From a multi-terms urn model to the self exciting NBD process and Hawkes process

Masato Hisakado*

Nomura Holdings, Inc., Otemachi 2-2-2,

Chiyoda-ku, Tokyo 100-8130, Japan

Kodai Hattori† and Shintaro Mori†

†Department of Mathematics and Physics,

Graduate School of Science and Technology, Hirosaki University

Bunkyo-cho 3, Hirosaki, Aomori 036-8561, Japan

(Dated: February 1, 2022)

Abstract

In this article we consider new multi terms urn process which has the correlation in the same term and the temporal correlation. The purpose is to clarify the relation between the urn model and the Hawkes process. As the correlation in the same term we use the Pólya urn model. As the temporal correlation we introduce the conditional initial condition. In the double scaling limit of this urn process, we can obtain the self exciting negative binomial distribution (SE-NBD) process which is a marked point process. In the standard continuous limit, this process becomes Hawkes process which has no correlation in the same time. The difference is the variance of the intensity function. We can observe the steady-non steady state phase transition. The critical point is the same for Hawkes process and this urn process. We obtain the power low distribution at the critical point. We apply the urn process and the Hawkes process to the empirical default data and estimate parameters. We can observe the urn process is better for the default portfolio than Hawkes process and confirm the self excitation.

Keywords: Hawkes process, Phase transition, Pólya urn model, Power low

I. INTRODUCTION

Anomalous diffusion is one of the most interesting topics in sociophysics and econophysics [1–3]. The models describing such phenomena have correlations [4–10] and show several types of phase transitions. In our previous work, we investigated voting models for an information cascade [11–18]. The model is one kind of the urn processes which represents the correlations and has two types of phase transitions. One is the information cascade transition, which is similar to the phase transition of the Ising model [13] that shows whether a distribution converges or not. The other is the convergence transition of the super-normal diffusion that corresponds to an anomalous diffusion [12, 19].

In financial engineering, several products have been invented to hedge risks. These products protect against a subset of the total loss on a credit portfolio in exchange for payments and provide valuable insights into market implications on default dependencies, clustering of defaults [20–22] . This aspect is important because the difficulties in managing credit events depend on correlations. To represent the clustering defaults of time series, the Hawkes process was applied recently [23–28]. As the events increase, the probability of the evens increases. It corresponds to the self-excitation and is one of the temporal correlation. This process has the phase transition between the steady-non steady state. The confirmation of the steady state is important for the finance and the risk management to hedge the risks because we can not manage the non steady state.

In our previous study, we discussed parameter estimation of the urn process which has the correlation in the same term and considered a multi-year case with a temporal correlation [17, 18]. We introduce the similar multi terms urn process and discuss the relation between new urn process and Hawkes process. In the limit of the parameters and continuous limit, we can obtain the Hawkes process and we study the properties of the phase transition. To confirm the effects of the correlation in the same time and the temporal correlation, we investigate empirical default data. We can confirmed the urn process fits better than the Hawkes process. The reason is the effects of the network which correspond to the variance of the intensity function.

The remainder of this article is organized as follows. In Section 2, we introduce a multi terms urn process. We discuss the relation to the Hawkes process and the phase transition. In section 3 we study the phase transition of this model. In section 4 we show the power low of the distribution function at the critical point. In section 5 we describe the application of the process to the empirical data of default history and confirm its parameters. Finally, the conclusions are presented in Section 6.

II. FROM MULTI TERM URN PROCESS TO HAWKES PROCESS

In this section we consider a multi terms urn process which has the correlations. In the 1-st term the urn has θ_0 red balls and $n_0 - \theta_0$ white balls. We sequentially take out balls. For example we take out a ball. After that we input the same collar ω balls into the urn. The number of the total balls increases ω. ω is the parameter for correlation in the same term [29]. We repeat the process N times in the first term. It is nothing but the Pólya urn model which has the beta binomial distribution (BBD).

We consider the time series of this process. In the $t+1$-th term the urn has $\theta_t = \theta_0 + \sum_i^t k_i \hat{d}_{t-i+1}\tilde{\omega}$ red balls and $n_0 - \theta_t$ white balls. The total number of initial balls is n_0. $\hat{d}$ is the discount factor, $X_i = k_i$ is the number of red balls taken out in i-th term and $\tilde{\omega}$ is one of the parameters for the temporal correlation. $\tilde{\omega}$ is the scale parameter for the added red balls. We sequentially take out balls as the 1-st term. After that we add the same collar ω balls into the urn. We repeat the process N times in this $t+1$-th term. This is also Pólya urn model as the 1-st term. In this model we use the two kinds of correlations. One is the correlation in the same term using the parameter ω and another is the temporal correlation using the parameter $\tilde{\omega}$ and $\hat{d}$ which is the kernel function. The temporal correlation decays as the time using the parameter $\hat{d}$.

In [17] we introduced the similar urn process with the correlation in the same term and temporal correlation. The difference is the initial condition of $t+1$-th term. The number of the red balls is the same, but the number of the white ball is $n_0 - \theta_0 + \sum_i^t (N - k_i)\hat{d}_{t-i+1}\tilde{\omega}$ where $X_i = k_i$ and the number of the total balls of the initial condition of each term is not constant, n_0.

When k_1 red balls has been taken at the 1-st term, $X_1 = k_1$, the probability in the 1-st term can be calculated as

$$\begin{aligned} P(X_1 = k_1) &= \frac{N!}{k_1!(N-k_1)!} \frac{\Pi_{i=0}^{k_1-1}(\theta_0 + i\omega)\Pi_{j=0}^{N-k_1-1}(n_0 - \theta_0 + j\omega)}{\Pi_{k=0}^{N-1}(n_0 + k_1\omega)} \\ &= \frac{N!}{k_1!(N-k_1)!} \frac{\Pi_{i=0}^{k_1-1}(p + i\rho/(1-\rho))\Pi_{j=0}^{N-k_1-1}(q + j\rho/(1-\rho))}{\Pi_{l=0}^{N-1}(1 + l\rho/(1-\rho))}, \end{aligned} \quad (1)$$

