Merton model and Poisson process with Log Normal intensity function

Masato Hisakado*

Nomura Holdings, Inc., Otemachi 2-2-2,

Chiyoda-ku, Tokyo 100-8130, Japan

Kodai Hattori† and Shintaro Mori†

†Department of Mathematics and Physics,

Graduate School of Science and Technology, Hirosaki University

Bunkyo-cho 3, Hirosaki, Aomori 036-8561, Japan

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Abstract

This study considers the Merton model. We show the Merton model becomes Poisson process with the log-normal distributed intensity function in the continuous limit. We discuss the relation between this model and Hawkes process. We apply this model to the default portfolios. Moreover we show why the beta distribution and Merton model have the similar properties.

Keywords: Phase transition, Merton model, Anomalous diffusion, Credit risk management

1. INTRODUCTION

Anomalous diffusion is one of the most interesting topics in sociophysics and econophysics [1–3]. The models describing such phenomena have a long memory [4–10] and show several types of phase transitions. In our previous work, we investigated voting models for an information cascade [11–17]. This model has two types of phase transitions. One is the information cascade transition, which is similar to the phase transition of the Ising model [13] that shows whether a distribution converges or not. The other is the convergence transition of the super-normal diffusion that corresponds to an anomalous diffusion [12, 18].

In recent years, there have been several studies regarding the time series of financial markets from the perspective of econophysics [24–28]. The important properties of these data are particularly the fat tailed distribution of returns, long memory in volatility, and multi-fractal nature. The market data of stock prices and foreign exchange have been used in most of these studies. The long memory of volatility is known as volatility clustering [28] and affects the risk management and especially the calculation of Value at Risk (VaR). It corresponds to the temporal correlation of time series. The temporal correlation of exponential decay is the short and the power decay is the medium and long memories. In fact, most of the asset have the short memory, but it is known that some assets has the long and medium memories.

We study a Bayesian estimation method using the Merton model. Under normal circumstances, the Merton model incorporates default correlation by the correlation of asset price movements (asset correlation), which is used to estimate the PD and the correlation. A Monte Carlo simulation is an appropriate tool to estimate the parameters, except under the limit of large homogeneous portfolios [21]. In this case, the distribution becomes a Vasicek distribution that can be calculated analytically [29].

In our previous study, we discussed parameter estimation using a beta-binomial distribution with default correlation and considered a multi-year case with a temporal correlation [17]. In the double scaling limit we can obtain the SE-NBD process and Hawkes process. In this paper we consider the limit of Merton model process. In this study, we discuss a phase transition when we use the Merton model and compare this model to the SE-NBD and Hawkes models.

The remainder of this paper is organized as follows. In Section II, we introduce the stochastic process of the Merton model and obtain the Poisson process in the limit. In Section III we discussion the normal-slow phase transition. In Section IV, we describe the application of the Bayesian estimation approach to the empirical data of default history using the Merton model and confirm its parameters. Finally, the conclusions are presented in Section V.

II ASSET CORRELATION AND DEFAULT CORRELATION

A. Introduction of the Model

In this section, we consider whether the time series of a stochastic process using the Merton model [30]. We take the limit of the process, and obtain the Poisson process with the log-normal intensity function.

Normal random variables, y_t, are hidden variables that explain the status of the economics, and y_t affects all obligors in the t-th term. Through y_t temporal correlation affects the obligors. To introduce the temporal correlation from different term, let $\{y_t, 1 \leq t \leq T\}$ be the time series of the stochastic variables of the correlated normal distribution with the following correlation matrix:

$$\Sigma \equiv \begin{pmatrix} 1 & d_1 & d_2 & \cdots & d_{T-1} \\ d_1 & 1 & d_1 & \ddots & \vdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & d_1 \\ d_{T-1} & \cdots & d_2 & d_1 & 1 \end{pmatrix}, \tag{1}$$

where $(y_1, \cdots, y_T)^T \sim \mathrm{N}_T(0, \Sigma)$.

