On the calculation of surface tension from measurements of pendant drops

By S. Fordham

Research Department, I.C.I. Ltd., Explosives Division, Stevenston, Ayrshire

(*Communicated by F. A. Freeth, F.R.S.—Received* 13 *November* 1947—*Revised* 15 *March* 1948)

The calculations presented are an extension of the work of Bashforth & Adams (1883) to give the shapes of pendant drops of liquids for values of β from $-0{\cdot}25$ to $-0{\cdot}6$ at intervals of 0·025. The results have been used to calculate, to an accuracy of 0·001 to 0·01 %, the constants needed for measuring surface and interfacial tensions by the method suggested by Andreas, Hauser & Tucker (1938), which has thus been made independent of calibration.

Introduction

Many methods have been proposed for the measurement of surface and interfacial tensions, and each may be said to have advantages for particular applications. Some are dynamic, and are ideally suited to following the changes occurring in a surface during the first fractions of a second after its formation. More commonly, however, measurements are made in a static manner, and are intended to indicate the properties of the surface in its equilibrium condition. The static methods all purport to measure the same property, but they may differ greatly in suitability for individual purposes. A valuable general discussion of these methods has been given by Adam (1938).

At the present time the basic method for determining surface tension involves the use of a capillary tube, by measuring either the capillary rise or else the hydrostatic pressure needed to restore the meniscus to its normal level. The results are calculated, following Sugden (1921), with the help of the tables of Bashforth & Adams (1883) for the shape of free surfaces of liquids. To obtain accuracy, however, extreme precautions are needed, and the method is not well adapted to routine use. A number of secondary methods have therefore been evolved, which are more rapid in operation. The maximum bubble pressure method of Sugden (1922) has a theoretical foundation, being based also on the tables of Bashforth & Adams. The drop-weight method of Harkins & Brown (1919) and the ring method of de Noüy (1919) are more empirical. All these techniques for measuring surface tension depend finally upon comparison with the capillary method using such liquids as benzene. Generally they involve enlarging and breaking the surface of the liquid, and therefore are not strictly static in nature.

Some time ago the author wished to carry out routine measurements of the surface tensions of some colloidal solutions. These solutions were characterized by relatively high viscosity, and by containing material of high molecular weight and thus of low speed of diffusion. A method was therefore required which did not depend on an extension of the surface, because under such conditions not only would the slow diffusion prevent the attainment of equilibrium, but also the high viscosity would make invalid any calibration with free-flowing liquids. A modification of the technique proposed by Wilhelmy (1863) was considered, in which the surface remained unchanged, but this was discarded in favour of a method more recently developed by Andreas, Hauser & Tucker (1938), based on measurements of the shape and size of pendant drops.

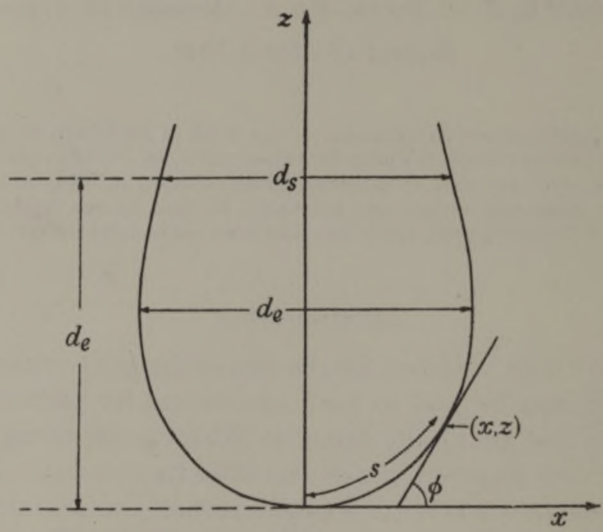

FIGURE 1. Pendant drop. $\beta = -0{\cdot}45$.

The use of pendant drops has been suggested by several workers, such as Worthington (1881, 1885) and Ferguson (1912), but proposals have usually involved difficult measurements, as of the position of points of inflexion on the profile of the drop. Andreas *et al.* suggested a simpler method of calculation. Taking the origin at the bottom of the drop, and co-ordinates x and z in a horizontal and vertical direction respectively, the equation for the profile of a liquid surface of revolution may be written as

$$\frac{1}{\rho}+\frac{\sin\phi}{x} = 2+\beta z, \tag{1}$$

where ρ is the radius of curvature at the point (x, z), ϕ is the slope at the point (x, z), and

$$\beta = -g\sigma b^2/\gamma, \tag{2}$$

where σ is the density of the liquid, γ is the surface tension, and b is the radius of curvature at the origin. The unit of length adopted in equation (1) is b, so that x, z and β are non-dimensional. It will thus be seen that β determines the shape of a drop, and b its size. The negative sign for β for a pendant drop follows the convention adopted by Bashforth & Adams (1883).

Andreas *et al.* avoided the actual determination of β and b by a method, illustrated in figure 1, in which the maximum diameter d_e was measured, and also the diameter d_s at a horizontal plane distant d_e from the bottom of the drop. Denoting the ratio d_s/d_e by S, it is clear that S is a function of β, whilst for a given S, d_e is proportional to b. Thus, by (2),

$$\gamma = g\sigma d_e^2/H,$$

where
$$1/H = 1/\beta(d_e/b)^2 = 1/4\beta x_e^2 \tag{3}$$

is a function of $S = d_s/d_e$. This function, connecting $1/H$ and S, was obtained by experiments with water, the surface tension of which was known from capillary measurements.

There is no reason why the relation between S and $1/H$ should not be calculated, and the method thus placed on a firm theoretical foundation and made independent of experimental calibration. Although Andreas *et al.* realized this, they tended to doubt the accuracy of available mathematical methods; but with sufficient care, numerical integrations may be carried out to any finite degree of accuracy which may be required.

Method of calculation

At any point (x, z),
$$\frac{d\phi}{ds} = \frac{1}{\rho} \tag{4}$$

or, by equation (1),
$$\frac{d\phi}{ds} = 2 + \beta z - \frac{\sin\phi}{x}, \tag{5}$$

$$\frac{dx}{ds} = \cos\phi, \tag{6}$$

$$\frac{dz}{ds} = \sin\phi, \tag{7}$$

where s = the arc distance of the point (x, z) from the origin at the bottom of the drop. The unit of length in these equations is again b, the radius of curvature at the bottom of the drop.

For each value of β, simultaneous finite difference integration gives ϕ, x and z in terms of s, after the first values have been obtained by series expansions. At each step an estimate is made of the next entry in the finite difference table of $d\phi/ds$ against s, and the value of ϕ calculated by the integration formula (expressed in terms of backward differences from the end of the interval of integration)

$$\phi_s = \phi_{s-\delta s} + \delta s(\phi'_s - \tfrac{1}{2}\Delta\phi'_s - \tfrac{1}{12}\Delta^2\phi'_s - \tfrac{1}{24}\Delta^3\phi'_s - \tfrac{19}{720}\Delta^4\phi'_s - \tfrac{3}{160}\Delta^5\phi'_s). \tag{8}$$

This value of ϕ by equations (6) and (7) gives the next entries in the tables of dx/ds and dz/ds, so that a formula similar to (8) may be used to calculate x and z. These figures, substituted in equation (5), should give a value for $d\phi/ds$ equal to that assumed. The manner of correcting for differences between the calculated and assumed values of $d\phi/ds$ is given by Bashforth & Adams (1883), who also give tables of results for many values of β, of which $-0{\cdot}3$, $-0{\cdot}4$, $-0{\cdot}5$ and $-0{\cdot}6$ are relevant to the present problem.