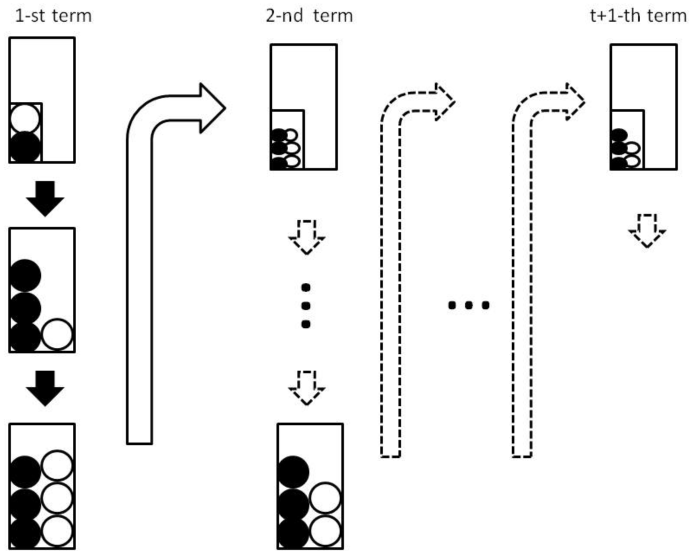


FIG. 1. Image of the multi term urn process. Each term process is the Pólya urn model. The conclusion of each term affects the initial condition of the next terms as the temporal correlation.

where $p = \theta_0/n_0$, $q = 1 - \theta_0/n_0$, and $\omega/n_0 = \rho/(1-\rho)$. It is known as the beta binomial distribution BBD(α, β,N) where $p = \alpha/(\alpha+\beta)$ and $q = \beta/(\alpha+\beta)$. Here α and β corresponds to the parameters of the beta distribution in the continuous limit of BBD. Note that $\rho = 1/(1+\alpha+\beta)$ plays the role of the correlation in this term [29].

Here we set the double scaling limit $N/n_0 = \Delta$, $N \to \infty$ and $n_0 \to \infty$. We can obtain

$$P(X_1 = k_1) \sim \mathrm{NBD}(X_1 = k_1|K_0, M_0/K_0) = \frac{(K_0 + k_1 - 1)!}{k_1!(K_0 - 1)!}(\frac{K_0}{K_0 + M_0})^{K_0}(\frac{M_0}{K_0 + M_0})^{k_1}, \tag{2}$$

where $M_0 = \theta_0 N/n_0 = \theta_0\Delta$ and $K_0 = \theta_0/\omega$. This is the negative binomial distribution (NBD), $NBD(X_1 = k_1|K_0, M_0/K_0)$. The parameter $M_0/K_0 = \omega N/n_0 = \omega\Delta$ is related to the correlation in this term. The mean and the variance of NBD is M_0 and $M_0 + M_0^2/K_0$, respectively.

The negative binomial distribution $\mathrm{NBD}(X_1 = k_1|K_0, M_0/K_0)$ has another face:

$$\begin{aligned} \mathrm{NBD}(X_1 = k_1|K_0, M_0/K_0) &= \int_0^\infty \mathrm{Poisson}(k_1|\lambda) \cdot \mathrm{Gamma}(\lambda|K_0, M_0/K_0)d\lambda, \\ &= \int_0^\infty \frac{\lambda^{k_1}e^{-\lambda}}{k_1!}\frac{\lambda^{\dot{K_0}-1}}{\Gamma(K_0)(M_0/K_0)^{K_0}}e^{-\lambda K_0/M_0}d\lambda, \end{aligned} \tag{3}$$

where $Poisson(k_1|\lambda)$ is the Poisson process and $Gamma(\lambda|K_0, M_0/K_0)$ is the Gamma distribution and λ is the intensity function. In multi-term model $lambda_t$ is the intensity function at the t-th term. $Gamma(\lambda|K_0, M_0/K_0)$ has the average M_0 and the variance M_0^2/K_0. It means that the urn process in the 1-st term corresponds to the Poisson process with the intensity function λ which has the Gamma distribution in the double scaling limit. The intensity function has the variance comparing the Poisson process which has the constant intensity function. We call this the NBD process.

Next we define the $t+1$-th term with the temporal correlation as [17] using the conditional probability. We define the conditional probability of $t+1$ th term,

$$P(X_{t+1} = k_{t+1}|X_0 = k_0, \cdots, X_t = k_t) = NBD(X_{t+1} = k_{t+1}|K_t, M_t/K_t), \tag{4}$$

where k_i is the history of the number of red balls which we took out. The conditional probability is defined by the update of the parameters K_t and M_t. The difference between the 1-st term and $t+1$-th term is only the number of white balls in the initial condition of the term. Other conditions are same as 1-st term condition. We can obtain for the parameters at the $t+1$-th term for the intensity function,

$$\begin{aligned} M_t = \theta_t N/n_0 &= \frac{\theta_0 + \sum_i^t k_i \hat{d}_{t+1-i}\tilde{\omega}}{n_0} N = (\theta_0 + \sum_i^t k_i \hat{d}_{t+1-i}\tilde{\omega})\Delta \\ &= M_0 + M_0/L_0 \sum_i^t k_i \hat{d}_{t+1-i}, \end{aligned} \tag{5}$$

where $M_0 = \theta_0 N/n_0 = \theta_0\Delta$, $K_0 = \theta_0/\omega$, $L_0 = \theta_0/\tilde{\omega}$ and $\omega N/n_0 = \omega\Delta = M_0/K_0$. We can obtain other parameters,

$$K_t = \theta_t/\omega = \frac{\theta_0 + \sum_i^t k_i \hat{d}_{t+1-i}\tilde{\omega}}{\omega} = K_0 + K_0/L_0 \sum_i^t k_i \hat{d}_{t+1-i}, \tag{6}$$

and

$$M_t/K_t = \omega N/n_0 = \omega\Delta = M_0/K_0. \tag{7}$$

We call the process the discrete self exciting negative binomial distribution (SE-NBD) process. The self exciting is introduced by the conditional probability, Eq.(4). Note that in all process parameter M_t/K_t is constant M_0/K_0. It corresponds to that the correlation in the same term does not depend on the term t. By this condition the process has the reproductive property of NBD. The mean of the intensity function is M_t and the variance is M_t^2/K_t. As ω increases, K_t decreases, and the variance of the intensity function increases. In this mean the correlation in the same term affects the variance of the intensity function. Note that in the limit $K_0 \to \infty (\omega \to 0)$ with Δ fixed , the variance becomes 0 and the intensity function becomes the delta function. It is nothing but the discrete Hawkes process [24]. We show the image of the intensity function in Fig.2 (a). L_0 plays the role of the parameter for the temporal correlation as $\tilde{\omega}$. In the limit $L_0 \to \infty$ ($\tilde{\omega} \to 0$), K_t does not depend on t . In this case the distribution of the taken out red balls becomes NBD in each term.

