Chapter 4: Electric Flux and Gauss's Law

4.1 Introduction

We have seen in chapter 3 that determining the electric field of a continuous charge distribution can become very complicated for some charge distributions. It would be desirable if we could find a simpler way to determine the electric field of a charge distribution. It turns out that if a certain symmetry exists in the charge distribution it is possible to determine the electric field by means of Gauss's law. To understand Gauss's law we must first understand the concept of electric flux.

4.2 Electric Flux

Flux is a quantitative measure of the number of lines of a vector field that passes perpendicularly through a surface. Figure 4.1(a), shows an electric field **E** passing through a portion of a surface of area A. The area of the surface is represented by a

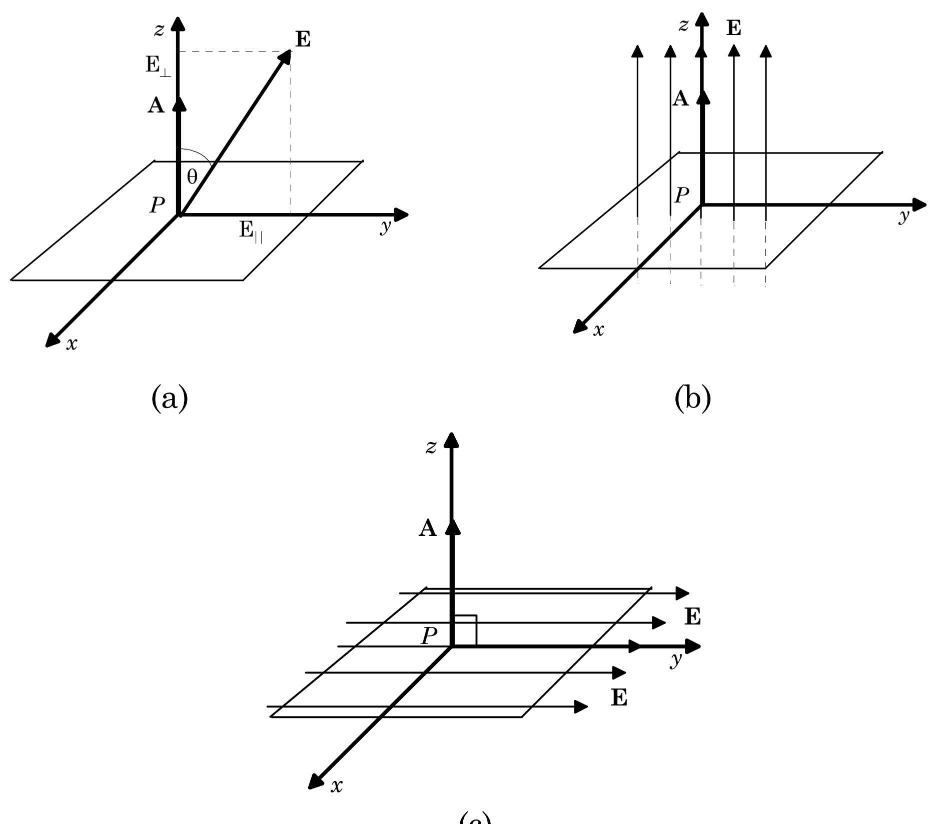

Figure 4.1 Electric flux.

vector **A**, whose magnitude is the area A of the surface, and whose direction is perpendicular to the surface. That an area can be represented by a vector was shown in Chapter 1. *The electric flux is defined to be*

$$\Phi_E = \boldsymbol{E} \bullet \boldsymbol{A} = EA\cos\theta \tag{4.1}$$

and is a quantitative measure of the number of lines of ***E*** *that pass normally through the surface area* ***A***. The number of lines represents the strength of the field. The vector **E**, at the point P of figure 4.1(a), can be resolved into the components, $E_\perp$ the component perpendicular to the surface, and $E_{||}$ the parallel component. The perpendicular component is given by

$$E_\perp = E\cos\theta \tag{4.2}$$

while the parallel component is given by

$$E_{||} = E\sin\theta \tag{4.3}$$

The parallel component $E_{||}$ lies in the surface itself and therefore does not pass through the surface, while the perpendicular component $E_\perp$ completely passes through the surface at the point P. The product of the perpendicular component $E_\perp$ and the area A

$$E_\perp A = (E\cos\theta)A = EA\cos\theta = \mathbf{E} \bullet \mathbf{A} = \Phi_E \tag{4.4}$$

is therefore a quantitative measure of the number of lines of **E** passing normally through the entire surface area **A**. If the angle θ in equation 4.1 is zero, then **E** is parallel to the vector **A** and all the lines of **E** pass normally through the surface area A, as seen in figure 4.1b. If the angle θ in equation 4.1 is 90^0 then **E** is perpendicular to the area vector **A**, and none of the lines of **E** pass through the surface A as seen in figure 4.1c. The concept of flux is a very important one, and one that will be used frequently later.

Example 4.1

Electric flux. An electric field of 500 V/m makes an angle of 30.0^0 with the surface vector, which has a magnitude of 0.500 m^2. Find the electric flux that passes through the surface.

Solution

The electric flux passing through the surface is given by equation 4.1 as

$$\Phi_E = \mathbf{E} \bullet \mathbf{A} = EA\cos\theta$$
$$\Phi_E = (500\text{ V/m})(0.500\text{ m}^2)\cos 30.0^0$$
$$\Phi_E = 217\text{ V m}$$

Notice that the unit of electric flux is a volt times a meter.

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4.3 Gauss's Law for Electricity

It was pointed out in section 4.2 that an electric flux was a quantitative measure of the number of electric field lines passing normally through an area. The electric flux was illustrated in figure 4.1 and defined in equation 4.1 as

$$\Phi_E = \mathbf{E} \bullet \mathbf{A} = EA\cos\theta \tag{4.1}$$

Let us now consider the amount of electric flux that emanates from a positive point charge. Figure 4.2 shows a positive point charge surrounded by an imaginary spherical surface called a Gaussian surface. Let us measure the amount of electric flux through the sphere. The direction of the **E** field is different at every point, however, so equation 4.1 cannot be used in its present form. Instead, the spherical

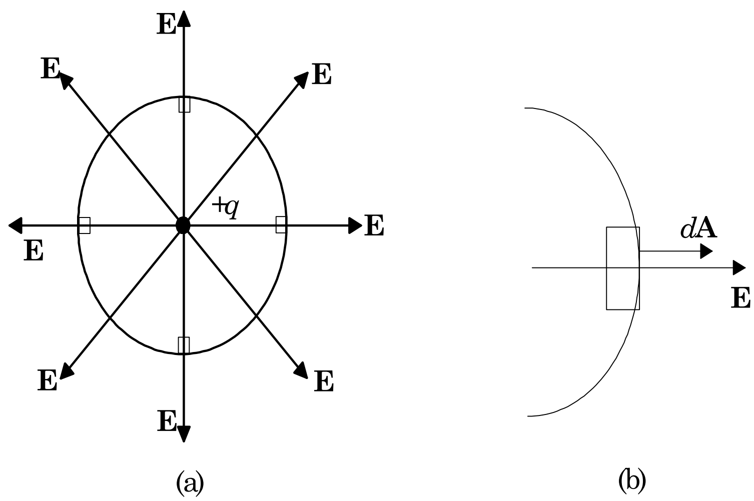

Figure 4.2 Gauss's law for electricity.

surface is broken up into a large number of infinitesimal surface areas $d\mathbf{A}$, figure 4.2(b), and the infinitesimal amount of flux $d\Phi_E$ through each of these little areas is computed, i.e.,

$$d\Phi_E = \mathbf{E} \bullet d\mathbf{A} \tag{4.5}$$

The total flux out of the Gaussian surface becomes the sum or integral of all the infinitesimal fluxes $d\Phi_E$, through all the infinitesimal areas $d\mathbf{A}$, that is

$$\Phi_E = \oint d\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} \tag{4.6}$$

The integral symbol $\oint$ means that the integration is performed over the entire closed surface that the flux is passing through. The electric vector **E** is everywhere radial from the point charge q, and $d\mathbf{A}$ is also everywhere radial, hence

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \oint E\, dA \cos 0^0 \tag{4.7}$$

Thus,

$$\Phi_E = \oint E\, dA \tag{4.8}$$

But it was found in chapter 3 that the electric field of a point charge was given by equation 3-2 as

$$E = kq/r^2 \tag{3-2}$$

Substituting the electric field of a point charge, from equation 3-2, into equation 4.8, we get for the electric flux through the spherical surface

$$\Phi_E = \oint k\frac{q}{r^2}\, dA \tag{4.9}$$

where r is the radius of the spherical Gaussian surface, and for that sphere it is a constant. The terms k and q are also constants and they can, therefore, be taken outside of the integral sign. Thus,

$$\Phi_E = k\frac{q}{r^2} \oint dA \tag{4.10}$$

But the integral of all the elements of area dA is equal to the entire surface area of the sphere. Since the surface area of a sphere is $4\pi r^2$, we have

$$\oint dA = 4\pi r^2$$

Thus, the electric flux emanating from a point charge becomes

$$\Phi_E = k\frac{q}{r^2}4\pi r^2 \tag{4.11}$$

$$\Phi_E = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}4\pi r^2 \tag{4.12}$$

where use has been made of the fact that k in equation 4.11 is equal to $1/(4\pi\varepsilon_o)$. Hence, *the electric flux associated with a point charge is*

$$\Phi_E = \frac{q}{\varepsilon_o} \tag{4.13}$$

Equation 4.13 is Gauss's law for electricity, and it says that the electric flux Φ_E that passes through a surface surrounding the point charge q is a measure of the amount *of charge q contained within the Gaussian surface.* Although equation 4.13 was derived for a point charge, it is true, in general, for any kind of charge distribution. Since Φ_E was initially defined in equation 4.6, it can be combined with equation 4.13 *into the generalization of Gauss's law as*

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o} \tag{4.14}$$

Where q is now the net charge contained within the Gaussian surface. Equation 4.14 was derived on the basis of a point charge. If the charge is distributed over a volume V, with a volume charge density ρ, then the charge q can be written as

$$q = \oint \rho \, dV \tag{4.15}$$

and Gauss's law can also be written as

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{1}{\varepsilon_o} \oint \rho \, dV \tag{4.16}$$

If the total charge q within the Gaussian surface is known, Gauss's law in the form of equation 4.14 can be used. When only the charge density ρ is known, then Gauss's law in the form of equation 4.16 is used.

When Φ_E is a positive quantity, the Gaussian surface surrounds a source of positive charge, and electric flux diverges out of the surface. If the point charge is negative then the electric field would go inward, through the Gaussian surface, to the point charge. The vector **E** would, therefore, make an angle of 180^0 with the area vectors $d\mathbf{A}$ and their dot product would be

$$\mathbf{E} \bullet d\mathbf{A} = E \, dA \cos 180^0 = -E \, dA$$

Thus, the flux passing through the Gaussian surface would be negative. *Hence, whenever Φ_E is negative, the Gaussian surface surrounds a negative charge distribution and the electric flux converges into the Gaussian surface.* If there is no enclosed charge, $q = 0$, and hence, $\Phi_E = 0$. In this case, whatever electric flux enters one part of a Gaussian surface, the same amount must leave somewhere else. These different possibilities are shown for an electric dipole in figure 4.3. Thus, Gaussian surface 1 shows the electric flux diverging from the positive point charge. Gaussian surface 2 shows the electric flux converging into the negative point charge, and Gaussian surface 3 shows no enclosed charge, and the amount of electric field entering normally through the surface is equal to the amount leaving the surface normally.

Let us now consider some examples of the application of Gauss's law to determine the electric field of various symmetric charge distributions.

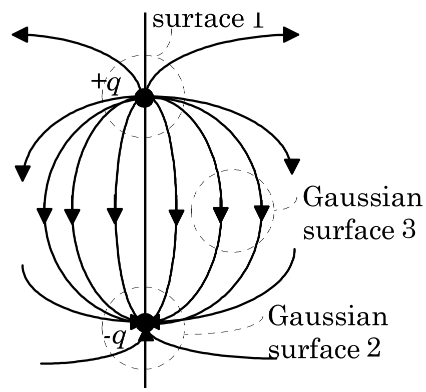

Figure 4.3 Gaussian surfaces and an electric dipole.

4.4 The Electric Field of a Spherically Symmetric Uniform Charge Distribution

Figure 4.4(a) shows a spherically symmetric distribution of total electric charge q. We assume that the distribution is uniform over the sphere of radius R and has a volume charge density ρ (C/m^3). Let us find the electric field **E** at (a) points outside

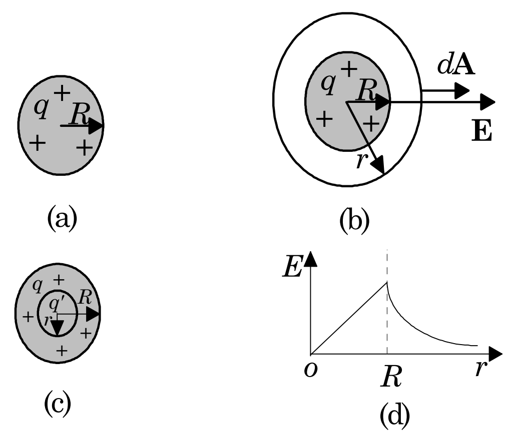

Figure 4.4 A spherically symmetric distribution of electric charge.

the sphere, that is for $r > R$ and (b) for points inside the sphere, that is for $r < R$.

(a) The electric field outside the uniform charge distribution.

To determine the electric field outside the spherical distribution of charge we first must draw a Gaussian surface. The shape of the Gaussian surface depends on the symmetry of the problem. Since the inherent symmetry in this problem is spherical, we draw a spherical Gaussian surface of radius r around the spherical charge distribution as shown in figure 4.4b. Gauss's law, equation 4.14, is applied to this spherical surface as

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o} \tag{4.14}$$

From the symmetry of the problem, the electric field **E** must be radially outward from the charge distribution. (Since the charge distribution is spherical, the most probable direction of the electric field is also spherical, that is, there is no reason to assume that the electric field is more likely to point in one direction than another, that is, there is no reason to assume that the electric field should point in the x-direction rather than in some other direction.) The element of area d**A** of the spherical Gaussian surface is perpendicular to the spherical surface and points outward in the radial direction. Hence the angle between the electric field vector **E** and the element of area vector d**A** is 0^o. Gauss's law therefore becomes

$$\oint \boldsymbol{E} \bullet d\boldsymbol{A} = \oint E\, dA \cos 0^o = \frac{q}{\varepsilon_o}$$

$$\oint E\, dA = \frac{q}{\varepsilon_o} \tag{4.17}$$

But again, from the symmetry of the problem, the magnitude of the electric field will be a constant value at the location r of the Gaussian surface. (Again, there is no reason to assume that the magnitude of the electric field is greater in the x-direction than in the y-direction, or any other direction. Hence, because of the spherical nature of the charge distribution the magnitude of the electric field will be constant anywhere on the Gaussian surface.) Therefore, the electric field E can be taken outside of the integral to yield

$$E \oint dA = \frac{q}{\varepsilon_o}$$

But $\oint dA$ is the sum of all the infinitesimal elements of area distributed over the entire surface of the Gaussian sphere and is equal to the total surface area of the Gaussian sphere. Since the surface area of a sphere is $4\pi r^2$ we have

$$\oint dA = 4\pi r^2 \tag{4.18}$$

Therefore Gauss's law becomes

$$E \oint dA = E 4\pi r^2 = \frac{q}{\varepsilon_o}$$

$$E = \frac{1}{4\pi r^2}\frac{q}{\varepsilon_o} = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}$$

But from Coulomb's law

$$\frac{1}{4\pi\varepsilon_o} = k$$

Hence, the electric field E outside the spherical charge distribution is found to be

$$E = k\frac{q}{r^2} \tag{4.19}$$

Thus outside a spherical charge distribution q, the electric field looks like the electric field of a point charge.

