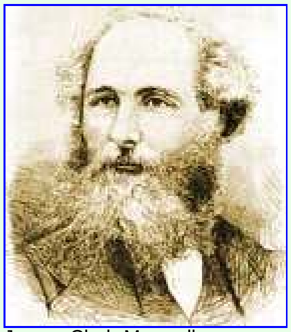
home | index | units | counting | geometry | algebra | trigonometry & functions | calculus
analysis | sets & logic | number theory | recreational | misc | nomenclature & history | physics



James Clerk Maxwell

Final Answers

Maxwell's Equations

(James Clerk Maxwell's "Great Guns")

The aim of exact science is to reduce the problems of nature to the determination of quantities by operations with numbers.
James Clerk Maxwell (1831-1879)
On Faraday's Lines of Force (1856)

- Clarifications: Vector calculus (Heaviside) & *microscopic* view (Lorentz).
- The vexing problem of units is a thing of the past if you stick to SI units.
- The Lorentz force on a test particle defines the local electromagnetic fields.
- Electrostatics (1785): The study of the electric field due to static charges.
- Electric capacity is an electrostatic concept (adequate at low frequencies).
- Electrostatic multipoles: The multipole expansion of an electrostatic field.
- Birth of electromagnetism (1820): Electric currents create magnetic fields.
- Biot-Savart Law: The *static* magnetic induction due to steady currents.
- Magnetic scalar potential: A *multivalued* static scalar field.
- Magnetic monopoles do not exist : A law stating a fact not yet disproved.
- Ampère's law (1825): The law of *static* electromagnetism.
- Faraday's law (1831): Electric circulation *induced* by magnetic flux change.
- Self-induction received by a circuit from the magnetic field it produces.
- Ampère-Maxwell law: Dynamic generalization (1861) of *Ampère's law*.
- Putting it all together: Historical paths to Maxwell's *electromagnetism*.
- Maxwell's equations unify electricity and magnetism dynamically (1864).
- Continuity equation: Maxwell's equations imply *conservation of charge*.
- Waves: Predicted by Faraday, Maxwell & FitzGerald. Observed by Hertz.
- Electromagnetic energy density and the flux of the *Poynting vector*.
- Planar electromagnetic waves: The simplest type of electromagnetic waves.
- Maxwell-Bartoli radiation pressure. First detected by *P. Lebedev* in 1899.
- Electromagnetic potentials are postulated to obey the *Lorenz gauge*.
- Solutions to Maxwell's equations, as *retarded* or *advanced* potentials.
- Electrodynamic fields corresponding to *retarded* potentials.
- Electrodynamic fields corresponding to *advanced* potentials.
- The gauge of retarded potentials: is it *really* the Lorenz gauge?
- Power radiated by an accelerated charge: The Larmor formula (1897).
- Lorentz-Dirac equation for the motion of a point charge is of *third* order.

Articles formerly on this page:

- Electric and magnetic dipoles: Dipolar solutions of Maxwell's equations.

- Static distributions of magnetic dipoles can be emulated by steady currents.
- Static distributions of electric dipoles are equivalent to charge distributions.
- Field at center of a uniformly magnetized or polarized sphere of *any* size.
- Sign reversal in *magnetic* and *electric* fields from matching dipoles.
- Relativistic dipoles: A moving magnet develops an electric moment.

Related articles on this site:

- Tesla.
- The Vacuum.
- Electronics 101.
- Philosophy and Science.
- Electromagnetic dipoles.
- Electric motors and generators.
- Amber, compass and lightning.
- Light | Optics | Waves | Lasers
- Electromagnetic properties of matter.
- Vector calculus and differential forms.
- Differential Equations | Linear Filters
- Special Relativity | General Relativity
- Lorentz Lagrangian of a charged particle.
- On the Origins of the Theory of Relativity.
- Humor: Watt Hertz Faraday, then Gauss away?
- The Schrödinger Equation: Its derivation and some of its implications.
- Photonics: Photons (quanta of light) are *both* wavelike and corpuscular.
- The *de Broglie* celerity (u) is inversely proportional to a particle's speed.
- The Lorenz Gauge, due to Ludwig Lorenz (1829-1891) *not* H.A. Lorentz.
- Two "del" symbols: ¶ for partial derivatives, and Ñ for Hamilton's *nabla*.

Related Links (Outside this Site)

Faraday's Law & Lenz's Rule, by Carl R. (Rod) Nave, Georgia State U.
History of Classical Electromagnetism, by *Jeff Biggus*.
Ampère et l'histoire de l' électricité (*Blondel, Wolff, Wronecki, Pouyllau, Usal*).
On the Notation of Maxwell's Field Equations, by *André Waser* (June 2000)
Classical Electromagnetism by *Richard Fitzpatrick* (UT Austin).
The Greatest Equations Ever, by *Robert P. Crease* (Physics World, 2004).
Ampère, Gauss & Weber (*21st Century Science & Technology Magazine*)
Integral and Differential Forms of Maxwell's Equations.
Retarded and advanced potentials, by *Richard Fitzpatrick*.
Heaviside-Lorentz Units by *J. B. Calvert*.
The Theory of the Electron (H. A. Lorentz, 1892) by *Fritz Rohrlich* (1962).
A Gallery of Electromagnetic Personalities by *L.S. Taylor*.
Self-Force & Radiation Reaction by *Luca Bombelli*, University of Mississippi.
Some Cut-off Methods for the Electron Self-Energy by Jan Rzewuski (1949).

Causality and the Wave Equation by Kevin S. Brown (January 2007).
Maxwell's Equations and Self-Bending Light by *Nancy Owano* (2012-04-21).

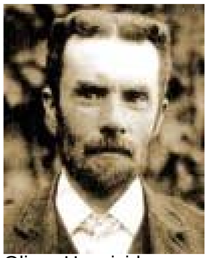
DMOZ: Electromagnetism

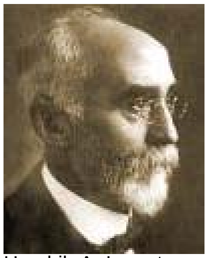
Video: MIT OpenCourseWare Electricity & Magnetism by Walter Lewin.
Visualizing Electricity and Magnetism : Physics 8.02 at MIT.
Maxwell's Equations (Mechanical Universe #39) by *David Goodstein* (CalTech)

"The Story of Electricity" by *Jim Al-Khalili* : 1 | 2 | 3 |
The Greatest Victorian Mathematical Physicist by Pr. Raymond Flood.

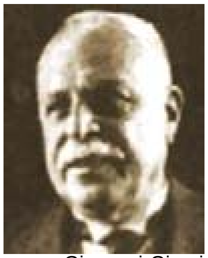
Electromagnetism

The following modern presentation of electromagnetism incorporates three clarifications which came only many years *after* Maxwell's original work (1864):


Oliver Heaviside


Hendrik A. Lorentz

- Vectorial notations and differential operators are used, which were developed by Oliver Heaviside (1850-1925) after 1880.
- Arguably, the original equations of Maxwell (1864) were essentially the so-called macroscopic equations, which describe electromagnetism in a *dense medium*. The *microscopic* approach (which is now standard) is due to H.A. Lorentz (1853-1928). Lorentz showed how the introduction of densities of polarization and magnetization reduces the macroscopic equations ("in matter") to the more fundamental microscopic ones ("in vacuum") stated below.
- Except in the first article, we consider only *one flavor* of electromagnetic quantities and use only the MKSA units introduced by Giovanni Giorgi (1871-1950) in 1901, which are the basis of *all* modern SI electrical units: ampere (A), ohm (W), coulomb (C), volt (V), tesla (T), farad (F), henry (H), weber (Wb)...


Giovanni Giorgi

(2005-07-22) The *Former* Problem with Electromagnetic Units

A science which hesitates to forget its founders is lost.
Alfred North Whitehead (1861-1947)

This article is of historical interest only. You are advised to skip it if you were blessed with an education entirely grounded on Giorgi's electrodynamic units (SI units based on the MKSA system).

The first consistent system of *mechanical* units was the meter-gram-second system advocated by Carl Friedrich Gauss in 1832. It was used by Gauss and Weber (c.1850) in the first definitions of electromagnetic units in absolute terms.

However, the term *Gaussian system* now refers to a particular mix of electrical C.G.S. units (discussed below) once dominant in theoretical investigations.

James Clerk Maxwell himself was instrumental in bringing about the *cgs system* in 1874 (centimeter-gram-second). Two sets of electrical units were made part of the system. An enduring confusion results from the fact that the quantities measured by these different units have different definitions (in modern terms, for example, the magnetic quantity now denoted **B** could be either **B** or c**B**). Following Maxwell's own vocabulary, it's customary to speak of either *electrostatic units* (esu) or *electromagnetic units* (emu). However, one must appreciate the obscure fact that these two are not only different system of units, they are different *traditions* in which symbols may have different meanings...

At first, no C.G.S. electromagnetic units had a specific name. On August 25th, 1900, the International Electrical Congress (IEC) adopted 2 names :

- *Gauss* for the CGS unit of magnetic field (**B**) : 1 G = 10^{-4} T.
- *Maxwell* for the CGS unit of magnetic flux (F) : 1 Mx = 10^{-8} Wb.

The *maxwell*, still known as a *line of force*, is called *abweber* (abWb) using the later naming of CGS electrical units after their MKSA counterparts. Likewise, the *gauss* (1 maxwell per square centimeter) is also called *abtesla* (abT). For *electrostatic* CGS units (esu) the prefix *stat-* is used instead...

In 1930, the Advisory Committee on Nomenclature of the IEC adopted the gilbert (Gb) as a CGS-emu unit equal to the *magnetomotive force* around the border of a surface through which flows a current of (1/4p) abA. The relevant values in SI units are:

1 abA = 10 A
1 Gb = (10/4p) A-t = 0.795774715459... A-t
1 A-t = 1 A

The last expression is to say that no distinction is made in SI units beween an ampere-turn and an ampere. Although the gilbert seems obsolete, the *oersted* (equal to one gilbert per cntimeter) is still very much alive in the trade as a unit of magnetization (density of magnetic dipole moment per unit volume) and/or *magnetic field strength* (the vectorial quantity usually denoted **H**). The oersted was introduced by the IEC in the plenary convention at Oslo, in 1930.

