Polya with Memory Kernel

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Model

- Memory kernel

$$d(t), t=0,1,\cdots, d(0)=1, d(t)\geq d(t+1)$$

$$D(t)\equiv\sum_{s=1}^{t} d(t-s)=\sum_{s=0}^{t-1} d(s)$$

$$S(t)\equiv\sum_{s=1}^{t} X(s)d(t-s)$$

$$z(t)\equiv\frac{S(t)}{D(t)}$$

$$P(X(t+1)=1|z(t)=z)=f(z)=\begin{cases}(1-p)q+pz & \text{Linear Model}\\ (1-p)q+p\sigma(\lambda(z-1/2)) & \text{Non-Linear Model}\end{cases}$$

$$P(X(t+1)=0|z(t)=z)=1-f(z)$$

EXP decay

$$d(s)=e^{-s/\tau}$$

$$D(t)=\sum_{s=0}^{t-1} d(s)=\frac{1-e^{-t/\tau}}{1-e^{-1/\tau}}$$

$$S(t)=\sum_{s=1}^{t} X(s)d(t-s)$$

$$z(t)\equiv\frac{S(t)}{D(t)}$$

$$d(t+1-s)=e^{-1/\tau}d(t-s)$$

$$S(t+1)=\sum_{s=1}^{t+1} X(s)d(t+1-s)=S(t)e^{-1/\tau}+X(t+1)$$

$$D(t+1)=D(t)e^{-1/\tau}+1$$

$$D(t)\sim\frac{1}{1-e^{-1/\tau}}\simeq\tau, t>>\tau>>1$$

$$D(t)=\frac{1-e^{-t/\tau}}{1-e^{-1/\tau}}=\frac{t/\tau}{1/\tau}\simeq t, \tau>>t>>1$$

$$z(t+1)=\frac{S(t+1)}{D(t+1)}=\frac{X(t+1)}{D(t+1)}+\frac{e^{-1/\tau}S(t)}{D(t+1)}=\frac{X(t+1)}{D(t+1)}+\frac{e^{-1/\tau}D(t)}{D(t+1)}\cdot z(t)=\frac{1}{D(t+1)}X(t+1)+\frac{D(t+1)-1}{D(t+1)}z(t)$$

$$\Delta z(t)\equiv z(t+1)-z(t)=\frac{X(t+1)-z(t)}{D(t+1)}$$

$$E(\Delta z(t+1)|z(t)=z)=\frac{E(X_{t+1}|z(t)=z)-z}{D(t+1)}=\frac{f(z)-z}{D(t+1)}$$

$$V(\Delta z(t+1)|z(t)=z)=\frac{1}{D(t+1)^2}V(X(t+1)|z(t)=z)=\frac{f(z)(1-f(z))}{D(t+1)^2}$$

$$dz = A(z)dt + B(z)dW, A(z) = \frac{f(z)-z}{D(t+1)}, B(z) = \frac{\sqrt{f(z)(1-f(z))}}{D(t+1)}$$

- Fokker-Planck eq.

$$\partial_t P(z,t) = -\frac{\partial}{\partial z} A(z)P(z,t) + \frac{1}{2}\frac{\partial^2}{\partial z^2} B(z)^2 P(z,t)$$

- Stationary solution with reflecting boundary condition for contant τ

$$t >> \tau, D(t) \simeq \tau, A(z) = \frac{f(z)-z}{\tau}, B(z)^2 = \frac{f(z)(1-f(z))}{\tau^2}$$

$$P_{st}(z) \propto \frac{1}{B(z)^2}\exp\left(\int_{1/2}^{z} \frac{2A(y)}{B(y)^2} dy\right) = \frac{\tau^2}{f(z)(1-f(z))}\exp\left(\int_{1/2}^{z} \frac{2(f(y)-y)\tau}{f(y)(1-f(y))} dy\right)$$

Linear Case

$$f(z) = (1-p)q + pz \rightarrow f(z) - z = -(1-p)(z-q)$$

$$dz = -\frac{(1-p)(z-q)}{D(t+1)} dt + \frac{\sqrt{f(z)(1-f(z))}}{D(t+1)} dW$$

$$t >> \tau >> 1 \rightarrow D_r(t+1) \simeq \tau$$

$$dz = -\frac{(1-p)(z-q)}{\tau} dt + \frac{\sqrt{f(z)(1-f(z))}}{\tau} dW$$

- τ:constant, stationary solution

$$P_{st}(z) \propto \frac{\tau^2}{f(z)(1-f(z))}\exp\left(\int_{1/2}^{z} \frac{-2(1-p)(y-q)\tau}{f(y)(1-f(y))} dy\right)$$