In summary, we have obtained the discrete SE-NBD process X_t obeys NBD for M_t from the urn process,

$$X_{t+1} \sim \mathrm{NBD}\left(K_t, M_0/K_0\right), t \geq 0, \tag{8}$$

where

$$M_t = M_0 + M_0/L_0 \sum_{s=1}^{t} X_s \hat{d}_{t+1-s}, t \geq 1, \tag{9}$$

and

$$K_t = K_0 + K_0/L_0 \sum_{s=1}^{t} X_s \hat{d}_{t+1-s}, t \geq 1. \tag{10}$$

In the limit $K_0 \to \infty(\omega = 0)$ we can obtain the discrete Hawkes process, X_t obeys the Poisson process for M_t from the urn process,

$$X_{t+1} \sim \mathrm{Poisson}\left(M_t\right), t \geq 0, \tag{11}$$

where

$$M_t = M_0 + M_0/L_0 \sum_{s=1}^{t} X_s \hat{d}_{t+1-s}, t \geq 1, \tag{12}$$

In the limit $L_0 \to \infty(\tilde{\omega} = 0)$ the process becomes the NBD process which does not have the self excitation. Δ corresponds to the number of balls which are taken out in a term and means the interval between the terms. Here we introduce the counting process, $\tilde{X}_t = \sum_i X_i$. We set the double scaling limit, $\Delta = N/n_0 \to 0$, $\omega \to \infty$ with $\omega\Delta = \omega'$, as the continuous limit of the process $\tilde{X}_t$. Note that in this limit $\omega\Delta = \omega'$ is constant and the process have the reproductive property as the discrete SE-NBD process. We can obtain the mean of the intensity function at t, λ_t

$$E(\lambda_t|F_t) = \lim_{\Delta \to 0} \frac{E[\tilde{X}_{t+\Delta} - \tilde{X}_t|F_t]}{\Delta} = \lim_{\Delta \to 0} \frac{M_t}{\Delta} = (\theta_0 + \tilde{\omega} \sum_{i<t} k_i \hat{d}_{t-i}), \tag{13}$$

which corresponds to the Hazard function. The variance of the intensity of the distribution at time t is

$$Var(\lambda_t|F_t) = \lim_{\Delta \to 0} \frac{M_t^2/K_t}{\Delta} = \omega'(\theta_0 + \tilde{\omega} \sum_{i<t} \hat{d}_{t-i} k_i), \tag{14}$$

where F_t is the history of the number of red balls, $X_1 = k_1, \cdots, X_t = k_t$. In the continuous SE-NBD process the intensity function has the gamma distribution as the discrete SE-NBD process. We can confirm the intensity function becomes the delta function in the limit $\omega \to 0$ which corresponds to the continuous Hawkes process. In summary we can obtain in the continuous limit,

$$\tilde{X}_{t+\Delta} - \tilde{X}_t \sim \mathrm{NBD}\left(\theta_t \Delta/\omega', \omega'\right), t \geq 0, \tag{15}$$

where

$$\theta_t = \theta_0 + \tilde{\omega} \sum_{s<t} X_s \hat{d}_{t-s}, t \geq 0. \tag{16}$$

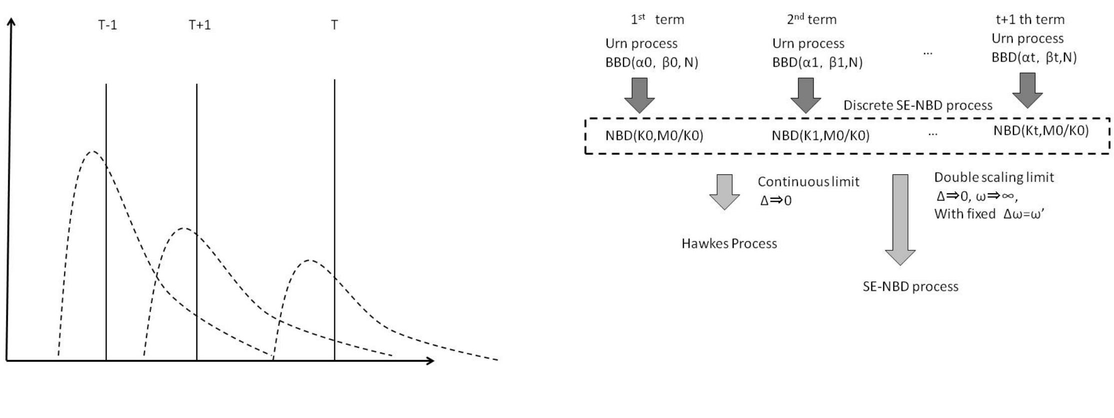
In the limit $\omega' \to 0$, the continuous SE-NBD process becomes the Hawkes process.

$$\tilde{X}_{t+\Delta} - \tilde{X}_t \sim \mathrm{Poisson}\left(\theta_t \Delta\right), t \geq 0, \tag{17}$$

where

$$\theta_t = \theta_0 + \tilde{\omega} \sum_{s<t} X_s \hat{d}_{t-s}, t \geq 0. \tag{18}$$

At last we show the road from the urn process to the Hawkes process in Fig. 2 (b).



(a) (b)

FIG. 2. The difference between the continuous SE-NBD process and Hawkes process. (a) Intensity function of the continuous SE-NBD process which is the gamma function (the dotted line) and the Hawkes process which is the delta function (the solid line) (b) From Beta binomial distribution (BBD) to Hawkes process. We can confirm the flow form BBD to Hawkes process though NBD process.