This method was used to obtain tables for all other values of β between $-0{\cdot}25$ and $-0{\cdot}6$, at intervals of 0·025; calculations were made to six decimal places, the interval δs being 0·1, and the results, given in table 1, are considered accurate to the fifth place, in view of the relatively short range of the integration. Constant checks were made by Weddle's equation, and by finite difference integration of the individual functions $d\phi/ds$, dx/ds and dz/ds over extended intervals by the central difference formula (expressed in terms of backward differences)

$$\int_{x_0}^{x_n} y\,dx = \delta x\left\{\sum_{k=0}^{n} y_k - \tfrac{1}{2}(y_0+y_n) - \tfrac{1}{24}(\Delta y_{n+1}+\Delta y_n-\Delta y_1-\Delta y_0) + \tfrac{11}{1440}(\Delta^3 y_{n+2}+\Delta^3 y_{n+1}-\Delta^3 y_2-\Delta^3 y_1)\ldots\right\}.$$

Further tables were then drawn up, giving x and z in terms of β for each value of the arc distance s, and differenced with respect to β, and the constancy of the higher differences showed that there were no random errors in the fifth decimal place of the calculated figures. These tables were interpolated to derive x and z in terms of s for all other values of β from $-0{\cdot}25$ to $-0{\cdot}6$ at intervals of 0·0125.

For further calculations, the Bessel and Stirling interpolation formulae were used except at the ends of the table, where it was necessary to use the Newton formula. For each value of β, the position of the maximum value of x was derived in terms of s by equating to zero the first differential coefficient obtained from the difference tables. This value was checked at intervals of 0·025 in β by inverse interpolation of the tables of ϕ to give s when $\phi = \frac{1}{2}\pi$. A further check against arithmetical mistakes was made by tabulating these values of s against β and noting the constancy of the higher differences. Values of x_e, the maximum radii, were then calculated by interpolation from the known values of s, and again checked by tabulation against β.

For each value of β, inverse interpolation was employed to find s when $z = 2x_e$, and direct interpolation then gave x_s, the radius at the selected plane. Checks were obtained by tabulating s and x_s against β, and noting the constancy of the higher differences. The values obtained for x_e and x_s are given in table 2, columns 2 and 3.

In each case $S = x_s/x_e$ was calculated, and also $1/H$ by equation (3). The values of S and $1/H$ thus found are given in table 2, columns 4 and 5.

Values of $1/H$ were calculated for S 0·66 (0·01) 1·00 by two methods. One method involved inverse interpolation to give the value of β corresponding to each selected value of S, followed by direct interpolation to give the appropriate value of $1/H$. The other method was by interpolation of a table of $1/H$ against S by Newton's formula for unequal intervals, checked periodically by the Lagrange formula to six terms. The former method gave the more reliable results, especially at the ends of the table, but the maximum difference between the values of $1/H$ within the range $S = 0{\cdot}68$ and $S = 0{\cdot}98$ was only 5 in the sixth decimal place. Using values obtained by the former method ordinary subtabulation gave $1/H$ for steps of 0·001 in S, between which linear interpolation is adequate. Table 3 gives the final results thus obtained; for convenience H also has been tabulated against S in table 4.

Accuracy of results

Since difference tables had been drawn up at all stages of the calculation, it was a simple matter to estimate the magnitude of random errors. Six figures had been retained during actual computation and the final tables of S and $1/H$ against β showed that between the limits of 0·68 and 0·98 for S, random errors did not exceed 4 and 3 respectively in the sixth decimal place. The final table of $1/H$ against S showed that the greatest possible random variation in $1/H$ was 7 in the sixth decimal place in the above range. These figures are based on the rather conservative estimate that a random error of ± 1 in a variable will cause random errors not exceeding ± 16 in the sixth difference (the maximum possible error is ± 32).

At the extreme ends of the table, this method of estimation could not be applied. When S exceeded 0·98, in particular, higher differences did not tend rapidly to zero and the accuracy in this region could not be estimated, although there was no reason to suppose that there was any considerable decrease. This was supported by the fact that the different methods used for interpolation did not give variations exceeding 8 in the sixth decimal place. Where S was less than 0·68, accuracy estimates could not be made from the difference tables, but since the higher differences were small, there was no reason to suppose that the errors were greater than in the middle portion of the table. The maximum error found on working back to randomly selected original values of S and $1/H$ was 5 in the sixth decimal place.

The possibility of systematic errors in the calculation was also considered. The highest differences considered in equation (8) were the fifth, and these affected the sixth decimal place of the integrands ϕ, x and z only when β was numerically less than 0·3 and s exceeded 2·5. The largest errors from neglect of the sixth and higher differences would occur with the lowest numerical values of β, and thus of S. Closer consideration of this indicated that the maximum possible error in $1/H$ from this cause was not greater than 1 in the fifth place of decimals when S was less than 0·68, and considerably less elsewhere.

Considering all these possible causes of error, it is considered that the accuracy of the entries in table 3 may be taken to be as follows:

$S = 0{\cdot}66$ to $0{\cdot}68$; maximum error $\pm 0{\cdot}00003 = \pm 0{\cdot}003\,\%$.
$S = 0{\cdot}68$ to $0{\cdot}98$; maximum error $\pm 0{\cdot}00001 = \pm 0{\cdot}001$ to $0{\cdot}003\,\%$.
$S = 0{\cdot}98$ to $1{\cdot}00$; maximum error $\pm 0{\cdot}00003 = \pm 0{\cdot}01\,\%$.

It is concluded that these tables are adequate for all likely experimental purposes, and that the method should be capable of giving results which are as reliable as those obtained by the use of capillary elevations.

It should, however, be pointed out that the attainment of such accuracy would entail very stringent demands upon experimental techniques. For example, if the diameters of the drop are measured to an accuracy of 0·1 %, the error of the calculated surface tension may be almost 0·7 %. For this reason the agreement between the calculated table and the measured values of Andreas *et al.*—the maximum difference is about 1·4 % when $S = 0{\cdot}78$—is considered satisfactory.

Note to tables. In order to avoid possible misreading the headings of the tables have been written in full as s/b, x/b, etc., instead of the dimensionless form used in the text.

