Field Arithmetic - Encyclopedia Academia

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Field Arithmetic

Field Arithmetic	
Richting	Wiskunde
Jaar	MWIS

Academiejaar 2023 - 2024

Test

1. (6 points) Let K be the fraction field of the domain $\mathbb{F}_{125}[s,t]/(s^2-t^3)$. Show that the quadratic form $f=X_1^2-3X_2^2+sX_3^2-3sX_4^2$ over K is anisotropic and that it represents every element of K.
2. (6 points) Decide which of the following statements are correct:
    1. 4 is a cube in $\mathbb{Q}_3$.
    2. 6 is a cube in $\mathbb{Q}_7$.
3. (8 points) Let v and w be two valuations on a field K with $v(K^\times)=\mathbb{Z}$ and $w(K^\times)=\mathbb{Q}$. Show that there exists $x\in K$ such that $v(x)<v(x-1)$ and $w(x)<-73$.

Examen

Let K be a field.

1. (12 points) Decide for **three** of the following four statements whether they are correct:
    1. Every pair of cubic forms in 63 variables over $\mathbb{F}_{729}(X,Y)$ is isotropic.
    2. There exists a $\mathbb{Z}$-valuation v on $\mathbb{Q}(\sqrt[4]{5})$ whose residue field is real.
    3. For every non-trivial valuation on $\mathbb{C}$, also its restriction to $\mathbb{Q}$ is non-trivial.
    4. For every archimedean ordering P on K and every non-trivial automorphism σ of K, we have $\sigma(P)\neq P$.
2. (8 points) Assume that K is an extension of $\mathbb{Q}$ and let v be a complete $\mathbb{Z}$-valuation on K with $\mathrm{char}(\kappa_v)=2$. Show that $d\in\mathbb{Z}, d\equiv 1 \bmod 8 \Rightarrow d\in K^{\times 2}$.

3. (6 points) Assume that KK is real and let X(K)X(K) be the set of orderings of KK. For a∈K×a∈K× and P∈X(K)P∈X(K), let
   signP(a)={1−1 if a∈P, if a∈−P.

   signP(a)={1 if a∈P,−1 if a∈−P.

   Show that the induced map
   Φ:K×/K×2→{−1,1}X(K),aK×2↦(signP(a))P∈X(K)

   Φ:K×/K×2→{−1,1}X(K),aK×2↦(signP(a))P∈X(K)

   is injective if and only if (∑K2)×=K×2(∑K2)×=K×2.
4. (12 points) Show the following:
   1. There is a maximal subfield KK of RR in which 2 is not a square.
   2. The field KK is uniquely ordered with |K×/K×2|=4|K×/K×2|=4.
   3. Every polynomial of odd degree in K[X]K[X] has a root in KK.
   4. The choice of KK in (a) does not determine whether 3 is a square in KK. (Bonus, can add 2 points to this exercise if not all other questions are correct)
5. (12 points) Determine all prime numbers pp for which (90,105)Qp(90,105)Qp is a division algebra. Conclude that (90,105)Q (−17,7)Q(90,105)Q≄(−17,7)Q.

Academiejaar 2022 - 2023

Testen

Media:FA22-test1.pdf
Media:FA22-test2.pdf
Media:FA22-test3.pdf

Examen

Media:FA22-exam.pdf

Academiejaar 2016 - 2017

Test

KK is a field and PP an ordering on KK.

1. Let KK the fraction field of F25[X,Y]/(X5+Y+1)F25[X,Y]/(X5+Y+1) If QQ is an anisotropic form of degree 55 in 2525 variables over KK, then show that QQ represents every element of KK.
2. Define O={x∈K|a≤Px≤Pb for some a,b∈Z}O={x∈K|a≤Px≤Pb for some a,b∈Z}
   - Show thath OO is an evaluation ring of KK.
   - Give an example where OO is a proper evaluation ring (i.e. not a field).

3. Let Ω be an algebraic closure of K. Show that there exists a real field $R \subseteq \Omega$ containing K such that $P = R^2 \cap K$ and $\Omega = R(\sqrt{-1})$. (Hint: Look at the ordered fields (K', P') with $K \subseteq K' \subseteq \Omega$.)
4. If p is an odd prime number, then show the following:
   - $\forall a,b,c \in \mathbb{Z} \setminus p\mathbb{Z}$: the quadratic form $aX^2+bY^2+cZ^2$ is isotropic over $\mathbb{Q}_p$.
   - Every 5-dimensional quadratic form over $\mathbb{Q}_p$ is isotropic. (You may assume that the form is given by $aU^2+bV^2+cX^2+dY^2+eZ^2$ where $a,b,c,d,e \in \mathbb{Q}_p^\times$.)
5. Let v be a $\mathbb{Z}$ on K. Show that
   - For any valuation w on K with $\mathfrak{m}_v \subseteq \mathcal{O}_w$ we have $\mathcal{O}_w \subseteq \mathcal{O}_v$.
   - If v is complete, then $1+\mathfrak{m}_v \subseteq K^{\times n}$ for all $n \in \mathbb{N}$ not divisible by the characteristic of the residue field κ_v.
   - Is v is complete, then it is the unique $\mathbb{Z}$-valuation on K.

1ste zit

1. Answer the following questions:
   - What does the Chevalley-Warning Theorem say about isotropy of a form over a finite field.
   - Explain why for an extension F/E, the norm $F \to E$ is surjective.
2. Formulate the Artin-Schreier Theorem which characterises real fields and give the main steps of the proof.
3. Let $f \in \mathbb{C}[X,Y]$ be irreducible and let F be the function field of the curve $f(x,y)=0$ over $\mathbb{C}$ (F is the field of fractions of the domain $\mathbb{Q}[X,Y]/(f)$) Show that:
   - F has diophantine dimension 1.
   - $\mathrm{Br}(F)=0$
4. F is a field and A a central simple F-algebra.
   - Explain the relationship between the index and the degrees of splitting fields over A
   - Let $A = D_1 \otimes_F D_2$, with D_1, D_2 central F-division algebras of coprime degrees. Show that A is a division algebra.
5. Answer the following questions:
   - Show that $k_2(\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $k_2(\mathbb{C})=0$.
   - Use Milnor's exact sequence to compute $k_2(\mathbb{R}[X])$
6. Let $F=\mathbb{R}(X)$
   - Every central simple F-algebra splits over $F(\sqrt{-1})$ and is equivalent to an F-quaternion algebra.
   - The classes of F-quaternion algebras $(-1, X-r)_F$ with $r \in \mathbb{R}$ and $(-1,-1)_F$ generate $\mathrm{Br}(F)$.

7.
    - Given $f \in \mathbb{Z}[X]$, what is the relationship between the following statements:
        - f has a root modulo every positive prime integer n
        - f has a root in $\mathbb{Z}_p$ for every prime p.
    - Plan a proof for the following statement: The equation $(X^2-13)(X^2-17)(X^2-221)=0$ has no integer solution, but it has a solution modulo every positive prime integer.

Categorieën:

- Wiskunde
- MWIS