III. PHASE TRANSITION OF A NEW POD PROCESS

In this section we consider the phase transition of the SE-NBD process. Here we set the average $\bar{v}$ of the intensity function. The mean field approximation of Eq.(16) is

$$\bar{v} = \theta_0/(1-\tilde{\omega}\hat{T}), \tag{19}$$

where $\hat{T} = \sum^{\infty} \hat{d}_i$.

In the limit $\tilde{\omega} \to 0$, the temporal correlation is 0 and the process is the simple NBD process and the phase transition disappears. In the continuous limit we show in the horizontal axis in Fig. 2.

The SE-NBD process which includes the Hawkes process as the branching process. The branching ratio is

$$\nu = \tilde{\omega}\hat{T}, \tag{20}$$

and the condition for the steady state is

$$\nu = \tilde{\omega}\hat{T} < 1. \tag{21}$$

The phase transition between the steady state to the non steady state is at $\nu = 1$ which is the critical point. The transition point is same as the Hawkes process [27, 28].

The parameter ν is also called effective reproduction number when we discuss an infectious disease. It is the number of the patients that one patient has given in the infection model. If the effective reproduction number is above 1, the number of patients increase to infinity and it is the non-steady state.

In the SE-NBD process, the intensity function has the variance. On the other hand, the Hawkes process, the intensity function is the delta function. The variance of the intensity function is the origin of the variance of the branching ratios. In this mean the SE-NBD process has the mixture of the several branching ratios. In fact, [31] shows that the effective reproduction number depends on the environments for COVID-19. The mixture of the branching ratios affects not only the expected value of the intensity function but also the variance of the intensity function. Hence, the SE-NBD process which has the gamma distribution of the intensity function may be useful. We confirm it in the section V.

In this article we consider the exponential and power decay cases which corresponds to the short memory and long memories [30] as the kernel function. When we consider the exponential decay case $\hat{d} = e^{-\beta t}$, the condition for the steady state is $M_0/L_0 < \beta$. When we consider the power decay case $\hat{d} = 1/(1+t)^{\gamma}$, the condition for the steady state is $M_0/L_0 < \gamma - 1$. In the section 5 we use the exponential kernel for the empirical default data.

IV. POWER LOW DISTRIBUTION AT THE CRITICAL POINT

We start from a discrete SE-NBD process, $\{X_t\}, t = 1, \cdots$. Here, $X_t \in \{0, 1, \cdots\}$ represents the size of the event at time t. This is the process which we have obtained from the urn process. The event corresponds to that the red ball is taken from the urn. X_t obeys NBD for M_t.

$$X_{t+1} \sim \text{NBD}\left(\frac{M_t}{\omega} = K_t, \omega\Delta\right), t \geq 0,$$
$$M_t = M_0 + n\sum_{i=1}^{t} X_s h(t-i), t \geq 1,$$

where $n = M_0/(1-r)L_0$. We adopt the exponential decay kernel function, $h(t) = (1-r)\hat{d} = (1-r)r^t, 0 \leq r < 1$. In addition, we replace the normalization factor $(1-r)$ of $h(t)$ with $1/\tau$ to ensure that $\int_0^{\infty} h(t)dt = \frac{1}{\tau}\int_0^{\infty} e^{-t/\tau}dt = 1$.

The stochastic process $\{X_t\}, t = 1, \cdots$ is non-Markovian, we focus on the time evolution of the intensity function M_t. M_t satisfies the next recursive equation,

$$M_{t+1} = r(M_t - M_0) + M_0 + nh(0)X_{t+1}.$$

Here, we use the relation $\sum_{i=1}^{t+1} X_i h(t+1-i) = X_{t+1}h(0) + r\sum_{i=1}^{t} X_s h(t-i)$. The stochastic difference equation for the excess intensity $\hat{z}_t \equiv M_t - M_0$ is

$$\hat{z}_{t+1} - \hat{z}_t = (r-1)\hat{z}_t + nh(0)X_{t+1}, z_0. \tag{22}$$

We take the continuous time limit as the section II. We divide the unit-time interval by the infinitesimal time intervals with width $dt = \Delta$. The decreasing factor r^t during the interval dt is replaced with $r^{dt} = e^{-dt/\tau} \simeq 1 - dt/\tau + o(dt/\tau)$.

X_{t+1} is the noise for the time interval $[t, t+1]$, it is necessary to prepare the noise for the infinitesimal interval $[t, t+dt)$. For the purpose, we use the reproductive property of NBD. If X_{t+1}, the noise for the interval $[t, t+1)$, obeys $\text{NBD}(\theta_t/\omega, \omega)$, the noise for $[t, t+dt)$, is denoted as $d\hat{\xi}^{NB}_{(\theta_t/\omega',\omega')}(t)$. Here applying the double scaling limit the parameter changes form ω to ω'. As Eq.(15) the stochastic difference equation (SDE) then becomes

$$d\hat{z}_t = \hat{z}_{t+dt} - \hat{z}_t = -\frac{1}{\tau}\hat{z}_t dt + \frac{n}{\tau} d\hat{\xi}^{NBD}_{(\frac{\theta_0+\hat{z}_t}{\omega'},\omega')}(t). \tag{23}$$

The state-dependent NBD noise $\hat{\xi}^{NBD}_{\theta_t/\omega,\omega}$ defines the noise for the infinitesimal time interval $[t, t+dt)$ with the next probabilistic rules,

$$d\hat{\xi}^{NBD}_{(\theta_t/\omega',\omega')}(t) \equiv \hat{\xi}^{NBD}_{(\theta_t/\omega',\omega')}(t+dt) - \hat{\xi}^{NBD}_{(\theta_t/\omega',\omega')}(t) \sim \mathrm{NBD}(\theta_t dt/\omega', \omega').$$

When one denotes the timing and the size of i-th event as t_i and k_i and the number of events before t as $\hat{N}(t)$, we can rewrite the state-dependent NBD noise $\hat{\xi}^{NBD}_{(\frac{\theta_0+\hat{z}_t}{\omega'},\omega')}(t)$ as

$$\hat{\xi}^{NBD}_{(\frac{\theta_0+\hat{z}_t}{\omega'},\omega')}(t) = \sum_{i=1}^{\hat{N}(t)} k_i \delta(t - t_i).$$