In this paper, we consider two cases of temporal correlation: exponential decay, $d_i = \theta^i, 0 \leq \theta \leq 1$, and power decay, $d_i = 1/(i+1)^{-\gamma}, \gamma \geq 0$. The exponential decay corresponds to short memory and the power decay corresponds to intermediate and long memories [31]. In the case of the exponential decay, we can write

$$y_{t+1} = \theta y_t + \sqrt{1-\theta^2}\xi_{t+1}, \tag{2}$$

where ξ_t is i.i.d. and the correlation between y_t and y_{t+1} is r. The first term corresponds to the temporal correlation decay.

Without loss of generality, we assume that the number of obligors in the t-th term is constant and denote it as N. The asset correlation, ρ_A, is the parameter that describes the correlation between the value of the assets of the obligors in the same term. We consider the i-th asset value, $\hat{U}_{it}$, at term t, to be

$$\hat{U}_{it} = \sqrt{\rho_A} y_t + \sqrt{1-\rho_A}\epsilon_{it}, \tag{3}$$

where $\epsilon_{it} \sim \mathrm{N}(0,1)$ is i.i.d. The first term is the effect of the economics and the second term is the effect of the individual obligor. By this formulation, the equal-time correlation of U_{it} is ρ_A.

The discrete dynamics of the process is described by

$$X_{it} = 1_{\hat{U}_{it} \le Y}, \tag{4}$$

where Y is the threshold and $1 \le i \le N$. When $X_{it} = 1(0)$, the i-th obligor in the t-th term is default (non-default). Eq.(4) corresponds to the conditional default probability for $y_t = y$ as

$$G(y) \equiv \mathrm{P}(X_t = 1 | y_t = y) = \Phi\left(\frac{Y - \sqrt{\rho_A} y}{\sqrt{1-\rho_A}}\right), \tag{5}$$

where $\Phi(x)$ is the standard normal distribution, $G(y_t)$ is the distribution of the default probability during the t-th term in the portfolio, and the average PD is $p' = \Phi(Y)$, which is the long term average of PD. It is the original process.

Next we consider the limit of this process. We set $y_t = y$ here. The non-conditional distribution of number of the default is

$$\begin{aligned} P[X_t = k_t] &= \int_{-\infty}^{\infty} \frac{N!}{k_t!(N-k_t)!} G(y)^{k_t}(1-G(y))^{N-k_t}\phi(y)dy, \\ &= \int_{-\infty}^{\infty} \frac{N!}{k_t!(N-k_t)!} \Phi\left(\frac{\Phi^{-1}(p') - \sqrt{\rho_A}y}{\sqrt{1-\rho_A}}\right)^{k_t} (1-\Phi\left(\frac{\Phi^{-1}(p') - \sqrt{\rho_A}y}{\sqrt{1-\rho_A}}\right))^{N-k_t}\phi(y)dy, \end{aligned} \tag{6}$$

where $\phi(y)$ is the normal distribution, $1/\sqrt{2\pi}e^{-y^2/2}$. Here we use the approximation for the normal distribution,

$$\Phi(x) \approx \frac{1}{1+e^{-\beta x}}, \tag{7}$$

the logistic function and $\beta \sim 1.3$. Using this approximation, we can obtain

$$G(y) = \Phi\left(\frac{\Phi^{-1}(p') - \sqrt{\rho_A}y}{\sqrt{1-\rho_A}}\right) \equiv p \sim \frac{1}{1+((1-p')/p')^{1/\sqrt{1-\rho_A}} e^{\beta\sqrt{\rho_A}y/\sqrt{1-\rho_A}}}, \tag{8}$$

where we use the relation,

$$Y = \Phi^{-1}(p') \approx \log \frac{p'}{1-p'}. \tag{9}$$

In the case $p = G(y) << 1$, we can obtain,

$$G(y) = p'^{1/\sqrt{1-\rho_A}} e^{-\frac{\sqrt{\rho_A}}{\sqrt{1-\rho_A}}\beta y}. \tag{10}$$