(b) The electric field inside the uniform charge distribution.

To find the electric field inside the spherical charge distribution, we draw a new spherical Gaussian surface of radius r that encloses an amount of charge q' as shown in figure 4.4c. The amount of flux passing through this Gaussian surface is found by modifying Gauss's law, equation 4.14, as

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q'}{\varepsilon_o} \tag{4.20}$$

where q' is the amount of charge now contained within the new Gaussian surface and is less than the total spherical charge q. If the charge distribution is uniform, then the volume charge density ρ is the same for the total charge as it is for the charge enclosed in the new Gaussian surface. That is,

$$\rho = \frac{q}{V} = \frac{q'}{V'} \tag{4.21}$$

where q is the total charge enclosed in the total volume V of charge, while q' is the charge enclosed in the volume V' of charge. From equation 4.21 the amount of charge enclosed within the new Gaussian surface can be written as

$$q' = \frac{V'}{V}q \tag{4.22}$$

But the volume of the total spherical charge is

$$V = \tfrac{4}{3}\pi R^3$$

and the volume of the charge in the new Gaussian surface is given by

$$V' = \tfrac{4}{3}\pi r^3$$

Replacing these volumes in equation 4.22 gives

$$q' = \frac{V'}{V}q = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}q$$

and the charge q' contained within the Gaussian surface is

$$q' = \frac{r^3}{R^3}q \tag{4.23}$$

Replacing the enclosed charge q', equation 4.23, into Gauss's law, equation 4.20 we obtain

$$\oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q'}{\varepsilon_o} = \frac{r^3}{\varepsilon_o R^3} q \tag{4.24}$$

Again, from the symmetry of the problem, the electric field **E** must be radially outward from the charge distribution. The element of area d**A** of the spherical Gaussian surface is perpendicular to the spherical surface and also points outward in the radial direction. Hence the angle between the electric field vector **E** and the element of area vector d**A** is 0^o. Gauss's law therefore becomes

$$\begin{gathered}\oint \boldsymbol{E} \bullet d\boldsymbol{A} = \oint E\, dA \cos 0^o = \frac{r^3}{\varepsilon_o R^3} q \\ \oint E\, dA = \frac{r^3}{\varepsilon_o R^3} q\end{gathered} \tag{4.25}$$

But again, from the symmetry of the problem, the magnitude of the electric field will be a constant value at the location r of the Gaussian surface. Therefore, the electric field E can be taken outside of the integral to yield

$$E \oint dA = \frac{r^3}{\varepsilon_o R^3} q$$

But $\oint dA$ is the sum of all the infinitesimal elements of area distributed over the entire surface of the Gaussian sphere and hence is equal to the total surface area of the Gaussian sphere. Since the surface area of a sphere is $4\pi r^2$ we have

$$\oint dA = 4\pi r^2$$

Therefore Gauss's law becomes

$$\begin{gathered}E \oint dA = E(4\pi r^2) = \frac{r^3}{\varepsilon_o R^3} q \\ E = \frac{1}{4\pi r^2} \frac{r^3}{\varepsilon_o R^3} q = \frac{1}{4\pi\varepsilon_o} \frac{r}{R^3} q\end{gathered}$$

But from Coulomb's law

$$\frac{1}{4\pi\varepsilon_o} = k$$

Hence, the electric field E inside the spherical charge distribution is found to be

$$E = k \frac{r}{R^3} q \tag{4.26}$$

Equation 4.26 says that the electric field intensity E inside the charge distribution is directly proportional to the radial distance r from the center of the charge distribution, whereas the electric field E outside the charge distribution, equation

4.19, is inversely proportional to the square of the radial distance r from the center of the charge distribution.

Note that when r, the radius of the Gaussian surface, is equal to R, the radius of the charge distribution, in equation 4.26 (for the electric field intensity E inside the charge distribution), the electric field intensity E at the surface of the charge distribution becomes

$$E = k\frac{r}{R^3}q = k\frac{R}{R^3}q = k\frac{q}{R^2}$$

Also note that when r the radius of the Gaussian surface is equal to R the radius of the charge distribution in equation 4.19 (the electric field intensity E outside the charge distribution), the electric field intensity E at the surface of the charge distribution becomes

$$E = k\frac{q}{r^2} = k\frac{q}{R^2}$$

which shows that the two results agree at the edge of the charge distribution as they must.

A plot of the electric field intensity E as a function of the radial distance r is shown in figure 4.4(d). It shows the linear relation between E and r within the charge distribution and the inverse square relationship outside the charge distribution.

Example 4.2

Electric field of a spherical charge distribution. A spherical charge distribution of 7.52×10^{-5} C has a radius of 5.25×10^{-3} m. Find the electric field intensity E at (a) 5.00 cm from the charge and (b) 3.25×10^{-3} m inside the charge.

Solution

(a) The electric field intensity E outside a uniform charge distribution is found from equation 4.19 as

$$E = k\frac{q}{r^2}$$

$$E = (9.00 \times 10^9 \text{ N m}^2/\text{C}^2)\frac{(7.52 \times 10^{-5} C)}{(0.050 \text{ m})^2}$$

$$E = 2.71 \times 10^8 \text{ N/C}$$

(b) The electric field intensity E inside a uniform charge distribution is found from equation 4.26 as

$$E = k\frac{r}{R^3}q$$

$$E = (9.00 \times 10^9 \text{ N m}^2/\text{C}^2)\frac{(3.25 \times 10^{-3} C)}{(5.25 \times 10^{-3} \text{ m})^3}(7.52 \times 10^{-5} C)$$

$$E = 1.52 \times 10^{10} \text{ N/C}$$

To go to this Interactive Example click on this sentence.

4.5 The Electric Field of a Spherically Symmetric Nonuniform Charge Distribution

This is very similar to the problem studied in section 4.4 except that the charge distribution is nonuniform. A picture of the charge distribution is the same as in figure 4.4(a) except that the charge distribution is nonuniform. Let us assume that the charge distribution over the sphere of radius R is given by

$$\rho = \rho_o / r \tag{4.27}$$

where ρ_o is a constant having units of C/m^2 and r is the distance from the center of the charge distribution. *Notice that in this problem the charge density varies inversely with the distance r from the center 0 of the charge distribution, to R the radius of the charge distribution, whereas in section 4.4 the charge density was constant throughout the entire charge distribution.* Let us find the electric field **E** at (a) points outside the sphere, that is for $r > R$ and (b) for points inside the sphere, that is for $r < R$.

(a) The electric field outside the nonuniform charge distribution.

To determine the electric field outside the nonuniform spherical distribution of charge is essentially the same problem as in section 4.4 for a uniform spherical distribution of charge. A spherical Gaussian surface of radius r is drawn around the spherical charge distribution just as it was in figure 4.4(b). Gauss's law, equation 4.14, is applied to this spherical surface

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o} \tag{4.14}$$

Again, from the symmetry of the problem, the electric field **E** must be radially outward from the charge distribution and the element of area d**A** of the spherical Gaussian surface also points outward in the radial direction. Hence the angle between the electric field vector **E** and the element of area vector d**A** is 0^o. Gauss's law therefore becomes

$$\oint \boldsymbol{E} \bullet d\boldsymbol{A} = \oint E\, dA \cos 0^o = \frac{q}{\varepsilon_o}$$

$$\oint E\, dA = \frac{q}{\varepsilon_o} \tag{4.28}$$

But again, from the symmetry of the problem, the magnitude of the electric field will be a constant value at the location r of the Gaussian surface. Therefore, the electric field E can be taken outside of the integral to yield

$$E \oint dA = \frac{q}{\varepsilon_o}$$

But $\oint dA$ is the sum of all the infinitesimal elements of area distributed over the entire surface of the Gaussian sphere and is equal to the total surface area of the sphere. Since the surface area of a sphere is $4\pi r^2$ we have

$$\oint dA = 4\pi r^2 \tag{4.29}$$

Therefore Gauss's law becomes

$$E \oint dA = E(4\pi r^2) = \frac{q}{\varepsilon_o}$$
$$E = \frac{1}{4\pi r^2}\frac{q}{\varepsilon_o} = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}$$

But from Coulomb's law

$$\frac{1}{4\pi\varepsilon_o} = k$$

Hence, the electric field E outside the spherical charge distribution is found to be

$$E = k\frac{q}{r^2} \tag{4.30}$$

Notice that q is the total amount of charge contained within the nonuniform spherical charge distribution and when calculating the electric field outside the charge distribution it is not necessary to know how that charge is distributed. *Thus outside a spherical nonuniform charge distribution, the electric field looks like the electric field of a spherical uniform charge distribution, which looks like the electric field of a point charge.*

To obtain the value of the total charge q contained in the nonuniform spherical charge distribution when only the volume charge density is known, the total charge is obtained from

$$q = \int \rho \, dV \tag{4.31}$$

But the charge density ρ is not a constant but is given by equation 4.27 as $\rho = \rho_o/r$. Replacing this value in equation 4.31 we get

$$q = \int \frac{\rho_o}{r} dV \tag{4.32}$$

Now dV is the element of volume of a sphere and is given by

$$dV = 4\pi r^2 \, dr \tag{4.33}$$

(The element of volume of a sphere can be visualized as a sphere of area $4\pi r^2$ times a small element of radial distance dr. The product of the area of the sphere and the element of radial distance will generate a small element of spherical volume.) Hence the *total* charge q contained in the nonuniform distribution is

$$q = \int \frac{\rho_o}{r} dV = \int \frac{\rho_o}{r} 4\pi r^2 dr \tag{4.34}$$
$$q = \int \rho_o 4\pi r \, dr$$
$$q = 4\pi\rho_o \int_o^R r \, dr$$
$$q = 4\pi\rho_o \frac{r^2}{2} \Big|_o^R$$
$$q = 2\pi\rho_o R^2 \tag{4.35}$$

Equation 4.35 gives the total amount of charge that is contained within the nonuniform charge distribution and can be easily calculated when the constant ρ_o *and the radius of the charge R is known.*

(b) The electric field inside the nonuniform spherical charge distribution.

To find the electric field inside the nonuniform spherical charge distribution is also similar to the electric field inside the uniform spherical charge distribution. A new spherical Gaussian surface of radius $r < R$ is drawn within the nonuniform charge distribution as it was in figure 4.4(c). The amount of flux passing through this Gaussian surface is found by Gauss's law, equation 4.14, as

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q'}{\varepsilon_o} \tag{4.36}$$

where q' is the amount of charge contained within the new Gaussian surface and is less than the total spherical charge q. To obtain the value of the charge q' contained within the spherical Gaussian surface we modify equation 4.31 as

$$q' = \int \rho dV \tag{4.37}$$

But the charge density ρ is not a constant but is given by equation 4.27 as $\rho = \rho_o/r$. Replacing this value in equation 4.37 we get

$$q' = \int \frac{\rho_o}{r} dV$$

Now dV is the element of volume of the spherical Gaussian surface within the charge distribution and is given by

$$dV = 4\pi r^2 \, dr$$

Where r is now the radius of the Gaussian surface. Hence the total charge q' contained in this Gaussian surface is

$$q' = \int \frac{\rho_o}{r} dV = \int \frac{\rho_o}{r} 4\pi r^2 dr$$
$$q' = \int \rho_o 4\pi r \, dr$$
$$q' = 4\pi\rho_o \int_o^r r \, dr$$

$$q' = 4\pi\rho_o \frac{r^2}{2} \Big|_o^r$$
$$q' = 2\pi\rho_o r^2 \tag{4.38}$$

Equation 4.38 gives the total amount of charge that is contained within the Gaussian surface that lies within the nonuniform charge distribution.

Replacing the enclosed charge q', equation 4.38, into Gauss's law, equation 4.36 we get

$$\oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q'}{\varepsilon_o} = \frac{2\pi\rho_o r^2}{\varepsilon_o} \tag{4.39}$$

Again, from the symmetry of the problem, the electric field **E** must be radially outward from the charge distribution. The element of area d**A** of the spherical Gaussian surface is perpendicular to the spherical surface and also points outward in the radial direction. Hence the angle between the electric field vector **E** and the element of area vector d**A** is 0^o. Gauss's law therefore becomes

$$\oint \boldsymbol{E} \bullet d\boldsymbol{A} = \oint E\, dA \cos 0^o = \frac{2\pi\rho_o r^2}{\varepsilon_o}$$
$$\oint E\, dA = \frac{2\pi\rho_o r^2}{\varepsilon_o}$$

But again, from the symmetry of the problem, the magnitude of the electric field E will be a constant value at the location r of the Gaussian surface and can be taken outside of the integral to yield

$$E \oint dA = \frac{2\pi\rho_o r^2}{\varepsilon_o}$$

But $\oint dA$ is the sum of all the infinitesimal elements of area distributed over the entire surface of the Gaussian sphere and hence is equal to the total surface area of the Gaussian sphere. Since the surface area of a sphere is $4\pi r^2$ we have

$$\oint dA = 4\pi r^2$$

Therefore Gauss's law becomes

$$E \oint dA = E(4\pi r^2) = \frac{2\pi\rho_o r^2}{\varepsilon_o}$$

Hence, the electric field E inside the spherical charge distribution is found to be

$$E = \frac{\rho_o}{2\varepsilon_o} \tag{4.40}$$

Equation 4.40 says that the electric field intensity E is a constant within the charge distribution, whereas the electric field E outside the charge distribution, equation 4.30, is inversely proportional to the square of the radial distance r from the center of *the charge distribution*. (It should be noted that not all nonuniform charge distributions will give a constant electric field within the charge distribution. It occurred here because the charge density was given by $\rho = \rho_o/r$. If the charge density had been given by $\rho = \rho_o/r^2$ the electric field within the charge distribution would not be constant.)

Example 4.3

The electric field of a nonuniform spherical charge distribution. A nonuniform spherical charge distribution has a volume charge density $\rho = \rho_o/r$, where $\rho_o = 2.50 \times 10^{-6}$ C/m^2, and has a radius of 7.87×10^{-3} m. Find the electric field intensity E at (a) $r = 2.78$ m and (b) $r = 3.55 \times 10^{-3}$ m.

