Merton model and Poisson process with Log Normal intensity function

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Abstract

This study considers the Merton model. We show the Merton model becomes Poisson model with the log-normal distributed intensity function in the continuous limit. We discuss the relation between this model and Hawkes process. We apply this model to the default portfolios. Moreover we show why the beta distribution and Vasicek model have the similar.

Keywords: Phase transition, Merton model, Anomalous diffusion, Credit risk management

1. INTRODUCTION

Anomalous diffusion is one of the most interesting topics in sociophysics and econophysics [1–3]. The models describing such phenomena have a long memory [4–10] and show several types of phase transitions. In our previous work, we investigated voting models for an information cascade [11–17]. This model has two types of phase transitions. One is the information cascade transition, which is similar to the phase transition of the Ising model [13] that shows whether a distribution converges or not. The other is the convergence transition of the super-normal diffusion that corresponds to an anomalous diffusion [12, 18].

In recent years, there have been several studies regarding the time series of financial markets from the perspective of econophysics [24–28]. The important properties of these data are particularly the fat tailed distribution of returns, long memory in volatility, and multi-fractal nature. The market data of stock prices and foreign exchange have been used in most of these studies. The long memory of volatility is known as volatility clustering [28] and affects the risk management and especially the calculation of Value at Risk (VaR). It corresponds to the temporal correlation of time series. The temporal correlation of exponential decay is the short and the power decay is the medium and long memories. In fact, most of the asset have the short memory, but it is known that some assets has the long and medium memories.

In this study, we investigate the time series of asset values using long default data and show the long memory of the hidden time series of asset value.

We study a Bayesian estimation method using the Merton model. Under normal circumstances, the Merton model incorporates default correlation by the correlation of asset price movements (asset correlation), which is used to estimate the PD and the correlation. A Monte Carlo simulation is an appropriate tool to estimate the parameters, except under the limit of large homogeneous portfolios [21]. In this case, the distribution becomes a Vasicek distribution that can be calculated analytically [29].

In our previous study, we discussed parameter estimation using a beta-binomial distribution with default correlation and considered a multi-year case with a temporal correlation [17]. A non-equilibrium phase transition, similar to that of the Ising model, occurs when the temporal correlation decays by a power law. In this study, we discuss a phase transition when we use the Merton model. When the power index is less than one, the estimator distribution of the PD converges slowly to the delta function. In contrast, when the power index is greater than one, the convergence is the same as that of the normal case. When the distribution converges slowly, the estimation of the PD with limited data becomes time-consuming.

To confirm the decay form of the temporal correlation, we investigate empirical default data. We verify the estimation of the power index in the slow-convergence range. This demonstrates that even if adequate historical data are available, accurate estimation of the parameters of the PD, asset correlation, and temporal correlation is time-consuming.

The remainder of this paper is organized as follows. In Section 2, we introduce the stochastic process of the Merton model and obtain the Poisson process in the double scaling limit. In Section 3 we discussion the normal-slow phase transition. In Section 4, we describe the application of the Bayesian estimation approach to the empirical data of default history using the Merton model and confirm its parameters. Finally, the conclusions are presented in Section 5.

2. ASSET CORRELATION AND DEFAULT CORRELATION

In this section, we consider whether the time series of a stochastic process using the Merton model [30]. We take the limit of the process, and obtain the Poisson process with the log-normal intensity function.

Normal random variables, y_t, are hidden variables that explain the status of the economics, and y_t affects all obligors in the t-th term. To introduce the temporal correlation from different term, let $\{y_t, 1 \leq t \leq T\}$ be the time series of the stochastic variables of the correlated normal distribution with the following correlation matrix:

$$\Sigma \equiv \begin{pmatrix} 1 & d_1 & d_2 & \cdots & d_{T-1} \\ d_1 & 1 & d_1 & \ddots & \vdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & d_1 \\ d_{T-1} & \cdots & d_2 & d_1 & 1 \end{pmatrix}, \quad (1)$$

where $(y_1, \cdots, y_T)^T \sim \mathrm{N}_T(0, \Sigma)$.

We consider two cases of temporal correlation: exponential decay, $d_i = \theta^i, 0 \leq \theta \leq 1$, and power decay, $d_i = 1/(i+1)^{-\gamma}, \gamma \geq 0$. The exponential decay corresponds to short memory and the power decay corresponds to intermediate and long memories [31]. Without loss of generality, we assume that the number of obligors in the t-th term is constant and denote it as N.