Here we take the limit $p', p \to 0$ and $N \to \infty$ with the condition, fixed $\lambda_0 = Np'^{1/\sqrt{1-\rho_A}}$. Note that the relation p and p' is Eq.(8) We define the number of the defaults at the term t, $\lambda(y_t) = \lambda(t)$ as

$$\lambda(y) = Np'^{1/\sqrt{1-\rho_A}} e^{-\frac{\sqrt{\rho_A}}{\sqrt{1-\rho_A}}\beta y} = \lambda_0 e^{-\frac{\sqrt{\rho_A}}{\sqrt{1-\rho_A}}\beta y} \tag{11}$$

The expected value of $\lambda(y)$ is

$$\bar{\lambda} = \int_{-\infty}^{\infty} \lambda(y)\phi(y)dy = \lambda_0 e^{\frac{\rho_A}{2(1-\rho_A)}\beta^2} = \lambda_0 e^{\alpha^2/2}, \tag{12}$$

where $\alpha = \frac{\sqrt{\rho_A}}{\sqrt{1-\rho_A}}\beta$.

Here we change the variable from y to $\hat{y} = -y$. The standard normal distribution has the symmetry in y, so we can change the variable without the loss of generality. We can obtain

$$\lambda(\hat{y}) = \lambda_0 e^{\frac{\sqrt{\rho_A}}{\sqrt{1-\rho_A}}\beta \hat{y}} = \lambda_0 e^{\alpha \hat{y}}. \tag{13}$$

In the exponential decay case we can obtain another representation

$$\lambda(\hat{y}_{t+1}) = \lambda_0 e^{\alpha \hat{y}_{t+1}} = \lambda_0^{1-\theta} e^{\sqrt{1-\theta^2}\alpha \hat{\xi}_{t+1}} \lambda(\hat{y}_t)^{\theta}, \tag{14}$$

using Eq.(2) and $\hat{\xi}_t = -\xi_t$.

In this limit the we can calculate Eq.(6) as

$$\begin{aligned} P[X_t = k_t] &\sim \int_{-\infty}^{\infty} \frac{\lambda(\hat{y})^{k_t} e^{-\lambda(\hat{y})}}{k_t!} \phi(\hat{y}) d\hat{y} \\ &= \int_{0}^{\infty} \frac{\lambda(\hat{y})^{k_t} e^{-\lambda(\hat{y})}}{k_t!} f(\lambda) d\lambda, \end{aligned} \tag{15}$$

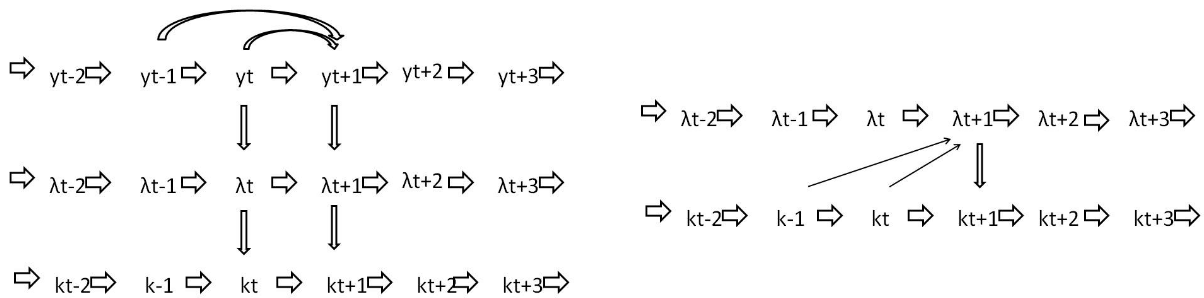
where

$$f(\lambda) = \phi(\hat{y})\frac{d\hat{y}}{d\lambda} = \frac{1}{\sqrt{2\pi}\alpha\lambda} e^{-\frac{(\log\lambda - \log\lambda_0)^2}{2\alpha^2}} \tag{16}$$

Then the distribution of the intensity function is the log normal distribution. The expected values is $\bar{\lambda} = \lambda_0 e^{\alpha^2/2}$ and the variance is $\bar{V} = \bar{\lambda}^2(e^{\alpha^2} - 1)$. Note that the expected value is the function of ρ_A in this process. On the other hand, in the original process the expected value of the probability of defaults does not depends on the function of ρ_A.