1 A/m = 4 p 10^{-3} Oe = 0.0125663706... Oe

1 Oe = 79.5774715459... A/m

Electrodynamic units are now based on a proper *independent* electrical unit, as advocated by the Italian engineer Giovanni Giorgi (1871-1950) in 1901: The addition of the *ampère* to the MKS system has turned it into a consistent 4-dimensional system (MKSA) which is the foundation for modern SI units.

Paradoxically, this mess comes from a great clarification of Maxwell's: The ratio of the emu value to the esu value of a given field is equal to the speed of light (c = 299792458 m/s). Scholars from bygone days should be credited for accomplishing so much *in spite of* such confusing systems.

Physics Forum: Relation between H and B fields, and D and E fields

(2005-07-15) The Lorentz Force

The Lorentz force on a test particle *defines* the electromagnetic field(s).

The expression of the Lorentz force introduced here defines dynamically the fields which are governed by Maxwell's equations, as presented further down. Neither of these two statements is a logical consequence of the other.

> Such a definition is anachronistic: The concept of an electromagnetic field is due to Michael Faraday (1791-1867) while the modern expression of the force exerted by electromagnetic fields on a moving electric charge was devised in 1892 by H.A. Lorentz (1853-1928).

In *electrostatics*, the electric field **E** present at the location of a particle of charge q summarizes the influence of all other electric charges, by stating that the particle is submitted to an electrostatic force equal to q **E**. This defines **E**.

This *concept* may be extended to *magnetostatics* for a moving test particle. More generally, the electromagnetic fields need not be constant in the following expression of the force acting on a particle of charge q moving at velocity **v**.

The Lorentz Force (1892)

$$\mathbf{F} = q\ (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

The average force exerted per unit of volume may thus be expressed in terms of the *density of charge* r and the *density of current* **j**.

Density of Force

d**F** / dV = r **E** + **j** ´ **B**

Another way to define the *magnetic field* **B** (best called "magnetic induction") would involve the concept of a pointlike magnetic dipole. This may look less elementary now, but it's essentially how **B** was *first* quantified by Coulomb:

Torque on a Magnetic Dipole **m**

m ´ **B**

Potential Energy of a Dipole **m**

- **m . B**

The force exerted on a dipole is **grad** (**m.B**). It vanishes in a uniform field.

(2005-07-18) Electrostatics:

On the electric field from static charges.

Coulomb's *inverse square law* translates into the local differential property of the field expressed by Gauss, namely: div **E** = r/e_o

The SI unit of electric charge is named after the French military engineer Charles Augustin de Coulomb (1736-1806). Using a *torsion balance*, Coulomb discovered, in 1785, that the electrostatic force between two charged particles is proportional to each charge, and inversely proportional to the square of the distance between them. In modern terms, *Coulomb's Law* reads:

Electrostatic Force between Two Charged Particles

$$|| \mathbf{F} || = \frac{|\, q\ q' \,|}{4pe_o\ r^2}$$

The coefficient of proportionality denoted 1/4pe_o (to match the modern conventions about the rest of electromagnetism) is called *Coulomb's constant* and is roughly equal to 9 10^9 if SI units are used (forces in newtons, electric charges in coulombs and distances in meters). More precisely, the modern definitions of the units of electricity (ampere) and distance (meter) give *Coulomb's constant* an exact value in SI units whose digits are the same as the square of the speed of light (itself exactly equal to 299792458 m/s because of the way the meter is defined nowadays):

$$\frac{1}{4pe_o} \;=\; 8.9875517873681764 \;´\; 10^{9} \text{ m / F} \;\;»\;\; 9 \;´\; 10^{9} \;\; \text{N . m}^{2} / \text{C}^{2}$$

The *direction* of the electrostatic force is on the line joining the two charges. The force is *repulsive* between charges of the same sign (both negative or both positive). It's *attractive* between charges of *unlike* signs.

In the language of fields introduced above, all of the above is summarized by the following expression, which gives the *electrostatic* field **E** produced at position **r** by a motionless particle of charge q located at the origin:

Electrostatic Field of a Point Charge at the Origin

$$\boxed{\mathbf{E} \;=\; \frac{q\,\mathbf{r}}{4pe_o\, r^{3}}}$$

Since $\mathbf{r} / r^{3}$ is the *opposite* of the gradient of 1/r, we may rewrite this as :

$$\mathbf{E} \;=\; -\,\mathbf{grad}\; f \qquad\qquad \text{where} \quad f \;=\; \frac{q}{4pe_o\, r}$$

The additivity of forces means that the contributions to the local field **E** of many remote charges are additive too. The *electrostatic potential* f we just introduced may thus be computed additively as well. This leads to the following formula, which reduces the computation of a *three-dimensional* electrostatic field to the integration of a *scalar* over any *static* distribution of charges:

The Electrostatic Field **E** *and Scalar Potential* f

$$\boxed{\mathbf{E} \;=\; -\,\mathbf{grad}\; f \qquad \text{where} \quad f(\mathbf{r}) \;=\; ÒÒÒ\; \frac{r\,(\mathbf{s})}{4pe_o \,\|\, \mathbf{r} - \mathbf{s} \,\|}\; d^{3}\mathbf{s}}$$

The above *static* expression of **E** would have to be completed with a dynamic quantity (namely **-¶A/¶**t, as discussed below) in the *nonstatic* case governed by the full set of Maxwell's equations. Also, the *dynamical* scalar potential f involves a more delicate integration than the above one.

In 1813, Gauss bypassed both dynamical caveats with a *local* differential expression, also valid in *electrodynamics* :

$$\boxed{\text{div}\,\mathbf{E} \;=\; \frac{r}{e_o}}$$

A similar differential relation had been obtained by *Lagrange* in

1764 for Newtonian gravity (which also obeys an inverse square law). This can be established with elementary methods...

One way to do so is to approximate *any* distribution of charges by a sum of pointlike sources: For each point charge q, we can check that the above electric field has a zero divergence away from the source. Then, we observe that our relation does hold *on the average* in any tiny sphere centered on the source, because the integral of the divergence is the flux of **E** through the surface of such a sphere, which is readily seen to be equal to q / e_o ■

Gauss's Theorem of Electrostatics (1813)

In electrostatics, we call *Gauss's Theorem* the integral equivalent of the above differential relation, namely:

$$Q / e_o \quad = \quad \text{ÒÒÒ}_V \; (r / e_o) \; dV \quad = \quad \text{ÒÒ}_S \; \mathbf{E} \,.\, d\mathbf{S}$$

This states that the *outward flux* of the electric field **E** through a surface bounding any given volume is equal to the electric charge Q contained in that volume, divided by the permittivity e_o.

The next section features a typical example of the use of *Gauss's Theorem*.

Another nice consequence is that the field *outside* any distribution of charge with spherical symmetry has the same expression as the field which would be produced if the entire charge was concentrated at the center.

> This property of inverse square laws was first discovered by Newton (using elementary methods) in the context of gravitation, by working out the Newtonian field *outside* an homogeneous spherical shell (incidentally, the field *inside* such a shell is zero). This means that a celestial body with perfect spherical symmetry would behave exactly like a point of the same mass located at its center.

(2005-07-20) **Electric Capacity** [electrostatics, or *low* frequency]
The static charges on conductors are proportional to their potentials.

Consider an horizontal foil carrying a *supercifial charge* of s C/m^2. Let's limit ourselves to points that are close enough to [the center of] the plate to make it look practically infinite. Symmetries imply that the field is vertical (the electrical flux through any vertical surface vanishes) and that its value depends only on the altitude z above the plate (also, if it's E at altitude z, then it's -E at altitude -z).

Let's apply [Gauss's theorem] to a vertical cylinder whose horizontal bases are above and below the foil, each having area S. This *pillbox* contains a charge sS and the flux out of it is 2 E S. Therefore, we obtain for E a *constant* value, which does *not* depend on the distance z to the plate: $E = ½\ s/e_o$.

> Of course, this constant static field produced by an infinite plate under an inverse square law (electrostatics or Newtonian gravitation) may also be worked out using [elementary methods]. It's just more tedious.

Capacitor consisting of two parallel plates :

For two parallel foils with opposite charges, the situation is the *superposition* of two distributions of the type we just discussed: This means an electric field which vanishes outside of the plates, but has twice the above value between them.

Assuming a small enough distance d between two plates of a large surface area S, the above analysis is supposed to be good enough for most points between the plates (what happens close to the edges is thus negligible). The whole thing is called a *capacitor* and the following quantity is its *electric capacity*.

Capacity of Two Parallel Plates

$$C = \frac{e_o\, S}{d}$$

Because $E = s / e_o = q / Se_o = -¶f/¶z$, the difference U between the potentials f of the two plates is $qd / Se_o = q/C$. In other words:

Charge on a Capacitor's Plate

$$q = C\, U$$

This is a general relation. In a static (or *nearly* static) situation, the potential is the same throughout the conductive material of each plate. The proportionality between the field and its sources imply that the charge q on one plate is proportional to the difference of potential between the two plates. We *define* the capacity as the relevant coefficient of proportionality.

Permittivity of Dielectric Materials :

The above holds only if the space between the capacitor's plates is empty (*air* being a fairly good approximation for emptiness). In practice, a dielectric material may be used instead, which behaves *nearly* as the vacuum would if it had a different permittivity. This turns the above formula into the following one. In electrodynamics, the [permittivity] e may depend *a lot* on frequency.

$$C = \frac{e\, S}{}$$

d

A *capacity* is e times a geometrical factor, homogeneous to a *length*.

The SI unit of capacity is called the *farad* (1 F = 1 C/V) in honor of Michael Faraday. It's such a large unit that only its submultiples (mF, nF, pF) are used.