Solution

(a) The electric field E outside the spherical nonuniform charge distribution is given by equation 4.30 as

$$E = k\frac{q}{r^2} \tag{4.30}$$

However, before we can solve for E we must determine the charge q. To do this we use equation 4.35 as

$$q = 2\pi\rho_o R^2 \tag{4.35}$$

$$q = 2\pi(2.50 \times 10^{-6}\ \text{C/m}^2)(7.87 \times 10^{-3}\ \text{m})^2$$

$$q = 9.73 \times 10^{-10}\ \text{C}$$

Replacing this value of q back into equation 4.30 gives

$$E = k\frac{q}{r^2} \tag{4.30}$$

$$E = (9.00 \times 10^9\ \text{N m}^2/\text{C}^2)\frac{(9.73 \times 10^{-10} C)}{(2.78\ \text{m})^2}$$

$$E = 1.13\ \text{N/C}$$

(b) The electric field E inside the spherical nonuniform charge distribution is given by equation 4.40 as

$$E = \frac{\rho_o}{2\varepsilon_o} \tag{4.40}$$

$$E = \frac{2.50 \times 10^{-6} C/m^2}{2(8.85 \times 10^{-12} C^2/N\,m^2)}$$

$$E = 1.41 \times 10^5\ \text{N/C}$$

To go to this Interactive Example click on this sentence.

4.6 The Electric Field of an infinite line of Charge

Using Gauss's law, let us determine the electric field intensity **E** at a distance r from an infinite line of charge lying on the x-axis. A small portion of the infinite line of charge is shown in figure 4.5(a). We assume that the charge is uniformly

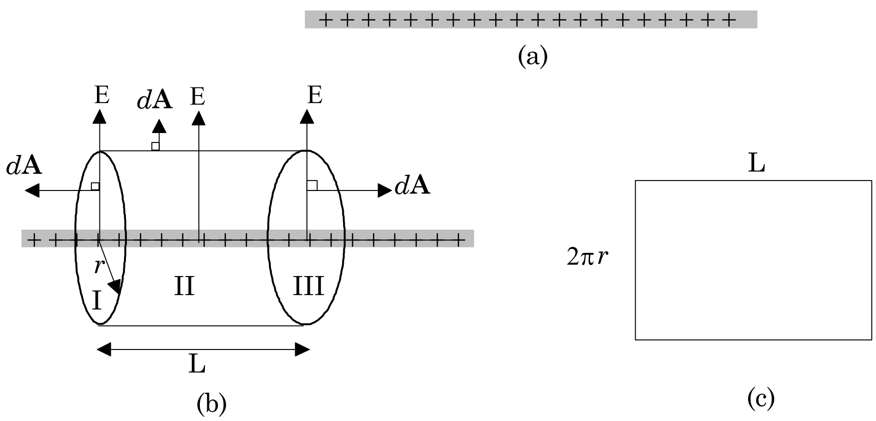

Figure 4.5 The electric field of an infinite line of charge.

distributed and has a linear charge density λ. (Recall that the linear charge density λ is the charge per unit length, that is, $\lambda = q/L$.) To determine the electric field we must first draw a Gaussian surface. In the previous examples of Gauss's law we used a spherical Gaussian surface because of the spherical symmetry of the problem. However, the line of charge does not have any spherical symmetry and a spherical Gaussian surface cannot be used. The line of charge does have a cylindrical symmetry however. Therefore, we draw a cylindrical Gaussian surface of radius r and length L around the infinite line of charge as shown in figure 4.5b. Gauss's law for the total flux emerging from the Gaussian cylinder, equation 4.14, is applied.

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o}$$

The integral in Gauss's law is over the entire Gaussian surface. We can break the entire cylindrical Gaussian surface into three surfaces. Surface I is the end cap on the left-hand side of the cylinder, surface II is the main cylindrical surface, and surface III is the end cap on the right-hand side of the cylinder as shown in figure 4.5b. The total flux Φ through the entire Gaussian surface is the sum of the flux through each individual surface. That is,

$$\Phi = \Phi_{\mathrm{I}} + \Phi_{\mathrm{II}} + \Phi_{\mathrm{III}} \tag{4.41}$$

where

Φ_{I} is the electric flux through surface I
Φ_{II} is the electric flux through surface II
Φ_{III} is the electric flux through surface III

Hence, Gauss's law becomes

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o} \tag{4.42}$$

Notice that the normal integral sign $\int$ is now used instead of, $\oint$ because each integration is now over only a portion of the entire closed surface. Along cylindrical surface I, d**A** is everywhere perpendicular to the surface and points toward the left as shown in figure 4.5b. The electric field intensity vector **E** lies in the plane of the end cylinder cap and is everywhere perpendicular to the surface vector d**A** of surface I and hence $\theta = 90^0$. Therefore the electric flux through surface I is

$$\Phi_I = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I E\, dA \cos\theta = \int_I E\, dA \cos 90^0 = 0$$

Surface II is the cylindrical surface itself. As can be seen in figure 4.5(b), **E** is everywhere perpendicular to the cylindrical surface pointing outward, and the area vector d**A** is also perpendicular to the surface and also points outward. Hence **E** and d**A** are parallel to each other and the angle θ between **E** and d**A** is zero. Hence, the flux through surface II is

$$\Phi_{II} = \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{II} E\, dA \cos\theta = \int_{II} E\, dA \cos 0^o = \int_{II} E\, dA$$

Along cylindrical surface III, d**A** is everywhere perpendicular to the surface and points toward the right as shown in figure 4.5(b). The electric field intensity vector **E** lies in the plane of the end cylinder cap and is everywhere perpendicular to the surface vector d**A** of surface III and therefore $\theta = 90^0$. Hence the electric flux through surface III is

$$\Phi_{III} = \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{III} E\, dA \cos\theta = \int_{III} E\, dA \cos 90^0 = 0$$

Combining the flux through each portion of the cylindrical surfaces, equation 4.42 becomes

$$\Phi_E = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o}$$

$$\Phi_E = 0 + \int_{II} E\, dA + 0 = \frac{q}{\varepsilon_o}$$

From the symmetry of the problem, the magnitude of the electric field intensity E is a constant for a fixed distance r from the line of charge, that is, there is no reason to assume that E in the z-direction is any different than E in the y-direction. Hence, E can be taken outside the integral sign to yield

$$\int_{II} E\,dA = E\int_{II} dA = \frac{q}{\varepsilon_o}$$

The integral $\int dA$ represents the sum of all the elements of area dA, and that sum is just equal to the total area of the cylindrical surface. It is easier to see the total area if we unfold the cylindrical surface as shown in figure 4.5(c). One length of the surface is L, the length of the cylinder, while the other length is the unfolded circumference $2\pi r$ of the end of the cylindrical surface.[1] The total area A is just the product of the length times the width of the rectangle formed by unfolding the cylindrical surface, that is, $A = (L)(2\pi r)$. Hence, the integral of dA is

$$\int dA = A = (L)(2\pi r)$$

Thus, Gauss's law becomes

$$E\int_{II} dA = E(L)(2\pi r) = \frac{q}{\varepsilon_o}$$

Solving for the magnitude of the electric field E intensity we get

$$E = \frac{q}{2\pi\varepsilon_o rL}$$

But $q/L = \lambda$ the linear charge density. Therefore

$$E = \frac{\lambda}{2\pi\varepsilon_o r}$$

Also, since $k = 1/(4\pi\varepsilon_o)$ then $2k = 1/(2\pi\varepsilon_o)$. Hence the magnitude of the electric field intensity E for an infinite line of charge is

$$E = \frac{2k\lambda}{r} \tag{4.43}$$

Notice that this is the same result we obtained in the last chapter by a direct addition of the electric fields of all the elements of charge. The solution by Gauss's law, however, is much simpler.

Example 4.4

The electric field of an infinite line of charge. An infinite line of charge carries a linear charge density of 2.55×10^{-5} C/m. Find the electric field E at a distance of 15.5 cm from the line of charge.

[1]You can try this by taking a sheet of paper and rolling it into a cylinder. The area of that cylinder is found by unrolling the piece of paper into its normal rectangular shape. The area of the rectangle is the product of its length times its width. The length is just the length of the page. The width was the circular portion of the cylinder. But the circumference of a circle is $2\pi r$. So the unrolled width of the paper is equal to the circumference of the circle of the rolled paper.

Solution

The electric field intensity E for an infinite line of charge is found by equation 4.43 as

$$E = \frac{2k\lambda}{r}$$

$$E = \frac{2(9.00 \times 10^{9} N\ m^2/C^2)(2.55 \times 10^{-5} C/m)}{0.155\ m}$$

$$E = 2.96 \times 10^6 \text{ N/C}$$

To go to this Interactive Example click on this sentence.

4.7 The Electric Field of an Infinite Plane Sheet of Charge

Using Gauss's law, let us determine the electric field intensity **E** at a distance r in front of an infinite plane sheet of charge as shown in figure 4.6(a). We assume that the charge is uniformly distributed over the sheet and has a surface charge density σ. (Recall that the surface charge density σ is the charge per unit area, that is, $\sigma = q/A$.) To determine the electric field we must first draw a Gaussian surface. The type of Gaussian surface drawn depends on the symmetry of the problem. We draw a cylindrical Gaussian surface through the sheet of charge

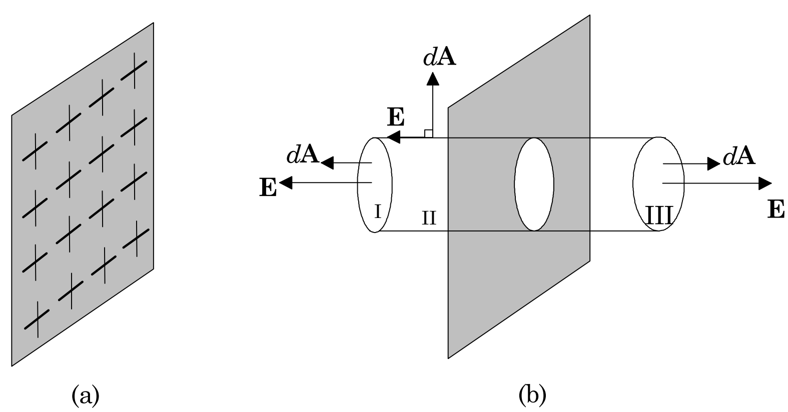

Figure 4.6 The electric field in front of an infinite plane sheet of charge.

as shown in figure 4.6(b). Gauss's law for the total flux emerging from the Gaussian cylinder, equation 4.14, is

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o}$$

The integral in equation 4.14 is over the entire Gaussian surface. As before, we can break the entire surface of the cylinder into three surfaces. Surface I is the end cap on the left-hand side of the cylinder, surface II is the main cylindrical surface itself, and surface III is the end cap on the right-hand side of the cylinder as shown in figure 4.6(b). The total flux Φ_E through the Gaussian surface is the sum of the flux through each individual surface. That is,

$$\Phi = \Phi_I + \Phi_{II} + \Phi_{III}$$

where

Φ_I is the electric flux through surface I
Φ_{II} is the electric flux through surface II
F_{III} is the electric flux through surface III

Gauss's law becomes

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o} \tag{4.44}$$

Surface I is the end cap on the left-hand side of the cylinder and as can be seen from figure 4.6b, the electric field vector **E** points toward the left and since the area vector d**A** is perpendicular to the surface pointing outward it also points to the left. Hence **E** and d**A** are parallel to each other and the angle θ between **E** and d**A** is zero. Hence, the flux through surface I is

$$\Phi_I = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I E\, dA \cos\theta = \int_I E\, dA \cos 0^0 = \int_I E\, dA$$

Along cylindrical surface II, d**A** is everywhere perpendicular to the surface as shown in figure 4.6(b). **E** lies in the cylindrical Gaussian surface and is everywhere perpendicular to the surface vector d**A** on surface II and therefore $\theta = 90^0$. Hence the electric flux through surface II is

$$\Phi_{II} = \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{II} E\, dA \cos\theta = \int_{II} E\, dA \cos 90^o = 0$$

Surface III is the end cap on the right-hand side of the cylinder and as can be seen from figure 4.6b the electric field vector **E** points toward the right and since the area vector d**A** is perpendicular to the surface pointing outward it also points to the right. Hence **E** and d**A** are parallel to each other and the angle θ between **E** and d**A** is zero. Hence, the flux through surface III is

$$\Phi_{III} = \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{III} E\, dA \cos\theta = \int_{III} E\, dA \cos 0^o = \int_{III} E\, dA$$

The total flux through the Gaussian surface is equal to the sum of the fluxes through the individual surfaces. Hence, Gauss's law becomes

$$\Phi_E = \Phi_I + \Phi_{II} + \Phi_{III}$$
$$\Phi_E = \int_I E\, dA + 0 + \int_{III} E dA = \frac{q}{\varepsilon_o}$$

But E has the same magnitude in integral I and III and hence the total flux can be written as

$$\Phi_E = 2\int E\, dA = \frac{q}{\varepsilon_o}$$

But the magnitude of E is a constant in the integral and can be factored out of the integral to yield

$$2E\int dA = \frac{q}{\varepsilon_o}$$

But $\int dA = A$ the area of the end cap of the Gaussian cylinder, hence

$$2E\, A = \frac{q}{\varepsilon_o}$$

A is the magnitude of the area of the Gaussian end cap surface and q is the charge enclosed within that Gaussian surface. Solving for the electric field intensity

$$E = \frac{q}{2\varepsilon_o A} \tag{4.45}$$

But the surface charge density σ is defined as the charge per unit area, i.e.,

$$\sigma = q\,/A \tag{4.46}$$

Combining equation 4.46 with 4.45, gives for the electric field intensity

$$E = \frac{\sigma}{2\varepsilon_o} \tag{4.47}$$

Thus, the electric field in front of an infinite sheet of charge is given by equation 4.47 in terms of the surface charge density and the permittivity of free space. Notice that the electric field is a constant depending only upon the surface charge density σ and not on the distance from the sheet of charge. Also note that in many practical problems with a finite sheet of charge, we can use the result of the infinite sheet of charge as a good approximation if the electric field is found very close to the finite sheet of charge. At a very close distance, the finite sheet of charge can look like an infinite sheet of charge. Finally note that this is the same result we obtained in equation 3-97 of the last chapter. However the solution obtained by Gauss's law is much simpler.

Example 4.5

The electric field of an infinite sheet of charge. Find the electric field in front of an infinite sheet of charge carrying a surface charge density of 3.58×10^{-12} C/m^2.

Solution

The electric field in front of an infinite sheet of charge is found from equation 4.47 as

$$E = \frac{\sigma}{2\varepsilon_o}$$

$$E = \frac{3.58 \times 10^{-12}\, C/m^2}{2(8.85 \times 10^{-12}\, C^2/N\, m^2)}$$

$$E = 0.202 \text{ N/C}$$

To go to this Interactive Example click on this sentence.