The asset correlation, ρ_A, is the parameter that describes the correlation between the value of the assets of the obligors in the same term. We consider the i-th asset value, $\hat{U}_{it}$, at term t, to be

$$\hat{U}_{it} = \sqrt{\rho_A} y_t + \sqrt{1-\rho_A}\epsilon_{it}, \tag{2}$$

where $\epsilon_{it} \sim \mathrm{N}(0,1)$ is i.i.d. By this formulation, the equal-time correlation of U_{it} is ρ_A. The discrete dynamics of the process is described by

$$X_{it} = 1_{\hat{U}_{it} \leq Y}, \tag{3}$$

where Y is the threshold and $1 \leq i \leq N$. When $X_{it} = 1(0)$, the i-th obligor in the t-th term is default (non-default). Eq. (3) corresponds to the conditional default probability for $y_t = y$ as

$$G(y) \equiv \mathrm{P}(X_t = 1 | y_t = y) = \Phi\left(\frac{Y - \sqrt{\rho_A} y}{\sqrt{1-\rho_A}}\right), \tag{4}$$

where $\Phi(x)$ is the standard normal distribution, $G(y_t)$ is the distribution of the default probability during the t-th term in the portfolio, and the average PD is $p' = \Phi(Y)$, which is the long term average of PD.

Here we extend to the multi-term model. The number of the companies is N and the conditional probabilities are $G(y)$. The distribution of number of the default is

$$\begin{aligned} P[X_t = k_t] &= \int_{-\infty}^{\infty} \frac{N!}{k_t!(N-k_t)!} G(y)^{k_t} (1 - G(y))^{N-k_t} \phi(y) dy, \\ &= \int_{-\infty}^{\infty} \frac{N!}{k_t!(N-k_t)!} \Phi\left(\frac{\Phi^{-1}(p') - \sqrt{\rho_A} y}{\sqrt{1-\rho_A}}\right)^{k_t} (1 - \Phi\left(\frac{\Phi^{-1}(p') - \sqrt{\rho_A} y}{\sqrt{1-\rho_A}}\right))^{N-k_t} \phi(y) dy, \end{aligned} \tag{5}$$

where $\phi(y)$ is the normal distribution, $1/\sqrt{2\pi} e^{-y^2/2}$. Here we use the approximation for the normal distribution,

$$\Phi(x) \approx \frac{1}{1 + e^{-\beta x}}, \tag{6}$$

the logistic function and $\beta \sim 1.3$. Using this approximation, we can obtain

$$G(y) = \Phi\left(\frac{\Phi^{-1}(p') - \sqrt{\rho_A} y}{\sqrt{1-\rho_A}}\right) = p \sim \frac{1}{1 + ((1-p')/p')^{1/\sqrt{1-\rho_A}} e^{\beta \sqrt{\rho_A} y / \sqrt{1-\rho_A}}}, \tag{7}$$

where we use the relation,

$$Y = \Phi^{-1}(p') \approx \log \frac{p'}{1-p'}. \tag{8}$$

In the case $p << 1$, we can obtain,

$$G(y) = p'^{1/\sqrt{1-\rho_A}} e^{-\frac{\sqrt{\rho_A}}{\sqrt{1-\rho_A}}\beta y}. \tag{9}$$

Here we take the limit $p', p \to 0$ and $N \to \infty$ with the condition, fixed $\lambda_0 = Np'^{1/\sqrt{1-\rho_A}}$. Then, we define

$$\lambda(y) = Np'^{1/\sqrt{1-\rho_A}} e^{-\frac{\sqrt{\rho_A}}{\sqrt{1-\rho_A}}\beta y} = \lambda_0 e^{-\frac{\sqrt{\rho_A}}{\sqrt{1-\rho_A}}\beta y} \tag{10}$$

The expected value of $\lambda(y)$ is

$$\bar{\lambda} = \int_{-\infty}^{\infty} \lambda(y)\phi(y)dy = \lambda_0 e^{\frac{\rho_A}{2(1-\rho_A)}\beta^2} = \lambda_0 e^{\alpha^2/2}, \tag{11}$$

where $\alpha = \frac{\sqrt{\rho_A}}{\sqrt{1-\rho_A}}\beta$.