Capacitor Dielectric (Video)

(2008-03-24) Electrostatic Multipole Expansion

Consider the electric field created by *static* charges located near the origin. The electric potential f(**r**) seen by an observer located at position **r** is:

$$ f(\mathbf{r}) \;=\; \text{òòò} \; \frac{r\,(\mathbf{s})}{4pe_o \,\|\, \mathbf{r} - \mathbf{s} \,\|} \; d^3\mathbf{s} $$

If r > s, we may expand 1 / || **r - s** || using the Legendre polynomials P_n :

$$ (\, r^2 - 2\,rs\;\cos q + s^2\,)^{-½} \;=\; \overset{\text{¥}}{\underset{n\,=\,0}{\text{å}}} \; \frac{s^n \;\; P_n(\cos q)}{r^{\,n+1}} $$

q is the angle between **s** and **r**. The Legendre polynomials (A008316) are:

$$ P_n(x) \;=\; (2-1/n)\; x\; P_{n-1}(x) \;-\; (1-1/n)\; P_{n-2}(x) $$

$$ P_0(x) = 1 $$
$$ P_1(x) = x $$
$$ 2\,P_2(x) = -1 + 3x^2 $$
$$ 2\,P_3(x) = -3x + 5x^3 $$
$$ 8\,P_4(x) = 3 - 30x^2 + 35x^4 $$
$$ 8\,P_5(x) = 15x - 70x^3 + 63x^5 $$
$$ 16\,P_6(x) = -5 + 105x^2 - 315x^4 + 231x^6 $$
$$ 16\,P_7(x) = -35x + 315x^3 - 693x^5 + 429x^7 $$

Let's define the *electric multipole moment* (of order n) as the following function of the *unit vector* **u** (where $\cos\theta = \mathbf{u.s} / s$).

$$Q_n(\mathbf{u}) = \iiint \rho(\mathbf{s})\ s^n\ P_n(\cos\theta)\ d^3\mathbf{s}$$

This yields the so-called *multipole expansion* of the electrostatic potential:

$$\varphi(\mathbf{r}) = \varphi(r\,\mathbf{u}) = \frac{1}{4\pi\varepsilon_o} \sum_{n=0}^{\infty} \frac{Q_n(\mathbf{u})}{r^{n+1}}$$

Note that the convergence of this series is not guaranteed unless the above basic [Legendre expansion](#) converges for all values of θ. So, it may not be valid inside a sphere whose radius equals the distance from the origin to the most distant source (i.e., $r > s$ is "safe").

The first term (n=0) corresponds to the field created by a point charge (equal to the sum of all the charges in the distribution) according to [Coulomb's law](#). The second term (n=1) corresponds to the field created by an *electric dipole moment* **P**, as discussed [elsewhere on this site](#) in full details (including non-static cases).

$$Q_1(\mathbf{u}) = \mathbf{u} \cdot \mathbf{P}$$

The names of multipoles follow the Greek scheme used for [polygons](#) and other scientific things... The sequence starts with the "monopole moment" for n=0 (which is really the total electric charge) and the number of "poles" doubles at each step: Monopole, dipole, quadrupole (not "tetrapole"), octupole or octapole, [hexadecapole](#), dotriacontapole, tetrahexacontapole ("hexacontatetrapole" is *not* recommended) and octacosahectapole (128 poles, for n=7).

[Quadrupole](#) | [Electric Multipole Expansion](#)
[What's a hexacontatetrapole, anyway](#)? by Timothy Gay [tetrahexacontapole, rather]

(2008-04-03) **The Birth of Electromagnetism (Ørsted, 1820)**

A steady current produces a steady magnetic field.

Electricity and magnetism were known as [separate phenomena](#) for centuries.

In 1752, [Benjamin Franklin](#) (1706-1790) performed his famous (and dangerous) [electric kite experiment](#) which established firmly that lightning is an *electrical* discharge. Franklin himself never wrote about the story but he proofread the account which [Joseph Priestley](#) (1733-1804) gave 15 years after the event. Priestley concludes that report with the comment: "This happened in June 1752, a month after the electricians in France had verified the same theory, but before he heard of anything they had done."

It's unclear who those "electricians in France" are, but the following text by Louis-Guillaume Le Monnier appears (in French) in the Encyclopédie of *Diderot and d'Alembert* (71818 articles in 35 volumes, the first 28 of which were edited by Diderot himself and published between 1751 and 1766).

> " A violent electric spark can modify a compass or magnetize small needles, according to the direction given to that spark. It has long been observed that a bolt of lightning (which is only a large electric spark) is able to magnetize all sorts of iron and steel tools stored in boxes and to give the nails in a ship enough magnetic properties to influence a compass at a fair distance. This formidable fluid has simply changed into true magnets some iron crosses of ancient belltowers that have been exposed several times to its powerful effects. "

Indeed, many people must have wondered why the needle of a compass goes haywire near a bolt of lightning. However, the havoc brought about by lightning may have precluded the proper investigation of this comparatively delicate aspect.


Domenico Romagnosi

In 1802, the Italian jurist Domenico Romagnosi (1761-1835) experimented with a voltaic pile to charge capacitors. He observed that their sudden discharges would deflect a nearby magnetic needle. This raw observation was reported in newspapers. Although Romagnosi didn't explictly mentioned the connection between magnetism and electric current, at least two others did it for him when they described his experiments:

- **Essai théorique et expérimental sur le Galvanisme** (1804) p. 340 by Giovanni Aldini (1762-1834).
- **Manuel du Galvanisme** (1805) by Joseph Izarn (1766-1847).

The crucial fact that a *steady* electric current does produce magnetism was finally established, by a Danish scholar, who became famous for that:


Hans Christian Ørsted

On April 21, 1820, the Danish physicist Hans Christian Oersted (1777-1851) was preparing demonstrations for one of his lectures at the University of Copenhagen. He noticed that a compass needle was deflected when a large electrical current was flowing in a nearby wire. This precise instant marks the birth of *electromagnetism*, the study of the *interrelated* phenomena of electricity and magnetism.

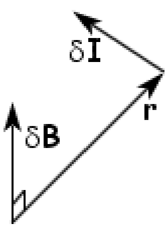
Contrary to popular belief, the discovery of Ørsted was not entirely a chance accident (R.C. Stauffer, 1953). As early as 1812, Ørsted had published speculations that electricity and magnetism were connected. So, when the experimental evidence came to him, he was prepared to make the best of it.

> François Arago (1786-1853; X1803) was the first person to build an electromagnet, in September 1820, by placing iron in a wire coil.

(2008-01-04) **Biot-Savart Law of Magnetostatics (1820)**

The magnetic field produced by a static distribution of electric currents.

Experimentally, Ørsted had found that a given current in a straight wire creates in its *immediate vicinity* a magnetic field which seems inversely proportional to the distance from the wire. The French physicists *Jean-Baptiste Biot* and *Félix Savart* proposed that the contribution of each *piece* of the wire actually varies inversely as the *square* of the distance to the observer. Over the entire length of the wire, such contributions do add up to a total field which varies inversely as the distance from the wire. The Biot-Savart law can be precisely stated as follows:



Contribution to the Magnetostatic Field at the Origin of a Current Element d**I** *at Position* **r**.

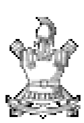
$$d\mathbf{B} = \frac{\mu_0 \; \mathbf{r} \times d\mathbf{I}}{4\pi \, r^3}$$

Jean-Baptiste Biot

In this, d**I** is the quantity (current multiplied by the small length it travels) which results from integrating the current density **j** (current per unit of surface) over a small element of *volume*. In particular, for a thin wire circuit whose length element d**s** is traversed by a total current I (counted positively in the direction of d**s**) we have d**I** = I d**s**.

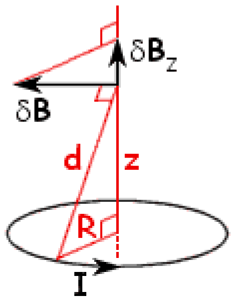
> The Biot-Savart law is for steady currents only. For changing currents, a term that falls off as 1/r must be added, as specified below.

Note that we're using the vector **r** which goes from the location of interest to the sources. This is a convenient viewpoint for practical computations which seek to obtain a magnetic field at a specific point from remote distributions of current. However, *many* authors take the opposite viewpoint (opposite sign of **r**) to describe the field produced at a remote location by currents located at the origin.




On October 30, 1820, the Biot-Savart law was presented to the *Académie des Sciences* jointly by the physicist Jean-Baptiste Biot (1774-1862; X1794) and his protégé Félix Savart (1791-1841) who is also remembered for the logarithmic unit of musical interval named after him (1000 savarts per decade, or about 301.03 savarts per octave). A rounded version of the *savart* unit (exactly 1/301 of an octave) was called *eptaméride* in an earlier scheme devised by the acoustician Joseph Sauveur (1653-1716).

In many practical applications, the magnetic field is known to have a simple symmetry and Ampère's Law (below) may yield the value of the magnetic field throughout space without tedious integrations (just like the theorem of Gauss easily yields the electrostatic field in cases with spherical, planar or cylindrical symmetries). One example where no such shortcut is available is that of the magnetic induction on the axis of a circular current loop:



In that case, all radial contributions cancel out, so the resulting magnetic induction **B** is oriented along the axis ($\mathbf{B} = \mathbf{B}_z$).

Because of the similarity of the relevant triangles, the contribution dB_z is R/d times what's given by the above law:

$$dB_z \;=\; (R / d)\; dB \;=\; (R / d)\; (m_o I / 4pd^2)\; ds$$

As the elements ds simply add up to the circumference (2pR) we obtain:

$$B_z \;=\; (R / d)\; (m_o I / 4pd^2)\; (2pR) \;=\; ½\, m_o I\; R^2 / d^3$$

In particular, the field at the center of the loop (d = R) is: $B_z = m_o I / 2R$.

Helmholtz Coil

Consider two coils (or two loops) like the above, sharing the same vertical axis. Let their respective altitudes be $+a$ and $-a$. By the previous result, the magnetic induction B (on the axis) at altitude z is:

$$B \;=\; ½\, m_o I\; R^2 \;\{\; [\, R^2 + (a\text{-}z)^2 \,]^{-3/2} \;+\; [\, R^2 + (a\text{+}z)^2 \,]^{-3/2} \;\}$$

The second derivative of this expression with respect to z at z = 0 is:

$$B''(0) \;=\; 3\, m_o I\; R^2\; [\, 4a^2 - R^2 \,]\; (\, R^2 + a^2 \,)^{-7/2}$$

The value $a = ½\, R$ is thus the largest for which the magnetic induction has a *single* maximum along the vertical axis, in the center of the apparatus (for larger values of a, B″ is positive at the center z = 0, which indicates a minimum there).