TABLE 1. CALCULATED SHAPES OF DROPS

		ϕ deg.				
s/b	ϕ rad.	deg.	min.	sec.	x/b	z/b
			$\beta = -0{\cdot}25$			
0	0		0		0	0
0·1	0·099,97	5	43	40	0·099,83	0·005,00
0·2	0·199,75	11	26	41	0·198,67	0·019,92
0·3	0·299,16	17	8	26	0·295,54	0·044,60
0·4	0·398,01	22	48	16	0·389,48	0·078,75
0·5	0·496,14	28	25	35	0·479,61	0·121,97
0·6	0·593,35	33	59	48	0·565,10	0·173,78
0·7	0·689,50	39	30	20	0·645,19	0·233,60
0·8	0·784,42	44	56	39	0·719,20	0·300,78
0·9	0·877,97	50	18	14	0·786,57	0·374,64
1·0	0·970,00	55	34	37	0·846,80	0·454,42
1·1	1·060,38	60	45	20	0·899,52	0·539,35
1·2	1·148,99	65	49	57	0·944,43	0·628,66
1·3	1·235,71	70	48	2	0·981,35	0·721,56
1·4	1·320,41	75	39	14	1·010,18	0·817,29
1·5	1·402,99	80	23	8	1·030,91	0·915,08
1·6	1·483,34	84	59	21	1·043,62	1·014,25
1·7	1·561,34	89	27	28	1·048,44	1·114,10
1·8	1·636,84	93	46	42	1·045,59	1·214,04
1·9	1·709,72	97	57	34	1·035,34	1·313,49
2·0	1·779,79	101	58	29	1·018,01	1·411,96
2·1	1·846,86	105	49	2	0·993,98	1·509,01
2·2	1·910,65	109	28	21	0·963,64	1·604,28
2·3	1·970,84	112	55	16	0·927,46	1·697,49
2·4	2·027,00	116	8	18	0·885,92	1·788,44
2·5	2·078,53	119	5	28	0·839,53	1·877,02
2·6	2·124,67	121	44	4	0·788,87	1·963,23
2·7	2·164,33	124	0	25	0·734,55	2·047,18
2·8	2·196,01	125	49	20	0·677,26	2·129,13
2·9	2·217,58	127	3	29	0·617,78	2·209,52
3·0	2·225,96	127	32	18	0·557,09	2·289,00
3·1	2·216,67	127	0	22	0·496,39	2·368,47
3·2	2·183,23	125	5	24	0·437,36	2·449,18
			$\beta = -0{\cdot}275$			
0	0		0		0	0
0·1	0·099,97	5	43	40	0·099,83	0·005,00
0·2	0·199,73	11	26	36	0·198,67	0·019,92
0·3	0·299,08	17	8	9	0·295,54	0·044,60
0·4	0·397,82	22	47	36	0·389,49	0·078,73
0·5	0·495,75	28	24	16	0·479,63	0·121,93
0·6	0·592,69	33	57	31	0·565,15	0·173,69
0·7	0·688,46	39	26	45	0·645,29	0·233,44
0·8	0·782,88	44	51	19	0·719,39	0·300,54
0·9	0·875,78	50	10	43	0·786,89	0·374,27
1·0	0·967,03	55	24	24	0·847,33	0·453,89
1·1	1·056,47	60	31	52	0·900,34	0·538,65
1·2	1·143,96	65	32	39	0·945,65	0·627,76
1·3	1·229,38	70	26	18	0·983,09	0·720,45
1·4	1·312,60	75	12	23	1·012,60	0·815,97

TABLE 1 (*cont.*)

s/b	ϕ rad.	ϕ deg. deg.	min.	sec.	x/b	z/b
		$\beta = -0{\cdot}275$ (*cont.*)				
1·5	1·393,50	79	50	29	1·034,18	0·913,59
1·6	1·471,95	84	20	12	1·047,92	1·012,61
1·7	1·547,84	88	41	4	1·053,98	1·112,40
1·8	1·621,02	92	52	39	1·052,59	1·212,37
1·9	1·691,34	96	54	23	1·044,05	1·311,98
2·0	1·758,62	100	45	41	1·028,67	1·410,77
2·1	1·822,66	104	25	50	1·006,83	1·508,34
2·2	1·883,19	107	53	56	0·978,97	1·604,37
2·3	1·939,89	111	8	50	0·945,52	1·698,60
2·4	1·992,33	114	9	8	0·906,98	1·790,86
2·5	2·039,98	116	52	57	0·863,87	1·881,08
2·6	2·082,11	119	17	45	0·816,75	1·969,27
2·7	2·117,75	121	20	17	0·766,22	2·055,55
2·8	2·145,62	122	56	5	0·712,97	2·140,19
2·9	2·163,92	123	59	1	0·657,76	2·223,57
3·0	2·170,25	124	20	46	0·601,51	2·306,25
3·1	2·161,28	123	49	55	0·545,34	2·388,98
3·2	2·132,52	122	11	2	0·490,71	2·472,74
		$\beta = -0{\cdot}325$				
0	0		0		0	0
0·1	0·099,96	5	43	39	0·099,83	0·005,00
0·2	0·199,68	11	26	26	0·198,67	0·019,92
0·3	0·298,91	17	7	34	0·295,54	0·044,58
0·4	0·397,42	22	46	14	0·389,50	0·078,69
0·5	0·494,98	28	21	37	0·479,67	0·121,84
0·6	0·591,37	33	52	58	0·565,24	0·173,52
0·7	0·686,37	39	19	34	0·645,48	0·233,13
0·8	0·779,79	44	40	43	0·719,75	0·300,04
0·9	0·871,43	49	55	45	0·787,53	0·373,52
1·0	0·961,11	55	4	3	0·848,37	0·452,83
1·1	1·048,67	60	5	3	0·901,96	0·537,23
1·2	1·133,95	64	58	13	0·948,06	0·625,93
1·3	1·216,79	69	43	1	0·986,55	0·718,20
1·4	1·297,06	74	18	57	1·017,40	0·813,29
1·5	1·374,62	78	45	35	1·040,65	0·910,52
1·6	1·449,33	83	2	25	1·056,44	1·009,25
1·7	1·521,05	87	8	59	1·064,96	1·108,86
1·8	1·589,63	91	4	45	1·066,48	1·208,83
1·9	1·654,92	94	49	12	1·061,31	1·308,68
2·0	1·716,73	98	21	40	1·049,80	1·408,00
2·1	1·774,84	101	41	27	1·032,36	1·506,45
2·2	1·829,00	104	47	39	1·009,42	1·603,78
2·3	1·878,91	107	39	14	0·981,45	1·699,77
2·4	1·924,20	110	14	54	0·948,94	1·794,33
2·5	1·964,37	112	33	1	0·912,42	1·887,41
2·6	1·998,85	114	31	33	0·872,44	1·979,07
2·7	2·026,89	116	7	57	0·829,60	2·069,43
2·8	2·047,55	117	18	59	0·784,58	2·158,72
2·9	2·059,65	118	0	34	0·738,08	2·247,25
3·0	2·061,71	118	7	38	0·690,95	2·335,44
3·1	2·051,92	117	33	58	0·644,15	2·423,81
3·2	2·028,17	116	12	19	0·598,81	2·512 95