$\tau >> 1, z \simeq q \rightarrow f(z) \simeq q$

$$P_{st}(z) \propto \exp\left(\int_{q}^{z} \frac{-2(1-p)(y-q)\tau}{q(1-q)} dy\right) = \exp\left(-\frac{-(z-q)^2}{2\cdot q(1-q)/2(1-p)\tau}\right)$$

$$z(t) \sim N\left(q, \frac{q(1-q)}{2(1-p)\tau}\right)$$

- c.f.

$$X(t) \sim \mathrm{Ber}(q), S(t) = \sum_{s=1}^{t} X(s)e^{-(t-s)/\tau}, D(t) = \sum_{s=1}^{t} e^{-(t-s)/\tau} \sim \tau, t >> \tau$$

$$V(S(t)) = q(1-q)\sum_{s=1}^{t} e^{-2(t-s)/\tau} \simeq q(1-q)\frac{\tau}{2} \rightarrow V(z(t)) = V(S(t)/D(t)) \sim q(1-q)\frac{1}{2\tau}$$

- IVP Solution

$$dz = -\frac{(1-p)(z-q)}{\tau} dt + \frac{\sqrt{f(1-f)}}{\tau} dW$$

$$z_{cl}(t) = q + (z_0 - q)\exp\left(-\frac{(1-p)}{\tau}(t-t_0)\right), z(t_0) = z_0$$

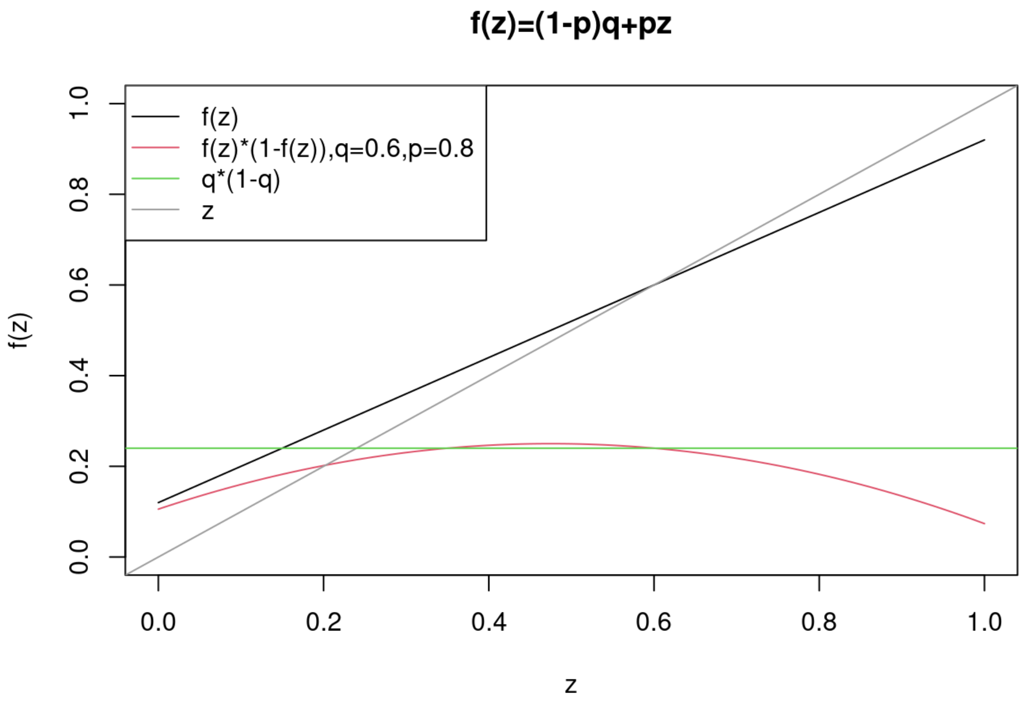
$$z(t) - q = \exp\left(-\frac{(1-p)}{\tau}(t-t_0)\right)u(t), u(t_0) = z_0 - q$$

$$du = e^{-\frac{1-p}{\tau}(t-t_0)}\frac{\sqrt{f(1-f)}}{\tau} dW, u(t_0) = z_0 - q$$

$$u(t) = \int_{t_0}^{t} e^{-\frac{1-p}{\tau}(s-t_0)}\frac{\sqrt{f(1-f)}}{\tau} dW(s) + (z_0 - q)$$