Here we change the variable from y to $\hat{y} = -y$. Then,

$$\lambda(\hat{y}) = \lambda_0 e^{\frac{\sqrt{\rho_A}}{\sqrt{1-\rho_A}}\beta \hat{y}} = \lambda_0 e^{\alpha \hat{y}}. \tag{12}$$

In this limit the we can calculate Eq.(5) as

$$\begin{aligned} P[X_t = k_t] &\sim \int_{-\infty}^{\infty} \frac{\lambda(\hat{y})^{k_t} e^{-\lambda(\hat{y})}}{k_t!} \phi(\hat{y}) d\hat{y} \\ &= \int_{0}^{\infty} \frac{\lambda(\hat{y})^{k_t} e^{-\lambda(\hat{y})}}{k_t!} f(\lambda) d\lambda, \end{aligned} \tag{13}$$

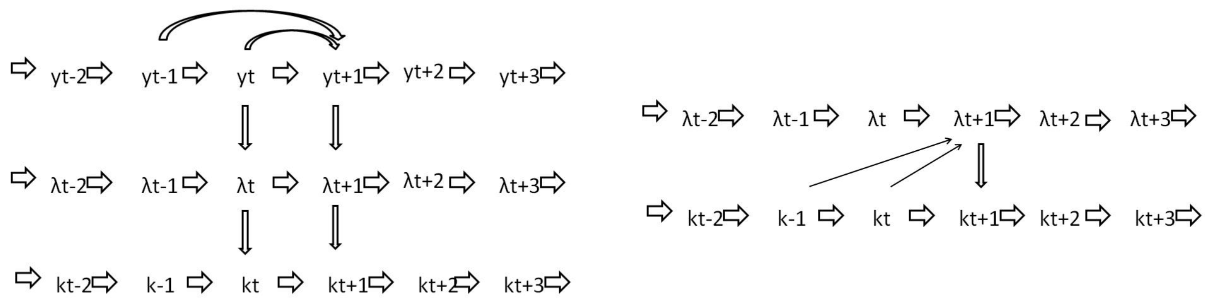
where

$$f(\lambda) = \phi(\hat{y})\frac{d\hat{y}}{d\lambda} = \frac{1}{\sqrt{2\pi}\alpha\lambda} e^{-\frac{(\log\lambda - \log\lambda_0)^2}{2\alpha^2}} \tag{14}$$

Then the distribution of the intensity function is log normal distribution. The expected values is $\bar{\lambda} = \lambda_0 e^{\alpha^2/2}$ and the variance is $\bar{V} = \bar{\lambda}^2(e^{\alpha^2} - 1)$. Note that the expected value is the function of ρ_A.

In the limit $\rho_A \to 0$, $\bar{\lambda} = \lambda_0$ and $\bar{V} = 0$, it is the Poisson process. In the limit $\rho_A \to 1$, $\alpha \to \infty$ and $f(\lambda) \sim 1/\lambda$ where $\bar{\lambda} \to \infty$ and $V \to \infty$.

We show the image of the relation among the variables of this process in Fig.1 (a). For the comparison we show that of SE-NBD and Hawkes processes in Fig.1 (b). In this process y_t is independent form the historical data of λ_t and k_t. In this process the temporal correlation is by time series of y_t. On the other hand in the NBD and Hawkes processes it is in between the intensity function λ_t and the number of defaults k_t.



(a) (b)

FIG. 1. The relation among the variables of (a) this processl and (b) SE-NBD and Hawkes process

Next we consider the impact analysis. It is the study of the the effect of the shock of the noise δ at term t. When the shock is added in $\hat{y}_t$, we consider the added expected value of the intensity function form $t+1$ to ∞.

At first we consider the case, exponential decay model $d_i = \theta^i, \theta \leq 1$:

We consider the ratio for the impact δ added,

$$\frac{\bar{\lambda}_\delta}{\bar{\lambda}} = \mathrm{E}[\log \frac{\lambda_0 e^{\alpha\delta+\alpha\hat{y}_t}}{\lambda_0 e^{\alpha\hat{y}_t}} \frac{\lambda_0 e^{\alpha\delta\theta+\alpha\hat{y}_t}}{\lambda_0 e^{\alpha\hat{y}_{t+1}}} \frac{\lambda_0 e^{\alpha\delta\theta^2+\alpha\hat{y}_t}}{\lambda_0 e^{\alpha\hat{y}_{t+1}}} \cdots] = \frac{\alpha\delta}{1-\theta}. \tag{15}$$

Then, in this case the impact is the finite.

