))$ gaat is gegeven door
     $$f(x)-P_{n-1}(x)=\frac{(x-x_1)(x-x_2)...(x-x_n)}{n!}f^{(n)}(c)$$
     met $c\in[a,b]$
   - Gebruik deze formule op te tonen dat de precisie van de trapeziumregel 1 bedraagt.
   - Toon aan dat $E_n/E_{2n}\approx 4$ als $n\to\infty$
   - Stel met vorige formule een verbeterde integratiemethode op en bewijs dat deze overeen komt met de methode van Simpson.
4. Beschouw volgende Runga-Kutta regel
   $$y_{n+1}=y_n+h[a_1f(t_n,y_n)+a_2f(t_n+p_1h,y_n+p_2hf(t_n,y_n))]$$
   - Bepaal de coefficienten a_1,a_2,p_1,p_2
   - Wat is de globale fout van deze Runga-Kutta methode?
5. Beschouw de 2D transportvergelijking van warmte met het FTCS-scheme
   $$T^{n+1}_{ij}=T^n_{ij}+\frac{\kappa\tau}{h_x^2}(T^n_{i+1,j}+T^n_{i-1,j}-2T^n_{ij})+\frac{\kappa\tau}{h_y^2}(T^n_{i,j+1}+T^n_{i,j-1}-2T^n_{ij})$$
   met h_x en h_y de afstand tussen je roosterpunten in x en y-richting en τ de grootte van je tijdstap, bepaal door middel van Von Neumann stabiliteitsanalyse de voorwaarde op τ voor stabiliteit.

Academiejaar 2008-2009 1ste zit

Bart Partoens

1. De regel van Simpson gebruikt een kwadratische interpolerende polynoom $P_2(x)$ om het integrandum te benaderen:
   $I(f)=\int_a^b f(x)dx \approx S_2=\int_a^b P_2(x)dx$

   De keuze van de interpolatiepunten x_1,x_2 en x_3 in het interval [a,b] bepaalt mee het resultaat. Met de conventie $x_1=a, x_2=(a+b)/2=c$ (middelpunt) en $x_3=b$, vinden we m.b.v. Lagrange interpolatie:
   $S_2=\int_a^b P_2(x)dx$

   $=f(a)\int_a^b \frac{(x-b)(x-c)}{(a-b)(a-c)}dx+f(c)\int_a^b \frac{(x-a)(x-b)}{(c-a)(c-b)}dx+f(b)\int_a^b \frac{(x-a)(x-c)}{(b-a)(b-c)}dx$

   $=\frac{1}{3}h[f(a)+4f(c)+f(b)]$

   .

   Het subscript bij S k.o.m. het aantal deelintervallen van lengte h.

   Voor de fout bij poynomiale interpolatie van een functie f(x) in het interval [a,b] door ene polynoom $P_{n-1}(x)$ van graad n-1 door de n punten $x_i \in [a,b]$ geldt dat voor elke $x \in [a,b]$ er een $c \in [a,b]$ waarvoor geldt:
   $f(x)-P_{n-2}(x)=\frac{(x-x_1)(x-x_2)\ldots(x-x_{n-1})(x-x_n)}{n!}f^{(n)}(c)$

   .

   Toon hiermee aan dat de precisie van de regel van Simpson 3 is, ondanks het feit dat men gebruik maakt van een kwadratische interpolerende polynoom.

2. Wanneer resulteert een fix-point iteratie in kubische convergentie?
3. Een methode om de advectie vergelijking
   $\frac{\partial a}{\partial t}=-c\frac{\partial a}{\partial x}$

   op te lossen is de methode van Lax:
   $a^{n+1}_j=\frac{1}{2}(a^n_{j+1}+a^n_{j-1})-\frac{c\tau}{2h}(a^n_{j+1}-a^n_{j-1})$

   ,

   met τ de tijdstap en h de afstand tussen de twee gridpunten.

   Voer een von Neumann stabiliteitsanalyse uit. Wanneer is deze methode stabiel?

4. Onderstel dat je een lineair stel Ax=b hebt opgelost op een computer met machine-precisie $\epsilon_{machine}=10^{-8}$. De relatieve fout in de bekomen oplossing $\tilde{x}$ is
   $\frac{||x-\tilde{x}||}{||\tilde{x}||}=10^{-3}$

   voor gegeven A en b.

   Onderstel nu dat je hetzelfde probleem oplost op een computer met machine-precisie $\epsilon_{machine}=10^{-16}$. Schat de nieuwe relatieve fout in de oplossing af.

Academiejaar 2007-2008 1ste zit

Bart Partoens

1. Bedenk een functie waarvoor de convergentiesnelheid r voor de bepaling van een nulpunt met de methode van Newton gegeven wordt door r = 3.
2. Een andere methode voor het oplossen van de advectie vergelijking
   $$\frac{\partial a}{\partial t}=-c.\frac{\partial a}{\partial x}$$

   is het 'leap-frog' schema
   $$a_j^{n+1}=a_j^{n-1}-c.\frac{\tau}{h}(a_{j+1}^n-a_{j-1}^n)$$

   Voer von Neumann stabiliteitsanalyse uit. Wanneer is deze methode stabiel?

3. Bepaal de coëfficiënten A_0, A_1 en x_1 zodat de kwadratuurformule
   $$I(f)=\int_{-1}^{1}f(x)dx=A_0f(-1)+A_1f(x_1)$$

   minstens precisie 2 heeft.

    1. Wat is het verschil tussen de begrippen *gevoeligheid* en *stabiliteit*.
    2. Toon aan dat de gevoeligheid van het oplossen van een lineair stelsel Ax=b bepaald wordt door het conditiegetal van matrix A.