4.8 The Electric Field Inside a Conducting Body

Let us take a spherical conductor, such as a solid aluminum ball of radius R. A positively charged rod is touched to the sphere giving it a positive charge q. Let us find the electric field intensity **E** inside the spherical conductor. When the rod is touched to the sphere, positive charge is distributed over the sphere. But since like charges repel each other there will be a force of repulsion on each of these charges tending to push them away from each other. Since the body is a conducting body, the charges are free to move and because of the repulsion, move as far away from each other as possible. However, the maximum distance that they can move from each other is just to the surface of the sphere. Hence, the charge q becomes distributed over the surface of the sphere. That is, there is no charge within the sphere itself. To determine E inside the conducting sphere, we draw a spherical Gaussian surface of radius r within the conducting sphere as shown in figure 4.7. Applying Gauss's law to determine the flux through the Gaussian sphere we get

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o}$$

But since we have just shown that $q = 0$ inside the sphere, Gauss's law becomes

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o} = 0$$

$$\oint \boldsymbol{E} \bullet d\boldsymbol{A} = 0 \tag{4.48}$$

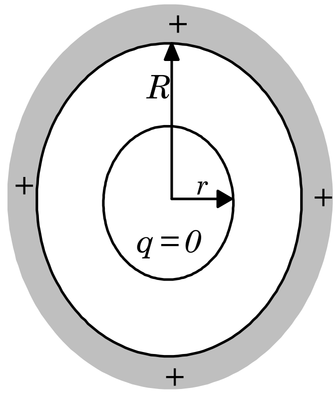

Figure 4.7 The electric field within a conducting body.

The only way that this can be true in general is for

$$E = 0 \tag{4.49}$$

Hence the electric field within a conductor is always equal to zero.

4.9 The Electric Field in front of an Infinite Charged Conducting Plate

Let us determine the electric field in front of an infinite charge conducting plate, carrying a surface charge density σ. A portion of the infinite plate is shown in figure 4.8(a). This problem is similar to the one studied in section 4.7 but there it was a

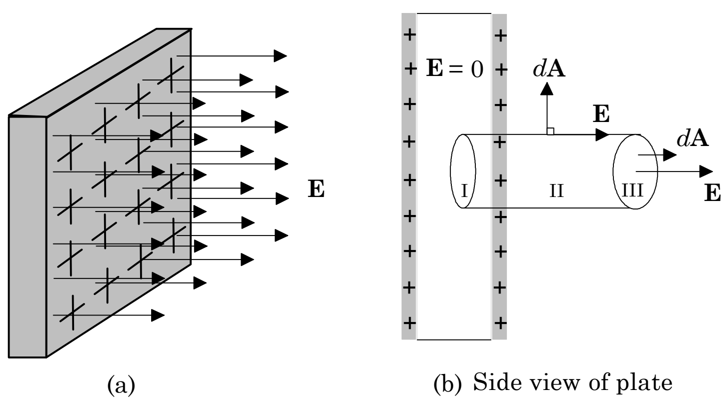

Figure 4.8 The electric field in front of an infinite charged conducting plate.

sheet of charge, here it is charge on a conducting plate. To find the electric field outside of the plate we start with Gauss's law. As in all the previous problems, we start by drawing a Gaussian surface. For the symmetry of this problem we pick a cylinder for the Gaussian surface as shown in figure 4.8(b). Notice that one end of the Gaussian cylinder lies within the conducting plate. Gauss's law is given by equation 4.14 as

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o} \tag{4.14}$$

The sum in equation 4.14 is over the entire Gaussian surface. As in the previous sections, we can break the entire surface of the cylinder into three surfaces. Surface I is the end cap on the left-hand side of the cylinder, surface II is the main cylindrical surface, and surface III is the end cap on the right-hand side of the cylinder as shown in figure 4.8(b). The total flux Φ through the Gaussian surface is the sum of the flux through each individual surface. That is,

$$\Phi = \Phi_I + \Phi_{II} + \Phi_{III}$$

where

Φ_I is the electric flux through surface I
Φ_{II} is the electric flux through surface II
Φ_{III} is the electric flux through surface III

Gauss's law becomes

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o}$$

Because the plate is a conducting body, E = 0 inside the conducting body as was shown in section 4.8, and *all charge must reside on the outer surface of the conducting body*. Since Gaussian surface I lies within the conducting body the electric field on Gaussian surface I is zero. Hence the flux through surface I is,

$$\Phi_I = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I E\, dA \cos\theta = \int_I (0)\ dA \cos\theta = 0$$

Along cylindrical surface II, $d\mathbf{A}$ is everywhere perpendicular to the surface as shown in figure 4.8(b). **E** lies in Gaussian surface II and is everywhere perpendicular to the surface vector $d\mathbf{A}$ on surface II and therefore $\theta = 90^0$. Hence the electric flux through surface II is

$$\Phi_{II} = \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{II} E\, dA \cos\theta = \int_{II} E\, dA \cos 90^o = 0$$

Surface III is the end cap on the right-hand side of the cylinder and as can be seen from figure 4.8(b), **E** points toward the right and since the area vector $d\mathbf{A}$ is perpendicular to the surface pointing outward it also points to the right. Hence **E** and $d\mathbf{A}$ are parallel to each other and the angle θ between **E** and $d\mathbf{A}$ is zero. Hence, the flux through surface III is

$$\Phi_{III} = \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{III} E\, dA \cos\theta = \int_{III} E\, dA \cos 0^o = \int_{III} E\, dA$$

The total flux through the Gaussian surface is equal to the sum of the fluxes through the individual surfaces. Hence, Gauss's law becomes

$$\Phi = \Phi_{\mathrm{I}} + \Phi_{\mathrm{II}} + \Phi_{\mathrm{III}}$$
$$\Phi_E = 0 + 0 + \int_{III} E dA = \frac{q}{\varepsilon_o}$$

But E is constant in every term of the sum in integral III and can be factored out of the integral giving

$$\Phi_E = E \int dA = \frac{q}{\varepsilon_o}$$

But $\int dA = A$ the area of the end cap, hence

$$EA = \frac{q}{\varepsilon_o}$$

A is the magnitude of the area of Gaussian surface III and q is the charge enclosed within Gaussian surface III. Solving for the electric field in front of the conducting plate gives

$$E = \frac{q}{\varepsilon_o A} \tag{4.50}$$

Instead of describing the electric field in terms of the charge on Gaussian surface III and the area of Gaussian surface III, it is convenient to express this result in terms of the surface charge density of the entire plate. The surface charge density σ is defined as the charge per unit area, i.e.,

$$\sigma = q/A \tag{4.51}$$

If the surface charge density is uniform, the surface charge density of Gaussian surface III is the same as the surface charge density of the entire plate. Hence, combining equation 4.51 with 4.50, gives for the electric field

$$E = \frac{\sigma}{\varepsilon_o} \tag{4.52}$$

Thus, the electric field in front of an infinite conducting plate is given by equation 4.52 in terms of the surface charge density σ on the plate and the permittivity of free space ε_o.

Example 4.6

The electric field of an infinite conducting plate. Find the electric field in front of an infinite conducting plate carrying a surface charge density of 8.70×10^{-7} C/m² on the plate.

Solution

The electric field in front of the conducting plate is found from equation 4.52 as

$$E = \frac{\sigma}{\varepsilon_o}$$

$$E = \frac{8.70 \times 10^{-10} \text{ C/m}^2}{\left(8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2\right)}$$

$$E = 0.983 \text{ N/C}$$

To go to this Interactive Example click on this sentence.

4.10 The Electric Field Between Two Oppositely Charged Parallel Conducting Plates

Let us determine the electric field between the two oppositely charged conducting plates, shown in figure 4.9(a), by Gauss's law. Note that this problem is very similar

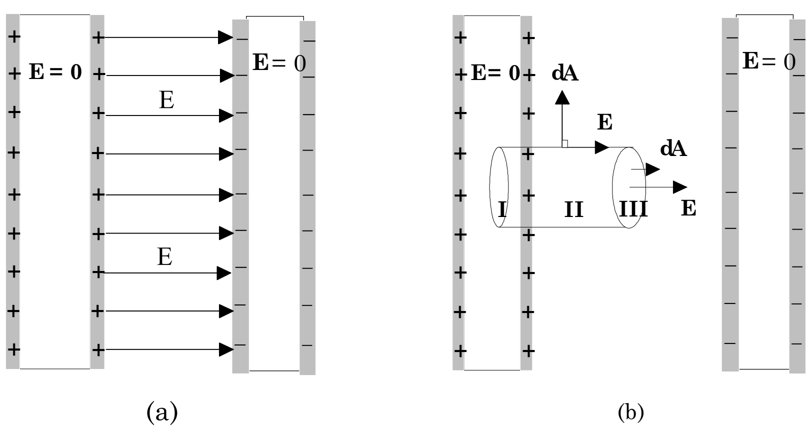

Figure 4.9 The electric field between two oppositely charged conducting plates.

to the problem in section 4.9. We start by drawing a Gaussian surface. For the symmetry of this problem we pick a cylinder for the Gaussian surface as shown in the diagram. One end of the Gaussian surface lies within one of the parallel conducting plates while the other end lies in the region between the two conducting plates where we wish to find the electric field. Gauss's law is given by equation 4.14 as

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o} \tag{4.14}$$

The sum in equation 4.14 is over the entire Gaussian surface. As in the previous sections, we can break the entire surface of the cylinder into three surfaces. Surface I is the end cap on the left-hand side of the cylinder, surface II is the main cylindrical surface, and surface III is the end cap on the right-hand side of the cylinder as shown in figure 4.9(b). The total flux Φ through the Gaussian surface is the sum of the flux through each individual surface. That is,

$$\Phi = \Phi_{\mathrm{I}} + \Phi_{\mathrm{II}} + \Phi_{\mathrm{III}}$$

where

Φ_{I} is the electric flux through surface I
Φ_{II} is the electric flux through surface II
Φ_{III} is the electric flux through surface III

Gauss's law becomes

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o}$$

Because the plate is a conducting body, all charge must reside on its outer surface, hence E = 0, inside the conducting body. Since Gaussian surface I lies within the conducting body the electric field on surface I is zero. Hence the flux through surface I is,

$$\Phi_I = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I E\, dA \cos\theta = \int_I (0)\; dA \cos\theta = 0$$

Along cylindrical surface II, $d\mathbf{A}$ is everywhere perpendicular to the surface as shown in figure 4.9(b). $\mathbf{E}$ lies in the cylindrical surface, pointing toward the right, and is everywhere perpendicular to the surface vector $d\mathbf{A}$ on surface II, and therefore $\theta = 90^0$. Hence the electric flux through surface II is

$$\Phi_{II} = \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{II} E\, dA \cos\theta = \int_{II} E\, dA \cos 90^o = 0$$

Surface III is the end cap on the right-hand side of the cylinder and as can be seen from figure 4.9b $\mathbf{E}$ points toward the right and since the area vector $d\mathbf{A}$ is perpendicular to the surface pointing outward it also points to the right. Hence $\mathbf{E}$ and $d\mathbf{A}$ are parallel to each other and the angle θ between $\mathbf{E}$ and $d\mathbf{A}$ is zero. Hence, the flux through surface III is

$$\Phi_{III} = \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{III} E\, dA \cos\theta = \int_{III} E\, dA \cos 0^o = \int_{III} E\, dA$$

The total flux through the Gaussian surface is equal to the sum of the fluxes through the individual surfaces. Hence, Gauss's law becomes

$$\Phi = \Phi_{I} + \Phi_{II} + \Phi_{III}$$
$$\Phi_E = 0 + 0 + \int_{III} E dA = \frac{q}{\varepsilon_o}$$

But **E** is constant in every term of the sum in integral III and can be factored out of the integral giving

$$\Phi_E = E \int dA = \frac{q}{\varepsilon_o}$$

But $\int dA = A$ the area of the end cap, hence

$$EA = \frac{q}{\varepsilon_o}$$

A is the magnitude of the area of Gaussian surface III and q is the charge enclosed within Gaussian surface III. Solving for the electric field E between the conducting plates gives

$$E = \frac{q}{\varepsilon_o A} \tag{4.53}$$

Equation 4.53 describes the electric field in terms of the charge on Gaussian surface III and the area of Gaussian surface III. *If the surface charge density on the plates is uniform, the surface charge density of Gaussian surface III is the same as the surface charge density of the entire plate. Hence the ratio of q/A for the Gaussian surface is the same as the ratio q/A for the entire plate.* Thus q in equation 4.53 can also be interpreted as the total charge q on the plates, and A can be interpreted as the total area of the conducting plate. Therefore, *equation 4.53 gives the electric field E between the conducting plates in terms of the charge q on the plates, the area A of the plates, and the permittivity ε_o of the medium between the plates.*

Instead of describing the electric field in terms of the charge q on the plates and the area A of the plates, it is sometimes convenient to express this result in terms of the surface charge density σ. Since the surface charge density σ is defined as the charge per unit area, i.e.,

$$\sigma = q/A$$

the electric field between the oppositely charged conducting plates can also be expressed as

$$E = \frac{\sigma}{\varepsilon_o} \tag{4.54}$$

Thus, the electric field between the oppositely charged conducting plates is also given by equation 4.54 in terms of the surface charge density σ and the permittivity of free space ε_o, the medium between the plates. The configuration of the two oppositely charged conducting plates is called a parallel plate capacitor and we will return to this in chapter 6 when we discuss capacitors.

Example 4.7

The electric field between two oppositely charged conducting plates. Find the electric field between two oppositely charged circular conducting plates of 5.00 cm radius, if a charge of 8.70×10^{-10} C is placed on the plates.

Solution

The area of the plate is

$$A = \pi r^2 = \pi(0.0500\text{ m})^2 = 7.85 \times 10^{-3}\text{ m}^2$$

The electric field between the conducting plates is found from equation 4.53 as

$$E = \frac{q}{\varepsilon_o A}$$

$$E = \frac{8.70 \times 10^{-10} C}{(8.85 \times 10^{-12} C^2/N\,m^2)(7.85 \times 10^{-3} m^2)}$$

$$E = 1.25 \times 10^4\text{ N/C}$$

To go to this Interactive Example click on this sentence.

4.11 The Electric Field Associated with Charged Concentric Conducting Cylinders

Let us determine the electric field **E** associated with the charged concentric conducting cylinders shown in figure 4.10(a). The inner cylinder is a solid cylinder while the outer cylinder is a cylindrical shell. A charge $+q$ is placed on the inner cylinder of radius r_a. Since the outer cylinder is a conducting body, the positive charge on the inner cylinder will induce a negative charge on the inside of the outer cylinder, radius r_b. That is, the inside of the outer cylinder was originally neutral. When the positive charge is placed on the inner cylinder, it attracts negative charge to the inside of the outer cylinder leaving the outside of the outer cylinder, radius r_c, positive. Hence, the electric field **E** goes from the positive charge ($+q$) on the inner cylinder to the negative charge ($-q$) on the inside of the outer cylinder and then goes from the positive charge ($+q$) on the outer cylinder outward. The charge distribution and electric field is shown in figure 4.10(b). Let us find the magnitude of the electric field for (a) $r < r_a$, (b) $r_a < r < r_b$ (c) $r_b < r < r_c$ (d) $r > r_c$.