- Approximatio $f(1-f) \simeq q(1-q)$

```
p<-0.8
q<-0.6
f<-function(z) (1-p)*q+p*z
f2<-function(z) ((1-p)*q+p*z)*(1-((1-p)*q+p*z))
curve(f,0,1,ylim=c(0,1),xlab="z",ylab="f(z)",main="f(z)=(1-p)q+pz")
curve(f2,0,1,add=TRUE,col=2)
abline(a=0,b=1,col=8)
abline(h=q*(1-q),col=3)
legend("topleft",legend=c("f(z)","f(z)*(1-f(z)),q=0.6,p=0.8","q*(1-q)","z"),lty=1,pch=-1,col=c(1,2,3,8))
```



$$z(t) = q + (z_0 - q)e^{-\frac{1-p}{\tau}(t-t_0)} + \frac{\sqrt{q(1-q)}}{\tau}\int_{t_0}^{t} e^{\frac{1-p}{\tau}(s-t)} dW(s)$$

$$E(z(t)) = q + (z_0 - q)e^{-\frac{1-p}{\tau}(t-t_0)}$$

$$V(z(t)) = \frac{q(1-q)}{\tau^2}\int_{t_0}^{t} e^{2\frac{1-p}{\tau}(s-t)} ds = \frac{q(1-q)}{2(1-p)\tau}(1 - e^{2\frac{(1-p)}{\tau}}(t_0 - t)) \to \frac{q(1-q)}{2(1-p)\tau}$$

$$z(t) \sim N(E(z(t)), V(z(t)))$$

$$C(t) = E(X(t+1)|z(t_0)=1) - E(X(t+1)|z(t_0)=0) = E[f(z(t))|z(t_0)=1] - E[f(z(t))|z(t_0)=0]$$

$$= p \cdot (E[z(t)|z(t_0)=1] - E[z(t)|z(t_0)=0]) = pe^{-\frac{(1-p)}{\tau}(t-t_0)}$$

- $\tau = t^{\alpha}, 0 < \alpha < 1$:$t >> \tau(t) >> 1$.

$$D(t) = \frac{1 - e^{-t/\tau(t)}}{1 - e^{-1/\tau(t)}} \simeq \frac{1}{1 - e^{-1/\tau(t)}} \simeq \tau(t)$$

$$dz = \frac{-(1-p)(z-q)}{\tau(t)}dt + \frac{\sqrt{q(1-q)}}{\tau(t)}dW$$

$$z - q = y, z = q + y, y(t_0) = z(t_0) - q$$

$$dy = -\frac{(1-p)}{t^{\alpha}}ydt \to d\log y = -\frac{1-p}{t^{\alpha}} = -d\left(\frac{1-p}{1-\alpha}t^{1-\alpha}\right)$$

$$\log\left(\frac{y(t)}{y(t_0)}\right) = -\left(\frac{1-p}{1-\alpha}\right)(t^{1-\alpha} - t_0^{1-\alpha})$$

$$y(t) = y(t_0)e^{-\left(\frac{1-p}{1-\alpha}\right)(t^{1-\alpha}-t_0^{1-\alpha})}$$

$$z(t) = q + (z(t_0) - q)e^{-\left(\frac{1-p}{1-\alpha}\right)(t^{1-\alpha}-t_0^{1-\alpha})} = z_{cl}(t)$$

$$dy = -\frac{(1-p)}{t^\alpha}ydt + \frac{\sqrt{q(1-q)}}{t^\alpha}dW$$

$$y = y_{cl}(t)u, y_{cl}(t) = y(t_0)e^{-\left(\frac{1-p}{1-\alpha}\right)(t^{1-\alpha}-t_0^{1-\alpha})}, u(t_0) = 1$$

$$dy = dy_{cl}u + y_{cl}du \to du = y_{cl}^{-1}(t)\frac{\sqrt{q(1-q)}}{t^\alpha}dW$$

$$u(t) = \int_{t_0}^{t} y_{cl}^{-1}(s)\frac{\sqrt{q(1-q)}}{s^\alpha}dW + 1$$

$$y(t) = y_{cl}(t)u(t) = y_{cl}(t) + \int_{t_0}^{t} e^{-\left(\frac{1-p}{1-\alpha}\right)(t^{1-\alpha}-s^{1-\alpha})}\frac{\sqrt{q(1-q)}}{s^\alpha}dW$$

$$z(t) = z_{cl}(t) + \int_{t_0}^{t} e^{-\left(\frac{1-p}{1-\alpha}\right)(t^{1-\alpha}-s^{1-\alpha})}\frac{\sqrt{q(1-q)}}{s^\alpha}dW$$

$$C(t) = E(X(t+1)|z(t_0) = 1) - E(X(t+1)|z(t_0) = 0) = E[f(z(t))|z(t_0) = 1] - E[f(z(t))|z(t_0) = 0]$$

$$= p \cdot (E[z(t)|z(t_0) = 1] - E[z(t)|z(t_0) = 0]) = pe^{-\left(\frac{1-p}{1-\alpha}\right)(t^{1-\alpha}-t_0^{1-\alpha})}$$