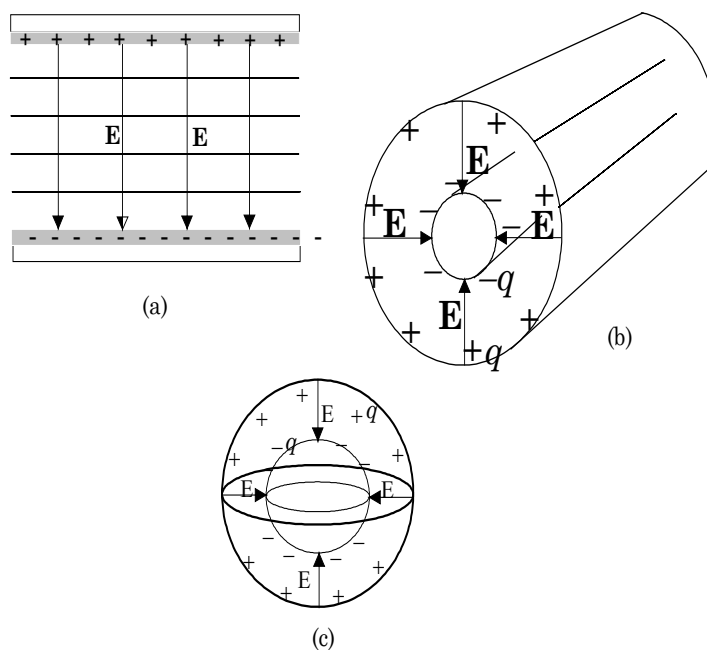


# Chapter 6: Capacitance

## 6.1 Introduction

Whenever two nearby conductors of any size or shape, carry equal and opposite charges, the combination of these conducting bodies is called a capacitor. Because the isolated conducting bodies have equal but opposite charges on them, an electric field exists in the space between them. The importance of the capacitor lies in the fact that energy can be stored in the electric field between the two conducting bodies. Some simple examples of capacitors are shown in figure 6.1.



**Figure 6.1** Some simple capacitors.

Figure 6.1(a) is a parallel plate capacitor which consists of two metal plates separated by a distance  $d$ . A positive charge,  $+q$ , is placed on one of the plates, let us say the top one, and a charge  $-q$  is placed on the bottom conducting plate. Neglecting any edge corrections, there is a uniform electric field  $\mathbf{E}$  between the two charged plates.

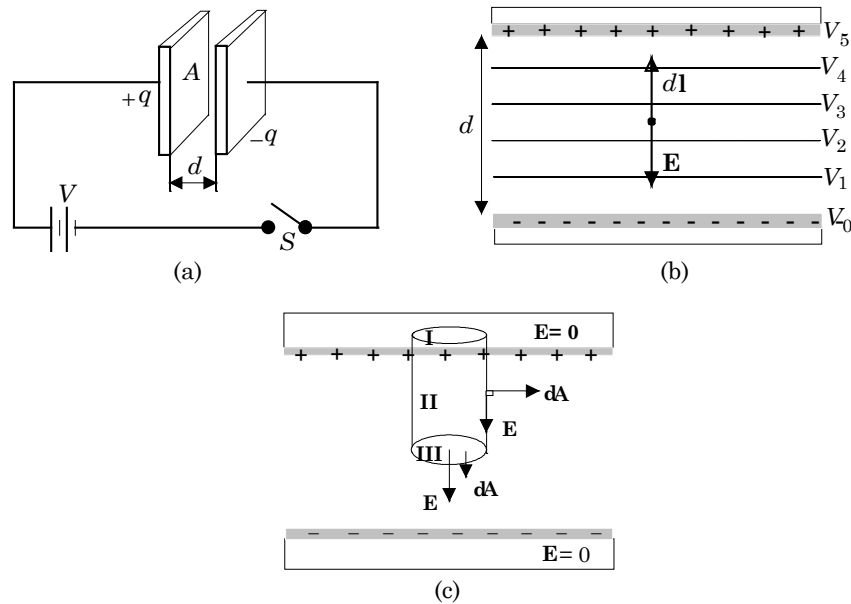
Figure 6.1(b) is a coaxial cylindrical capacitor. As the name implies, it consists of two coaxial cylinders. The inner cylinder has a negative charge  $-q$  placed on it, while a positive charge  $+q$  is placed on the outer cylinder. The electric field fills the space between the cylinders as shown in figure 6.1(b).