Next we consider the case, $d_i = 1/(i+1)^\gamma, i = 1, 2, \cdots$, where $\gamma \geq 0$ is the power index. We consider the ratio for the impact δ added,

$$\frac{\bar{\lambda}_\delta}{\bar{\lambda}} = \mathrm{E}[\log \frac{\lambda_0 e^{\alpha\delta+\alpha\hat{y}_t}}{\lambda_0 e^{\alpha\hat{y}_t}} \frac{\lambda_0 e^{\alpha\delta/2^\gamma+\alpha\hat{y}_t}}{\lambda_0 e^{\alpha\hat{y}_{t+1}}} \frac{\lambda_0 e^{\alpha\delta/3^\gamma+\alpha\hat{y}_t}}{\lambda_0 e^{\alpha\hat{y}_{t+1}}} \cdots \tag{16}$$

When $\gamma > 1$, we can obtain

$$\frac{\bar{\lambda}_\delta}{\bar{\lambda}} < \alpha\delta(1 + \frac{1}{\gamma - 1}). \tag{17}$$

Then the impact is finite.

When $gamma = 1$, we can obtain

$$\frac{\bar{\lambda}_\delta}{\bar{\lambda}} \sim \alpha\delta \log T. \tag{18}$$

Then the impact is infinite.

When $gamma = 1$, we can obtain

$$\frac{\bar{\lambda}_\delta}{\bar{\lambda}} \sim \alpha\delta T^{1-\gamma}. \tag{19}$$

Then the impact is infinite.

In summary the impact is finite when $\gamma > 1$ and the impact is infinite when $\gamma \leq 1$.

In the next section we consider the phase transition for the impact.

III. NORMAL-SLOW PHASE TRANSITION

We are interested in the effect of the temporal correlation. Here we consider the variance of λ_T, $V(\lambda_T)$ the variance of λ of the term T is

$$\mathrm{V}(\Sigma^T \lambda(\hat{y}_i)) = T\bar{V} + 2\bar{V}\sum_{i=1}^{T-1} d_i(T-i).$$

The second term is from the temporal correlation. We consider the behavior of the second term in limit $T \to \infty$.

2.1 Exponential temporal correlation

In this subsection, we study $V(\lambda_T)$ for exponential decay model $d_i = \theta^i, \theta \leq 1$:

$$V(\lambda_T) \simeq \bar{V}T + +2\bar{V}\sum_{i=1}^{T-1} \theta^i(T-i). \tag{20}$$

The first term on the right-hand side (RHS) behave as $\propto T$ and, thus, this is the normal diffusion. In the case that $\theta \neq 1$, the second term is

$$2\bar{V}[T\frac{1-\theta^{T-1}}{1-\theta} + \frac{(T-1)\theta^{T-1}(1-\theta)-(1-\theta^{T-1})\theta}{(1-\theta)^2} \propto T$$

and it also the normal diffusion. We conclude that as the number of data samples increases, the variance increases as T. When $\theta = 1$, there is no temporal correlation decay case and all obligors are correlated ρ_A. Hence, there is no phase transition for $\theta < 1$.

2.2 Power temporal correlation

In this subsection, we consider power decay case $d_i = 1/(i+1)^\gamma, i = 1, 2, \cdots,$ where $\gamma \geq 0$ is the power index. The power correlation affects the number of defaults for long periods of time. Ranges $\gamma \leq 1$ and $\gamma > 1$ are called long memory and intermediate memory, respectively [31]. In contrast, the exponential decay affects short periods of time and is called short memory. The asymptotic behavior of $V(\lambda_T))$ is given as:

$$\mathrm{V}(\lambda_T) \simeq \bar{V}T + 2\bar{V}T\sum_{i=1}^{T-1}(i+1)^{-\gamma}(T-i).$$

2.2.1) $\gamma > 1$ case

We can obtain

$$\begin{aligned}\mathrm{V}(\lambda_T) &\simeq \bar{V}T + +2\bar{V}\sum_{i=1}^{T-1}(T-i)/(i+1)^{\gamma} \\ &\simeq \bar{V}T + +2\bar{V}T^{-\gamma+2}/(\gamma-1).\end{aligned} \tag{21}$$

The first term is the normal diffusion and the second term is the $T^{-\gamma+2}$, where $\gamma > 1$. It is the slower than the the normal diffusion. Hence, the significant terms are the first term which is the normal diffusion.