This configuration where the separation between the two loops is equal to their radius ($2a = R$) is known as a *Helmholtz coil*. It yields a magnetic induction which is *almost uniform* near the center of the coil. Namely:

$$B \;=\; (4/5)^{3/2}\; m_o I \,/\, R \;=\; 0.71554...\; m_o I \,/\, R$$

Wikipedia: Biot-Savart Law

(2008-05-12) **Magnetic Scalar Potential (in a *current-free* region)**

A *multivalued* function whose gradient is the *magnetostatic* induction.

In a *current-free* region of space, a scalar potential can be defined (called the *magnetic scalar potential*) whose negative gradient is the magnetostatic induction given by the Biot-Savart law.

For a simply-connected region, such a potential is well-defined (up to a uniform additive constant). Otherwise, an essential ambiguity arises whenever the region contains loops which are interlocked with loops of outside current. In that case a continuous potential can only be defined modulo a certain number of discrete quantities (each of which corresponds to one interlocking outside current).

The *magnetic scalar potential* V for the induction **B** created by a loop of thin wire is simply proportional to the current I in that loop and to the solid angle Ω subtended by the *south side* of that loop at the location of the observer :

$$\boxed{\begin{aligned} \mathbf{B} &= - \,\mathbf{grad}\, V \\ V &= - \frac{\mu_0 I}{4\pi}\, \Omega \end{aligned}}$$

The solid angle Ω is defined modulo 4π, which is consistent with the aforementioned "ambiguity". The sign convention is such that the south side of a small loop is seen at a solid angle which exceeds a multiple of 4π by a small positive quantity.

This is just a nice way to express the Biot-Savart law while making it clear that, in static distributions, all currents must circulate in *closed loops* (div **j** = 0). Neither this approach nor the Biot-Savart law itself can deal with dynamic distributions where local electric charges may vary according to the inbound flux of current.

(2008-03-10) **There are no magnetic monopoles ! (Peregrinus, 1269)**

The magnetic field (*magnetic induction* **B**) has vanishing divergence.

It's a simple matter to establish with elementary methods that the above Biot-Savart law describes a field with zero divergence: First, we can verify directly (using Cartesian expressions) that the divergence of the Biot-

Savart field vanishes at any nonzero distance from its source d**I**.

> We could also remark that the Biot-Savart expression is proportional to the rotational of the vector field dI / r. As such, it has zero divergence.

Then, we may check that d**B** has zero flux through *any* tiny sphere centered on d**I** (this is true because of a trivial symmetry argument). Thus, the divergence of the Biot-Savart field is identically zero, even at the very location of a source!

> By contrast, that second part of the argument does not hold with spheres centered on an elementary electric charge for the Coulomb field. This is why the divergence of the electric field turns out to be proportional to the local density of electric charge (Gauss's Law).

The magnetic field may well have sources other than electrical currents (including the dipole moments related to the intrinsinc spins of *point particles* which are part of the modern quantum picture). Nevertheless, all sources ever observed yield magnetic fields with no divergence. Like all scientific facts, this can be stated as a *law* which holds until disproved by experiment:

Gauss-Weber Law
(Pèlerin's Law, 1269)

$$\boxed{\operatorname{div} \mathbf{B} = 0}$$

In the vocabulary of multipoles, only monopole fields have nonzero divergence (in particular, any dipolar field is divergence free). Thus, the vanishing divergence of **B** is often expressed by stating that *there are no magnetic monopoles*.

This was first stated in 1269 by the French scholar Peter Peregrinus (*Pierre Pèlerin de Maricourt*) who first described magnetic poles and observed that a magnetic pole could not be isolated (they always come in opposite pairs).

This law has survived all modern experimental tests so far and it is postulated to remain valid in the general nonstatic case. It is arguably the *oldest* of the four equations of Maxwell. Unfortunately, unlike the other three (Gauss's Law, Faraday's Law, Ampère-Maxwell Law) it has no universally accepted name... It's sometimes referred to as the "magnetic Gauss law", which is rather awkward. Calling it the "Gauss-Weber Law" is perfectly acceptable because the name of Gauss is universally associated with the *electric* counterpart of the law while the *magnetic* flux so governed (see next paragraph) is closely associated with the name of Wilhelm Eduard Weber (1804-1891) a younger colleague of Gauss after whom the SI unit of magnetic flux (Wb) is named. However, I argue that the law ought to be called *Pèlerin's law* (or the Law of Peregrinus).

Because of that law, the *magnetic flux* (F) enclosed by a given oriented loop is well-defined as the flux of the magnetic induction **B** through *any* surface which is bordered (and oriented) by that loop.

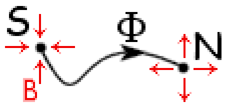
> On the other hand, two open surfaces with the same border need not

have the same "electric flux" through them, because div E isn't zero.

Searching for magnetic monopoles

A famous argument of Paul Dirac shows that the existence of even a single *true* magnetic monopole in the Universe would imply a quantizarion of electric charge everywhere (as observed). Many physicists do not yet rule out the existence of magnetic monopoles (like any proper physical law, Pèlerin's law only holds until proven wrong experimentally).

A true magnetic monopole would be completely surrounded by a closed surface traversed by a nonzero *total magnetic flux*. The two ends of a thin flux tube do not qualify as monopoles, because the *return flux* through the cross-section of the tube balances exactly the nonzero flux traversing the rest of any closed surface enclosing one pole (but not the other). For example, the magnetic flux which flows from north to south *outside* a long bar magnet is exactly balanced by the flux of the strong field which flows from south to north *inside* the magnet itself.



Mathematically, we may envision an *ideal flux tube* (often dubbed a *Dirac string*) as the infinitely thin version of the above, namely a line carrying, within itself, a *finite* magnetic flux from one of its extremities (the south pole) to the other (the north pole). The total magnetic flux (F) through a cross-section is *constant* along such a *Dirac string*.

In the Summer of 2009, two independent teams found that actual flux tubes in some so-called *spin ices* could have cross-sections small enough to fit in the spaces between the atoms of the crystal. Such tubes behave like the *ideal* Dirac strings presented above. The whole thing looks as though some of the cells in the crystal contain a magnetic monopole while an opposite monopole is found nearby, possibly several cells away...

- Dirac Strings and Magnetic Monopoles in Spin Ice Dy2Ti2O7
  Jonathan Morris, Alan Tennant et al.
  Helmholtz-Zentrum Berlin für Materialien und Energie, Germany.
- Magnetic Coulomb Phase in the Spin Ice Ho2Ti2O7
  Tom Fennel, P.P. Deen, A.R. Wildes et al.
  Institut Laue-Langevin de Grenoble, France.

Those exciting discoveries do not violate Pèlerin's law (magnetic poles still come only in pairs, connected by thin flux tubes). Unfortunately, they were heralded in *press releases*, *review articles* and *popular magazines* as a "discovery of magnetic monopoles". So, a new urban legend was born which makes is *slightly* more difficult to teach basic science...

- Observing Monopoles in a Magnetic Analog of Ice (M.J.P. Gingras).
- Magnetic monopoles detected in a real magnet for the first time (2009-09-03)
- 'Magnetic charge' measured in spin ice (PhysicsWorld, 2009-10-15).

(2008-04-25) Ampère's law:
The *static* version (1825)
The magnetic circulation is m_o times the enclosed current.


André-Marie Ampère

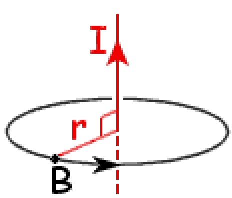
What Gauss did in 1813 for the Coulomb law of 1785, André-Marie Ampère (1775-1836) did in 1825 for the Biot-Savart law of 1820. Unlike the law of Gauss, *Ampére's law* only holds in the *static* case. It had to be amended by Maxwell in 1861 for the *dynamic* case. Here's Ampère's *static* law (1825) in differential form:

$$\mathbf{rot\ B} = m_o\,\mathbf{j}$$

By the Kelvin-Stokes formula, the circulation of a vector around an oriented loop is equal to the flux of its rotational (*curl*) through any smooth oriented surface bordered by that loop. This yields *Ampère's law in integral form* :

$$m_o\ I\ \ ^{\text{o}}\ \ m_o\ \grave{\text{O}}\grave{\text{O}}_S\ \mathbf{j\,.\,dS} \ = \ \grave{\text{O}}_{\P S}\ \mathbf{B\,.\,dr}$$

The simplest (and most fundamental) direct application of Ampère's law is to *retrieve* the experimental fact which prompted the formulation of the Biot-savart law to begin with, namely that the magnetic induction **B** due to a long straight wire is inversely proportional to the distance from that wire:



Indeed, consider a circular loop of radius r whose axis is a straight wire carrying a current I. For reasons of symmetry, the magnetic induction **B** on that loop is tangent to it. Its projection on the oriented tangent is a constant B (see sign conventions). The magnetic circulation is 2pr B and *Ampère's law* gives:

$$2pr\ B = m_o\,I \qquad \text{or, equivalently:} \qquad B = m_o\,I\,/\,2pr$$

Another popular (and important) application of *Ampère's law* yields the magnetic field due to an infinitely long *solenoid* (of arbitrary cross-section) : For a long solenoid consisting of n loops of wire *per unit of height* (each carrying the same current I) the magnetic induction vanishes outside and has the following value *inside* the solenoid:

$$B = m_o\ n\,I$$

This can be established by noticing *first* that the *direction* of the magnetic induction **B** must be everywhere *vertical* (i.e., parallel to the axis of a solenoid with *horizontal* cross-section). That is so because the horizontal contribution of each element of current is exactly cancelled by the horizontal contribution from its mirror image with respect to the horizontal plane of the observer.

We may then apply *Ampère's law* to any rectangular loop with two vertical sides and two horizontal ones (on which the circulation of **B** is zero, because it's perpendicular to the line element). This establishes that the magnetic field is constant inside the solenoid and constant outside of it, with the difference between the two equal to the value advertised above. (The fact that the constant value of the induction outside of the solenoid must be zero is just common sense, or else the magnetic energy of the solenoid *per unit of height* would be infinite.)

Sneak Preview :

In 1861, Maxwell was able to amend the *static* law of Ampère into the following generalization, which holds in *all cases* (including changing charge distributions).

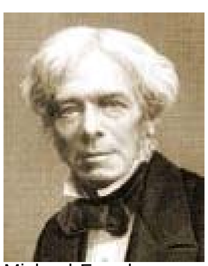
Ampère-Maxwell Law (1861)

$$\boxed{\textbf{rot B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0\, \mathbf{j}}$$

We shall *postpone* the discussion of this crowning achievement (which made the entire structure of electromagnetism consistent) so we can present *first* a key breakthrough made by *Faraday* on August 29, 1831 (when *James Clerk Maxwell* was 2 months old): The law of magnetic induction.