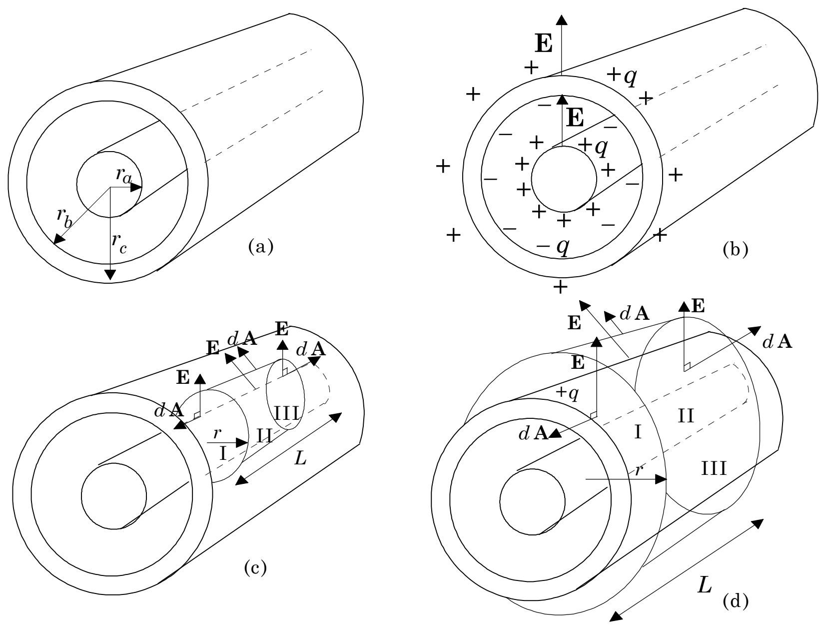

Figure 4.10 The electric field of charged concentric conducting cylinders.

(a) The electric field for $r < r_a$ is found by inspection by using the result of section 4.8. Since the inner cylinder is a solid conducting body the electric field **E** within the inner conducting cylinder is equal to zero, i.e., $\mathbf{E} = 0$.

(b) To determine the electric field between the two conducting cylinders, $r_a < r < r_b$ we draw a cylindrical Gaussian surface of radius r and length L between the inner cylinder and the outer cylinder as shown in figure 4.10(c). We assume that the charge is uniformly distributed and has a surface charge density σ. Gauss's law for the total flux emerging from the Gaussian cylinder, equation 4 -14, is applied.

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o}$$

The integral in Gauss's law is over the entire Gaussian surface. As we did in the previous sections, we break the entire cylindrical Gaussian surface into three surfaces. Surface I is the end cap on the left-hand side of the cylinder, surface II is the main cylindrical surface, and surface III is the end cap on the right-hand side of the cylinder as shown in figure 4.10(c). The total flux Φ through the entire Gaussian surface is the sum of the flux through each individual surface. That is,

$$\Phi = \Phi_{\mathrm{I}} + \Phi_{\mathrm{II}} + \Phi_{\mathrm{III}}$$

where

Φ_{I} is the electric flux through surface I
Φ_{II} is the electric flux through surface II
Φ_{III} is the electric flux through surface III

Hence, Gauss's law becomes

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o}$$

Along cylindrical surface I, $d\mathbf{A}$ is everywhere perpendicular to the surface and points toward the left as shown in figure 4.10(c). The electric field intensity vector $\mathbf{E}$ lies in the plane of the end cylinder cap and is everywhere perpendicular to the surface vector $d\mathbf{A}$ of surface I and hence $\theta = 90^0$. Therefore the electric flux through surface I is

$$\Phi_I = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I E\, dA \cos\theta = \int_I E\, dA \cos 90^o = 0$$

Surface II is the cylindrical surface itself. As can be seen in figure 4.10(c), $\mathbf{E}$ is everywhere perpendicular to the cylindrical surface pointing outward, and the area vector $d\mathbf{A}$ is also perpendicular to the surface and also points outward. Hence $\mathbf{E}$ and $d\mathbf{A}$ are parallel to each other and the angle θ between $\mathbf{E}$ and $d\mathbf{A}$ is zero. Hence, the flux through surface II is

$$\Phi_{II} = \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{II} E\, dA \cos\theta = \int_{II} E\, dA \cos 0^o = \int_{II} E\, dA$$

Along cylindrical surface III, $d\mathbf{A}$ is everywhere perpendicular to the surface and points toward the right as shown in figure 4.5(b). The electric field intensity vector $\mathbf{E}$ lies in the plane of the end cylinder cap and is everywhere perpendicular to the surface vector $d\mathbf{A}$ of surface III and therefore $\theta = 90^0$. Hence the electric flux through surface III is

$$\Phi_{III} = \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{III} E\, dA \cos\theta = \int_{III} E\, dA \cos 90^o = 0$$

Combining the flux through each portion of the cylindrical surfaces we get

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o}$$
$$\Phi_E = 0 + \int_{II} E\, dA + 0 = \frac{q}{\varepsilon_o}$$

From the symmetry of the problem, the magnitude of the electric field intensity E is a constant for a fixed distance r from the axis of the inner cylinder. Hence, E can be taken outside the integral sign to yield

$$\int_{II} E\, dA = E \int_{II} dA = \frac{q}{\varepsilon_o}$$

The integral $\int dA$ represents the sum of all the elements of area dA, and that sum is just equal to the total area of the cylindrical surface. As we showed in section 4.6, the total area of the cylindrical surface can be found by unfolding the cylindrical surface. One length of the surface is L, the length of the cylinder, while the other length is the unfolded circumference $2\pi r$ of the end of the cylindrical surface. The total area A is just the product of the length times the width of the rectangle formed by unfolding the cylindrical surface, that is, $A = (L)(2\pi r)$. Hence, the integral of dA is

$$\int dA = A = (L)(2\pi r)$$

Thus, Gauss's law becomes

$$E\int_{II} dA = E(L)(2\pi r) = \frac{q}{\varepsilon_o}$$

Solving for the magnitude of the electric field intensity E we get

$$E = \frac{q}{2\pi\varepsilon_o rL} \tag{4.55}$$

Since the inner cylinder is a conducting body, the charge is distributed uniformly over it. Hence the surface charge density is given by $\sigma = q/A$ and the charge itself can be written as

$$q = \sigma A \tag{4.56}$$

But the area A is the area of the inner cylindrical surface that contains the charge and is given by

$$A = L(2\pi r_a) \tag{4.57}$$

Hence the charge q on the inner cylinder can be written as

$$q = \sigma A = \sigma L(2\pi r_a) \tag{4.58}$$

Replacing the charge q, equation 4.58, into equation 4.55 we get

$$E = \frac{q}{2\pi\varepsilon_o rL} = \frac{\sigma L(2\pi r_a)}{2\pi\varepsilon_o rL}$$

$$E = \frac{\sigma r_a}{\varepsilon_o r} \tag{4.59}$$

Equation 4.59 gives the magnitude of the electric field intensity E between the two concentric cylinders at a distance r from the axis of the inner cylinder.

(c) The electric field for $r_b < r < r_c$ is found by inspection by using the result of section 4.8. Since the outer cylindrical shell is a conducting body the electric field E within that conducting cylindrical shell must be equal to zero, i. e., $E = 0$.

(d) The electric field for $r > r_c$ is found by now drawing a much larger cylindrical Gaussian surface of radius r and length L around the outer cylinder as shown in figure 4.10(d). Gauss's law for the total flux emerging from the Gaussian cylinder, equation 4.14, is applied.

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o}$$

The integral in Gauss's law is over the entire Gaussian surface. The entire cylindrical Gaussian surface is again broken up into three surfaces. Surface I is the end cap on the left-hand side of the cylinder, surface II is the main cylindrical surface, and surface III is the end cap on the right-hand side of the cylinder as shown in figure 4.10(d). The total flux Φ through the entire Gaussian surface is the sum of the flux through each individual surface. That is,

$$\Phi = \Phi_{\mathrm{I}} + \Phi_{\mathrm{II}} + \Phi_{\mathrm{III}}$$

Hence, Gauss's law becomes

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o}$$

Along cylindrical surface I, $d\mathbf{A}$ is everywhere perpendicular to the electric field intensity vector $\mathbf{E}$ and hence $\theta = 90^0$. Therefore the electric flux through surface I is

$$\Phi_I = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I E\, dA \cos\theta = \int_I E\, dA \cos 90^o = 0$$

Surface II is the cylindrical surface itself. As can be seen in figure 4.10(d), $\mathbf{E}$ is everywhere parallel to the area vector $d\mathbf{A}$. Hence the angle θ between $\mathbf{E}$ and $d\mathbf{A}$ is zero. The flux through surface II is

$$\Phi_{II} = \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{II} E\, dA \cos\theta = \int_{II} E\, dA \cos 0^o = \int_{II} E\, dA$$

Along cylindrical surface III, $d\mathbf{A}$ is everywhere perpendicular to the electric field intensity vector $\mathbf{E}$ and therefore $\theta = 90^0$. Hence the electric flux through surface III is

$$\Phi_{III} = \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{III} E\, dA \cos\theta = \int_{III} E\, dA \cos 90^o = 0$$

Combining the flux through each portion of the cylindrical surfaces we get

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o}$$
$$\Phi_E = 0 + \int_{II} E\, dA + 0 = \frac{q}{\varepsilon_o}$$

The magnitude of the electric field intensity E is a constant for a fixed distance r from the axis of the inner cylinder and can be taken outside the integral sign to yield

$$\int_{II} E\, dA = E \int_{II} dA = \frac{q}{\varepsilon_o}$$

The integral $\int dA$ represents the sum of all the elements of area dA, and that sum is just equal to the total area of the cylindrical surface. The total area A is just the product of the length times the width of the rectangle formed by unfolding the cylindrical surface, that is, $A = (L)(2\pi r)$. Hence, the integral of dA is

$$\int dA = A = (L)(2\pi r)$$

Thus, Gauss's law becomes

$$E \int_{II} dA = E(L)(2\pi r) = \frac{q}{\varepsilon_o}$$

Solving for the magnitude of the electric field E intensity we get

$$E = \frac{q}{2\pi\varepsilon_o rL} \tag{4.60}$$

The total charge q within the Gaussian surface can be broken up into three parts: q_1 the positive charge on the inner cylinder; q_2 the negative induced charge on the inside of the outer cylinder; and q_3 the induced positive charge on the outside of the outer cylinder. Since q_2 and q_3 are equal but opposite charges their sum adds up to zero. Hence, *the total charge* contained within the Gaussian surface is just the original charge placed on the inner cylinder. Hence the surface charge density is given by $\sigma = q/A$. The charge itself can be written as $q = \sigma A$ where A is the area of the inner cylindrical surface and is again given by $A = L(2\pi r_a)$ Hence the charge q on the inner cylinder is the same as found previously in equation 4.58, that is,

$$q = \sigma L(2\pi r_a)$$

and the electric field is found as in section b as

$$E = \frac{q}{2\pi\varepsilon_o rL} = \frac{\sigma L(2\pi r_a)}{2\pi\varepsilon_o rL}$$

$$E = \frac{\sigma r_a}{\varepsilon_o r} \tag{4.61}$$

Equation 4.60 gives the magnitude of the electric field intensity E outside the two concentric cylinders at a distance r from the axis of the inner cylinder. Note that this is the same equation as equation 4.59 but there r referred to a distance between the cylinders and now it refers to the distance outside the cylinders. If the outer cylinder were connected to ground, how would this change the entire problem, and in particular, the electric field E outside the cylinders?

Example 4.8

The electric field between two conducting cylinders. Find the electric field halfway between the two charged concentric conducting cylinders shown in figure 4.10(a) if a charge $q = 1.50 \times 10^{-6}$ C is placed on the inner cylinder. The radius of the inner cylinder is r_a = 2.50 cm, the radius of the outer cylinder is r_b = 5.00 cm, and the length of the cylinder is L = 25.0 cm.

Solution

The charge $+q$ is placed on the inner cylinder and this induces a charge of $-q$ on the inner surface of the outer conducting cylinder. Hence, an electric field now exists between the two cylinders. The surface charge density σ on the inner cylinder is defined as the charge per unit area, that is,

$$\sigma = \frac{q}{A}$$

The area A of the inner cylinder is found from equation 4.57 as

$$A = L\,(2\pi r_a)$$
$$A = (0.25 \text{ m})(2\pi)(0.025 \text{ m})$$
$$A = 3.93 \times 10^{-2} \text{ m}^2$$

The surface charge density can now be calculated as

$$\sigma = \frac{q}{A}$$
$$\sigma = \frac{1.50 \times 10^{-6} \text{ C}}{3.93 \times 10^{-2} \text{ m}^2}$$
$$\sigma = 3.82 \times 10^{-5} \text{ C/m}^2$$

The electric field halfway between the two cylinders (r = 3.75 cm) is found from equation 4.59 as

$$E = \frac{\sigma r_a}{\varepsilon_o r}$$
$$E = \frac{(3.82 \times 10^{-5} \text{ C/m}^2)(0.025 \text{ m})}{(8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2)(0.0375 \text{ m})}$$
$$E = 2.88 \times 10^6 \text{ N/C}$$

To go to this Interactive Example click on this sentence.

4.12 The Differential Form of Gauss's Law and Gauss's Divergence Theorem

We have seen in our analysis of Gauss's law that if a simple symmetry was present in a problem, Gauss's law could be used to give a very simple solution to what is essentially a much more difficult problem. But this is a highly restrictive condition. Most problems in nature do not have this simplified symmetry. That is, we have picked Gaussian surfaces such that **E** was either perpendicular or parallel to the surface so the element of flux **E**•d**A** was either equal to $E\,dA$ or zero. Also the magnitude of the electric field intensity E was able to be considered as a constant over the Gaussian surface and hence could be taken outside of the integral ($\int E\,dA = E\int dA$) and solved for. Without this inherent symmetry the integral cannot be evaluated.

Is there any way that we can still use Gauss's law where this simplifying symmetry is absent? The answer is yes. We can solve any problem if we let the Gaussian surface shrink to such a small region, an infinitesimal region, that the electric field intensity E is almost a constant over that infinitesimal region. That is, instead of an integral form of Gauss's law we will develop a differential form of Gauss's law. With this new technique we will not be able to determine the electric field intensity E directly as we did in the integral form of Gauss's law, but we will get a relation that shows how E varies spatially. This new relation allows the relatively simple solution of problems that could otherwise be extremely complicated.