2.2.2) $\gamma = 1$ case

$\mathrm{V}(\lambda_T)$ behaves as

$$\mathrm{V}(\lambda_T) \simeq \bar{V}T + 2\bar{V}\sum_{i=1}^{T-1}(T-i)/(i+1). \tag{22}$$

The RHS of Eq. (22) is evaluated as

$$RHS \simeq \bar{V}T + +2\bar{V}T\log T - T + 2] \sim T\log T. \tag{23}$$

In conclusion, $\mathrm{V}(\lambda_T)$ behaves asymptotically as

$$\mathrm{V}(\lambda_T) \sim T\log T \tag{24}$$

and becomes the anomalous faster diffusion.

2.2.3) $\gamma < 1$ case

$\mathrm{V}(\lambda_T)$ is calculated as

$$\mathrm{V}(\lambda_T) \simeq \bar{V}T + 2\bar{V}[\frac{1}{(1-\gamma)(2-\gamma)]}T^{-\gamma+2} \sim T^{-\gamma+2}. \tag{25}$$

Thus, we can conclude that $\mathrm{V}(\lambda_T)$ behaves as

$$\mathrm{V}(\lambda_T) \sim T^{-\gamma+2}, \tag{26}$$

which is the anomalous faster diffusion.

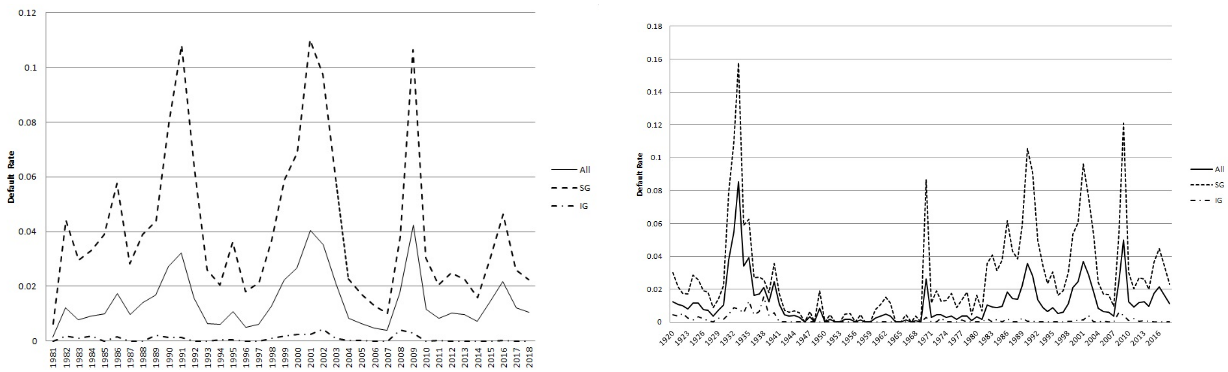
In conclusion, a phase transition occurs when the temporal correlation decays by the power law. When the power index, γ, is less than one, the variance , $\mathrm{Z}(\lambda_T)$, anomalous diffusion. Conversely, when the power index, γ, is greater than one, it is the normal diffusion. This phase transition is called a "super-normal transition" [12, 18], which is the transition between long memory and intermediate memory. In that study, when the power index was less than one, the estimator does not converge the steady state [17].

3. ESTIMATION OF PARAMETERS

As discussed in the previous section, whether temporal correlation obeys an exponential decay or a power decay is an important issue because there exists a super-normal transition in the latter case. Further, the appearance of a transition affects whether we can estimate the PD.

First, the S&P default data from 1981 to 2018 [32] are used. The average PD is 1.51 % for all ratings, 3.90 % for speculative grade (SG) ratings, and 0.09% for investment grade (IG) ratings. The SG rating represents ratings under BBB-(Baa3) and IG represents that above BBB-(Baa3). In Fig. 2 (a) we show the historical default rate of the S&P. The solid and dotted lines correspond to the speculative grade and investment grade samples, respectively. We use Moody's default data from 1920 to 2018 for 99 years [33]. It includes the Great Depression in 1929 and Great Recession in 2008. The average default rate is 1.50% for all of the ratings, 3.70% for speculative ratings, and 0.14% for investment grade. In Fig. 2 (b), we show the historical default rate of Moody's.