TABLE 1 (*cont.*)

s/b	ϕ rad.	ϕ deg. deg.	min.	sec.	x/b	z/b
			$\beta = -0{\cdot}35$			
0	0		0		0	0
0·1	0·099,96	5	43	38	0·099,83	0·004,99
0·2	0·199,65	11	26	21	0·198,67	0·019,92
0·3	0·298,82	17	7	17	0·295,54	0·044,58
0·4	0·397,22	22	45	33	0·389,51	0·078,68
0·5	0·494,60	28	20	18	0·479,69	0·121,79
0·6	0·590,71	33	50	42	0·565,28	0·173,43
0·7	0·685,33	39	16	0	0·645,57	0·232,97
0·8	0·778,24	44	35	24	0·719,93	0·299,78
0·9	0·869,25	49	48	15	0·787,85	0·373,14
1·0	0·958,15	54	53	53	0·848,89	0·452,30
1·1	1·044,78	59	51	41	0·902,77	0·536,51
1·2	1·128,95	64	41	3	0·949,26	0·625,01
1·3	1·210,52	69	21	27	0·988,27	0·717,06
1·4	1·289,33	73	52	22	1·019,78	0·811,94
1·5	1·365,23	78	13	19	1·043,86	0·908,97
1·6	1·438,09	82	23	46	1·060,66	1·007,52
1·7	1·507,75	86	23	16	1·070,40	1·107,03
1·8	1·574,07	90	11	15	1·073,36	1·206,96
1·9	1·636,88	93	47	11	1·069,87	1·306,89
2·0	1·695,99	97	10	24	1·060,29	1·406,41
2·1	1·751,21	100	20	12	1·045,04	1·505,23
2·2	1·802,26	103	15	43	1·024,55	1·603,10
2·3	1·848,88	105	55	58	0·999,32	1·699,86
2·4	1·890,70	108	19	44	0·969,82	1·795,40
2·5	1·927,28	110	25	30	0·936,60	1·889,72
2·6	1·958,11	112	11	29	0·900,22	1·982,86
2·7	1·982,53	113	35	26	0·861,26	2·074,96
2·8	1·999,76	114	34	40	0·820,40	2·166,23
2·9	2·008,86	115	5	56	0·778,33	2·256,95
3·0	2·008,67	115	5	19	0·735,84	2·347,47
3·1	1·997,92	114	28	20	0·693,84	2·438,22
3·2	1·975,15	113	10	5	0·653,36	2·529,66
			$\beta = -0{\cdot}375$			
0	0		0		0	0
0·1	0·099,95	5	43	37	0·099,83	0·004,99
0·2	0·199,63	11	26	16	0·198,67	0·019,92
0·3	0·298,74	17	7	0	0·295,54	0·044,57
0·4	0·397,02	22	44	52	0·389,51	0·078,66
0·5	0·494,21	28	18	59	0·479,71	0·121,75
0·6	0·590,05	33	48	26	0·565,33	0·173,34
0·7	0·684,29	39	12	25	0·645,67	0·232,82
0·8	0·776,71	44	30	7	0·720,12	0·299,53
0·9	0·867,08	49	40	48	0·788,17	0·372,76
1·0	0·955,21	54	43	46	0·849,41	0·451,77
1·1	1·040,90	59	38	21	0·903,57	0·535,80
1·2	1·123,98	64	23	58	0·950,46	0·624,09
1·3	1·204,27	69	0	0	0·989,98	0·715,92
1·4	1·281,63	73	25	55	1·022,15	0·810,58
1·5	1·355,89	77	41	12	1·047,05	0·907,41
1·6	1·426,91	81	45	21	1·064,87	1·005,79

TABLE 1 (*cont.*)

s/b	ϕ rad.	ϕ deg. deg.	min.	sec.	x/b	z/b
		$\beta = -0{\cdot}375$ (*cont.*)				
1·7	1·494,53	85	37	49	1·075,82	1·105,17
1·8	1·558,60	89	18	4	1·080,21	1·205,05
1·9	1·618,96	92	45	33	1·078,38	1·305,02
2·0	1·675,41	95	59	38	1·070,72	1·404,71
2·1	1·727,76	98	59	35	1·057,64	1·503,84
2·2	1·775,76	101	44	37	1·039,61	1·602,19
2·3	1·819,14	104	13	44	1·017,10	1·699,62
2·4	1·857,56	106	25	50	0·990,63	1·796,05
2·5	1·890,64	108	19	33	0·960,71	1·891,46
2·6	1·917,91	109	53	18	0·927,93	1·985,94
2·7	1·938,82	111	5	11	0·892,88	2·079,59
2·8	1·952,72	111	52	57	0·856,20	2·172,62
2·9	1·958,85	112	14	2	0·818,58	2·265,27
3·0	1·956,37	112	5	30	0·780,79	2·357,86
3·1	1·944,35	111	24	11	0·743,66	2·450,71
3·2	1·921,86	110	6	53	0·708,13	2·544,18
		$\beta = -0{\cdot}425$				
0	0		0		0	0
0·1	0·099,95	5	43	35	0·099,83	0·004,99
0·2	0·199,58	11	26	6	0·198,67	0·019,91
0·3	0·298,57	17	6	25	0·295,55	0·044,56
0·4	0·396,63	22	43	31	0·389,52	0·078,62
0·5	0·493,44	28	16	20	0·479,74	0·121,66
0·6	0·588,73	33	43	54	0·565,42	0·173,16
0·7	0·682,21	39	5	17	0·645,86	0·232,51
0·8	0·773,64	44	19	34	0·720,48	0·299,03
0·9	0·862,75	49	25	56	0·788,79	0·372,01
1·0	0·949,34	54	23	35	0·850,44	0·450,71
1·1	1·033,18	59	11	48	0·905,17	0·534,37
1·2	1·114,07	63	49	54	0·952,83	0·622,25
1·3	1·191,84	68	17	14	0·993,37	0·713,64
1·4	1·266,31	72	33	14	1·026,85	0·807,84
1·5	1·337,31	76	37	19	1·053,39	0·904,23
1·6	1·404,69	80	28	57	1·073,21	1·002,23
1·7	1·468,28	84	7	33	1·086,56	1·101,32
1·8	1·527,92	87	32	36	1·093,79	1·201,04
1·9	1·583,45	90	43	29	1·095,27	1·301,02
2·0	1·634,68	93	39	36	1·091,41	1·400,93
2·1	1·681,42	96	20	18	1·082,66	1·500,54
2·2	1·723,45	98	44	48	1·069,49	1·599,66
2·3	1·760,53	100	52	16	1·052,41	1·698,19
2·4	1·792,38	102	41	45	1·031,95	1·796,07
2·5	1·818,69	104	12	12	1·008,64	1·893,31
2·6	1·839,10	105	22	22	0·983,07	1·989,98
2·7	1·853,24	106	10	58	0·955,83	2·086,20
2·8	1·860,68	106	36	34	0·927,55	2·182,12
2·9	1·860,99	106	37	36	0·898,89	2·277,92
3·0	1·853,72	106	12	38	0·870,56	2·373,82
3·1	1·838,49	105	20	16	0·843,31	2·470,04
3·2	1·814,98	103	59	26	0·817,93	2·566,76