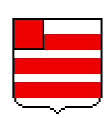
(2005-07-19) Faraday's Law of Electromagnetic Induction (1831)

On the electric circulation *induced* around a *varying* magnetic flux.


Michael Faraday

Michael Faraday (1791-1867) was the son of a blacksmith, and a bookbinder by trade. Effectively, he would remain *mathematically illiterate*, but he became an exceptionally brilliant experimental scientist who would lay the conceptual foundations that occupied several generations of mathematical minds. In 1810, Faraday started attending the lectures that Humphry Davy (1778-1829) had been giving at the Royal Institution of London since 1801. In December 1811, Faraday became an assistant of Davy, whom he would eventually surpass in knowledge and influence. Faraday was elected to the Royal Society in 1824, in spite of the jealous opposition of Sir Humphry Davy (who was its president from 1820 to 1827). In February 1833, Faraday became the first *Fullerian Professor of Chemistry* at the Royal Institution The chair was endowed


Humphry Davy



by his mentor and supporter [John "Mad Jack" Fuller] (1757-1834).

John Fuller

Arguably, the greatest of Faraday's many scientific contributions was the *Law of Induction* which he formulated in 1831. After explaining the 1820 observation of [Ørsted] in terms of what we now call the magnetic *field*, Faraday did much more than invent the electric motor. Eventually, he opened entirely new vistas for physics. He proposed that light itself was an electromagnetic phenomenon and lived to be proven right mathematically by his young friend, *James Clerk Maxwell*.

Faraday's Law (1831)

$$\mathbf{rot\ E} + \frac{\partial\,\mathbf{B}}{\partial\, t} = \mathbf{0}$$

Heinrich Friedrich "Emil Khristianovich" [Lenz] (1804-1865).
Lenz's Law (1833).


Heinrich Lenz

The magnetic flux...
F = **B . S**
dF = d**B .** S + **B .** dS
First term = Magnetic Induction. Second Term = Lorentz Force.

(2008-04-02) Self-Inductance

On the electric induction produced in a circuit by its own magnetic field.

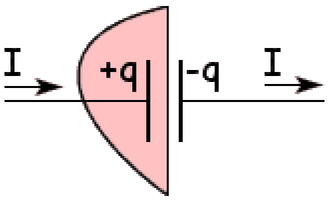

Joseph Henry, 1875

[Joseph Henry] (1797-1878) discovered the [law of induction] independently of Faraday and he went on to remark that the magnetic field created by a changing current in any circuit would induce in the circuit itself an electromotive force which tends to oppose the change in current.

(2008-04-30) **Ampère's law generalized by Maxwell (1861)**

The Ampère-Maxwell law holds even with changing charge distributions.

A simple way to show that the above static version of *Ampère's law* fails in the presence of changing electric fields is to consider how a capacitor breaks the flux of current it receives from a conducting wire: An open flat surface between the capacitor's two plates has no current flowing through it, unlike a surface *with the same border* which the wire happens to penetrate.



In 1861, Maxwell realized that, since electric charge is conserved, a difference in the flux of current through two surfaces sharing the same border must imply a change in the total electric charge q contained in the volume *between* those two surfaces.

By Gauss's theorem, this translates into a changing flux of the *electric* field through the closed surface formed by the two aforementioned open surfaces. More precisely, and remarkably, the "missing" flux of the current density $\mathbf{j}$ is exactly balanced by the flux of the vector e_o ¶**E**/¶t.

Maxwell identified this as the density of a quantity he called *displacement current*. He saw that the sum of the actual current and the displacement current was *divergence-free*. This made that sum a prime candidate for taking on the role played by the ordinary density of current in the static version of Ampère's law. Therefore, Maxwell proposed that *Ampère's law* should be amended accordingly:

$$\textbf{rot B} = m_o \left(\mathbf{j} + e_o \, ¶\mathbf{E}/¶t \right)$$

Putting the fields on one side and the sources on the the other, we obtain:

Ampère-Maxwell Law (1861)

$$\boxed{\textbf{rot B} - \frac{1}{c^2}\frac{¶\,\mathbf{E}}{¶\,t} = m_o\,\mathbf{j}}$$

At this point, we merely *define* c as a convenient constant satisfying:

$$e_o\, m_o\, c^2 = 1$$

The paramount fact that c turns out to be the speed of light will be seen to be a consequence of putting all of Maxwell's equations together...

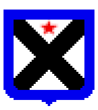
(2005-07-18)   **On the History of Maxwell's Equations**

The 4 basic laws of electricity and magnetism, discovered one by one.

Gauss' Electric Law = Coulomb's Law
Gauss's Magnetic Law.
Faraday's Law of Induction.
Ampère's Law.

(2005-07-09)   **Maxwell's Equations Unify Electricity and Magnetism**

They predicted *electromagnetic* waves before Hertz demonstrated them.



I have also a paper afloat, with an electromagnetic theory of light, which, till I am convinced to the contrary, I hold to be great guns.
James Clerk Maxwell   (1831-1879)
[letter to Charles H. Cay (1841-1869) dated January 5, 1865]

Maxwell's equations   govern the electromagnetic quantities defined above:

- The *electric field* **E**   (in V/m or N/C).
- The *magnetic induction* **B**   (in teslas; T or Wb/m^2).
- The *density of electric charge* r   (in C/m^3)
- The *density of electric current* **j**   (in A/m^2)

Maxwell's Equations (1864) *in modern vectorial form* :

$$\textbf{rot E} + \frac{¶\,\mathbf{B}}{¶\,t} = \mathbf{0} \qquad \text{div}\,\mathbf{E} = \frac{r}{e_o}$$

$$\textbf{rot B} - \frac{1}{c^2}\frac{¶\,\mathbf{E}}{¶\,t} = m_o\,\mathbf{j} \qquad \text{div}\,\mathbf{B} = 0$$

The three electromagnetic *constants* involved are tied by one equation:

$$e_o\, m_o\, c^2 = 1$$

- e_o is the *electric permittivity of the vacuum*   (in F/m)
- m_o is the magnetic permeability of the vacuum   (in H/m or N/A^2)

- c is the *speed of light in a vacuum*, best called Einstein's constant.

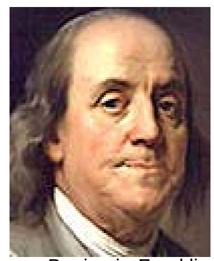
(2005-07-09) **Continuity Equation & Franklin-Watson Law** (1746)

The *continuity equation* expresses the conservation of electric charge.

A direct consequence of Maxwell's equations is the following relation, which expresses the *conservation of electric charge* (HINT: div **rot B** vanishes). This conclusion holds *if and only if* the 3 aforementioned electromagnetic constants are related as advertised: $\varepsilon_o \mu_o c^2 = 1$

Continuity Equation

$$\operatorname{div} \mathbf{j} + \frac{\partial \rho}{\partial t} = 0$$


Benjamin Franklin

Historically, the relation is reversed: The conservation of electric charge had been formulated before 1746, independently by Benjamin Franklin (1706-1790) and William Watson (1715-1787). This was more than a century before Maxwell used it to generalize Ampère's law into the proper equation which made the whole theoretical structure perfect.

(2005-07-09) **Electromagnetic Radiation : From light to *radio waves*.**

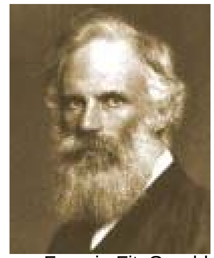
Electromagnetic fields propagate at the speed of light (c).

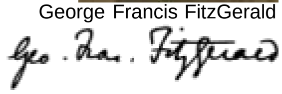
Using the identity **rot rot V** = **grad** div **V** - Δ**V** when $\rho = 0$ and $\mathbf{j} = \mathbf{0}$, Maxwell's equations imply that *any* electromagnetic component ψ verifies:

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = \Delta \psi$$

This *wave equation* shows that electromagnetism propagates at *celerity* c in a vacuum. Thus, Maxwell's equations support the *electromagnetic theory of light* which Michael Faraday had proposed well before all the evidence was in. (He engaged in such speculations in 1846, at the end of one of his famous lectures at the *Royal Institution*, because he had run out of things to say that particular Friday night!)

In 1883, the Irish physicist George FitzGerald (1851-1901) remarked that an oscillating current ought to generate electromagnetic radiation (*radio waves*). FitzGerald is also remembered for his 1889 hypothesis that all moving objects are foreshortened in the direction of motion (the *relativistic* FitzGerald-Lorentz contraction).


George Francis FitzGerald



The propagation of *radio waves* was first demonstrated experimentally in 1888, by Heinrich Rudolf Hertz (1857-1894).

(2005-07-15) Electromagnetic Energy & Poynting Vector

The Lorentz force transfers energy between the field and the charges.

The power **F.v** of the Lorentz force is q **E.v**. Thus, the power received by the electric charges per unit of volume is **E.j**. The charge carriers may then convert the power so received from the local electromagnetic field into other forms of energy (including the kinetic energy of particles).

> Conversely, E.j can be negative, in which case there is a transfer of energy from the charge carriers to the field. One process can be seen as a time-reversal of the other. In this, it is essential to retain both the retarded and advanced solutions of Maxwell's equations; the motion of the sources and the changes in the field may cause each other !