For our Gaussian surface, we draw an infinitesimal box of sides dx, dy, and dz, around the point $P(x,y,z)$, as shown in figure 4.11. Let us determine the total electric flux emanating from this box. Applying Gauss's law to the infinitesimal box, we get

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o} \tag{4.14}$$

The charge q can be expressed in terms of the volume charge density as

$$q = \int \rho\, dV$$

Hence,

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{1}{\varepsilon_o}\int \rho dV \tag{4.62}$$

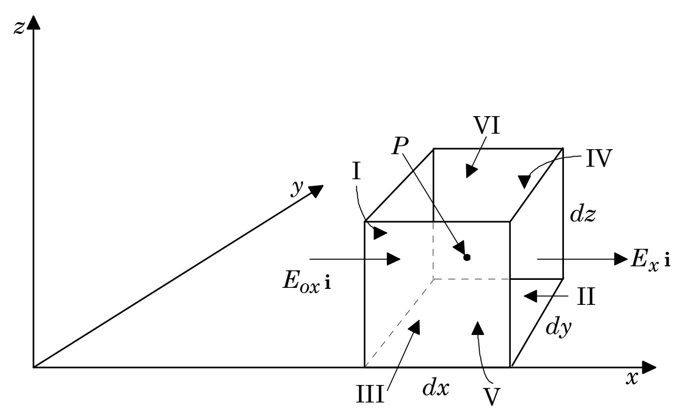

Figure 4.11 Differential Gaussian surface.

We break the Gaussian box into 6 surfaces, the 6 sides of the box. Therefore the net flux through the infinitesimal box can be written as

$$\Phi = \Phi_{I} + \Phi_{II} + \Phi_{III} + \Phi_{IV} + \Phi_{V} + \Phi_{VI} \tag{4.63}$$

and hence

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{III} \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{IV} \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{V} \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{VI} \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o} \tag{4.64}$$

Let us first consider the flux through surface I. The electric field intensity at the first surface is given by

$$\mathbf{E}_o = \mathbf{i}\, E_{ox} + \mathbf{j}\, E_{oy} + \mathbf{k}\, E_{oz} \tag{4.65}$$

and the element of area of surface I is given by

$$d\mathbf{A} = -\,\mathbf{i}\; dy\; dz \tag{4.66}$$

The minus sign for the element of area is used because the outward direction of surface I is in the negative x-direction. The flux through surface I is

$$\Phi_I = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} = \int_I (\boldsymbol{i}E_{ox} + \boldsymbol{j}E_{oy} + \boldsymbol{k}E_{oz}) \bullet (-\boldsymbol{i}\; dy\; dz) \tag{4.67}$$

$$\Phi_I = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} = -\int_I E_{ox}\; dy\; dz \tag{4.68}$$

since $\mathbf{i} \bullet \mathbf{i} = 1$ and $\mathbf{i} \bullet \mathbf{j} = \mathbf{i} \bullet \mathbf{k} = 0$. The electric field intensity emanating from surface II is given by

$$\mathbf{E} = \mathbf{i}\, E_x + \mathbf{j}\, E_y + \mathbf{k}\, E_z \tag{4.69}$$

and the element of area of surface II is given by

$$d\mathbf{A} = \mathbf{i}\; dy\; dz \tag{4.70}$$

The plus sign for the element of area is used because the outward direction for surface II is in the positive x-direction. The flux through surface II is

$$\Phi_{II} = \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{II} (\boldsymbol{i}E_x + \boldsymbol{j}E_y + \boldsymbol{k}E_z) \bullet (\boldsymbol{i}\; dy\; dz) \tag{4.71}$$

$$\Phi_{II} = \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{II} E_x\; dy\; dz \tag{4.72}$$

since $\mathbf{i} \bullet \mathbf{i} = 1$ and $\mathbf{i} \bullet \mathbf{j} = \mathbf{i} \bullet \mathbf{k} = 0$. Now E_x is the electric field intensity emanating from surface II. If there were no charge in the box then $E_x = E_{ox}$. But in general there could be some charge in the box and therefore the electric field intensity E_x coming out of the box would be greater than the electric field intensity E_{ox} entering the box. We can relate the electric field intensity E_x coming out of the box to the electric field intensity E_{ox} entering the box by

$$E_x = E_{ox} + (\Delta E)_x \tag{4.73}$$

where $(\Delta E)_x$ is the change in the electric field intensity E as we traverse the box in the x-direction. If we know how the electric field intensity varies with x, we can find the change in E_x in the x-direction $(\Delta E)_x$ as

$$(\Delta E)_x = \frac{\partial E_x}{\partial x} dx \tag{4.74}$$

That is, the change in E_x, $(\Delta E)_x$, is equal to the rate at which E_x changes with x, $(\partial E_x/\partial x)$, times the distance moved in the x-direction, (dx). Hence the value of the electric field intensity emanating from surface II can be written as

$$E_x = E_{ox} + \frac{\partial E_x}{\partial x} dx \tag{4.75}$$

Replacing the value of the electric field intensity E_x from equation 4.75 into equation 4.72 gives for the flux out of surface II

$$\Phi_{II} = \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} = \int_{II} E_x \, dy \, dz = \int_{II} \left[E_{ox} + \frac{\partial E_x}{\partial x} dx \right] dy \, dz \tag{4.76}$$

The total change in flux in the x-direction $(\Phi_E)_x$ is found as

$$(\Phi_E)_x = \Phi_1 + \Phi_2 = \int_I \boldsymbol{E} \bullet d\boldsymbol{A} + \int_{II} \boldsymbol{E} \bullet d\boldsymbol{A} \tag{4.77}$$

$$(\Phi_E)_x = -\int_I E_{ox} dy \, dz + \int_{II} E_x dy \, dz$$

$$(\Phi_E)_x = -\int_I E_{ox} dy \, dz + \int_{II} \left[E_{ox} + \frac{\partial E_x}{\partial x} dx \right] dy \, dz \tag{4.78}$$

$$(\Phi_E)_x = \int \left[-E_{ox} + E_{ox} + \frac{\partial E_x}{\partial x} dx \right] dy \, dz$$

$$(\Phi_E)_x = \int \frac{\partial E_x}{\partial x} dx \, dy \, dz \tag{4.79}$$

Equation 4.79 gives the total change in electric flux in the x-direction. A similar analysis for the change in electric flux in the y and z-directions yields

$$(\Phi_E)_y = \int \frac{\partial E_y}{\partial y} dx \, dy \, dz \tag{4.80}$$

$$(\Phi_E)_z = \int \frac{\partial E_z}{\partial z} dx \, dy \, dz \tag{4.81}$$

Hence the total flux through the entire infinitesimal box is

$$\Phi_E = (\Phi_E)_x + (\Phi_E)_y + (\Phi_E)_z$$

$$\Phi_E = \int \frac{\partial E_x}{\partial x} dx\, dy\, dz + \int \frac{\partial E_y}{\partial y} dx\, dy\, dz + \int \frac{\partial E_z}{\partial z} dx\, dy\, dz$$
$$\Phi_E = \int \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right] dx\, dy\, dz$$

But $dx\, dy\, dz = dV$ the volume of the infinitesimal box. Therefore

$$\Phi_E = \int \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right] dV \tag{4.82}$$

But as shown in chapter 1

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \nabla \bullet \boldsymbol{E} \tag{4.83}$$

the divergence of **E**. Hence the electric flux through the Gaussian box can also be written as

$$\Phi_E = \int [\nabla \bullet \boldsymbol{E}] dV \tag{4.84}$$

Equating the electric flux from equation 4.62 with the electric flux from equation 4.84 yields

$$\Phi_E = \int [\nabla \bullet \boldsymbol{E}] dV = \frac{1}{\varepsilon_o} \int \rho dV \tag{4.85}$$

Since the integration is over the volume dV of the box in both integrals they can be combined into the one integral

$$\int \left[\nabla \bullet \boldsymbol{E} - \frac{\rho}{\varepsilon_o} \right] dV = 0 \tag{4.86}$$

For the integral in equation 4.86 to be equal to zero for all values of **E** and r the entire term in brackets must equal zero. That is,

$$\nabla \bullet \boldsymbol{E} - \frac{\rho}{\varepsilon_o} = 0$$

or

$$\nabla \bullet \boldsymbol{E} = \frac{\rho}{\varepsilon_o} \tag{4.87}$$

Equation 4.87 is Gauss's law in differential form. It says that the divergence of **E** is a measure of the volume charge density ρ contained in the infinitesimal box surrounding the point *P*. *Equation 4.87 is also the first of Maxwell's equations of electromagnetism, and much more will be said about it in chapters 10 and 11 .*

Example 4.9

*The divergence of **E**.* A certain electric field vector is given by

$$\mathbf{E} = \frac{\mathbf{i}Cx}{(x^2 + y^2)^{3/2}} + \frac{\mathbf{j}Cy}{(x^2 + y^2)^{3/2}}$$

Find the divergence of **E**.

Solution

The divergence of **E** is found from equation 4.89 as

$$\nabla \bullet \boldsymbol{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}$$

$$\nabla \bullet \boldsymbol{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \frac{\partial\left[Cx(x^2+y^2)^{-\frac{3}{2}}\right]}{\partial x} + \frac{\partial\left[Cy(x^2+y^2)^{-\frac{3}{2}}\right]}{\partial y}$$

$$\nabla \bullet \boldsymbol{E} = \left[Cx(-\tfrac{3}{2})(2x)(x^2+y^2)^{-\frac{5}{2}} + C(x^2+y^2)^{-\frac{3}{2}}\right] + \left[Cy(-\tfrac{3}{2})(2y)(x^2+y^2)^{-\frac{5}{2}} + C(x^2+y^2)^{-\frac{3}{2}}\right]$$

$$\nabla \bullet \boldsymbol{E} = \left[\frac{Cx(-3x)}{(x^2+y^2)^{\frac{5}{2}}} + \frac{C}{(x^2+y^2)^{\frac{3}{2}}}\right] + \left[\frac{Cy(-3y)}{(x^2+y^2)^{\frac{5}{2}}} + \frac{C}{(x^2+y^2)^{\frac{3}{2}}}\right]$$

$$\nabla \bullet \boldsymbol{E} = \left[\frac{-3Cx^2 - 3Cy^2}{(x^2+y^2)^{\frac{5}{2}}} + \frac{2C}{(x^2+y^2)^{\frac{3}{2}}}\right]$$

$$\nabla \bullet \boldsymbol{E} = \left[\frac{-3C(x^2+y^2)}{(x^2+y^2)^{\frac{5}{2}}} + \frac{2C}{(x^2+y^2)^{\frac{3}{2}}}\right]$$

$$\nabla \bullet \boldsymbol{E} = \left[\frac{-3C}{(x^2+y^2)^{\frac{3}{2}}} + \frac{2C}{(x^2+y^2)^{\frac{3}{2}}}\right]$$

$$\nabla \bullet \boldsymbol{E} = \frac{-C}{(x^2+y^2)^{\frac{3}{2}}}$$

Since the quantity $(x^2 + y^2)^{3/2}$ is always greater than zero, $\nabla\bullet$ **E** will be either positive or negative depending upon the sign of C. If C is a positive quantity then $\nabla\bullet$ **E** < 0, and there is a convergence of the vector **E** into a region. If C is a negative quantity, then $\nabla\bullet$ **E** > 0, and there is a divergence of the vector **E** from the region. A more detailed description of the physical significance of the divergence will be given in the next section.

Another important result, that will be used frequently in the study of electromagnetic theory is obtained by equating the electric flux through a Gaussian surface, equation 4.14 to the electric flux found in equation 4.84. Thus,

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o} \tag{4.14}$$

$$\Phi_E = \int [\nabla \bullet \boldsymbol{E}]dV \tag{4.84}$$

$$\oint \boldsymbol{E} \bullet d\boldsymbol{A} = \int [\nabla \bullet \boldsymbol{E}]dV \tag{4.88}$$

Equation 4.88 is called Gauss's divergence theorem. It relates the value of ***E*** *over a closed surface to the divergence of* ***E*** *over a volume.* We will use this result in chapter 10.

4.13 The Divergence and Its Physical Significance

It was shown in section 1.12, that if you take the dot product of the del operator with any vector **E** the result is given by

$$\nabla \bullet \boldsymbol{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{4.89}$$

Equation 4.89 is called the divergence of ***E****.* In this section we will discuss the physical significance of the divergence. The earliest use of vector analysis was in the application of the motion of fluids. Such terminology as divergence and curl were used to describe the fluid. It will enhance our understanding if we analyze some fluid motion. However, the basic concepts belong to the study of fields in general, and in particular they will apply to the electric and magnetic fields that we will study in this course. Everywhere that we use the velocity vector **v** you can also use the electric field vector **E** and the magnetic field vector **B**.

Let us consider the flow of fluid from a faucet into a sink, as seen in figure 4.12. The water is flowing out of the faucet at a velocity $\mathbf{v}_{\text{in}}$ in the x-direction. The stopper is closed preventing any water from leaving the sink. Let us apply equation 4.89 to this problem. The vector **E** in equation 4.89 is replaced with the velocity vector **v** of the water. Hence the divergence equation becomes

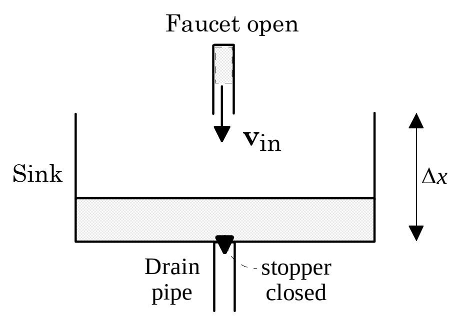

Figure 4.12 Illustration of convergence.

$$\nabla \bullet \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{4.90}$$

However the flow of water from the faucet is only in the x-direction. Hence, $v_y = v_z = 0$. Equation 4.90 becomes

$$\nabla \bullet \boldsymbol{v} = \frac{\partial v_x}{\partial x} = \frac{dv_x}{dx} \tag{4.91}$$

$$\nabla \bullet \boldsymbol{v} = \lim_{\nabla x \to o} \frac{\Delta v_x}{\Delta x} = \frac{v_{\text{out}} - v_{\text{in}}}{\Delta x} \tag{4.92}$$

But as can be seen from figure 4.12, the stopper is closed and thus the velocity of the water out of the sink, v_{out}, is equal to zero. Therefore, equation 4.92 becomes

$$\nabla \bullet \boldsymbol{v} = \frac{-v_{\text{in}}}{\Delta x} \tag{4.93}$$

Because of the minus sign, the divergence is negative, that is,

$$\nabla \bullet \boldsymbol{v} < 0 \tag{4.94}$$

Let us analyze what this means physically. When the stopper is closed, no water can leave the sink. Yet the faucet is open, pouring more water into the sink. Since the water cannot leave the sink it is piling up or converging into the sink. *Hence, whenever the quantity* $\nabla \bullet \boldsymbol{v}$ *is less than zero, water is converging into the sink.*

Let us now consider the case shown in figure 4.13. In this case the faucet has been closed but the stopper has been opened. No water is entering the sink and hence v_{in} is now equal to zero. However, because the stopper has been opened, water is pouring out of the sink through the drain pipe, and v_{out} is no longer equal to zero. The divergence of **v**, equation 4.92, now becomes

$$\nabla \bullet \boldsymbol{v} = \frac{v_{\text{out}}}{\Delta x} \tag{4.95}$$

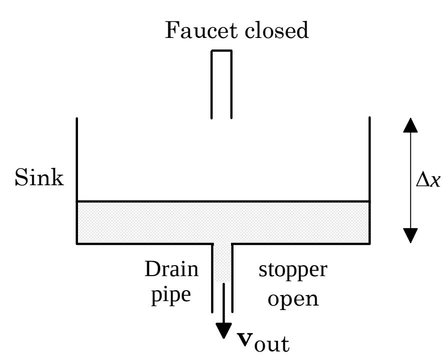

Figure 4.13 Illustration showing divergence.