TABLE 1 (*cont.*)

s/b	ϕ rad.	ϕ deg. deg.	min.	sec.	x/b	z/b
			$\beta = -0{\cdot}45$			
0	0		0		0	0
0·1	0·099,94	5	43	35	0·099,83	0·004,99
0·2	0·199,55	11	26	1	0·198,67	0·019,91
0·3	0·298,49	17	6	8	0·295,55	0·044,55
0·4	0·396,43	22	42	50	0·389,53	0·078,60
0·5	0·493,06	28	15	1	0·479,76	0·121,62
0·6	0·588,07	33	41	38	0·565,46	0·173,07
0·7	0·681,18	39	1	42	0·645,95	0·232,35
0·8	0·772,10	44	14	17	0·720,66	0·298,78
0·9	0·860,59	49	18	29	0·789,11	0·371,64
1·0	0·946,40	54	13	29	0·850,96	0·450,18
1·1	1·029,32	58	58	32	0·905,97	0·533,65
1·2	1·109,13	63	32	54	0·954,01	0·621,32
1·3	1·185,64	67	55	56	0·995,06	0·712,48
1·4	1·258,68	72	7	1	1·029,19	0·806,45
1·5	1·328,07	76	5	34	1·056,54	0·902,62
1·6	1·393,64	79	50	59	1·077,35	1·000,41
1·7	1·455,24	83	22	45	1·091,89	1·099,33
1·8	1·512,70	86	40	17	1·100,53	1·198,94
1·9	1·565,86	89	43	1	1·103,64	1·298,88
2·0	1·614,53	92	30	20	1·101,67	1·398,85
2·1	1·658,52	95	1	35	1·095,06	1·498,63
2·2	1·697,64	97	16	4	1·084,31	1·598,04
2·3	1·731,65	99	12	59	1·069,93	1·697,00
2·4	1·760,31	100	51	30	1·052,46	1·795,46
2·5	1·783,34	102	10	40	1·032,44	1·893,43
2·6	1·800,44	103	9	28	1·010,47	1·990,99
2·7	1·811,31	103	46	49	0·987,13	2·088,22
2·8	1·815,62	104	1	38	0·963,04	2·185,28
2·9	1·813,05	103	52	48	0·938,87	2·282,31
3·0	1·803,32	103	19	21	0·915,30	2·379,49
3·1	1·786,21	102	20	32	0·893,03	2·476,98
3·2	1·761,59	100	55	55	0·872,79	2·574,91
			$\beta = -0{\cdot}475$			
0	0		0		0	0
0·1	0·099,94	5	43	34	0·099,83	0·004,99
0·2	0·199,53	11	25	55	0·198,67	0·019,91
0·3	0·298,40	17	5	50	0·295,55	0·044,55
0·4	0·396,23	22	42	9	0·389,54	0·078,58
0·5	0·492,67	28	13	41	0·479,78	0·121,57
0·6	0·587,41	33	39	22	0·565,51	0·172,98
0·7	0·680,14	38	58	9	0·646,05	0·232,20
0·8	0·770,57	44	9	1	0·720,84	0·298,53
0·9	0·858,43	49	11	4	0·789,42	0·371,26
1·0	0·943,48	54	3	26	0·851,47	0·449,64
1·1	1·025,47	58	45	19	0·906,76	0·532,93
1·2	1·104,21	63	15	59	0·955,19	0·620,39
1·3	1·179,47	67	34	43	0·996,74	0·711,32
1·4	1·251,08	71	40	54	1·031,52	0·805,06
1·5	1·318,87	75	33	56	1·059,67	0·901,00
1·6	1·382,66	79	13	13	1·081,46	0·998,58

Table 1 (*cont.*)

s/b	ϕ rad.	ϕ deg.			x/b	z/b
		deg.	min.	sec.		
			$\beta = -0{\cdot}475$ (*cont.*)			
1·7	1·442,28	82	38	12	1·097,19	1·097,32
1·8	1·497,58	85	48	17	1·107,22	1·196,80
1·9	1·548,38	88	42	57	1·111,96	1·296,67
2·0	1·594,52	91	21	33	1·111,86	1·396,67
2·1	1·635,81	93	43	30	1·107,38	1·496,56
2·2	1·672,05	95	48	6	1·099,04	1·596,20
2·3	1·703,05	97	34	39	1·087,35	1·695,51
2·4	1·728,57	99	2	24	1·072,85	1·794,45
2·5	1·748,39	100	10	32	1·056,11	1·893,04
2·6	1·762,26	100	58	12	1·037,71	1·991,33
2·7	1·769,92	101	24	33	1·018,26	2·089,42
2·8	1·771,15	101	28	46	0·998,36	2·187,42
2·9	1·765,73	101	10	8	0·978,67	2·285,47
3·0	1·753,50	100	28	6	0·959,84	2·383,68
3·1	1·734,37	99	22	18	0·942,56	2·482,17
3·2	1·708,37	97	52	57	0·927,50	2·581,03
			$\beta = -0{\cdot}525$			
0	0		0		0	0
0·1	0·099,93	5	43	33	0·099,83	0·004,99
0·2	0·199,48	11	25	45	0·198,67	0·019,91
0·3	0·298,24	17	5	16	0·295,55	0·044,54
0·4	0·395,84	22	40	47	0·389,55	0·078,54
0·5	0·491,91	28	11	3	0·479,82	0·121,48
0·6	0·586,10	33	34	51	0·565,60	0·172,81
0·7	0·678,07	38	51	2	0·646,24	0·231,88
0·8	0·767,51	43	58	31	0·721,20	0·298,02
0·9	0·854,13	48	56	17	0·790,05	0·370,51
1·0	0·937,65	53	43	24	0·852,49	0·448,58
1·1	1·017,82	58	18	59	0·908,34	0·531,49
1·2	1·094,40	62	42	17	0·957,53	0·618,53
1·3	1·167,19	66	52	29	1·000,09	0·709,00
1·4	1·235,98	70	48	59	1·036,14	0·802,25
1·5	1·300,58	74	31	5	1·065,89	0·897,71
1·6	1·360,83	77	58	12	1·089,63	0·994,83
1·7	1·416,55	81	9	45	1·107,70	1·093,18
1·8	1·467,58	84	5	11	1·120,49	1·192,34
1·9	1·513,76	86	43	56	1·128,46	1·292,02
2·0	1·554,92	89	5	26	1·132,06	1·391,95
2·1	1·590,91	91	9	8	1·131,80	1·491,94
2·2	1·621,54	92	54	27	1·128,21	1·591,87
2·3	1·646,66	94	20	48	1·121,84	1·691,67
2·4	1·666,09	95	27	37	1·113,25	1·791,29
2·5	1·679,68	96	14	20	1·103,00	1·890,77
2·6	1·687,27	96	40	24	1·091,71	1·990,13
2·7	1·688,72	96	45	25	1·079,97	2·089,43
2·8	1·683,95	96	28	59	1·068,39	2·188,76
2·9	1·672,90	95	51	0	1·057,59	2·288,18
3·0	1·655,58	94	51	29	1·048,21	2·387,73
3·1	1·632,12	93	30	48	1·040,86	2·487,46
3·2	1·602,70	91	49	40	·036,15	2·587.35