The quantity **E.j** may be expressed in terms of the electromagnetic fields by dotting into $-\mathbf{E}/\mu_0$ both sides of the Ampère-Maxwell equation:

$$-\mathbf{E}\cdot\mathbf{rot\,B}/\mu_0 + \epsilon_0\,\mathbf{E}\cdot\partial\mathbf{E}/\partial t = -\mathbf{E}\cdot\mathbf{j}$$

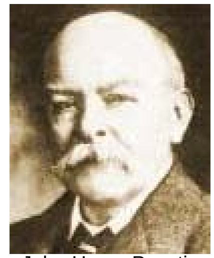
The identity $\mathbf{E}.\mathbf{rot\,B} = \mathbf{B}.\mathbf{rot\,E} - \mathrm{div}\,\mathbf{E}\times\mathbf{B}$ and Faraday's law yield:

$$-\mathbf{E}\cdot\mathbf{rot\,B} = \mathbf{B}\cdot\partial\mathbf{B}/\partial t + \mathrm{div}\,\mathbf{E}\times\mathbf{B}$$

Plugging that into the previous equation, we obtain an important relation:

Electromagnetic Balance of Energy Density : Poynting Theorem (1884)

$$\mathrm{div}\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\right) + \frac{\partial}{\partial t}\left(\frac{\epsilon_0\,\mathbf{E}^2 + \mathbf{B}^2/\mu_0}{2}\right) = -\mathbf{E}\cdot\mathbf{j}$$


John Henry Poynting

This is due to a pupil of Maxwell, John Henry Poynting (1852-1914). $\mathbf{S} = \mathbf{E}´\mathbf{B} / m_o$ is the *Poynting vector*.

In the above, the right-hand side is the *opposite* of the power delivered by the field to the sources, per unit of volume. So, it's the density of the power released by the sources to the field. The left-hand side is thus consistent with the following energy for the electromagnetic field:

Electromagnetic Energy Density

$$\boxed{{}^1/_2 \; e_o \; (\mathbf{E}^2 + c^2 \mathbf{B}^2)}$$

The above *Poynting theorem* states that, the variation of this energy in a given volume comes from power that is either delivered directly by inside sources or radiated through the surface, as the flux of the *Poynting vector*.

In the context of Classical Field Theory, the above is the *Hamiltonian density*, whereas the *Lagrangian density* of the electromagnetic field is a Lorentz scalar (a mere pseudo-scalar like **E**.**B** won't do) namely:

Lagrangian Density

$$\boxed{{}^1/_2 \; e_o \; (\mathbf{E}^2 - c^2 \mathbf{B}^2)}$$

Identifying the above with the usual formulas for the Hamiltonian (H=T+U) and the Lagrangian (L=T-U) we may think of the square of **E** as a *kinetic* term (T) and the square of **B** as a *potential* term (U). The analogy is more compelling when a special *gauge* is used which makes the electrostatic potential (f) vanish everywhere, as is the case for the standard Lorenz gauge in the particular case of a crystal of magnetic dipoles. For in such cases, the electric field consists entirely of time-derivatives of **A**...

The above is for the electromagnetic field by itself. In the presence of charges which interact with the field in the form of a density of Lorentz forces, the corresponding Lagrangian density of interaction should be added:

Lagrangian Density

$$\boxed{{}^1/_2 \; e_o \; (\mathbf{E}^2 - c^2 \mathbf{B}^2) - (r\, f - \mathbf{j}.\mathbf{A})}$$

Still missing are all the non-electromagnetic terms which are needed to determine correct expressions of the conjugate momenta and Hamiltonian density...

(2005-07-15) **Electromagnetic Planar Waves** (Progressive Waves)

The simplest solutions to *Maxwell's equations*, away from all sources.

In the absence of electromagnetic sources (r = 0, **j** = **0**) we may look for electromagnetic fields whose values *do not depend* on the y and z cartesian coordinates. A solution of this type is called a *progressive planar wave* and it may be established directly from the above *equations of Maxwell*, without invoking the electromagnetic potentials introduced below.

Indeed, when all derivatives with respect to y or z vanish, the 8 *scalar* relations which express Maxwell's equations in cartesian coordinates become:

$$\frac{\P B_x}{\P x} = 0 \qquad\qquad \frac{\P E_x}{\P x} = 0$$

$$0 = \frac{1}{c^2}\frac{\P E_x}{\P t} \qquad\qquad 0 = -\frac{\P B_x}{\P t}$$

$$-\frac{\P B_z}{\P x} = \frac{1}{c^2}\frac{\P E_y}{\P t} \qquad\qquad -\frac{\P E_z}{\P x} = -\frac{\P B_y}{\P t}$$

$$\frac{\P B_y}{\P x} = \frac{1}{c^2}\frac{\P E_z}{\P t} \qquad\qquad \frac{\P E_y}{\P x} = -\frac{\P B_z}{\P t}$$

To solve this, we introduce the new variables $u = t - x/c$ and $v = t + x/c$

For any quantity y, the two expressions of the differential form dy yield the expressions of the partial derivatives with respect to the new variables :

$$dy = \frac{\P y}{\P t}dt + \frac{\P y}{\P x}dx = \frac{\P y}{\P u}du + \frac{\P y}{\P v}dv$$

$$dt = {}^1/_2 \,(dv + du) \qquad \text{and} \qquad dx = {}^c/_2 \,(dv - du)$$

$$\text{Therefore,} \quad \begin{cases} \dfrac{\P y}{\P u} = \dfrac{1}{2}\left(\dfrac{\P y}{\P t} - c\,\dfrac{\P y}{\P x} \right) \\[2ex] \dfrac{\P y}{\P v} = \dfrac{1}{2}\left(\dfrac{\P y}{\P t} + c\,\dfrac{\P y}{\P x} \right) \end{cases}$$

We may apply this back and forth when y is one of the cartesian components of **E** or **B**, using the above relations between those. For example:

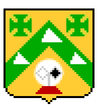
$$\frac{\P E_y}{\P u} = \frac{1}{2}\frac{\P E_y}{\P t} - \frac{c}{2}\frac{\P E_y}{\P x} = -\frac{c^2}{2}\frac{\P B_z}{\P x} + \frac{c}{2}\frac{\P B_z}{\P t} = c\,\frac{\P B_z}{\P u}$$

Thus, $E_y - c\,B_z$ doesn't depend on u. Likewise, neither does $E_z + c\,B_y$

Similarly, both $E_y + c\,B_z$ and $E_z - c\,B_y$ do not depend on v.

(2009-12-13) Radiation Pressure (Maxwell 1871, Lebedev 1899)
Electromagnetic waves (or stationary fields) exert a mechanical pressure.

In 1871, Maxwell himself predicted this as a consequence of his own equations. In 1876, Adolfo Bartoli (1851-1896) remarked that the existence of radiation pressure is also an unavoidable consequence of thermodynamics. (Radiation pressure is thus sometimes called *Maxwell-Bartoli pressure*.) *Maxwell-Bartoli pressure* was first demonstrated experimentally by Pyotr Lebedev in 1899.



In 1873, Sir William Crookes (1832-1919) believed that he had demonstrated radiation pressure when he came up with the so-called radiometer (or light-mill) displayed on his coat-of-arms. This ain't so, despite what many sources still state. Radiation pressure is too weak to turn the vanes of such a radiometer and its theoretical torque *opposes* the *observed* rotation! (The dark sides of the vanes are actually *receeding*.) Crookes' radiometer is actually a subtle heat engine in which the rarefied gas in the glass enclosure plays an essential rôle (it wouldn't work in a hard vacuum). The moving torque is due to what's called the "thermal creep" of the gas molecules near the edges of the vanes, where a substantial temperature gradient is maintained... This was first correctly explained by Osborne Reynolds (1842-1912) in a paper which Maxwell refereed the year he died (1879). Maxwell published immediately his own paper on the subject, giving credit to Reynolds for the key idea but criticizing his mathematics (the Reynolds paper itself was only published in 1881).


Pyotr Lebedev

The first proper measurement of *radiation pressure* was made in 1899 by Pyotr Lebedev (1866-1912). In 1901, the *pressure of light* was measured at Dartmouth by Nichols and Hull to an accuracy of about 0.6% (the original Nichols radiometer is at the Smithsonian). To avoid the aforementioned effect (dominant in Crookes radiometers) a *Nichols radiometer* must operate in a *high vacuum*.

(2005-07-13) Electromagnetic Potentials & Lorenz Gauge

Devised by Ludwig *Lorenz* in 1867 [when H.A. Lorentz was only 14].

Since Maxwell's equations assert that the divergence of **B** vanishes, there is necessarily a *vector potiental* **A** of which **B** is the rotational (or curl).

$$\boxed{\mathbf{B} \;=\; \mathbf{rot}\ \mathbf{A}}$$

Faraday's law now reads **rot** [**E** + ¶**A**/¶t] = 0 . The square bracket is the gradient of a *scalar potential*, called -f for compatibility with electrostatics:

$$\boxed{\mathbf{E} \;=\; -\ \mathbf{grad}\ f - \P\mathbf{A}/\P t}$$

These two equations *do not* uniquely determine the potentials, as the same fields are obtained with the following substitutions of the potentials, valid for *any* smooth scalar field y.

$$\mathbf{A} \;\neg\; \mathbf{A} + \mathbf{grad}\ y \qquad\qquad f \;\neg\; f - \P y/\P t$$

This leeway can be used to make sure the following equation is satisfied, as proposed by Ludwig Lorenz in 1867. (Watch the spelling... There's no "t".)

The Lorenz Gauge (1867)

$$\boxed{\mathrm{div}\ \mathbf{A} + \frac{1}{c^2}\,\frac{\P f}{\P t} \;=\; 0}$$

The *Lorenz Gauge* does not eliminate the above type of leeway. It restricts it to a free field y propagating at celerity c, according to the *wave equation* :

$$Dy - \frac{1}{c^2}\,\frac{\P^2 y}{\P t^2} \;=\; 0$$

The two Maxwell equations which don't involve electromagnetic sources are equivalent to the above definitions of **E** and **B** in terms of electromagnetic potentials. *Using the Lorenz Gauge*, the other two equations boil down to the following relations between the electromagnetic sources and the potentials:

D'Alembert's Equations

$$\boxed{\begin{array}{l} Df - \dfrac{1}{c^2}\,\dfrac{\P f}{\P t} \;=\; -\dfrac{r}{e_o} \\[2ex] D\mathbf{A} - \dfrac{1}{c^2}\,\dfrac{\P \mathbf{A}}{\P t} \;=\; -\,m_o\,\mathbf{j} \end{array}}$$

Without the *Lorenz Gauge*, more complicated relations would hold:

$$D f \; - \; \frac{1}{c^2} \frac{\P\, f}{\P\, t} \; = \; -\frac{r}{e_o} \; - \frac{\P}{\P\, t} \left(\text{div}\, \mathbf{A} + \frac{1}{c^2} \frac{\P\, f}{\P\, t} \right)$$

$$D\mathbf{A} \; - \; \frac{1}{c^2} \frac{\P\, \mathbf{A}}{\P\, t} \; = \; - m_o\, \mathbf{j} \; + \; \mathbf{grad} \left(\text{div}\, \mathbf{A} + \frac{1}{c^2} \frac{\P\, f}{\P\, t} \right)$$

Formerly viewed as a mere mathematical convenience (which Maxwell himself didn't like at all) the *Lorenz gauge* is now considered fundamental, because quantum theory assigns a *physical* significance to the potentials.