In the limit $\rho_A \to 0$, $\bar{\lambda} = \lambda_0$ and $\bar{V} = 0$, it is the Poisson process. In the limit $\rho_A \to 1$, $\alpha \to \infty$ and $f(\lambda) \sim 1/\lambda$ where $\bar{\lambda} \to \infty$ and $V \to \infty$.

We show the image of the relation among the variables of this process in Fig.1 (a). For the comparison we show that of SE-NBD and Hawkes processes in Fig.1 (b). In this process y_t is independent form the historical data of λ_t and k_t. In this process the temporal correlation is by time series of y_t and the hidden process. y affects the number of defaults through the intensity function λ_t. On the other hand in the NBD and Hawkes processes there is the feedback between the intensity function λ_t and the number of defaults k_t.



(a) (b)

FIG. 1. The relation among the variables of (a) this process and (b) SE-NBD and Hawkes process

B. Impact Analysis

Next we consider the impact analysis. It is the study of the the effect of the shock of the noise δ at term t. When the shock is added in $\hat{y}_t$, we consider the added expected value of the intensity function form $t+1$ to ∞.

i. *Exponential decay case*

At first we consider the case, exponential decay model $d_i = \theta^i, \theta \leq 1$. We consider the ratio for the impact δ added at the time $t \to \infty$,

$$\frac{\bar{\lambda}_{\delta\infty}}{\bar{\lambda}_{\infty}} = \mathrm{E}[\log \frac{\lambda_0 e^{\alpha\delta+\alpha\hat{y}_t}}{\lambda_0 e^{\alpha\hat{y}_t}} \frac{\lambda_0 e^{\alpha\delta\theta+\alpha\hat{y}_{t+1}}}{\lambda_0 e^{\alpha\hat{y}_{t+1}}} \frac{\lambda_0 e^{\alpha\delta\theta^2+\alpha\hat{y}_{t+2}}}{\lambda_0 e^{\alpha\hat{y}_{t+2}}} \cdots] = \frac{\alpha\delta}{1-\theta}. \tag{17}$$

Then, in this case the impact is the finite.

ii. *Power decay case*

Next we consider the case, $d_i = 1/(i+1)^\gamma, i = 1, 2, \cdots$, where $\gamma \geq 0$ is the power index. We consider the ratio for the impact δ added,

$$\frac{\bar{\lambda}_{\delta\infty}}{\bar{\lambda}_{\infty}} = \mathrm{E}[\log \frac{\lambda_0 e^{\alpha\delta+\alpha\hat{y}_t}}{\lambda_0 e^{\alpha\hat{y}_t}} \frac{\lambda_0 e^{\alpha\delta/2^\gamma+\alpha\hat{y}_{t+1}}}{\lambda_0 e^{\alpha\hat{y}_{t+1}}} \frac{\lambda_0 e^{\alpha\delta/3^\gamma+\alpha\hat{y}_{t+2}}}{\lambda_0 e^{\alpha\hat{y}_{t+2}}} \cdots] \tag{18}$$

When $\gamma > 1$, we can obtain

$$\frac{\bar{\lambda}_{\delta\infty}}{\bar{\lambda}_{\infty}} < \alpha\delta(1+\frac{1}{\gamma-1}). \tag{19}$$

Then the impact is finite. When $\gamma = 1$, we can obtain

$$\frac{\bar{\lambda}_{\delta\infty}}{\bar{\lambda}_{\infty}} \sim \alpha\delta \log T. \tag{20}$$

Then the impact is infinite. When $\gamma < 1$, we can obtain

$$\frac{\bar{\lambda}_{\delta\infty}}{\bar{\lambda}_{\infty}} \sim \alpha\delta T^{1-\gamma}. \tag{21}$$

Then the impact is infinite.