The probability for the occurrence and non-occurrence of an event with size k during time interval dt is given as,

$$P\left(d\hat{\xi}^{NBD}_{(\frac{\theta_0+\hat{z}_t}{\omega'},\omega')} = k\right) = \begin{cases} 1 - \frac{\hat{z}_t+\theta_0}{\omega'}\ln(\omega'+1)dt & k = 0 \\ \frac{1}{k}\left(\frac{\hat{z}_t+\theta_0}{\omega'}\right)\left(\frac{\omega'}{\omega'+1}\right)^k dt & k \geq 1. \end{cases} \tag{24}$$

In the limit $\omega \to 0$, the probabilities becomes

$$\lim_{\omega\to 0} P\left(d\hat{\xi}^{NBD}_{(\frac{\theta_0+\hat{z}_t}{\omega'},\omega')} = k\right) = \begin{cases} 1 - (\hat{z}_t + \theta_0)dt & k = 0 \\ (\hat{z}_t + \theta_0)dt & k = 1 \\ 0 & k \geq 2. \end{cases}$$

k_i is restricted to be one or zero, the state dependent noise becomes the Poisson noise.

The SDE (23) is interpreted as

$$\hat{z}(t+dt) - \hat{z}(t) = \begin{cases} -\frac{1}{\tau}\hat{z}_t & \text{Prob.} = 1 - \frac{\hat{z}_t+\theta_0}{\omega'}\ln(\omega'+1)dt \\ \frac{nk}{\tau} & \text{Prob.} = \frac{1}{k}\left(\frac{\hat{z}_t+\theta_0}{\omega'}\right)\left(\frac{\omega'}{\omega'+1}\right)^k dt, k = 1, \cdots. \end{cases}$$

We adopt the same procedure to derive the master equation for the probability density function (PDF) of $\hat{z}_t$ in [27, 28]. We obtain

$$\frac{\partial}{\partial t}P_t(z) = \frac{1}{\tau}\frac{\partial}{\partial z}zP_t(z) + \sum_{k=1}^{\infty}\frac{1}{\omega' k}\left(\frac{\omega'}{\omega'+1}\right)^k \left\{(\theta_0 + z - \frac{nk}{\tau})P_t(z - \frac{nk}{\tau}) - (\theta_0 + z)P_t(z)\right\}. \tag{25}$$

We denote the Laplace representation of the steady state $P_{SS}(z)$ as $\tilde{P}_{SS}(s)$.

$$\tilde{P}_{SS}(s) \equiv \int_0^\infty P_{SS}(z)e^{-sz}dz.$$

The master equation for $\tilde{P}_{SS}(s)$ is

$$\left[\sum_{k=1}^{\infty} \frac{1}{\omega' k}\left(\frac{\omega'}{\omega'+1}\right)^k \left(e^{-\frac{nk}{\tau}s}-1\right)+\frac{s}{\tau}\right]\frac{d}{ds}\tilde{P}_{SS}(s) = \sum_{k=1}^{n} \frac{1}{\omega k'}\left(\frac{\omega'}{\omega'+1}\right)^k \theta_0 \left(e^{-\frac{nk}{\tau}s}-1\right)\tilde{P}_{SS}(s). \tag{26}$$

We obtain

$$\frac{d}{ds}\ln \tilde{P}_{SS}(s) = \theta_0 - \frac{\theta_0 s/\tau}{\sum_{k=1}^{\infty} \frac{1}{\omega' k}\left(\frac{\omega'}{\omega'+1}\right)^k (e^{-\frac{nk}{\tau}s}-1)+\frac{s}{\tau}}.$$

We integrate the equation with the initial condition $\tilde{P}_{SS}(0) = 1$.

$$\ln \tilde{P}_{SS}(s) = \theta_0 s - \int_0^s \frac{\theta_0 s'/\tau}{\sum_{k=1}^{\infty} \frac{1}{\omega' k}\left(\frac{\omega'}{\omega'+1}\right)^k (e^{-\frac{nk}{\tau}s'}-1)+\frac{s'}{\tau}}ds'.$$

We are interested in the large z behavior of $P_{SS}(z)$ near the critical point $n = 1$, we study the integral at $s \simeq 0$ and $n = 1-\epsilon, \epsilon << 1$. We expand $e^{\frac{nk}{\tau}s} = 1 - \frac{nks}{\tau} + \frac{1}{2}(\frac{nks}{\tau})^2 + o(s^2)$ and calculate the summation over k in the denominator of the integral. We obtain

$$\ln \tilde{P}_{SS}(s) \simeq \theta_0 s - \int_0^s \frac{\frac{2\theta_0\tau}{\omega'+1}}{\frac{2\tau}{\omega'+1}\epsilon + s'}ds'.$$

Near the critical point, the excess intensity distribution shows a power-law behavior with a non-universal exponent, up to an exponential truncation:

$$P_{SS}(z) \propto z^{-1+2\frac{\theta_0\tau}{\omega'+1}} e^{-\frac{2\tau\epsilon}{\omega'+1}z}. \tag{27}$$

The power-law exponent of the PDF of the excess intensity is $1 - \frac{2\theta_0\tau}{\omega'+1}$ and depends on ω' which is the correlation in the same time. In the limit $\omega' \to 0$, the result coincide with the result in [27, 28]. The power-law exponent increase with ω' and it converges to 1 in the limit $\omega \to \infty$. By the correlation in the same term alters the critical behavior. In addition, the length-scale of the exponential decay for the off-critical case is $(\omega'+1)/(2\tau\epsilon')$. It is also an increasing function of ω'.

V. PARAMETER ESTIMATION FOR THE DEFAULT DATA

We apply the parameter estimation for the default data. First, the S&P default data from 1981 to 2020 are used. The speculative grade (SG) rating represents ratings under BBB-(Baa3) and investment grade (IG) represents that above BBB-(Baa3). We use Moody's default data from 1920 to 2020 for 100 years. It includes the Great Depression in 1929 and Great Recession in 2008.