> In the Aharonov-Bohm effect, interference patterns produced by charged particles travelling outside of a solenoid are seen to depend on the value of a steady current through the solenoid, although the electromagnetic fields outside of the solenoid do not depend on it...

The Lorenz gauge is *relativistically covariant* (if it's true in one frame of reference it's true in all of them). This isn't the case for other popular gauges, including the *Coulomb gauge* (div **A** = 0) once favored by Maxwell. Such putative gauges are thus incompatible with the *objectivity* of potentials.

The expressions of the Lagrangian, Hamiltonian and *canonical momentum* of a charged particle in an electromagnetic field do depend explicitly on the potentials, although the *classical* Lorentz force derived from them does *not* depend on the choice of a gauge (see elsewhere on this site for a proof).

Canonical momentum of a particle of mass m, *charge* q *and velocity* **v**

$$\mathbf{p} \; = \; q\,\mathbf{A} \; + \; \frac{m\,\mathbf{v}}{Ö\overline{1 - \mathbf{v}^2/c^2}}$$

Lagrangian of a charged particle :

$$L \; = \; q\,(\,\mathbf{A}.\mathbf{v} - f\,) \; - \; m\,c^2\; Ö\overline{1 - \mathbf{v}^2 / c^2}$$

The Aharonov-Bohm Effect

(2005-07-15) Retarded and advanced potentials (& free photons)

General solutions of *Maxwell's equations* using the *Lorenz gauge*.

As shown above, the miraculous effect of the Lorenz gauge is that it effectively *decouples* electricity and magnetism to turn Maxwell equations into *parallel* diffential equations that can formally be solved using standard techniques (the [d'Alembert equations](#) are named after [Jean-le-Rond d'Alembert](#), who solved the related homogeneous [wave equation](#)). One relation equates second derivatives of the electric potential f to the electric density r. The other [vectorial] relation equates the same derivatives of each component of the vector potential **A** to the corresponding component of the density of current **j**. The mathematical solution for each component (and, therefore, for the whole thing) can be expressed as the sum of three terms said to be, respectively, *retarded*, *advanced* and *free* :

$$f \;=\; (1- a)\, f^{-} + a\, f^{+} + f^{o}$$
$$\mathbf{A} \;=\; (1- a)\, \mathbf{A}^{-} + a\, \mathbf{A}^{+} + \mathbf{A}^{o}$$

Usually, only a = 0 is considered, for the *causality* reasons discussed below.

> a = 1 is an alternate choice which reverses the arrow of time. In 1945, [Wheleer & Feynman](#) fantasized about the possibility of a = ½.

The free terms (superscripted ^{o}) are exactly what we have [already encountered](#) as the remaining degrees of freedom after imposing the *Lorenz gauge*. They correspond mathematically to solutions of the *homogeneous differential equations* (zero charges and currents characterize *free* space). Happily, the fact that they appear again here means that the choice of that gauge really involved no loss of generality. (This is not coincidental but we may pretend it is.)

The *retarded* terms are given by the following expressions, proposed by [Alfred-Marie Liénard](#) (1869-1958; [X](#)1887) in 1898 and by [Emil Wiechert](#) (1861-1928) in 1900. They're known as the *Liénard-Wiechert potentials*.

Electrodynamic Retarded Potentials $\mathbf{A}^{-}$ *and* f^{-}

$$f^{-}(t,\mathbf{r}) \;=\; \text{ÒÒÒ} \;\frac{r\,(\,t - \|\mathbf{r}\text{-}\mathbf{s}\| / c\;,\;\mathbf{s}\,)}{4pe_{o}\,\|\,\mathbf{r} - \mathbf{s}\,\|}\; d^{3}\mathbf{s}$$

$$\mathbf{A}^{-}(t,\mathbf{r}) \;=\; \text{ÒÒÒ} \;\frac{m_{o}\,\mathbf{j}\,(\,t - \|\mathbf{r}\text{-}\mathbf{s}\| / c\;,\;\mathbf{s}\,)}{4p\,\|\,\mathbf{r} - \mathbf{s}\,\|}\; d^{3}\mathbf{s}$$

This is similar to the expressions obtained in the static cases (electrostatics, magnetostatics) *except* that the fields we observe *here and now* depend on a prior state of the sources. The influence of the sources is delayed by the time it takes for the "news" of their motions to be broadcasted at speed c.

The so-called *advanced* potentials ($\mathbf{A}^{+}$ and f^{+}) are formally obtained by making c *negative* in the above *retarded* expressions (or *equivalently* by reversing the *arrow of time*). This is just like what we've already encountered in the case of [planar waves](#), with two possible directions of travel. However, the physical interpretation is not nearly as easy now that we're dealing with some causality relationship between the field and its "sources".

Advanced potentials make the situation here and now (potentials and/or fields) depend on the *future* state of remote "sources". Such a thing may be summarily dismissed as "unphysical" but this fails to make the issue go away. Indeed, quantum treatments of electromagnetic fields (photons in *Quantum Field Theory*) imply that a field can create some of its sources in the form of charged particle-antiparticle pairs. What seems to be lacking is the *coherence* of such creations because of statistical and/or thermodynamical considerations (which feature a pronounced arrow of time). I don't understand this. Nobody does...

What's clear, however, is that the distinction between past and future vanishes in *stationary* cases. This makes *advanced potentials* relevant and/or necessary, without the need for mind-boggling philosophical considerations.

We've only shown (admittedly skipping the mathematical details) that potentials that obey the *Lorenz gauge* would necessarily be given by the above formulas (possibly adding *advanced* and *free* components). Conversely, we ought to determine now what restrictions, if any, (pertaining to the sources ρ and $\mathbf{j}$) would make the above solutions verify the assumed *Lorenz gauge*. However, we shall postpone this discussion to present first a clarification of the physics...

(2005-08-21) Electrodynamic Fields *Caused* by Moving Sources

An expression derived from the Liénard-Wiechert retarded potentials.

Let ρ and $\mathbf{j}$ denote $\rho\,(t - R/c\,,\ \mathbf{s})$ and $\mathbf{j}\,(t - R/c\,,\ \mathbf{s})$.

As always, $R = \| \mathbf{r} - \mathbf{s} \|$ is the distance from a source (located at $\mathbf{s}$) to the observer (at $\mathbf{r}$). The following expressions of the fields then hold:

Electrodynamic fields obtained from retarded potentials :

$$\mathbf{E}(t,\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \iiint \left[\frac{\rho\,(\mathbf{r}-\mathbf{s})}{R^3} + \frac{(\partial\rho/\partial t)\,(\mathbf{r}-\mathbf{s})}{c\,R^2} - \frac{\partial\mathbf{j}/\partial t}{c^2 R} \right] d^3\mathbf{s}$$

$$\mathbf{B}(t,\mathbf{r}) = \frac{\mu_o}{4\pi} \iiint \left[\frac{\mathbf{j}\times(\mathbf{r}-\mathbf{s})}{R^3} + \frac{(\partial\mathbf{j}/\partial t)\times(\mathbf{r}-\mathbf{s})}{c\,R^2} \right] d^3\mathbf{s}$$

In the static case, only the first term of either expression subsists and we retrieve either the Coulomb law of electrostatics or the Biot-Savart law of magnetostatics.

A changing distribution of charges and currents generates the additional terms whose amplitudes dominate at large distances because they only decrease as 1/R. This is what makes *radio transmission* practical!

On 2009-09-06, Henryk Zajdel wrote: [edited summary]

> I just stumbled on your website. It is brilliant !
>
> However, [the above formulas] do not look right to me.
> Could you direct me to a publication where they are derived?
>
> Best regards,
> *Henryk Zajdel*, Katowice (Poland).

Thanks for the kind words, Henryk.

I find those expressions for the electromagnetic fields *caused* by dynamic sources very enlightening. Personally, I discovered them *after* establishing the dipolar solutions of Maxwell's equations, which strongly suggest such formulas. They are now known as Jefimenko's equations, in honor of Oleg D. Jefimenko (1922-2009). They were probably discovered privately many times. According to Kirk T. McDonald (1997) rhe first textbook which mentions them is the second edition of *Panofsky and Phillips* (1962).

Here's an outline of how those formulas can be derived from the well-known integrals giving the retarded potentials. In either of those integrals, t is a constant and so are the coordinates x,y,z of **r**. Differentiation with respect to x,y,z or t is thus performed by differentiating the *integrand*, which involves only numerical expressions of the following type (using the notations introduced at the outset):

$$k(R)\ f(\,t - R/c\ ,\ \mathbf{s}\,)$$

In this, k(R) is simply proportional to 1/R (but we may treat it like some unspecified function of R). Both factors depend on x,y,z only because R does. The function *f* depends on time; k doesn't. The chain rule yields:

$$\frac{\P f}{\P x} \;=\; \frac{\P f}{\P t}\ \frac{\P}{\P x}\left(t - \frac{R}{c}\right) \;=\; -\frac{1}{c}\ \frac{\P R}{\P x}\ \frac{\P f}{\P t}$$

$\P R / \P x$ is obtained by differentiating $R^2 = (\mathbf{r}\text{-}\mathbf{s})^2$. Namely:

$$R\,dR \;=\; (x\text{-}s_x)\,dx \;+\; (y\text{-}s_y)\,dy \;+\; (z\text{-}s_z)\,dz$$

$$\boxed{\frac{\P f}{\P x} \;=\; -\frac{x - s_x}{c\,R}\ \frac{\P f}{\P t}}$$

From this basic relation, and its counterparts along y and z, we obtain:

$$-\ \mathbf{grad}\, f \;=\; \frac{\P f}{\P t}\ \frac{\mathbf{r} - \mathbf{s}}{c\,R}$$

The same relations applied to the components $f_x\ f_y\ f_z$ of a vector **F** yield:

$$\mathbf{rot\ F} \;=\; \underline{\P\, \mathbf{F}}\ \text{´}\ \underline{\mathbf{r} - \mathbf{s}}$$