A concentric spherical capacitor is shown in figure 6.1(c). The inner sphere has a negative charge  $-q$  placed on it, and a positive charge  $+q$  is placed on the outer sphere. A spherical electric field exists between the two conducting spheres. In all of these cases, energy is stored in the electric field between the plates.

In the next few sections we will go into more detail on these capacitors. We will see that the study of capacitors is an excellent application of the concepts of the potential we established in chapter 5, and Gauss's law in chapter 4.

## 6.2 The Parallel Plate Capacitor

A parallel plate capacitor is connected to a battery as shown in figure 6.2(a). Although the actual charge carriers in a metal are electrons, we will continue with the convention introduced in your college physics course that it is the positive charges that are moving in the circuit. Recall that a flow of negative charges in one direction is equivalent to a flow of positive charges in the opposite direction. Hence,



**Figure 6.2** The parallel plate capacitor.

using the concepts of conventional current, positive electric charge flows from the battery of potential  $V$  to the left hand plate of the capacitor when the switch  $S$  is closed. The charge distributes itself over the left hand plate. This positive charge on the left hand plate induces an equal but negative charge on the right hand plate. The positive charge on the originally neutral right plate is pushed into the battery. The net result of applying the battery to the capacitor is that a charge of  $+q$  is deposited on the left plate and  $-q$  on the right plate and a uniform electric field has been set up between the plates. In effect, the battery has supplied energy to move positive charges from the right plate and transferred them, through the battery, to the left plate.

The relationship between the potential  $V$  across the plates, the electric field  $\mathbf{E}$  between the plates, and the plate separation  $d$  can be seen in figure 6.2(b). To find the potential difference between the parallel plates we use equation 5.25.

## Chapter 6: Capacitance

$$V_B - V_A = -\int \mathbf{E} \cdot d\mathbf{l} \quad (5.25)$$

Remember that equation 5.25 was derived as the work done per unit charge as a charge was moved up a potential hill from a point  $A$  at a lower potential to a point  $B$ , at a higher potential. In figure 6.2(b) the work is done as we move along a path from the lower potential at the negative plate to the higher potential at the upper plate. Hence the path  $d\mathbf{l}$  points upward while the direction of the electric field vector  $\mathbf{E}$  points downward into the lower plate. Therefore the angle  $\theta$  between  $\mathbf{E}$  and  $d\mathbf{l}$  is  $180^\circ$ , and equation 5.25 becomes

$$V_B - V_A = -\int \mathbf{E} \cdot d\mathbf{l} = -\int E dl \cos 180^\circ = -\int E dl(-1) = \int E dl \quad (6.1)$$

We defined the zero of potential to be at the negative plate, that is,  $V_A = 0$ , and the point  $B$  is the positive plate and hence we let  $V_B = V$  the potential between the two plates. Also, as shown in chapter 3, the electric field  $E$  between the parallel plates is a constant and can come out of the integral sign. The difference in potential between the plates now becomes

$$V = \int_0^d E dl = E(d - 0) \quad (6.2)$$

and the potential between the plates becomes

$$V = Ed \quad (6.3)$$

Let us determine the electric field  $E$  between the two oppositely charged conducting plates, shown in figure 6.2(c), by Gauss's law. Note that the solution for  $E$  is the same solution we found in section 4.10. We start by drawing a Gaussian cylinder, as shown in the diagram. One end of the Gaussian surface lies within one of the parallel conducting plates while the other end lies in the region between the two conducting plates where we wish to find the electric field. Gauss's law is given by equation 4-14 as

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\epsilon_0} \quad (4-14)$$

The sum in equation 4-14 is over the entire Gaussian surface. We break the entire surface of the cylinder into three surfaces. Surface I is the end cap on the topside of the cylinder, surface II is the main cylindrical surface, and surface III is the end cap on the bottom of the cylinder as shown in figure 6.2(c). The total flux  $\Phi$  through the Gaussian surface is the sum of the flux through each individual surface. That is,

$$\Phi = \Phi_I + \Phi_{II} + \Phi_{III} \quad (6.4)$$

where

$\Phi_I$  is the electric flux through surface I  
 $\Phi_{II}$  is the electric flux through surface II  
 $\Phi_{III}$  is the electric flux through surface III

## Chapter 6: Capacitance

Gauss's law becomes

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \int_I \mathbf{E} \cdot d\mathbf{A} + \int_{II} \mathbf{E} \cdot d\mathbf{A} + \int_{III} \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\epsilon_0} \quad (6.5)$$