We estimate the parameters p, ρ_A, θ and γ of the Merton model using the Bayesian method and Stan 2.19.2 in R 3.6.2 software. We explain the method and how to estimate the parameters in Appendix B [34] and summarize the results in Table I. We show ρ_D instead of ρ_A, as we need to compare it with that of the beta-binomial distribution model. The estimation of the parameters are the maximum a posteriori (MAP) estimation. A detailed explanation of the estimation procedure and rmd file is provided on GitHub [35]. We notice



(a) (b)

FIG. 2. (a): S&P Default Rate in 1981-2018. (b)Moody's Default Rate in 1920-2018. The solid and dotted lines respectively correspond to the speculative grade (SG) and investment grade (IG) of all the samples.

that the power index γ is smaller than one for all cases and the values are smaller than the phase transition point, $\gamma = 1$.

Here we introduce the MAP estimation using the beta binomial distribution which used the default correlation directly to compare the the MAP estimation using Merton model. Beta binomial model is known as the correlated binomial distribution and we apply it to defaults in the portfolio with the correlation [17].

The conclusions are shown in Table II for the exponential and power decay models. We confirmed small θ and large γ values, which represent small temporal correlation. Parameter γ for the power decay is larger than the phase transition point, $\gamma = 1$. The PD and default correlation are almost the same as the estimations by the exponential and power decay models. The reason behind this is that the power exponent γ is adequately large and there is only a small difference between the exponential and power decay models.

We can confirm that θ and γ both have large differences between the values estimated by the beta-binomial distribution and the Merton model. The reason for this is shown in Fig. **??** (a). We set d_{1A} and d_{1D} for ρ_A and ρ_D, respectively. From this, we can obtain the inequality

$$d_{1A} = \frac{d_{1A}\rho_A}{\rho_A} >> \frac{f(d_{1A}\rho_A)}{f(\rho_A)} = \frac{f(d_{1A}\rho_A)}{\rho_D} = d_{1D}.$$

Hence, the difference in the estimated parameter between the Merton model and the beta-binomial model becomes large. We can find the large convexity for the mapping function f.

TABLE I. MAP estimation of the parameters for the exponential and power decay models by the Merton model

No.	Model	Exponential decay p	ρ_A	θ	Power decay p	ρ_A	γ
1	Moody's 1920-2018	1.43%	17.8%	0.858	7.40%	38.7%	0.090
2	S&P 1981-2018	1.43%	6.4%	0.597	1.85%	12.0%	0.610
3	Moody's 1981- 2018	1.61%	7.1%	0.613	1.92%	12.4%	0.622
4	S&P 1990-2018	1.72%	7.5%	0.616	2.97%	12.5%	0.616
5	Moody's 1990-2018	1.92%	10.0%	0.678	2.40%	12.1%	0.624
6	Moody's 1920-2018 SG	3.00%	18.9%	0.838	6.15%	32.2%	0.146
7	S&P 1981-2018 SG	4.53%	8.7%	0.588	4.42%	11.7%	0.628
8	Moody's 1981-2018 SG	4.28%	9.4%	0.603	3.97%	11.5%	0.619
9	S&P 1990-2018 SG	4.93%	11.2%	0.639	5.40%	13.9%	0.626
10	Moody's 1990-2018 SG	4.51%	11.1%	0.648	6.09%	14.6%	0.619
11	Moody's 1920-2018 IG	0.04%	35.3%	0.891	3.40%	51.4%	0.102
12	S&P 1981-2018 IG	0.02%	25.8%	0.483	0.02%	20.3%	9.189
13	Moody's 1981-2018 IG	0.01%	21.9%	0.672	1.84%	33.8%	0.618
14	S&P 1990-2018 IG	0.01%	37.4%	0.712	1.63%	46.7%	0.630
15	Moody's 1990-2018 IG	0.01%	33.0%	0.794	3.44%	51.1%	0.003

Hence, θ and γ for a default correlation is much smaller than that for asset correlation.

Next, we discuss whether the correlation has a long memory. In Table III, we calculated the WAIC and WBIC for each model that uses the Merton model for the discussion. Using Moody's data from 1920, the power decay model is found to be superior to the exponential decay model. Therefore, it seems that the default rate has a long memory. As γ is less than 1 for long history data, the phase converges slowly. In other words, parameter estimation becomes difficult because the convergence speed becomes slow when the temporal correlation is the power decay.