TABLE 1 (*cont.*)

s/b	ϕ rad.	ϕ deg. deg.	min.	sec.	x/b	z/b
			$\beta = -0{\cdot}55$			
0	0		0		0	0
0·1	0·099,93	5	43	33	0·099,83	0·004,99
0·2	0·199,45	11	25	40	0·198,67	0·019,91
0·3	0·298,15	17	4	58	0·295,55	0·044,53
0·4	0·395,64	22	40	6	0·389,56	0·078,53
0·5	0·491,52	28	9	44	0·479,84	0·121,44
0·6	0·585,44	33	32	36	0·565,65	0·172,72
0·7	0·677,04	38	47	29	0·646,34	0·231,73
0·8	0·765,99	43	53	16	0·721,38	0·297,78
0·9	0·851,98	48	48	54	0·790,36	0·370,13
1·0	0·934,74	53	33	24	0·853,00	0·448,05
1·1	1·014,00	58	5	52	0·909,13	0·530,78
1·2	1·089,52	62	25	29	0·958,69	0·617,60
1·3	1·161,07	66	31	28	1·001,75	0·707,84
1·4	1·228,46	70	23	9	1·038,43	0·800,85
1·5	1·291,50	73	59	50	1·068,97	0·896,05
1·6	1·349,99	77	20	56	1·093,67	0·992,94
1·7	1·403,79	80	25	52	1·112,90	1·091,06
1·8	1·452,71	83	14	4	1·127,07	1·190,04
1·9	1·496,62	85	44	59	1·136,62	1·289,58
2·0	1·535,34	87	58	6	1·142,06	1·389,42
2·1	1·568,72	89	52	51	1·143,89	1·489,40
2·2	1·596,61	91	28	45	1·142,66	1·589,39
2·3	1·618,86	92	45	15	1·138,91	1·689,32
2·4	1·635,34	93	41	52	1·133,24	1·789,16
2·5	1·645,90	94	18	11	1·126,21	1·888,91
2·6	1·650,43	94	33	47	1·118,43	1·988,61
2·7	1·648,87	94	28	24	1·110,50	2·088,29
2·8	1·641,17	94	1	55	1·103,04	2·188,01
2·9	1·627,34	93	14	22	1·096,64	2·287,80
3·0	1·607,47	92	6	3	1·091,94	2·387,69
3·1	1·581,72	90	37	34	1·089,51	2·487,66
3·2	1·550,37	88	49	46	1·089,94	2·587,65
			$\beta = -0{\cdot}575$			
0	0		0		0	0
0·1	0·099,93	5	43	32	0·099,83	0·004,99
0·2	0·199,43	11	25	35	0·198,67	0·019,91
0·3	0·298,07	17	4	41	0·295,56	0·044,52
0·4	0·395,44	22	39	25	0·389,56	0·078,51
0·5	0·491,14	28	8	25	0·479,85	0·121,40
0·6	0·584,79	33	30	20	0·565,69	0·172,63
0·7	0·676,01	38	43	56	0·646,43	0·231,57
0·8	0·764,46	43	48	2	0·721,55	0·297,53
0·9	0·849,84	48	41	32	0·790,67	0·369,76
1·0	0·931,84	53	23	25	0·853,50	0·447,51
1·1	1·010,19	57	52	46	0·909,91	0·530,06
1·2	1·084,64	62	8	44	0·959,85	0·616,67
1·3	1·154,98	66	10	31	1·003,40	0·706,66
1·4	1·220,97	69	57	23	1·040,71	0·799,42
1·5	1·282,44	73	28	42	1·072,04	0·894,37
1·6	1·339,20	76	43	50	1·097,70	0·991,01

TABLE 1 (*cont.*)

s/b	ϕ rad.	ϕ deg. deg.	min.	sec.	x/b	z/b
		$\beta = -0{\cdot}575$ (*cont.*)				
1·7	1·391,09	79	42	12	1·118,08	1·088,90
1·8	1·437,93	82	23	14	1·133,60	1·187,68
1·9	1·479,58	84	46	25	1·144,74	1·287,05
2·0	1·515,89	86	51	14	1·151,99	1·386,78
2·1	1·546,70	88	37	10	1·155,89	1·486,70
2·2	1·571,89	90	3	46	1·157,00	1·586,69
2·3	1·591,33	91	10	36	1·155,87	1·686,68
2·4	1·604,90	91	57	14	1·153,09	1·786,64
2·5	1·612,50	92	23	21	1·149,25	1·886,57
2·6	1·614,05	92	28	42	1·144,95	1·986,47
2·7	1·609,53	92	13	10	1·140,80	2·086,39
2·8	1·598,95	91	36	46	1·137,41	2·186,33
2·9	1·582,37	90	39	46	1·135,37	2·286,31
3·0	1·559,94	89	22	40	1·135,29	2·386,30
3·1	1·531,88	87	46	12	1·137,73	2·486,27
3·2	1·498,48	85	51	23	1·143,25	2·586,11

TABLE 2. CALCULATED FACTORS FOR PENDANT DROPS

$-\beta$	x_e/b	x_s/b	S	$1/H$
0·2500	1·048,50	0·700,18	0·667,81	0·909,63
0·2625	1·051,39	0·716,78	0·681,74	0·861,56
0·2750	1·054,33	0·733,18	0·695,40	0·817,81
0·2875	1·057,34	0·749,44	0·708,80	0·777,81
0·3000	1·060,41	0·765,56	0·721,95	0·741,10
0·3125	1·063,54	0·781,59	0·734,89	0·707,27
0·3250	1·066,74	0·797,53	0·747,63	0·675,99
0·3375	1·070,02	0·813,42	0·760,19	0·646,97
0·3500	1·073,37	0·829,28	0·772,59	0·619,97
0·3625	1·076,81	0·845,13	0·784,84	0·594,78
0·3750	1·080,33	0·860,99	0·796,97	0·571,21
0·3875	1·083,95	0·876,89	0·808,98	0·549,10
0·4000	1·087,66	0·892,82	0·820,86	0·528,31
0·4125	1·091,49	0·908,83	0·832,66	0·508,72
0·4250	1·095,42	0·924,94	0·844,37	0·490,22
0·4375	1·099,48	0·941,17	0·856,01	0·472,71
0·4500	1·103,67	0·957,53	0·867,59	0·456,09
0·4625	1·108,00	0·974,05	0·879,11	0·440,30
0·4750	1·112,49	0·990,76	0·890,57	0·425,26
0·4875	1·117,15	1·007,68	0·902,01	0·410,90
0·5000	1·122,01	1·024,84	0·913,40	0·397,17
0·5125	1·127,08	1·042,28	0·924,76	0·384,01
0·5250	1·132,39	1·060,03	0·936,10	0·371,35
0·5375	1·137,98	1·078,13	0·947,41	0·359,16
0·5500	1·143,90	1·096,65	0·958,69	0·347,38
0·5625	1·150,20	1·115,62	0·969,93	0·335,95
0·5750	1·157,00	1·135,13	0·981,10	0·324,79
0·5875	1·164,44	1·155,28	0·992,13	0·313,83
0·6000	1·172,87	1·176,23	1·002,87	0·302,89