We estimate the parameters using the Bayes formula

$$P(K_0, M_0, L_0, \beta | k_0, k_1, \cdots, k_T) = \frac{P(r_T|K_T, M_T, L_0, \beta,))}{P(k_T)} \cdots \frac{P(k_0|K_0, M_0, L_0, \beta)}{P(k_0)} \times f(K_0, M_0, L_0, \beta), \tag{28}$$

where $f(K_0, M_0, L_0, \beta)$ is a prior distribution [17]. We use the uniform distribution for a prior distribution. We apply the maximum a posteriori (MAP) estimation of Eq.(28). If we use the NBD for the distribution P, it is the parameter estimation for discrete SE-NBD process which we introduced in the section II. If we use the Poisson distribution in stead of NBD, it is the parameter estimation for the discrete Hawkes process. We show the conclusion of the optimization using the discrete SE-NBD process, discrete Hawkes process and the NBD process in Table I, Table II, and Table IV. We apply the NBD to confirm where there is the self excitation. The Hawkes process is the case $K_0 \to \infty (\omega = 0)$ and the NBD process is the case $L_0 \to \infty (\tilde{\omega} = 0)$ as discussed in the section II.

When K_0 is large, it is nearly Hawkes process. When L_0 is large, it is nearly the NBD process which has no self exciting. The estimated K_0 is small for the SE-NBD process. Especially K_0 is small for IG. As in Fig. IV, we can obtain the much better AIC for SE-NBD process. It means that the intensity function has the variance is not the delta function as Hawkes process. In fact some are defaulted obligors which affect the other obigors and some are ones which do not affect the other. The former case corresponds to to the chain bankruptcy. We can consider that it depends on the network effects. The obligor which connects many obligores affect many other obligors. On the other hand, the AIC for the SE-NBD process is smaller than the AIC for the NBD process. Hence, we can confirm the self excitation in this historical credit data.

TABLE I. MAP estimation of the parameters for discrete SE-NBD process and discrete Hawkes process

		SE-NBD					Hawkes			
No.	Model	K_0	L_0	M_0/K_0	β	$\bar{v}$	M_0	L_0	β	$\bar{v}$
1	Moody's 1920-2020	0.28	6.17	18.89	2.94	58.35	3.4	3.55	15.98	86.85
2	S&P 1981-2020	1.06	27.80	18.95	16.08	71.22	13.3	15.76	18.40	83.65
3	Moody's 1981- 2020	1.03	32.12	22.55	15.97	82.79	17.2	21.09	16.69	91.72
4	S&P 1990-2020	1.51	62.52	23.04	16.23	78.64	29.2	45.37	13.01	81.82
5	Moody's 1990-2020	1.58	86.55	27.92	14.40	90.00	40.6	73.03	19.19	91.38
6	Moody's 1920-2020 SG	0.29	5.91	17.81	3.03	56.04	2.9	3.00	13.57	105.32
7	S&P 1981-2020 SG	1.05	25.66	17.90	16.22	69.90	12.0	13.93	17.57	85.29
8	Moody's 1981-2020 SG	1.02	30.57	21.65	15.99	80.65	15.4	18.53	15.71	92.38
9	S&P 1990-2020 SG	1.54	60.05	21.82	16.02	76.39	28.2	43.74	18.97	79.59
10	Moody's 1990-2020 SG	1.62	86.79	26.76	15.44	87.05	39.6	71.53	14.57	88.57
11	Moody's 1920-2020 IG	0.13	1.10	4.06	0.99	2.13	1.14	0.40	0.98	1.24
12	S&P 1981-2020 IG	0.39	2.87	3.06	15.32	2.05	1.2	2.67	14.68	2.05
13	Moody's 1981-2020 IG	0.28	6.07	5.37	1.26	2.36	1.7	6.02	14.63	2.34
14	S&P 1990-2020 IG	0.33	2.73	3.75	16.80	2.32	1.2	2.65	17.46	2.32
15	Moody's 1990-2020 IG	0.28	4.13	5.80	13.27	2.69	1.8	5.58	16.26	2.68

VI. CONCLUDING REMARKS

In this article we consider multi terms urn process which has the correlation in the same time and the temporal correlation. Each term is the Pólya urn model which represent the correlation in the same time. The temporal correlation represents the correlation effects form the previous terms. When the number of the red balls is much smaller, we can obtain the Poisson process with the gamma distribution intensity function, NBD process. We introduce the temporal correlation as the conditional distribution for the intensity function. It is equivalent to the self exciting negative binomial distribution (SE-NBD) with the con-

TABLE II. MAP estimation of the parameters for the NBD process

No.	Model	NBD K_0	M_0/K_0	$\bar{v}$
1	Moody's 1920-2020	0.47	80.64	37.86
2	S&P 1981-2020	1.54	38.61	59.62
3	Moody's 1981- 2020	1.52	45.76	69.78
4	S&P 1990-2020	2.07	34.37	71.29
5	Moody's 1990-2020	2.14	38.86	83.23
6	Moody's 1920-2020 SG	0.47	76.28	35.81
7	S&P 1981-2020 SG	1.55	37.14	57.57
8	Moody's 1981-2020 SG	1.53	44.00	67.45
9	S&P 1990-2020 SG	2.12	32.46	68.97
10	Moody's 1990-2020 SG	2.20	36.65	80.58
11	Moody's 1920-2020 IG	0.29	7.18	2.05
12	S&P 1981-2020 IG	0.46	4.42	2.05
13	Moody's 1981-2020 IG	0.37	6.30	2.33
14	S&P 1990-2020 IG	0.41	5.63	2.32
15	Moody's 1990-2020 IG	0.36	7.32	2.65

ditional parameters. We called this process discrete SE-NBD process. This process becomes the discrete Hawkes process with no correlation in the same time but with the temporal correlation. In the standard continuous limit of the discrete Hawkes process, we can obtain the Hawkes process. On the otherhand, if we take double scaling limit, we can obtain continuous SE-NBD process. The difference between continuous SE-NBD process and Hawkes process is the variance of the intensity function. In other word in the limit that the intensity function becomes the delta function the continuous SE-NBD process becomes the Hawkes process. The continuous SE-NBD process is one of the marked point process.