In Table IV, we show the AIC and BIC for each model using the beta-binomial distribution

TABLE II. Most likelihood estimate of the parameters for the exponential and power decay models by beta-binomial distribution

		Exponential decay			Power decay		
No.	Model	p	ρ_D	θ	p	ρ_D	γ
1	Moody's 1920-2018	0.96%	1.9%	0.044	0.94%	2.0%	4.7
2	S&P 1981- 2018	1.53%	0.8%	0.026	1.54%	0.8%	5.7
3	Moody's 1981-2018	1.53%	0.8%	0.022	1.52%	0.7%	5.9
4	S&P 1990-2018	1.66%	0.9%	0.023	1.64%	0.9%	5.7
5	Moody's 1990-2018	1.67%	0.9%	0.019	1.61%	0.8%	6.0
6	Moody's 1920-2018 SG	2.36%	3.8%	0.044	2.34%	4.1%	4.7
7	S&P 1981-2018 SG	4.16%	2.0%	0.026	4.20%	2.0%	5.7
8	Moody's 1981-2018 SG	4.18%	2.0%	0.022	4.35%	1.9%	6.0
9	S&P 1990-2018 SG	4.42%	2.5%	0.024	4.43%	2.6%	5.6
10	Moody's 1990-2018 SG	4.33%	2.3%	0.020	4.31%	2.2%	5.9
11	Moody's 1920-2018 IG	0.13%	0.8%	0.17	0.11%	0.9%	3.0
12	S&P 1981-2018 IG	0.11%	0.4%	0.12	0.09%	0.3%	3.6
13	Moody's 1981-2018 IG	0.10%	0.6%	0.05	0.09%	0.3%	4.6
14	S&P 1990-2018 IG	0.09%	0.4%	0.12	0.09%	0.4%	3.7
15	Moody's 1990-2018 IG	0.09%	0.4%	0.06	0.07%	0.7%	4.2

and compare them to the estimation using the Merton model. We obtain the same conclusion using Moody's data from 1920: the power decay model is superior to the exponential decay model. The parameter γ is not less than 1 for the power decay case when we use the beta-binomial distribution.

4. CONCLUDING REMARKS

This study considers the Merton model. We show the Merton model becomes Poisson model with the log-normal distributed intensity function in the continuous limit. We discuss

TABLE III. WAIC and WBIC for the exponential and power decay using the Merton model

No.	Model	Exponential decay WAIC	Exponential decay WBIC	Power decay WAIC	Power decay WBIC
1	Moody's 1920-2018	572.9	746.7	568.6	745.9
2	S&P 1981- 2018	271.5	332.9	272.1	334.8
3	Moody's 1981-2018	277.6	339.0	277.5	341.4
4	S&P 1990-2018	214.3	256.4	214.6	258.3
5	Moody's 1990-2018	219.5	262.1	219.6	264.7
6	Moody's 1920-2018 SG	564.3	731.9	560.2	733.8
7	S&P 1981-2018 SG	268.7	328.1	268.9	330.5
8	Moody's 1981-2018 SG	274.1	333.6	274.4	337.1
9	S&P 1990-2018 SG	212.0	253.9	212.8	255.2
10	Moody's 1990-2018 SG	217.3	260.4	218.3	262.8
11	Moody's 1920-2018 IG	247.6	351.2	244.9	351.2
12	S&P 1981-2018 IG	116.0	153.3	115.0	160.3
13	Moody's 1981-2018 IG	110.3	156.3	108.8	156.7
14	S&P 1990-2018 IG	87.6	114.3	86.8	117.7
15	Moody's 1990-2018 IG	81.7	111.4	82.5	114.5

the relation between this model and Hawkes process. We apply this model to the default portfolios. Moreover we show why the beta distribution and Vasicek model have the similar.

When the power index, γ, was larger than one, the estimator distribution of the PD converged normally. When the power index was less than or equal to one, the diffusion is anomalous faster diffusion. It is the super phase.This phase transition is called the "super-normal transition." For the case of exponential decay, there was no phase transition.

TABLE IV. AIC and BIC for the exponential and power decay using the beta-binomial distribution