In summary the impact is finite when $\gamma > 1$ and the impact is infinite when $\gamma \leq 1$. Even if the impact infinite, the average number of defaults converges, because the process does not have the absorption process. The Hawkes process and SE-NBD process are one of the absorption process. Hence, there is the absorption transition which is one kind of the non-equilibrium transition. In the next section we consider the phase transition of this process for the impact.

III. NORMAL-SLOW PHASE TRANSITION

We are interested in the effect of the temporal correlation. Here we consider the variance of λ_T, $V(\lambda_T)$ the variance of λ of the term T is

$$\mathrm{V}(\Sigma^T\lambda(\lambda_T=\hat{y}_i)) = T\bar{V} + 2\bar{V}\sum_{i=1}^{T-1} d_i(T-i).$$

The second term is from the temporal correlation. We consider the behavior of the second term in limit $T \to \infty$.

A. Exponential temporal correlation

In this subsection, we study $V(\lambda_T)$ for exponential decay model $d_i = \theta^i, \theta \leq 1$:

$$V(\lambda_T) \simeq \bar{V}T + +2\bar{V}\sum_{i=1}^{T-1}\theta^i(T-i). \tag{22}$$

The first term on the right-hand side (RHS) behave as $\propto T$, thus, this is the normal diffusion. In the case that $\theta \neq 1$, the second term is

$$2\bar{V}[T\frac{1-\theta^{T-1}}{1-\theta}+\frac{(T-1)\theta^{T-1}(1-\theta)-(1-\theta^{T-1})\theta}{(1-\theta)^2}] \propto T$$

and it also the normal diffusion. We conclude that as the number of data samples increases, the variance increases as T. When $\theta = 1$, there is no temporal correlation decay case and all obligors are correlated ρ_A. Hence, there is no phase transition for $\theta < 1$.

B. Power temporal correlation

In this subsection, we consider power decay case $d_i = 1/(i+1)^\gamma, i = 1, 2, \cdots,$ where $\gamma \geq 0$ is the power index. The power correlation affects the number of defaults for long periods of time. Ranges $\gamma \leq 1$ and $\gamma > 1$ are called long memory and intermediate memory, respectively [31]. In contrast, the exponential decay affects short periods of time and is called short memory. The asymptotic behavior of $V(\lambda_T))$ is given as:

$$\mathrm{V}(\lambda_T) \simeq \bar{V}T + 2\bar{V}T\sum_{i=1}^{T-1}(i+1)^{-\gamma}(T-i).$$

i. $\gamma > 1$ case

We can obtain

$$\begin{aligned}\mathrm{V}(\lambda_T) &\simeq \bar{V}T + +2\bar{V}\sum_{i=1}^{T-1}(T-i)/(i+1)^\gamma \\ &\simeq \bar{V}T + +2\bar{V}T^{-\gamma+2}/(\gamma-1).\end{aligned} \quad (23)$$

The first term is the normal diffusion and the second term is the $T^{-\gamma+2}$, where $\gamma > 1$. It is the slower than the the normal diffusion. Hence, the significant terms are the first term which is the normal diffusion.

ii. $\gamma = 1$ case

$\mathrm{V}(\lambda_T)$ behaves as

$$\mathrm{V}(\lambda_T) \simeq \bar{V}T + 2\bar{V}\sum_{i=1}^{T-1}(T-i)/(i+1). \quad (24)$$

The RHS of Eq. (24) is evaluated as

$$V(\lambda_T) \simeq \bar{V}T + +2\bar{V}T\log T - T + 2] \sim T\log T. \tag{25}$$

In conclusion, $V(\lambda_T)$ behaves asymptotically as

$$V(\lambda_T) \sim T\log T \tag{26}$$

and becomes the anomalous faster diffusion.