We can observe the steady-non steady state phase transition which is same kind of phase

TABLE III. AIC for the discrete SE-NBD process,discrete Hawkes process, and the NBD process

No.	Model	SE-NBD process AIC	Hawkes process AIC	NBD process AIC
1	Moody's 1920-2020	791.9	2193.1	904.0
2	S&P 1981- 2020	386.7	1010.6	407.9
3	Moody's 1981-2020	399.3	1186.1	420.6
4	S&P 1990-2020	316.3	923.0	323.5
5	Moody's 1990-2020	327.1	1098.6	332.4
6	Moody's 1920-2020 SG	781.5	2060.1	893.0
7	S&P 1981-2020 SG	383.2	975.8	405.1
8	Moody's 1981-2020 SG	396.5	1140.0	417.8
9	S&P 1990-2020 SG	313.9	894.0	321.0
10	Moody's 1990-2020 SG	325.1	1062.4	329.9
11	Moody's 1920-2020 IG	321.7	490.1	360.3
12	S&P 1981-2020 IG	150.0	197.6	153.2
13	Moody's 1981-2020 IG	156.8	257.1	156.8
14	S&P 1990-2020 IG	121.3	168.3	124.2
15	Moody's 1990-2020 IG	127.4	219.7	127.9

transition as the Hawkes process. We can observe the difference in the distribution of the intensity function at the critical point. The distribution functions of the both models have the power low at the critical point and have the different indexes. We apply the process to the default process and estimate parameters. We can observe the urn process is better for the default portfolio because of the network effects.

Appendix A.Multi dimensional case

We consider the discrete multi- dimensional SE-NBD process $X_t^{(i)}$ obeys NBD for $M_t^{(i)}$, where $i = 1, 2, \cdots, M$,

$$X_{t+1}^{(i)} \sim \mathrm{NBD}\left(K_t^{(i)}, M_0^{(i)}/K_0^{(i)}\right), t \geq 0, \tag{29}$$

where

$$M_t^{(i)} = M_0^{(i)} + \sum_{j=1}^{M} M_0^{(i)}/L_0^{(ij)} \sum_{s=1}^{t} X_s^{(j)} \hat{d}_{t+1-s}^{(i)}, t \geq 1, \tag{30}$$

and

$$K_t^{(i)} = K_0^{(i)} + \sum_{j=1}^{M} K_0^{(i)}/L_0^{(ij)} \sum_{s=1}^{t} X_s^{(j)} \hat{d}_{t+1-s}^{(i)}, t \geq 1. \tag{31}$$

Here we consider the strength form the sector j to i as $S^{(ij)}$,

$$S^{(ij)} = M_0^{(i)}/L_0^{(ij)} T^{(i)}, \tag{32}$$

where $T^{(i)} = \sum_{s=1}^{\infty} \hat{d}_{t+1-s}^{(i)}$. Here we introduce the matrix,

$$S \equiv \begin{pmatrix} S^{11} & S^{12} & \cdots & S^{1M} \\ S^{21} & S^{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ S^{M1} & \cdots & \cdots & S^{MM} \end{pmatrix}, \tag{33}$$

Here we set average $\bar{v}^{(i)}$ and the vector $\bar{\mathbf{v}} = \bar{v}^{(i)}$, we can obtain the equations about $v^{(i)}$ using Eq.(30),

$$\bar{\mathbf{v}} = \mathbf{M_0} + S\bar{\mathbf{v}}, \tag{34}$$

where $\mathbf{M_0} = M_0^{(i)}$. Solving the Eq.(34), we can obtain the equilibrium solution for $\bar{\mathbf{v}}$.

TO study the sensitivity, we consider the one step of Eq.(30),

$$M_t^{(i)} - M_{t-1}^{(i)} = \sum_{j=1}^{M} M_0^{(i)}/L_0^{(ij)} v_{t-1}^{(j)}, \tag{35}$$

where $v_{t-1}^{(j)}$ is the impact at $t-1$. In the matrix form

$$\mathbf{M}_t - \mathbf{M}_{t-1} = \hat{S}\mathbf{v}_{t-1} \tag{36}$$

where

$$\hat{\mathbf{S}} = \hat{S}^{(ij)} = M_0^{(i)}/L_0^{(ij)}, \tag{37}$$

The sum of the initial impact$\mathbf{v}_0$ is defines as

$$\mathbf{M}_\infty - \mathbf{M}_0 = \sum_{i=1}^{\infty} \hat{S}^i \mathbf{v}_0 \tag{38}$$

We consider the discrete multi- dimensional Hawkes process $X_t^{(i)}$ obeys Poisson for $M_t^{(i)}$, where $i = 1, 2, \cdots, M$,

$$X_{t+1}^{(i)} \sim \text{Poisson}\left(M_t^{(i)}\right), t \geq 0, \tag{39}$$

where

$$M_t^{(i)} = M_0^{(i)} + \sum_{j=1}^{M} M_0^{(i)}/L_0^{(ij)} \sum_{s=1}^{t} X_s^{(j)} \hat{d}_{t+1-s}^{(i)}, t \geq 1, \tag{40}$$

TABLE IV. AIC for the discrete SE-NBD process,discrete Hawkes process, and the NBD process

No.	Model	MD-SE-NBD process AIC	MD-Hawkes process AIC	SE-NBD process AIC
1	Building	369.76	377.56	38.65
2	Consumer	616.63	626.16	627.20
3	Energy	498.82	552.08	504.77
4	Financial Institution	402.54	408.21	414.26
5	Health	387.40	390.22	389.28
6	Hi-tech	278.24	276.42	290.52
7	Insurance	253.37	261.49	256.48
8	Leisure	543.29	570.96	559.42
9	Metal	556.09	582.94	566.31
10	real Estate	124.39	122.34	133.37
11	Telecommunication	327.39	331.41	345.65
12	Transport	359.57	357.73	378.34
13	Utility	252.26	266.78	270.09

Appendix B. Covariance density function

In this section we consider the covariance density function.