TABLE 3. TABLE OF $1/H$ IN TERMS OF S

S	0	1	2	3	4	5	6	7	8	9
0·66	0·938,28	0·934,54	0·930,82	0·927,12	0·923,45	0·919,79	0·916,16	0·912,55	0·908,95	0·905,38
0·67	0·901,83	0·898,30	0·894,78	0·891,29	0·887,82	0·884,36	0·880,92	0·877,51	0·874,11	0·870,73
0·68	0·867,37	0·864,03	0·860,70	0·857,39	0·854,10	0·850,83	0·847,58	0·844,34	0·841,12	0·837,92
0·69	0·834,73	0·831,56	0·828,41	0·825,27	0·822,15	0·819,05	0·815,96	0·812,89	0·809,83	0·806,79
0·70	0·803,76	0·800,75	0·797,76	0·794,78	0·791,82	0·788,87	0·785,94	0·783,02	0·780,11	0·777,22
0·71	0·774,35	0·771,49	0·768,64	0·765,81	0·763,00	0·760,19	0·757,41	0·754,63	0·751,87	0·749,12
0·72	0·746,39	0·743,67	0·740,97	0·738,28	0·735,60	0·732,93	0·730,28	0·727,64	0·725,02	0·722,40
0·73	0·719,80	0·717,22	0·714,64	0·712,08	0·709,53	0·707,00	0·704,47	0·701,96	0·699,46	0·696,97
0·74	0·694,49	0·692,02	0·689,57	0·687,13	0·684,70	0·682,28	0·679,88	0·677,48	0·675,10	0·672,73
0·75	0·670,37	0·668,03	0·665,69	0·663,37	0·661,05	0·658,75	0·656,46	0·654,18	0·651,91	0·649,65
0·76	0·647,40	0·645,16	0·642,94	0·640,72	0·638,51	0·636,32	0·634,13	0·631,95	0·629,79	0·627,63
0·77	0·625,49	0·623,35	0·621,22	0·619,11	0·617,00	0·614,90	0·612,81	0·610,74	0·608,67	0·606,61
0·78	0·604,57	0·602,53	0·600,50	0·598,48	0·596,47	0·594,47	0·592,48	0·590,49	0·588,52	0·586,56
0·79	0·584,60	0·582,65	0·580,72	0·578,79	0·576,87	0·574,96	0·573,05	0·571,16	0·569,27	0·567,39
0·80	0·565,53	0·563,66	0·561,81	0·559,97	0·558,13	0·556,30	0·554,48	0·552,66	0·550,86	0·549,06
0·81	0·547,27	0·545,49	0·543,71	0·541,95	0·540,19	0·538,44	0·536,69	0·534,96	0·533,23	0·531,51
0·82	0·529,79	0·528,08	0·526,38	0·524,69	0·523,00	0·521,32	0·519,65	0·517,99	0·516,34	0·514,69
0·83	0·513,05	0·511,42	0·509,79	0·508,17	0·506,56	0·504,96	0·503,36	0·501,76	0·500,18	0·498,60
0·84	0·497,03	0·495,46	0·493,90	0·492,34	0·490,80	0·489,26	0·487,72	0·486,19	0·484,67	0·483,16
0·85	0·481,65	0·480,15	0·478,65	0·477,16	0·475,67	0·474,20	0·472,72	0·471,26	0·469,80	0·468,34
0·86	0·466,90	0·465,45	0·464,02	0·462,59	0·461,16	0·459,74	0·458,33	0·456,92	0·455,52	0·454,12
0·87	0·452,73	0·451,34	0·449,96	0·448,58	0·447,21	0·445,84	0·444,48	0·443,13	0·441,78	0·440,44
0·88	0·439,10	0·437,77	0·436,44	0·435,12	0·433,80	0·432,49	0·431,18	0·429,88	0·428,58	0·427,29
0·89	0·426,00	0·424,71	0·423,44	0·422,16	0·420,89	0·419,63	0·418,37	0·417,12	0·415,87	0·414,62
0·90	0·413,38	0·412,14	0·410,91	0·409,68	0·408,46	0·407,24	0·406,02	0·404,81	0·403,60	0·402,40
0·91	0·401,21	0·400,01	0·398,82	0·397,64	0·396,46	0·395,28	0·394,11	0·392,94	0·391,77	0·390,61
0·92	0·389,46	0·388,31	0·387,16	0·386,01	0·384,87	0·383,74	0·382,60	0·381,47	0·380,35	0·379,22
0·93	0·378,10	0·376,99	0·375,88	0·374,77	0·373,66	0·372,56	0·371,46	0·370,37	0·369,28	0·368,19
0·94	0·367,11	0·366,02	0·364,94	0·363,87	0·362,80	0·361,73	0·360,66	0·359,60	0·358,54	0·357,48
0·95	0·356,43	0·355,38	0·354,33	0·353,28	0·352,24	0·351,20	0·350,16	0·349,13	0·348,09	0·347,06
0·96	0·346,04	0·345,01	0·343,99	0·342,97	0·341,95	0·340,93	0·339,92	0·338,90	0·337,89	0·336,88
0·97	0·335,88	0·334,87	0·333,87	0·332,87	0·331,86	0·330,86	0·329,87	0·328,87	0·327,87	0·326,88
0·98	0·325,88	0·324,89	0·323,89	0·322,90	0·321,91	0·320,92	0·319,92	0·318,93	0·317,94	0·316,95
0·99	0·315,95	0·314,96	0·313,96	0·312,96	0·311,96	0·310,95	0·309,94	0·308,93	0·307,92	0·306,90
1·00	0·305,88	0·304,84	0·303,81	0·302,76	—	—	—	—	—	—