2.2.3) $\gamma < 1$ case

$V(\lambda_T)$ is calculated as

$$V(\lambda_T) \simeq \bar{V}T + 2\bar{V}[\frac{1}{(1-\gamma)(2-\gamma)]}T^{-\gamma+2} \sim T^{-\gamma+2}. \tag{27}$$

Thus, we can conclude that $V(\lambda_T)$ behaves as

$$V(\lambda_T) \sim T^{-\gamma+2}, \tag{28}$$

which is also the anomalous faster diffusion.

In conclusion, a phase transition occurs when the temporal correlation decays by the power law. It is same as the original process which is before the limit. When the power index, γ, is less than one, the variance , $Z(\lambda_T)$, anomalous diffusion. Conversely, when the power index, γ, is greater than one, it is the normal diffusion. This phase transition is called a "super-normal transition" [12, 18], which is the transition between long memory and intermediate memory. In that study, when the power index was less than one, the estimator does not converge the steady state [17].

III. ESTIMATION OF PARAMETERS

As discussed in the previous section, whether temporal correlation obeys an exponential decay or a power decay is an important issue because there exists a super-normal transition in the latter case. Further, the appearance of a transition affects whether we can estimate the PD.

The S&P default data from 1981 to 2022 [32] are used. The average nuber of defualts is 70.3 for all ratings, 68.1 % for speculative grade (SG) ratings, and 2.18% for investment grade (IG) ratings. The SG rating represents ratings under BBB-(Baa3) and IG represents that above BBB-(Baa3). In Fig. 2 (a) we show the historical default rate of the S&P. The solid and dotted lines correspond to the speculative grade and investment grade samples, respectively.

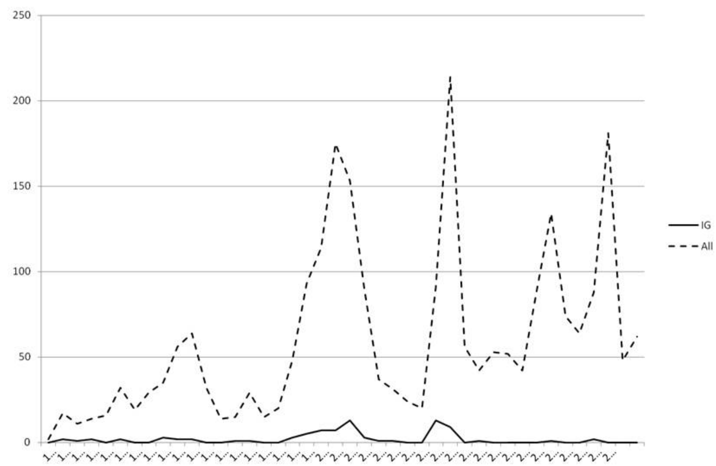


FIG. 2. S&P Default Rate in 1981-2022.

We estimate the parameters p, ρ_A, θ and γ of the Merton model using the Bayesian method and Stan 2.19.2 in R 3.6.2 software. We explain the method and how to estimate the parameters in Appendix B [34] and summarize the results in Table I. We show ρ_D instead of ρ_A, as we need to compare it with that of the beta-binomial distribution model. The estimation of the parameters are the maximum a posteriori (MAP) estimation. A detailed explanation of the estimation procedure and rmd file is provided on GitHub [35]. We notice that the power index γ is smaller than one for all cases and the values are smaller than the phase transition point, $\gamma = 1$.

Here we introduce the MAP estimation using the beta binomial distribution which used the default correlation directly to compare the the MAP estimation using Merton model. Beta binomial model is known as the correlated binomial distribution and we apply it to defaults in the portfolio with the correlation [17].