No.	Model	Exponential decay AIC	Exponential decay BIC	Power decay AIC	Power decay BIC
1	Moody's 1920-2018	746.8	780.8	746.8	780.9
2	S&P 1981- 2018	352.0	382.9	353.6	384.5
3	Moody's 1981-2018	362.2	395.0	363.4	396.2
4	S&P 1990-2018	285.0	315.5	285.8	316.3
5	Moody's 1990-2018	293.8	326.3	298.6	331.1
6	Moody's 1920-2018 SG	730.8	762.0	730.6	761.8
7	S&P 1981-2018 SG	346.8	366.7	348.8	368.7
8	Moody's 1981-2018 SG	356.0	385.9	360.4	390.3
9	S&P 1990-2018 SG	283.0	302.6	283.8	303.4
10	Moody's 1990-2018 SG	291.0	320.7	291.6	321.3
11	Moody's 1920-2018 IG	302.2	334.9	300.4	333.1
12	S&P 1981-2018 IG	134.9	164.5	136.1	165.7
13	Moody's 1981-2018 IG	142.3	173.7	140.2	171.7
14	S&P 1990-2018 IG	106.0	135.3	107.2	136.5
15	Moody's 1990-2018 IG	111.8	142.8	112.4	143.4

Appendix A. Temporal correlation for the Merton Model

We consider the assets of two obligors, $\hat{U}_{1t}$ and $\hat{U}_{2t}$, in year t to confirm temporal correlation. The assets of two obligors have correlation ρ_A. S_t is the global economic factor that affects the two obligors at t:

$$\begin{aligned} \hat{U}_{1t} &= \sqrt{\rho_A} S_t + \sqrt{1-\rho_A}\epsilon_{1t}, \\ \hat{U}_{2t} &= \sqrt{\rho_A} S_t + \sqrt{1-\rho_A}\epsilon_{2t}, \end{aligned} \tag{27}$$

where ϵ_1 and ϵ_2 are the individual factors for the obligors. Here, there is no correlation among ϵ_1, ϵ_2, and S_t because they are independent of each other. In the following year, $t+1$, the assets of the two obligors are $\hat{U}_{1t+1}$ and $\hat{U}_{2t+1}$. The assets have the same correlation, ρ_A, through the global factor S_{t+1}. We can write this as:

$$\begin{aligned}\hat{U}_{1t+1} &= \sqrt{\rho_A} S_{t+1} + \sqrt{1-\rho_A}\epsilon_{1t+1}, \\ \hat{U}_{2t+1} &= \sqrt{\rho_A} S_{t+1} + \sqrt{1-\rho_A}\epsilon_{2t+1}.\end{aligned} \tag{28}$$

The temporal correlation between t and $t+1$ is d_1. The correlation between $\hat{U}_{1t}$ and $\hat{U}_{2t+1}$ is $d_1\rho_A$. In the same way, we obtain the temporal correlation matrix, Eq.(1). It is same as that from the Bayesian estimation, which was introduced in [17], without differentiating between the asset and default correlations.

Appendix B. Bayesian estimation using the Merton model

In this Appendix, we explain the estimate of parameters using the Merton model [34]. There is a prior belief of the possible value of the PD. The prior belief is updated by observations while using the prior distribution as a weighting function. Here, we use the prior function, which is a uniform prior distribution.

To calculate the unconditional probability $P(X_1 = k_1, \cdots, X_T = k_T)$, we approximate the solution by Monte Carlo simulations and numerical integration. Here, the number of obligors and defaults in the t-th year are n_t and k_t, respectively, and they are observable variables. The likelihood is

$$P(X_1 = k_1, \cdots, X_T = k_T) \sim \sum_{i=1}^{n} \prod_{t=1}^{T} \frac{n_t!}{k_t!(n_t - k_t)!} G(S_t^i)^{k_t} (1 - G(S_t^i))^{(n_t - k_t)}, \tag{29}$$

where $G(S_t^i)$ is defined as the probability that an obligor will default in year t, which is conditional to the i-th path realization of all global factors such that

$$G(S_t^i) = \Phi\left(\frac{\Phi^{-1}(p) - S_t^i\sqrt{\rho_A}}{\sqrt{1-\rho_A}}\right), \tag{30}$$

where ρ_A is the asset correlation among obligors within a one year window and Φ is the cumulative normal distribution. S_t^i is the correlated multi-dimensional normal distribution and we use the MAP estimation to estimate the parameters. We use a normal copula [21].

We have estimated parameters using a beta-binomial distribution provided in Section 3 and [17]. One of the differences between using the Merton model and beta-binomial distribution is the default correlation and the asset correlation. The default correlation is defined by binary variables. In contrast, asset correlation is defined by continuous variables. The other difference is that one can calculate the parameters analytically when using the beta-binomial distribution. Hence, it is easier to estimate parameters when using the beta-binomial distribution than when using the Merton model. We estimate the parameters to be stable in Section 3, especially for IG samples, which have small PD. The estimation of IG samples using the Merton model is difficult.

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