In this case, there is no minus sign and hence the divergence is positive, that is,

$$\nabla \bullet \boldsymbol{v} > 0 \tag{4.96}$$

Let us analyze what this means physically. When the stopper is opened, the water in the sink can now pour out of the sink into the drain pipe. Because the faucet is closed, no new water can enter the sink to replace the water leaving. Hence all the water is leaving or diverging out of the sink. *Hence, whenever the quantity* $\nabla \bullet \boldsymbol{v}$ *is greater than zero, water is diverging out of the sink.*

The general case is considered in figure 4.14 where both the faucet and stopper are open. In general, if:

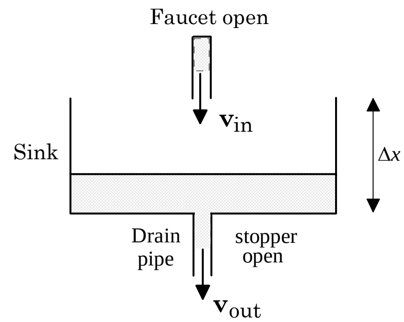

Figure 4.14 General case.

1) $v_{\text{out}} > v_{\text{in}}$ *then* $\nabla \bullet \boldsymbol{v} > 0$ *and there is divergence of water*. This case can be visualized as the faucet is not fully open and therefore relatively little new water is entering the sink. However, the stopper is fully open and water is flowing freely out of the sink. Hence, more water flows out of the sink then flows into it. Eventually all of the water in the sink will flow out of it, if the conditions are not changed.

2) $v_{\text{out}} < v_{\text{in}}$ *then* $\nabla \bullet \boldsymbol{v} < 0$ *and there is convergence of water*. In this case the stopper is open and water is flowing out of the sink, however the faucet is fully open and more new water is entering the sink then is flowing out at the bottom. Hence, there is convergence of water into the sink.

3) $v_{\text{out}} = v_{\text{in}}$ *then* $\nabla \bullet \boldsymbol{v} = 0$ *and this case is called nondivergence*. In this case both faucet and stopper are open and the rate at which new water enters the sink is equal to the rate at which old water flows out of the sink. Hence, the water is neither converging or diverging from the sink, and the situation is called nondivergence.

In general the divergence is a measure of the amount of the vector coming out of a volume. If the vector is the electric field intensity vector **E**, then $\nabla \bullet \mathbf{E}$ will represent the amount of the electric field leaving a volume. The amount of the electric field leaving the volume will turn out to be a measure of the quantity of electric charge within the volume.

Example 4.10

The divergence of ***E****.* In a certain region the electric field has a uniform value of 2 N/C in the *x*-direction. Find the divergence of **E**.

Solution

The electric field vector is given by the function **E** = 2**i**. The divergence is found from equation 4.91 as

$$\nabla \bullet \boldsymbol{E} = \frac{\partial E_x}{\partial x} = \frac{dE_x}{dx}$$
$$\nabla \bullet \boldsymbol{E} = \frac{d(2)}{dx} = 0$$

Because $\nabla \bullet \boldsymbol{E} = 0$*, the divergence is equal to zero. This means that the electric field is neither converging or diverging in the region.* This can be seen in figure 4.15 which shows the electric field. Notice that it is completely in the *x*-direction. The length of the electric field vectors are all the same indicating that the electric field is uniform throughout the region. Consider the box in the first quadrant. *As can be*

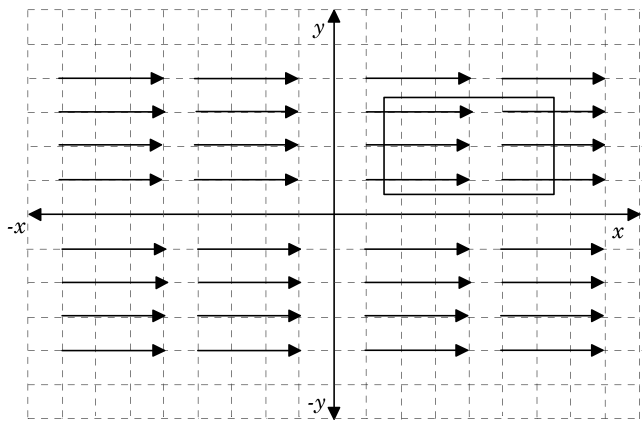

Figure 4.15 An example of nondivergence

seen, the electric field vectors entering the box are equal to the electric field vectors leaving the box, indicating that the amount of the electric filed entering the box is equal to the amount leaving the box. This is the characteristic of zero divergence, or nondivergence.

Example 4.11

*The divergence of **E**.* The electric field intensity in a certain region is given by the function **E** = 2*x***i**, find the divergence of **E**.

Solution

The divergence is found from equation 4.91 as

$$\nabla \bullet \boldsymbol{E} = \frac{\partial E_x}{\partial x} = \frac{dE_x}{dx}$$
$$\nabla \bullet \boldsymbol{E} = \frac{d(2x)}{dx} = 2 \text{ N/C}$$

Since $\nabla \bullet \boldsymbol{E}$ *is greater than zero, this represents a divergence of the electric field intensity and is shown in figure 4.16.* Because the electric field in the x-direction is a function of x, the electric field is not uniform in the x-direction. In fact for this case, at the location $x = 1$, the electric field is 2 N/C toward the right; at the location $x = 2$, the electric field is 4 N/C to the right; at $x = 3$, the electric field is 6 N/C to the right; and at any location x, the electric field is (2 N/C)x to the right. This is shown

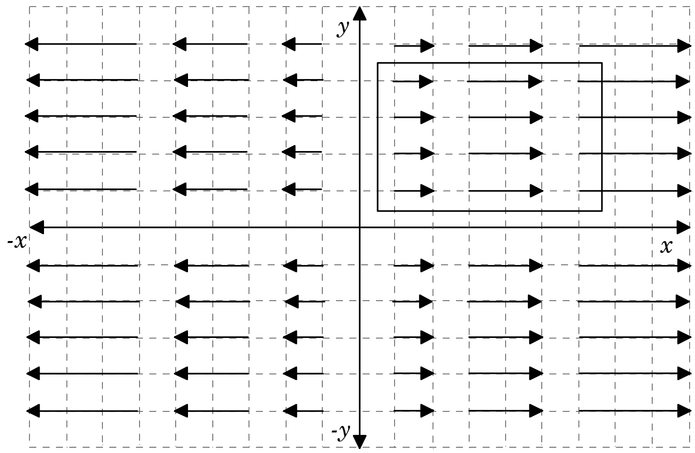

Figure 4.16 An example of divergence.

in figure 4.16 as the electric field vectors increasing with size as x increases. For negative values of x, the electric field is toward the left. Notice the box in the first quadrant and see that *the electric field vectors leaving the right side of the box are larger than the electric field vectors entering the box on the left side. This means that more of the electric field is leaving the box on the right hand side than is entering the box on the left hand side. Hence, the electric field is emanating or diverging away from the box. Hence, this is an example of divergence.*

Example 4.12

The divergence of ***E****.* The electric field in a certain region is given by the function

$$\mathbf{E} = -\,2x\mathbf{i},$$

find the divergence of **E**.

Solution

The divergence is found from equation 4.91 as

$$\nabla \bullet \boldsymbol{E} = \frac{\partial E_x}{\partial x} = \frac{dE_x}{dx}$$

$$\nabla \bullet \boldsymbol{E} = \frac{d(-2x)}{dx} = -2$$

Since $\nabla \bullet$ ***E*** *is less than zero, this represents convergence of the electric field* and is shown in figure 4.17. Because the electric field in the *x*-direction is a function of *x*,

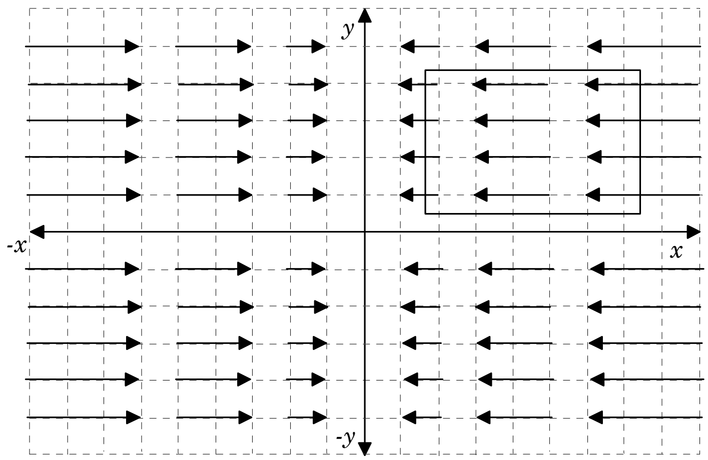

Figure 4.17 An example of convergence.

the electric field is not uniform in the *x*-direction. In fact for this case, at the location $x = 1$, the electric field is – 2 N/C toward the left; at the location $x = 2$, the electric field is – 4 N/C to the left; at $x = 3$, the electric field is – 6 N/C to the left; and at any location *x*, the electric field is – (2 N/C)*x* to the left. This is shown in figure 4.17 as the electric field vectors increasing in size as *x* increases. For negative values of *x*, the electric field is toward the right. Notice that this is the reverse of example 4.11. In example 4.11 the electric field vectors pointed toward the right in the first and fourth quadrants, and to the left in the second and third quadrants. In this example, the electric field vectors point toward the left in the first and fourth quadrants and toward the right in the second and third quadrants.

Notice the box in the first quadrant and see that *the electric field vectors entering the right side of the box are larger than the electric field vectors leaving the box on the left. This means that more of the electric field is entering the box on the right hand side than is leaving the box on the left hand side. Hence, the electric field is converging into the box. Hence, this is an example of convergence.*

Example 4.13

*The divergence of **E**.* The electric field intensity in a certain region is given by the function

$$\mathbf{E} = 2y\mathbf{i}$$

find the divergence of **E**.

Solution

The divergence is found from equation 4.91 as

$$\nabla \bullet \boldsymbol{E} = \frac{\partial E_x}{\partial x} = \frac{dE_x}{dx}$$
$$\nabla \bullet \boldsymbol{E} = \frac{d(2y)}{dx} = 0$$

Because $\nabla \bullet \boldsymbol{E} = 0$*, the divergence is equal to zero. This means that the electric field is neither converging or diverging in the region.* This can be seen in figure 4.18

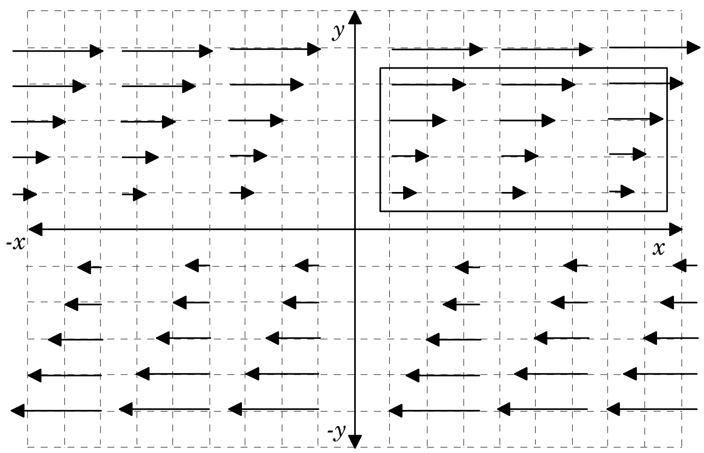

Figure 4.18

which shows the electric field vectors. Notice that this a very different case than studied in example 4.10. There, the electric field was uniform throughout the region. In this example, the electric field is constant in the x-direction for a particular value of x, but varies in the y-direction. That is for $y = 1$, the electric field in the x-direction is 2 N/C at every value of x; for $y = 2$ the electric field in the x-direction is 4 N/C for every value of x; and for $y = 3$, the electric field in the x-direction is 6 N/C for every value of x. Hence, the electric field in the x-direction is constant for a particular value of x, but that constant value varies for every value of y. The divergence is still equal to zero however, because, as can be seen in the box of figure 4.18, *the size of the electric field vectors leaving the box on the right hand side is equal to the size of the electric field vectors entering on the left hand side. Hence the amount of the electric filed entering the box is equal to the amount of the electric field leaving the box and the field is nondivergent.* We will show later, in section 9.11, that an electric field like figure 4.18 is an example of the curl of the electric field, i.e., $\nabla \times \mathbf{E}$.

Summary of Important Concepts

Flux is a quantitative measure of the number of lines of a vector field that passes perpendicularly through a surface. In particular, the **electric flux** is defined as the number of lines of the electric field intensity **E** that pass normally through the surface area **A**.

Gauss's law for electricity says that the electric flux Φ_E that passes through a surface surrounding electric charge q is a measure of the amount of charge q contained within the Gaussian surface. When Φ_E is a positive quantity, the Gaussian surface surrounds a source of positive charge, and electric flux diverges out of the surface. If the electric charge is negative then the electric field would go inward, through the Gaussian surface, to the electric charge. Thus, the flux passing through the Gaussian surface would be negative. Hence, whenever Φ_E is negative, the Gaussian surface surrounds a negative charge distribution and the electric flux converges into the Gaussian surface. If there is no enclosed charge, the electric flux Φ_E will be equal to zero. In this case, whatever electric flux enters one part of a Gaussian surface, the same amount must leave somewhere else.

The divergence ($\nabla\bullet$ E) of a vector - A measure of whether a vector field converges or diverges into an area or volume. Whenever the quantity $\nabla\bullet$ **E** is greater than zero, the vector field is diverging out of the area or volume. Whenever the quantity $\nabla\bullet$ **E** is less than zero, the vector field is converging into the area or volume. Whenever the quantity $\nabla\bullet$ **E** is equal to zero, the amount of the vector field converging into the area or volume is equal to the amount of the vector field diverging out of the area or volume. This case is called zero divergence or non-divergence. The amount of the electric field **E** leaving the volume is a measure of the quantity of electric charge within the volume.