¶ t   c R

Another relation (needed only in the next section) involves a *dot product* :

$$\text{div } \mathbf{F} \quad = \quad - \frac{¶\,\mathbf{F}}{¶\,t} \cdot \frac{\mathbf{r} - \mathbf{s}}{c\,R}$$

Handling the scaling part introduced above as k(R) is similar but less tricky conceptually, because k is *simply* a scalar function of a single argument (the distance R between source and observer) with a straight derivative k′. (As k is proportional to 1/R, we have k′(R) = -k/R.)

$$- \mathbf{grad}\, k \quad = \quad - k'(R) \frac{\mathbf{r} - \mathbf{s}}{R} \quad = \quad k \frac{\mathbf{r} - \mathbf{s}}{R^2}$$

We may now use the above identities to translate the expressions of the Liénard-Wiechert potentials into the advertised formulas with the following substitutions:

- f = r / e_o
- **F** = m_o **j**
- k = 1 / 4pR

$$f \quad = \quad ÒÒÒ\ k(R)\ f\,(t\text{-}R/c, \mathbf{s})\ \ d^3\mathbf{s}$$

$$\mathbf{A} \quad = \quad ÒÒÒ\ k(R)\ \mathbf{F}\,(t\text{-}R/c, \mathbf{s})\ \ d^3\mathbf{s}$$

- **B** = **rot A**
- **E** = - **grad** f - ¶**A**/¶t

The conclusion follows from two general identities of vector calculus and one trivial equation (expressing that k is time-independent) namely:

- **rot** (k **F**) = **grad** k ´ **F** + k **rot F**
- - **grad** (k *f*) = - *f* **grad** k - k **grad** *f*
- - ¶/¶t (k **F**) = - k ¶**F**/¶t

The first line yields the expression of **B**, the sum of the last two gives **E**.
■

(2010-12-06) Electrodynamic Fields *Causing* Sources to Move

An expression derived from the Liénard-Wiechert advanced potentials.

Let's now forget the aura of mystery traditonally associated with *advanced solutions*. Reversing the direction of time simply reverses causality. Bluntly, when the photons kick the electrons, the values of the fields are related to the values of the so-called "sources" at a later time (the *sources* are not the *causes* in this case; their name is misleading).

Now, ρ and $\mathbf{j}$ denote $\rho (t + R / c \, , \mathbf{s})$ and $\mathbf{j} (t + R / c \, , \mathbf{s})$.

Electrodynamic fields obtained from advanced potentials :

$$\mathbf{E}(t,\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \iiint \left[\frac{\rho \, (\mathbf{r} - \mathbf{s})}{R^3} - \frac{(\partial \rho / \partial t) \, (\mathbf{r} - \mathbf{s})}{c \, R^2} - \frac{\partial \mathbf{j} / \partial t}{c^2 R} \right] d^3\mathbf{s}$$

$$\mathbf{B}(t,\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint \left[\frac{\mathbf{j} \times (\mathbf{r} - \mathbf{s})}{R^3} - \frac{(\partial \mathbf{j} / \partial t) \times (\mathbf{r} - \mathbf{s})}{c \, R^2} \right] d^3\mathbf{s}$$

Compare this formally to the similar expressions for retarded potential and notice the changes of sign that occur in the second column but *not* the third! Thoses changes can be traced down to the beginning of the proof outlined above for retarded potentials, since for a function $f (t+R/c \, , \mathbf{s})$:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial t} \frac{\partial}{\partial x} (t + \frac{R}{c}) = + \frac{1}{c} \frac{\partial R}{\partial x} \frac{\partial f}{\partial t}$$

The corresponding change of sign (compared to retarded potentials) applies to the dynamical parts of **grad** φ or **rot A** but does not formally affect the $\partial \mathbf{A} / \partial t$ component of **E**.

One important consequence of such changes of signs is that it affects the distant fields in a way which reverses the sign of Larmor's formula. In other words, contrary to popular belief, an accelerated or decelerated charge need not radiate electromagnetic energy away. It does so only when the change of its motion is the cause of changing fields, not when it's the *result* of such changing fields. Electromagnetic energy always flows from cause to effect.

(2009-11-10) Gauge of classical retarded potentials :

Does the formulas for retarded potentials obey the Lorenz gauge ?

We may use the notation introduced in the [second part of] the previous section to investigate what gauge is obeyed by the expression of the retarded potentials.

We use the methods and the preliminary specific equations established in that section with yet another general identity of vector calculus:

$$\text{div} \, (k \, \mathbf{F}) = \mathbf{F} \, . \, \mathbf{grad} \, k + k \, \text{div} \, \mathbf{F}$$

(2005-08-11) Radiated Energy (Larmor Formula, 1897)

Accelerated [bound] charges radiate energy away, or do they?

Consider the dipolar solutions to Maxwell's equation (retarded spherical waves) presented elsewhere on this site. At a large distance, the dominant field components are proportional to the second derivatives $\mathbf{p}''$ or $\mathbf{m}''$. For an *electric* dipole, the dominant *far-field* component of the Poynting vector ($\mathbf{E}\times\mathbf{B} / \mu_0$) is thus in the radial direction of the normed vector $\mathbf{u}$:

$$\frac{\mu_0}{(4\pi r)^2 c} \left(\mathbf{u} \times \frac{d^2 \mathbf{p}}{dt^2} \right)^2 \mathbf{u}$$

This is a radial vector whose length is proportional to $\sin^2\theta = 1 - \cos^2\theta$ (where θ is the angle between $\mathbf{p}''$ and the direction of $\mathbf{u}$). Its flux through the surface of the sphere of radius r is the total power radiated away:

$$\frac{\mu_0}{(4\pi r)^2 c} \left(\frac{d^2 \mathbf{p}}{dt^2} \right)^2 \int_0^\pi (1 - \cos^2\theta) \, (2\pi r^2 \sin\theta) \, d\theta \quad = \quad \frac{\mu_0}{6\pi c} \, \| \mathbf{p}'' \|^2$$

Likewise, the total power radiated by a *magnetic* dipole is :

$$(\mu_0 / 6\pi c^3) \, \| \mathbf{m}'' \|^2$$

Let's use a *subterfuge* to compute the power radiated away by a *single* charge q near the origin: Place a charge $-q$ (a "witness") at the origin itself. At large distances, the resulting variable dipole $\mathbf{p} = q\,\mathbf{r}(t)$ would produce essentially the same *dynamic* field (at time t+r/c) as the lone moving charge q (as long as its acceleration does not vanish and its distance to the origin remains small). This translates into the following so-called *Larmor formula* (derived in 1897 by Joseph Larmor, 1857-1942):

Power radiated by a charge q

$$\boxed{\frac{\mu_0 q^2}{6\pi c} \left(\frac{d^2 \mathbf{r}}{dt^2} \right)^2}$$

Note that the above was obtained from field expressions based on retarded potentials which are appropriate when changing sources cause changing fields. If that causality relationship is reversed, the fields based on advanced potentials should be used instead. They yield a formula whose sign is the opposite of the above (which would indicate that an accelerated or decelerated charge *receives* energy). In other words, energy always flows from the cause to the effect.

The above argument skirts near-field difficulties, but it seems inadequate whenever the moving charge is not confined to the immediate vicinity of the artificial "witness" charge. In particular, we don't obtain a clear picture of what happens, in the long run, when a charge is subjected to a constant acceleration... It has been argued that no power would be lost away in this case, which (according to General Relativity) is equivalent to a motionless charge in a constant gravitational field. Even so, a *varying* gravity ought to make charges radiate (classically, at least).

A promising way out of that dilemma (2006-10-16) is to consider the *thermal* nature of the above exchange of energy, allowing the formula to hold, in some statistical way, as the classical counterpart of a quantum effect... Indeed, in 1976, W.G. Unruh found that an acceleration g (or, equivalently, a gravitational field) entails a *heat bath* whose temperature T is proportional to it :

Unruh's Temperature T (1976)

$$k\,T \;=\; \left(\frac{h}{4\pi^2 c}\right) g$$

(2005-08-09) The Lorentz-Dirac Equation

Classical Theory of the Electron. Strange inertia of charged particles.

The motion of an electron (point particle of charge q) submitted to a force **F** has been described in terms of the following 4-dimensional equation, where (primed) derivatives of the position **R** are with respect to the particle's *proper time* τ [defined via: $(c\,d\tau)^2 = (c\,dt)^2-(dx)^2-(dy)^2-(dz)^2$].

Lorentz-Dirac Equation (1938)

$$m\,\mathbf{R}'' \;=\; \mathbf{F} + \frac{\mu_o\, q^2}{6\pi\, c}\left[\;\mathbf{R}''' + \frac{|\,\mathbf{R}'\rangle\langle\mathbf{R}'\,|}{c^2}\,\mathbf{R}'''\;\right]$$

The *Abraham-Lorentz* equation is the non-relativistic version of this (using "absolute" time and retaining only the first term of the bracket).

$|\,\mathbf{R}'\rangle\langle\mathbf{R}'\,|$ is a square tensor (the product of the 4D velocity and its dual). The bracketed sum is only relevant for a point particle of nonzero charge. Its nature has been highly controversial since 1892, when H.A. Lorentz first proposed a *Theory of the Electron* derived microscopically from Maxwell's equations and from the expression of the electromagnetic force now named after him. Lorentz would only consider the electromagnetic part of the *rest mass* m (i.e., 3m/4). In 1938, Paul Dirac derived the above *covariantly*, for the total mass m.

Physically, the initial value of the acceleration (**R**″) in this third-order equation cannot be freely chosen (so the overall constraints are comparable to those of an ordinary newtonian equation). Almost all *mathematical* solutions are unphysical ones, which are dubbed *self-accelerating* or *runaway* because they would make the particle's energy grow indefinitely, even if no force was applied.

However, there could be *more than one* initial value of the acceleration which makes physical sense. The *wholly classical* Lorentz-Dirac equation thus allows a nondeterministic behavior more often associated with quantum mechanics.

The Lorentz-Dirac equation has other *weird* features, including the need for a so-called *preacceleration* which contradicts causality: The Lorentz-Dirac equation would require an electron to anticipate any impending pulse of force...

Order Reduction for the Lorentz-Dirac Type | Does a Uniformly Accelerating Charge Radiate?
Electromagnetism in GR by David Waite