Table 4. Table of H in terms of S

S	0	1	2	3	4	5	6	7	8	9
0·66	1·065,79	1·070,05	1·074,32	1·078,61	1·082,90	1·087,20	1·091,51	1·095,84	1·100,17	1·104,51
0·67	1·108,86	1·113,22	1·117,59	1·121,97	1·126,36	1·130,76	1·135,17	1·139,59	1·144,02	1·148,46
0·68	1·152,91	1·157,37	1·161,85	1·166,33	1·170,82	1·175,32	1·179,83	1·184,36	1·188,89	1·193,44
0·69	1·197,99	1·202,56	1·207,13	1·211,72	1·216,32	1·220,93	1·225,55	1·230,18	1·234,83	1·239,48
0·70	1·244,15	1·248,82	1·253,51	1·258,21	1·262,92	1·267,64	1·272,37	1·277,11	1·281,86	1·286,63
0·71	1·291,41	1·296,19	1·300,99	1·305,80	1·310,62	1·315,46	1·320,30	1·325,15	1·330,02	1·334,89
0·72	1·339,78	1·344,68	1·349,59	1·354,51	1·359,44	1·364,38	1·369,33	1·374,30	1·379,28	1·384,27
0·73	1·389,27	1·394,28	1·399,30	1·404,33	1·409,38	1·414,44	1·419,51	1·424,59	1·429,68	1·434,79
0·74	1·439,91	1·445,04	1·450,18	1·455,33	1·460,49	1·465,67	1·470,85	1·476,05	1·481,26	1·486,48
0·75	1·491,7	1·496,9	1·502,2	1·507,5	1·512,7	1·518,0	1·523,3	1·528,6	1·534,0	1·539,3
0·76	1·544,6	1·550,0	1·555,4	1·560,7	1·566,1	1·571,5	1·577,0	1·582,4	1·587,8	1·593,3
0·77	1·598,8	1·604,2	1·609,7	1·615,2	1·620,8	1·626,3	1·631,8	1·637,4	1·642,9	1·648,5
0·78	1·654,1	1·659,7	1·665,3	1·670,9	1·676,5	1·682,2	1·687,8	1·693,5	1·699,2	1·704,9
0·79	1·710,6	1·716,3	1·722,0	1·727,7	1·733,5	1·739,3	1·745,0	1·750,8	1·756,6	1·762,4
0·80	1·768,3	1·774,1	1·780,0	1·785,8	1·791,7	1·797,6	1·803,5	1·809,4	1·815,3	1·821,3
0·81	1·827,3	1·833,2	1·839,2	1·845,2	1·851,2	1·857,2	1·863,3	1·869,3	1·875,4	1·881,4
0·82	1·887,6	1·893,7	1·899,8	1·905,9	1·912,0	1·918,2	1·924,4	1·930,5	1·936,7	1·942,9
0·83	1·949,1	1·955,3	1·961,6	1·967,8	1·974,1	1·980,4	1·986,7	1·993,0	1·999,3	2·005,6
0·84	2·012,0	2·018,3	2·024,7	2·031,1	2·037,5	2·043,9	2·050,3	2·056,8	2·063,3	2·069,7
0·85	2·076,2	2·082,7	2·089,2	2·095,7	2·102,3	2·108,8	2·115,4	2·122,0	2·128,6	2·135,2
0·86	2·141,8	2·148,4	2·155,1	2·161,8	2·168,4	2·175,1	2·181,8	2·188,6	2·195,3	2·202,1
0·87	2·208,8	2·215,6	2·222,4	2·229,3	2·236,1	2·242,9	2·249,8	2·256,7	2·263,6	2·270,5
0·88	2·277,4	2·284,3	2·291,3	2·298,2	2·305,2	2·312,2	2·319,2	2·326,3	2·333,3	2·340,4
0·89	2·347,4	2·354,5	2·361,6	2·368,8	2·375,9	2·383,1	2·390,2	2·397,4	2·404,6	2·411,9
0·90	2·419,1	2·426,4	2·433,6	2·440,9	2·448,3	2·455,6	2·462,9	2·470,3	2·477,7	2·485,1
0·91	2·492,5	2·499,9	2·507,4	2·514,9	2·522,3	2·529,9	2·537,4	2·544,9	2·552,5	2·560,1
0·92	2·567,7	2·575,3	2·582,9	2·590,6	2·598,3	2·606,0	2·613,7	2·621,4	2·629,2	2·637,0
0·93	2·644,8	2·652,6	2·660,4	2·668,3	2·676,2	2·684,1	2·692,1	2·700,0	2·708,0	2·716,0
0·94	2·724,0	2·732,1	2·740,1	2·748,2	2·756,4	2·764,5	2·772,7	2·780,9	2·789,1	2·797,3
0·95	2·805,6	2·813,9	2·822,2	2·830,6	2·839,0	2·847,4	2·855,8	2·864,3	2·872,8	2·881,3
0·96	2·889,9	2·898,5	2·907,1	2·915,7	2·924,4	2·933,1	2·941,9	2·950,7	2·959,5	2·968,4
0·97	2·977,3	2·986,2	2·995,2	3·004,2	3·013,3	3·022,4	3·031,5	3·040,7	3·050,0	3·059,3
0·98	3·068,6	3·078,0	3·087,4	3·096,9	3·106,5	3·116,1	3·125,7	3·135,5	3·145,2	3·155,1
0·99	3·165,0	3·175,0	3·185,1	3·195,3	3·205,6	3·215,9	3·226,4	3·236,9	3·247,6	3·258,4
1·00	3·269,3	3·280,4	3·291,6	3·302,9	—	—	—	—	—	—

References

Adam, N. K. 1938 *The physics and chemistry of surfaces*, 2nd ed., pp. 363–388. Oxford: Clarendon Press.

Andreas, J. M., Hauser, E. A. & Tucker, W. B. 1938 *J. Phys. Chem.* **42**, 1001.

Bashforth, F. & Adams, J. C. 1883 *An attempt to test the theories of capillary action*. Cambridge University Press.

De Noüy, P. L. 1919 *J. Gen. Physiol.* **1**, 521.

Ferguson, A. 1912 *Phil. Mag.* **23**, 417.

Harkins, W. D. & Brown, F. E. 1919 *J. Amer. Chem. Soc.* **41**, 499.

Sudgen, S. 1921 *J. Chem. Soc.* p. 1483.

Sugden, S. 1922 *J. Chem. Soc.* p. 858.

Wilhelmy, L. 1863 *Ann. Phys., Lpz.*, **9**, 475.

Worthington, A. M. 1881 *Proc. Roy. Soc.* **32**, 362.

Worthington, A. M. 1885 *Phil. Mag.* **19**, 46.

Concerning the effect of compressibility on laminar boundary layers and their separation

By L. Howarth, *St John's College, Cambridge*

(*Communicated by Sir Geoffrey Taylor, F.R.S.—Received* 20 *November* 1947)

The theory of compressible flow in a laminar boundary layer has been developed for the case when the viscosity is assumed to be proportional to the absolute temperature and the Prandtl number is unity. (These assumptions may be compared with the empirical relations $\mu \propto T^{\frac{8}{9}}$ and $\sigma = 0{\cdot}715$ suggested by Cope.) It is shown that a transformation of the ordinate normal to the layer can lead to a simplified form of equation of motion very similar to the ordinary incompressible equation but modified by a multiplicative factor G in the pressure term. This factor is greater than unity at the boundary and tends to one at the outside of the layer.

Several particular solutions are considered including accelerated flow with a linearly increasing velocity and retarded flow along a flat plate with a linearly decreasing velocity.

The general implications of the theory are discussed and qualitative conclusions are drawn when the mainstream velocity starts from a stagnation point, rises to a maximum and subsequently falls. It is concluded that for such a velocity distribution increasing compressibility will reduce the skin friction, increase the boundary layer thickness and cause earlier separation as compared with the incompressible flow with the same mainstream velocity distribution and the kinematic viscosity corresponding to conditions at the stagnation point.

1. Introduction

The recent work of Cope & Hartree (1948) has made it abundantly clear that a complete study of compressible flow in boundary layers when allowance is made for the empirical temperature variation of viscosity and conductivity is a matter for modern electronic calculating machines. These authors have, in fact, initiated a study of the flow along a flat plate in the presence of a linear retarding pressure gradient by this means.

The empirical relations chosen by Cope (unpublished) for air in the temperature range 90° K to 300° K (the range important in wind tunnel experiments) are

$$\mu \propto T^{\frac{8}{9}} \quad \text{and} \quad \sigma = 0{\cdot}715, \tag{1}$$