The conclusions are shown in Table II for the exponential and power decay models. We confirmed small θ and large γ values, which represent small temporal correlation. Parameter

TABLE I. MAP estimation of the parameters for the exponential and power decay models by the Merton model

No.	Model	Exponential decay			Power decay		
		p	ρ_A	θ	p	ρ_A	γ
1	Moody's 1920-2018	1.43%	17.8%	0.858	7.40%	38.7%	0.090
2	S&P 1981-2018	1.43%	6.4%	0.597	1.85%	12.0%	0.610
3	Moody's 1981- 2018	1.61%	7.1%	0.613	1.92%	12.4%	0.622
4	S&P 1990-2018	1.72%	7.5%	0.616	2.97%	12.5%	0.616
5	Moody's 1990-2018	1.92%	10.0%	0.678	2.40%	12.1%	0.624
6	Moody's 1920-2018 SG	3.00%	18.9%	0.838	6.15%	32.2%	0.146
7	S&P 1981-2018 SG	4.53%	8.7%	0.588	4.42%	11.7%	0.628
8	Moody's 1981-2018 SG	4.28%	9.4%	0.603	3.97%	11.5%	0.619
9	S&P 1990-2018 SG	4.93%	11.2%	0.639	5.40%	13.9%	0.626
10	Moody's 1990-2018 SG	4.51%	11.1%	0.648	6.09%	14.6%	0.619
11	Moody's 1920-2018 IG	0.04%	35.3%	0.891	3.40%	51.4%	0.102
12	S&P 1981-2018 IG	0.02%	25.8%	0.483	0.02%	20.3%	9.189
13	Moody's 1981-2018 IG	0.01%	21.9%	0.672	1.84%	33.8%	0.618
14	S&P 1990-2018 IG	0.01%	37.4%	0.712	1.63%	46.7%	0.630
15	Moody's 1990-2018 IG	0.01%	33.0%	0.794	3.44%	51.1%	0.003

γ for the power decay is larger than the phase transition point, $\gamma = 1$. The PD and default correlation are almost the same as the estimations by the exponential and power decay models. The reason behind this is that the power exponent γ is adequately large and there is only a small difference between the exponential and power decay models.

We can confirm that θ and γ both have large differences between the values estimated by the beta-binomial distribution and the Merton model. The reason for this is shown in Fig. **??** (a). We set d_{1A} and d_{1D} for ρ_A and ρ_D, respectively. From this, we can obtain the

TABLE II. Most likelihood estimate of the parameters for the exponential and power decay models by beta-binomial distribution

		Exponential decay			Power decay		
No.	Model	p	ρ_D	θ	p	ρ_D	γ
1	Moody's 1920-2018	0.96%	1.9%	0.044	0.94%	2.0%	4.7
2	S&P 1981- 2018	1.53%	0.8%	0.026	1.54%	0.8%	5.7
3	Moody's 1981-2018	1.53%	0.8%	0.022	1.52%	0.7%	5.9
4	S&P 1990-2018	1.66%	0.9%	0.023	1.64%	0.9%	5.7
5	Moody's 1990-2018	1.67%	0.9%	0.019	1.61%	0.8%	6.0
6	Moody's 1920-2018 SG	2.36%	3.8%	0.044	2.34%	4.1%	4.7
7	S&P 1981-2018 SG	4.16%	2.0%	0.026	4.20%	2.0%	5.7
8	Moody's 1981-2018 SG	4.18%	2.0%	0.022	4.35%	1.9%	6.0
9	S&P 1990-2018 SG	4.42%	2.5%	0.024	4.43%	2.6%	5.6
10	Moody's 1990-2018 SG	4.33%	2.3%	0.020	4.31%	2.2%	5.9
11	Moody's 1920-2018 IG	0.13%	0.8%	0.17	0.11%	0.9%	3.0
12	S&P 1981-2018 IG	0.11%	0.4%	0.12	0.09%	0.3%	3.6
13	Moody's 1981-2018 IG	0.10%	0.6%	0.05	0.09%	0.3%	4.6
14	S&P 1990-2018 IG	0.09%	0.4%	0.12	0.09%	0.4%	3.7
15	Moody's 1990-2018 IG	0.09%	0.4%	0.06	0.07%	0.7%	4.2

inequality

$$d_{1A} = \frac{d_{1A}\rho_A}{\rho_A} >> \frac{f(d_{1A}\rho_A)}{f(\rho_A)} = \frac{f(d_{1A}\rho_A)}{\rho_D} = d_{1D}.$$

Hence, the difference in the estimated parameter between the Merton model and the beta-binomial model becomes large. We can find the large convexity for the mapping function f. Hence, θ and γ for a default correlation is much smaller than that for asset correlation.

