Thermodynamics III Cheatsheet

Christian Leser (cleser@ethz.ch)

1 Conduction

1.1 Fouriers Law

$$\boxed{\begin{array}{l} \vec{\dot{q}}'' = -\lambda \nabla T \\ \text{1D:} \quad \dot{q_x}'' = \frac{\dot{Q}}{A} = -\lambda \frac{dT}{dx} \end{array}}$$

$\vec{\dot{q}}'' =$ Heat flux, $\left[\frac{W}{m^2}\right]$

$\lambda =$ thermal conductivity, $\left[\frac{W}{mK}\right]$

1.2 Heat conduction equation

$$\boxed{\underbrace{\rho c \frac{\partial T}{\partial t}}_{\substack{\text{th. energy flow} \\ \text{into control volume}}} = \underbrace{\vec{\nabla}\left(\lambda \vec{\nabla} T\right)}_{\substack{\text{Change in thermal} \\ \text{energy storage}}} + \underbrace{\dot{Q}'''_{\text{source}}}_{\substack{\text{Thermal energy} \\ \text{generation}}}}$$

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

1.3 3D relations

Geometry	3D Fouriers Law	Heat conduction equation operators
Planar	$\underbrace{-\lambda \frac{\partial T}{\partial x}\hat{x}}_{\dot{q}''_x} \underbrace{-\lambda \frac{\partial T}{\partial y}\hat{y}}_{\dot{q}''_y} \underbrace{-\lambda \frac{\partial T}{\partial z}\hat{z}}_{\dot{q}''_z}$	$\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right)$
Cylinder	$\underbrace{-\lambda \frac{\partial T}{\partial r}\hat{r}}_{\dot{q}''_r} \underbrace{-\lambda \frac{\partial T}{r\partial \phi}\hat{\phi}}_{\dot{q}''_\phi} \underbrace{-\lambda \frac{\partial T}{\partial z}\hat{z}}_{\dot{q}''_z}$	$\frac{1}{r}\frac{\partial}{\partial r}\left(\lambda r \frac{\partial T}{\partial r}\right)$
Sphere	$\underbrace{-\lambda \frac{\partial T}{\partial r}\hat{r}}_{\dot{q}''_r} \underbrace{-\lambda \frac{\partial T}{r\partial \theta}\hat{\theta}}_{\dot{q}''_\theta} \underbrace{-\lambda \frac{\partial T}{r\sin(\theta)\partial \phi}\hat{\phi}}_{\dot{q}''_\phi}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda r^2 \frac{\partial T}{\partial r}\right)$

2 Convection

$$\boxed{\dot{q}'' = \frac{\dot{Q}}{A} = \alpha(T_\infty - T_w)}$$

$\alpha =$ heat transfer coefficient, $\left[\frac{W}{m^2K}\right]$

3 Thermal Resistance model

$$\dot{Q}_x = \dot{Q_x}''A = \frac{1}{R_{th}}\Delta T$$

Geometry	**Conduction** R_{th}	**Convection** R_{th}
Planar	$\frac{L}{\lambda A}$	$\frac{1}{\alpha A}$
Cylinder	$\frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi\lambda L}$	$\frac{1}{2\pi r_1 L\alpha}$
Sphere	$\frac{\frac{1}{R_1}-\frac{1}{R_2}}{4\pi\lambda}$	$\frac{1}{4\pi r^2\alpha}$

Fouling Resistance

$$R_f = \frac{R''_f}{A}$$

3.1 Critical thickness

$$r_{2,crit} = \frac{\lambda}{\alpha}$$

4 Dimensional Analysis

4.1 Dimensionless Numbers

Nusselt number:	$Nu \equiv \frac{\alpha L}{\lambda_{Fluid}} = \frac{\text{Heat transfer due to convection}}{\text{Heat conduction in the fluid}}$
Biot number:	$Bi \equiv \frac{\alpha L}{\lambda_{\text{Wall}}} = \frac{\text{Thermal resistance inside body}}{\text{Convective resistance at surface}}$
Reynolds number:	$Re \equiv \frac{\rho w D_h}{\mu} = \frac{\text{inertia}}{\text{viscous forces}}$
Prandtl number:	$Pr \equiv \frac{\mu c_p}{\lambda} = \frac{\text{Momentum diffusivity}}{\text{Thermal diffusivity}}$
Grashof number:	$Gr \equiv \frac{\beta \Delta T g L^3}{\left(\frac{\mu}{\rho}\right)^2} = \frac{\text{Buoyancy force}}{\text{Viscous force}}$
Peclet number:	$Pe \equiv Re \cdot Pr$
Rayleigh number:	$Ra \equiv Gr \cdot Pr$
Hydraulic diameter:	$D_h \equiv \frac{4A}{P_w}$

c_p = Heat capacity, $\left[\frac{J}{kgK}\right]$

β = isobaric volumetric thermal expansion coefficient, $\left[K^{-1}\right]$

g = acceleration of gravity = $9.81 \frac{m}{s^2}$

w = velocity of fluid, $\left[\frac{m}{s}\right]$

ΔT = temperature difference surface vs. fluid, $[K]$

L = characteristic length

$= \frac{\text{Volume}}{\text{Surface Area}} = \begin{cases} \frac{r}{3}, & \text{sphere} \\ \frac{r}{2}, & \text{cylinder} \end{cases}, [m]$