Summary of Important Equations

Linear charge density $\lambda = q/L$

surface charge density $\sigma = q/A$

volume charge density $\rho = q/V$

Electric flux $$\Phi_E = \mathbf{E} \bullet \mathbf{A} = EA\cos\theta \qquad (4.1)$$

Electric flux $$\Phi_E = \oint d\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} \qquad (4.6)$$

Gauss's law $$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o} \qquad (4.14)$$

Gauss's law $$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{1}{\varepsilon_o} \oint \rho\, dV \tag{4.16}$$

Charge distributed over a volume $$q = \oint \rho\, dV \tag{4.15}$$

Electric field *outside* of a uniform spherical charge distribution $$E = k\frac{q}{r^2} \tag{4.19}$$

Electric field *inside* a uniform spherical charge distribution $$E = k\frac{r}{R^3}q \tag{4.26}$$

Element of volume for a sphere $$q = \int \frac{\rho_o}{r} dV \tag{4.32}$$

Electric field for a nonuniform spherical charge distribution of the form $\rho = \rho_o/r$

Outside distribution $$E = k\frac{q}{r^2} \tag{4.30}$$

with $$q = 2\pi\rho_o R^2 \tag{4.35}$$

Inside distribution $$E = \frac{\rho_o}{2\varepsilon_o} \tag{4.40}$$

Electric field of an infinite line of charge $$E = \frac{2k\lambda}{r} \tag{4.43}$$

Electric field of an infinite sheet of charge $$E = \frac{\sigma}{2\varepsilon_o} \tag{4.47}$$

Electric field inside a conductor $$E = 0 \tag{4.49}$$

Electric field in front of an infinite conducting plate $$E = \frac{\sigma}{\varepsilon_o} \tag{4.52}$$

Electric field between two oppositely charged conducting plates $$E = \frac{q}{\varepsilon_o A} \tag{4.53}$$

Electric field between two oppositely charged conducting plates $$E = \frac{\sigma}{\varepsilon_o} \tag{4.54}$$

Electric field at a distance *r* between two concentric cylinders $$E = \frac{\sigma r_a}{\varepsilon_o r} \tag{4.59}$$

Electric field at a distance r outside two concentric cylinders
$$E = \frac{\sigma r_a}{\varepsilon_o r} \tag{4.61}$$

The divergence of E
$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \nabla \bullet \boldsymbol{E} \tag{4.83}$$

Electric flux through a Gaussian box
$$\Phi_E = \int [\nabla \bullet \boldsymbol{E}] dV \tag{4.84}$$

Gauss's law in differential form (Maxwell's first equation)
$$\nabla \bullet \boldsymbol{E} = \frac{\rho}{\varepsilon_o} \tag{4.87}$$

Gauss's divergence theorem
$$\oint \boldsymbol{E} \bullet d\boldsymbol{A} = \int [\nabla \bullet \boldsymbol{E}] dV \tag{4.88}$$

Questions For Chapter 4

1. Notice the great similarity between the solution of the electric field in front of a charged conducting plate (section 4.9) and the electric field between two oppositely charged conducting plates (Section 4.10). Why are the electric fields the same? Could the single conducting plate be viewed as only one part of two oppositely charged conducting plates, but one is so far away it is effectively at infinity?

2. Under what conditions can we consider the electric field of a line of charge to be the same as the electric field of a cylinder of charge?

Problems For Chapter 4

1. What is the magnitude of the electric flux Φ_E emanating from a point charge of 2.00 μC?

2. Find the total flux Φ_E passing through the sides of a cube 1.00 m on a side if a point charge of 5.00×10^{-6} C is located at its center.

3. Find the electric flux Φ_E through a hemisphere of radius R that is immersed in a uniform electric field directed in the z-direction as shown in the diagram.

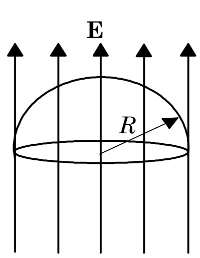

Diagram for problem 3.

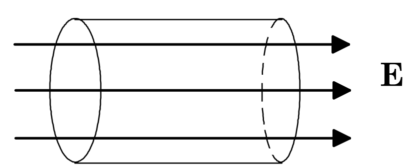

Diagram for problem 4.

4. Find the total electric flux Φ_E through a cylindrical surface that is immersed in a uniform electric field **E** that is parallel to the axis of the cylinder as shown in the diagram.

5. Find the electric flux Φ_E through a spherical surface of radius R that contains an electric dipole as shown in the diagram.

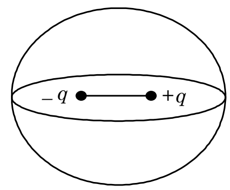

Diagram for problem 5.

6. The total electric flux Φ_E through a sphere is 2.82×10^5 N m²/C. What is the value of the enclosed charge q?

7. A charge q of 3.85 μC is at the center of a cube 2.54 cm on a side. Determine the electric flux Φ_E through one face of the Cube.

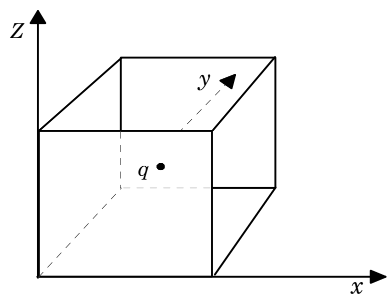

Diagram for problem 7.

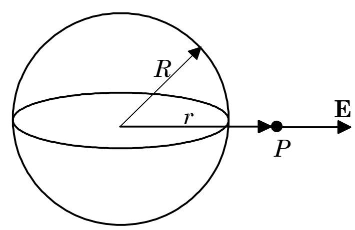

Diagram for problem 8.

8. The electric field intensity at the point P, a distance $r = 1.50$ m outside of a spherically charged body of radius $R = 0.45$ m, is $E = 1.40 \times 10^4$ N/C. Find the electric charge q on the sphere and the surface charge density σ.

9. Find the electric field E between the plates of a parallel plate capacitor of area $A = 2.00 \times 10^{-3}$ m² if a charge $q = 6.00$ μC is placed upon them.

10. A ring of charge of radius R carries a total charge q. Find the electric field E in the plane of the ring, at a distance r from the center of the ring, for r (a) inside the ring of charge and (b) outside the ring of charge. Indicate your assumptions.

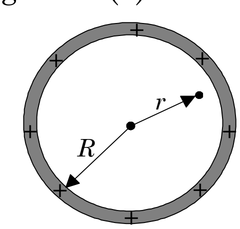

Diagram for problem 10.

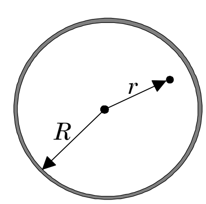

Diagram for problem 11.

11. A spherical shell of charge, of radius R, carries a total charge q. Find the electric field E at a distance r from the center of the shell, for r (a) inside the spherical shell of charge and (b) outside the spherical shell of charge.

12. A point charge q is placed at the center of two concentric conducting rings. The radius of the inner ring is r_1 while the radius of the outer ring is r_2. Find the electric field E in the plane of the rings for (a) $r < r_1$, (b) $r_1 < r < r_2$, and (c) $r > R_2$.

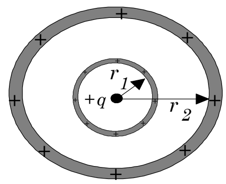

Diagram for problem 12.

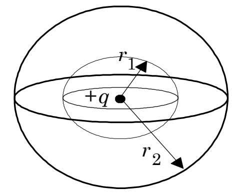

Diagram for problem 13.

13. A point charge q is placed at the center of two concentric conducting spheres. The radius of the inner sphere is r_1 while the radius of the outer sphere is r_2. Find the electric field E for (a) $r < r_1$, (b) $r_1 < r < r_2$, and (c) $r > r_2$.

14. A positively charge rod is touched to the outside of a hollow spherical conductor of outer radius r_b = 17.0 cm and inner radius r_a = 15.0 cm, leaving a surface charge density $\sigma = 1.51 \times 10^{-5}$ C/m^2. Find the electric field E for (a) $r < r_a$, (b) $r_a < r < r_b$, and (c) $r > r_b$.

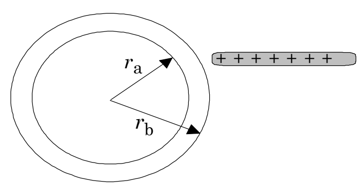

Diagram for problem 14.

15. Show that the total charge q' contained within a spherical charge distribution, equation 4-23,

$$q' = \frac{r^3}{R^3} q$$

can be derived from a more general approach by using

$$q' = \int \rho \, dV$$

where ρ is the constant volume charge density, and the element of volume of a sphere is $dV = 4\pi r^2 \, dr$.

16. Find the electric field E inside and outside of a nonuniform spherical charge distribution whose volume charge density is given by

$$\rho = \rho_o / r^2$$

17. A nonconducting sphere of charge, with a cavity in the center of the sphere, carries a uniform charge distribution ρ. Find the electric field E for (a) $r < a$, (b) $a < r < b$, and (c) $r > b$.

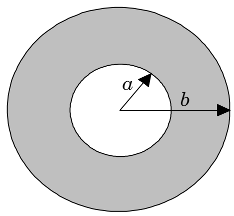

Diagram for problem 17.

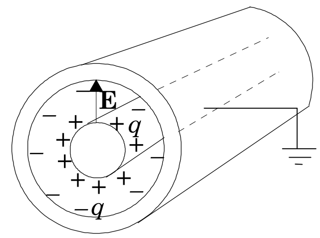

Diagram for problem 18 .

18. Using Gauss's law find the electric field outside the two coaxial cylinders in the diagram, if they are carrying equal but opposite charge.

19. Two large sheets of charge (assume they are infinite) each carry a surface charge density $\sigma = 2.85 \times 10^{-5}$ C/m². Find the electric field intensity E for (a) $0 < x < x_1$ (b) $x_1 < x < x_2$ (c) $x_2 < x$.

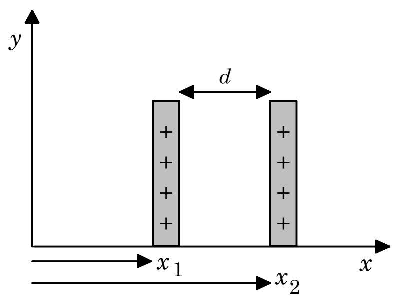

Diagram for problem 19.

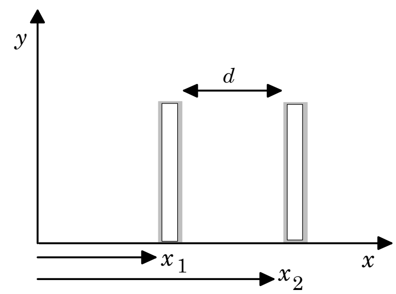

Diagram for problem 20.

20. Repeat problem 19 only now the two sheets of charge are replaced by two conducting plates carrying the surface charge densities $\sigma_1 = 2.85 \times 10^{-5}$ C/m² and $\sigma_2 = -2.85 \times 10^{-5}$ C/m²

21. A thin hollow metal sphere of radius $R = 15.0$ cm carries a charge $q = 6.28 \times 10^{-6}$ C. Find the electric field intensity E for (a) $r < R$ and (b) $r > R$.

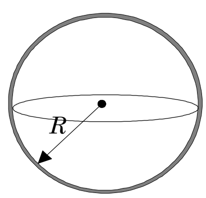

Diagram for problem 21.

22. Compare the electric field intensity E a short distance r outside a spherical conducting body with the electric field intensity E the same distance r outside a charged infinite conducting plate. Discuss the physical significance of the solution.

23. A charge + q is placed on the inner cylinder of two hollow coaxial cylindrical conducting shells. Find the electric field intensity E for (a) $r < r_a$ (b) $r_a < r < r_b$ (c) $r > r_b$

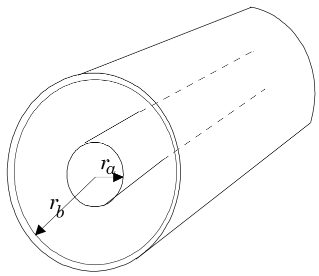

Diagram for problem 23.

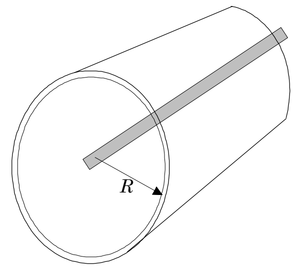

Diagram for problem 24.

24. A conducting cylindrical shell of radius R is placed around a very long line of charge of linear charge density λ. Find the electric field intensity E for (a) $r < R$ and (b) $r > R$.

25. Two concentric spherical conducting shells have radii r_a and r_b. A charge $+q$ is placed on the inner spherical shell. Find (a) the charge q on the inside of the outer shell, and (b) the charge q on the outside of the outer shell. Also find the electric field intensity E for (c) $r < r_a$, (d) $r_a < r < r_b$, and (e) $r > r_b$.

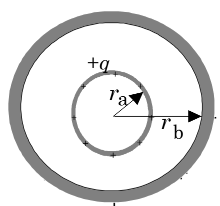

Diagram for problem 25.

26. Repeat problem 25 only this time the outer shell is grounded.

27. A charge $+q$ is placed on the inner cylinder and a charge of $-2q$ is placed on the outer coaxial cylinder. Find the electric field intensity E for (a) $r < r_a$, (b) $r_a < r < r_b$, and (c) $r > r_b$.

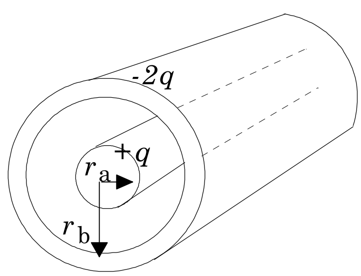

Diagram for problem 27.

28. Find the electric field intensity *E* outside a long conducting cylinder of charge density σ. What is the difference between a line of charge and a charged conducting cylinder? When can we relate the cylinder to a line of charge?

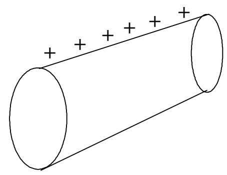

Diagram for problem 28.

29. In a certain region the electric field intensity is given by

$$\mathbf{E} = 2y\mathbf{i} + x\mathbf{j}$$

Find the divergence of **E**.

30. In a certain region the electric field intensity is given by

$$\mathbf{E} = 2x\mathbf{i} + y\mathbf{j}$$

Find the divergence of **E.**

31. Find the divergence of **E** for an infinite sheet of charge.

32. Find the divergence of **E** in the space between two oppositely charged conducting plates.

33. In some problems in electromagnetic theory it is much simpler to express the problem in polar coordinates rather than rectangular coordinates because of the radial and azimuthal symmetry inherent in the problem. In polar coordinates the del operator is given by

$$\nabla = \boldsymbol{r}_o \frac{\partial}{\partial r} + \boldsymbol{\theta}_o \frac{1}{r}\frac{\partial}{\partial \theta}$$

where $\mathbf{r}_o$ is a unit vector in the **r** direction and $\boldsymbol{\theta}_o$ is a unit vector in the direction of increasing θ. $\boldsymbol{\theta}_o$ is perpendicular to $\mathbf{r}_o$. Using the del operator in polar coordinates, find the divergence of **E** for a point charge.

34. Using the del operator in polar coordinates from problem 33, find $\nabla \bullet \mathbf{E}$ for the electric field at a distance r from an infinite line of charge.

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