Next, we discuss whether the correlation has a long memory. In Table **??**, we calculated the WAIC and WBIC for each model that uses the Merton model for the discussion. Using Moody's data from 1920, the power decay model is found to be superior to the exponential decay model. Therefore, it seems that the default rate has a long memory. As γ is less than 1 for long history data, the phase converges slowly. In other words, parameter estimation becomes difficult because the convergence speed becomes slow when the temporal correlation is the power decay.

IV. CONCLUDING REMARKS

This study considers the Merton model. We show the Merton model becomes Poisson model with the log-normal distributed intensity function in the limit. We discuss the relation between this model and Hawkes and SE-NBD processed. We apply this model to the default portfolios. Moreover we show why the beta distribution and Vasicek model have the similar.

When the power index, γ, was larger than one, the estimator distribution of the PD converged normally. When the power index was less than or equal to one, the diffusion is anomalous faster diffusion. It is the super phase.This phase transition is called the "super-normal transition." For the case of exponential decay, there was no phase transition.

APPENDIX A. KENSTEN PROCESS

Here we discuss the obligors which are independent from the economic index y_t. The PD of the oblogors is p'' and the distribution is the continuous uniform distribution from o to 1 $\eta(0,1)$. The obligors which are affected by the economic index y are group 1 and the those which are not affected are group 2. $G(y)$ corresponds to the conditional default probability as

$$G(y) = \mathrm{P}(X_t = 1|y_t = y) = \alpha\Phi\left(\frac{Y - \sqrt{\rho_A}y}{\sqrt{1-\rho_A}}\right) + (1-\alpha)p''z, \tag{29}$$

where the ratio of group 1 is α and z is the uniform distribution.

The distribution of number of the default is

$$P[X_t = k_t] = \int_{-\infty}^{\infty} \frac{N!}{k_t!(N-k_t)!} G(y)^{k_t}(1-G(y))^{N-k_t}\phi(y)dy. \tag{30}$$

Using the logistic function as the approximation, in the case $p << 1$, we can obtain,

$$G(y) = \alpha p'^{1/\sqrt{1-\rho_A}} e^{-\frac{\sqrt{\rho_A}}{\sqrt{1-\rho_A}}\beta y} + (1-\alpha)z, \tag{31}$$

where y is the standard normal distribution and z is the uniform distribution form 0 to 1.

Here we take the limit $p", p', p \to 0$ and $N \to \infty$ with the condition, fixed $\lambda_0 = Np''$. Then, we can obtain

$$\lambda(y,z) = \alpha\lambda_0 e^{-\frac{\sqrt{\rho_A}}{\sqrt{1-\rho_A}}\beta y} + (1-\alpha)\lambda_1 z \tag{32}$$

The expected value of $\lambda(y)$ is

$$\bar{\lambda} = \int_{-\infty}^{\infty}\int_0^1 \lambda(y,z)\phi(y)\eta(z)dydz = \lambda_0 e^{\alpha^2/2} + (1-\alpha)\lambda_1/2. \tag{33}$$

This process can be written as followings,

$$\lambda(\hat{y}_{t+1}) = \alpha\lambda_0^{1-r} e^{\sqrt{1-r^2}\alpha\xi_{t+1}}\lambda(\hat{y}_t)^r + (1-\alpha)\eta_t. \tag{34}$$

It is the extension of Kesten process.

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* hisakadom@yahoo.co.jp

† shintaro.mori@gmail.com

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