μ = fluid dynamic viscosity, $[Pas]$

ρ = fluid density, $\left[\frac{kg}{m^3}\right]$

A = cross-sectional area of flow, $\left[m^2\right]$

P_w = wetted perimeter of cross-section, $[m]$

$= 2\pi r$ for cylindrical tube

4.2 Forced and Free convection

forced

$$Nu = f(Re, Pr)$$

Average Nusselt number: $Nu = C \cdot Re^n \cdot Pr^{\frac{1}{3}}$

free

$$Nu = f(Gr, Pr)$$

Average Nusselt number: $Nu = C \cdot Gr^m \cdot Pr^m$

4.3 Heat transfer correlations

Channel flows, heating fluid

heat is supplied to fluid

$$Nu = 0.023 Re^0.8 Pr^0.4$$

Channel flows, cooling fluid

heat is removed from fluid

$$Nu = 0.023 Re^0.8 Pr^0.3$$

External flows

$$Nu = C Re^m Pr^0.3$$

5 Lumped capacitance method

$$M c_p \frac{\partial T}{\partial t} = \dot{Q}_{\text{source}} + \dot{Q}_{\text{in}} - \dot{Q}_{\text{out}}$$

5.1 Dependence of temperature on time

$$\boxed{\frac{T(t)}{T_0} = exp\left(-\frac{\alpha A}{M c_p}\right) = exp\left[-\frac{t}{\tau}\right], e^{-1} = 0.368}$$

A = surface of lumped body, $\left[m^2\right]$

M = mass of lumped body, $[kg]$

τ = time constant, $[s]$

6 Boundary layer analysis

Continuity equation

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

Momentum equation

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{Re} \frac{\partial^2 u^*}{\partial y^{*2}}$$

Energy equation

$$u^* \frac{\partial \Theta^*}{\partial x^*} + v^* \frac{\partial \Theta^*}{\partial y^*} = \frac{1}{Pe} \frac{\partial^2 \Theta^*}{\partial y^{*2}}$$

$x^* = \frac{x}{L}, y^* = \frac{y}{L}$

$u^* = \frac{u}{u_\infty}, v^* = \frac{v}{u_\infty}$

$\Theta^* = \frac{T - T_\infty}{T_W - T_\infty}$

6.1 boundary layer characteristics

Thickness of velocity bl $\frac{\delta}{L} \approx Re_L^{-\frac{1}{2}}$

	$\boldsymbol{\delta > \delta_T \Leftrightarrow \delta Pr >> 1}$	$\boldsymbol{\delta < \delta_T \Leftrightarrow \delta Pr << 1}$
Thickness temperature bl	$\frac{\delta_T}{L} \approx Pr^{-\frac{1}{3}} Re_L^{-\frac{1}{2}}$	$\frac{\delta_T}{L} \approx Pr^{-\frac{1}{2}} Re_L^{-\frac{1}{2}}$
Heat transfer	$\overline{\alpha} \approx \frac{\lambda}{L} Pr^{\frac{1}{3}} Re_L^{\frac{1}{2}}$	$\overline{\alpha} \approx \frac{\lambda}{L} Pr^{\frac{1}{2}} Re_L^{\frac{1}{2}}$
	$Nu \equiv \frac{\overline{\alpha} L}{\lambda} \approx Pr^{\frac{1}{3}} Re_L^{\frac{1}{2}}$	$Nu \equiv \frac{\overline{\alpha} L}{\lambda} \approx Pr^{\frac{1}{2}} Re_L^{\frac{1}{2}}$

7 Heat exchanger

7.1 Heat transfer from hot to cold

$$\boxed{\begin{aligned} \dot{q} &= AU\Delta T_{LM} \\ \Delta T_{LM} &= \frac{\Delta T_{out} - \Delta T_{in}}{\ln(\frac{\Delta T_{out}}{\Delta T_{in}})} \end{aligned}}$$

$A =$ Heat transfer surfache area, $\left[m^2\right]$

$U =$ overall heat transfer coefficient, $\left[\frac{W}{m^2 K}\right]$

7.2 Temperature in / efficiency of heat exchanger

parallel

$$\boxed{\begin{aligned} \Delta T(x) &= \Delta T_{(x=0)} exp\left(-U\left(\frac{1}{C_H} + \frac{1}{C_C}\right) Wx\right) \\ \varepsilon &= \frac{1 - e^{-N(1+\rho_S)}}{1 + \rho_S} \end{aligned}}$$

counter

$$\boxed{\begin{aligned} \Delta T(x) &= \Delta T_{(x=0)} exp\left(-U\left(\frac{1}{C_H} - \frac{1}{C_C}\right) Wx\right) \\ \varepsilon &= \frac{1 - e^{-N(1-\rho_S)}}{1 - \rho_S e^{-N(1-\rho_S)}} \end{aligned}}$$

$N = \frac{UA}{C_{\min}}$ $\quad \rho_S = \frac{C_{\min}}{C_{\max}}$ $\quad C_{\min} = min(C_h, C_c)$ $\quad C_{\max} = max(C_h, C_c)$ $\quad W =$ width of heat transfer surface, $[m]$

Note: $C = \dot{m} c_p \quad \varepsilon = \frac{\dot{q}}{\dot{q}_{\max}} = \frac{C_H(T_{\text{H, in}} - T_{\text{H, out}})}{C_{\min}(T_{\text{H, in}} - T_{\text{C, in}})}$