$$Cov[X_t, X_{t'}] = E[X_t, X_{t'}] - E[X_t]E[X_{t'}] = E[X_t, X_{t'}] - \bar{v}^2\Delta^2 = C^*(t,t')\Delta^2, \qquad (41)$$

where $C^*(t,t')$ is the covariance density function. Using Eq.(24) we can obtain for $t = t'$

$$E[X_t, X_{t'}] = E[X_t^2] = (1+\omega')\bar{v}\Delta. \qquad (42)$$

Hence, the covarinace decompose the density function becomes

$$C^*(t,t') = (1+\omega')\bar{v}\delta(t-t') + C(t,t'), \qquad (43)$$

$\delta(t)$ is the delta function. If we set $\tau = t' - t$ we can rewrite Eq.(41) using Eq.(43),

$$E[X_t, X_{t+\tau}] = [C(\tau) + \bar{v}^2 + (1+\omega')\bar{v}t\delta(\tau)]\Delta^2. \qquad (44)$$

Here we define $g(x) = \tilde{\omega}\hat{d}(x)$ and set $\tau > 0$.

$$\begin{aligned}
E[X_t X_{t+\tau}] &= [C(\tau) + \bar{v}^2]\Delta^2 \\
&= \theta_0\bar{v}\Delta^2 + \lim_{\Delta\to 0}\sum_{i=0}^{\infty} g(i\Delta)E[X_t X_{t+\tau-(i+1)\Delta}], \\
&= \theta_0\bar{v}\Delta^2 + \lim_{\Delta\to 0}\sum_{i=0}^{\infty} g(i\Delta)[C(\tau-(i+1)\Delta) + \bar{v}^2 + (\omega'+1)\theta_t\delta(\tau-(i+1)\Delta)]\Delta^3 \\
&= \theta_0\bar{v}\Delta^2 + \Delta^2\int_0^{\infty} g(w)[C(\tau-w) + \bar{v}^2 + (\omega'+1)\theta_t\delta(\tau-w)]dw \\
&= \theta_0\bar{v}\Delta^2 + \Delta^2(\omega'+1)\bar{v}g(\tau) + \Delta^2\int_0^{\infty} g(w)[C(\tau-w) + \bar{v}^2]dw \\
&= \bar{v}^2\Delta^2 + \Delta^2(\omega'+1)\bar{v}g(\tau) + \Delta^2\int_0^{\infty} g(w)[C(\tau-w)]dw \qquad (45)
\end{aligned}$$

where we use the mean field equation. We can obtain the integral equation

$$C(\tau) = \bar{v}^2 + (\omega'+1)\bar{v}g(\tau) + \int_0^{\infty} g(w)[C(\tau-w)]dw \qquad (46)$$

* hisakadom@yahoo.co.jp

† shintaro.mori@gmail.com

[1] G. Galam, *Stat. Phys.* **61**, 943 (1990).

[2] G. Galam, *Inter J. Mod. Phys. C* **19(03)**, 409 (2008).

[3] N. M. Mantegna and H. E. Stanley, *Introduction to Econophysics: Correlations and Complexity in Finance*, Cambridge University Press (2000).

[4] D. Brockmann, L. Hufinage, and T. Geisel, *Nature* **439**, 462 (2006).

[5] I. T. Wong, M. L. Gardel, D. R. Reichman, E. R. Weeks, M. T. Valentine, A. R. Bausch, and D. A. Weitz, *Phys. Rev. Lett.* **92**, 178101 (2004).

[6] Y. Gefen, A. Aharony, and S. Alexander, *Phys. Rev. Lett.* **50**, 77 (1983).

[7] R. Metzler and J. Klafter, *Phys. Rep.* **339**, 1 (2000).

[8] S. Hod and U. Keshet, *Phys. Rev. E* **70** 11006 (2004).

[9] T. Huillet, *J. Phys. A* **41** 505005 (2008).

[10] G. Schütz and S. Trimper, *Phys. Rev. E* **70** 045101 (2004).

[11] S Mori and M. Hisakado, *J. Phys. Soc. Jpn.* **79**, 034001 (2010).

[12] M. Hisakado and S. Mori, *J. Phys. A* **43**, 31527 (2010).

[13] M. Hisakado and S. Mori, *J. Phys. A* **44**, 275204 (2011).

[14] M. Hisakado and S. Mori, *J. Phys. A* **45**, 345002 (2012).

[15] M. Hisakado and S. Mori, *Physica A* **417**, 63 (2015).

[16] M. Hisakado and S. Mori, *Physica. A* **108**, 570 (2016).

[17] M. Hisakado and S. Mori, *Physica A,* **544** 123480 (2020).

[18] S. Mori, M.Hisakado, and K. Nakayama, arXiv:2105.10653, (2021).

[19] S. Hod and U. Keshet, *Phys. Rev. E* **70**, 11006 (2004).

[20] S. Mori, K. Kitsukawa, and M. Hisakado, *Quant. Fin.* **10**, 1469 (2010).

[21] S. Mori, K. Kitsukawa, and M. Hisakado, *J. Phys. Sco.Jpn.* **77**, 114802 (2008).

[22] P. J. Schönbucher, *Cresit Derivatives Pricing Models:Models, Pricing, and Inplementation* John Wiley & Sons, Ltd. (2003).

[23] A. G. Hawkes, *Biometrica* **58** 83 (1971).

[24] M. Kirchner, *Quant Fin* **17** 57 (2017).

[25] P. Blanc, J. Donier, and J. -P. Bouchard, *Quant FIn* **17** 171 (2017).

[26] E. Errais, K. Giesecke, L.R. Goldberg, *SIAM J. Fin Math* **1** 642 (2010)

[27] K.Kanazawa and D. Sornett *Rhys. Rev.Lett* **125** 138301 (2020).

[28] K.Kanazawa and D. Sornett *Rhys. Rev. Research* **2** 033442 (2020).

[29] M. Hisakado, K. Kitsukawa, S. and Mori, *J. Phys. A* **39**, 15365 (2006).

[30] I. Florescu, M. C. Mariani, H. E. Stanley, and F. G. Viens (Eds.) *Handbook of High-Frequency Trading and Modeling in Finance*, John Wiley& Sons (2016).

[31] H.Nishiura, et.al. *Closed environments facilitate secondary transmission of coronavirus disease2019(COVID-19) MedRxiv* (2020)