Because the plate is a conducting body, all charge must reside on its outer surface, hence  $E = 0$ , inside the conducting body. Since Gaussian surface I lies within the conducting body the electric field on surface I is zero. Hence the flux through surface I is,

$$\Phi_I = \int_I \mathbf{E} \cdot d\mathbf{A} = \int_I E dA \cos \theta = \int_I (0) dA \cos \theta = 0 \quad (6.6)$$

Along cylindrical surface II,  $d\mathbf{A}$  is everywhere perpendicular to the surface.  $\mathbf{E}$  lies in the cylindrical surface, pointing downward, and is everywhere perpendicular to the surface vector  $d\mathbf{A}$  on surface II, and therefore  $\theta = 90^\circ$ . Hence the electric flux through surface II is

$$\Phi_{II} = \int_{II} \mathbf{E} \cdot d\mathbf{A} = \int_{II} E dA \cos \theta = \int_{II} E dA \cos 90^\circ = 0 \quad (6.7)$$

Surface III is the end cap on the bottom of the cylinder and as can be seen from figure 6.2(c),  $\mathbf{E}$  points downward and since the area vector  $d\mathbf{A}$  is perpendicular to the surface pointing outward it also points downward. Hence  $\mathbf{E}$  and  $d\mathbf{A}$  are parallel to each other and the angle  $\theta$  between  $\mathbf{E}$  and  $d\mathbf{A}$  is zero. Hence, the flux through surface III is

$$\Phi_{III} = \int_{III} \mathbf{E} \cdot d\mathbf{A} = \int_{III} E dA \cos \theta = \int_{III} E dA \cos 0^\circ = \int_{III} E dA \quad (6.8)$$

The total flux through the Gaussian surface is equal to the sum of the fluxes through the individual surfaces. Hence, using equations 6.6, 6.7, and 6.8, Gauss's law becomes

$$\begin{aligned} \Phi &= \Phi_I + \Phi_{II} + \Phi_{III} \\ \Phi_E &= 0 + 0 + \int_{III} E dA = \frac{q}{\epsilon_0} \end{aligned} \quad (6.9)$$

But  $E$  is constant in every term of the sum in integral III and can be factored out of the integral giving

$$\Phi_E = E \int dA = \frac{q}{\epsilon_0}$$

But  $\int dA = A$  the area of the end cap, hence

$$E A = \frac{q}{\epsilon_0}$$

$A$  is the magnitude of the area of Gaussian surface III and  $q$  is the charge enclosed within Gaussian surface III. Solving for the electric field  $E$  between the conducting plates gives

## Chapter 6: Capacitance

$$E = \frac{q}{\epsilon_o A} \quad (6.10)$$

Equation 6.10, as it now stands, describes the electric field in terms of the charge enclosed by Gaussian surface III and the area of Gaussian surface III. *If the surface charge density on the plates is uniform, the surface charge density enclosed by Gaussian surface III is the same as the surface charge density of the entire plate. Hence the ratio of  $q/A$  for the Gaussian surface is the same as the ratio  $q/A$  for the entire plate.* Thus  $q$  in equation 6.10 can also be interpreted as the total charge  $q$  on the plates, and  $A$  can be interpreted as the total area of the conducting plate. Therefore, equation 6.10 is rewritten as

$$E = \frac{q}{\epsilon_o A} \quad (6.11)$$

where  $E$  is the electric field between the conducting plates and is given in terms of the charge  $q$  on the plates, the cross-sectional area  $A$  of the plates, and the permittivity  $\epsilon_o$  of the medium between the plates.

Substituting equation 6.11 into equation 6.3 gives

$$V = \frac{qd}{\epsilon_o A} \quad (6.12)$$

Upon solving for  $q$

$$q = \left( \frac{\epsilon_o A}{d} \right) V \quad (6.13)$$

Notice from equation 6.13 that the charge  $q$  on the plate is directly proportional to the potential  $V$  between the plates, which is of course supplied by the battery. The greater the battery voltage  $V$ , the greater will be the charge  $q$  on the plate; the smaller the voltage  $V$ , the smaller the charge  $q$ .

Let us now look carefully at the term in parenthesis in equation 6.13. Notice that it is a constant and is a function of the geometry of the capacitor. As you recall from chapter 2,  $\epsilon_o$  is called the permittivity of free space and has approximately the same value for air as for a vacuum. This term is a function of the medium between the plates, which in this case is air.  $A$ , in equation 6.13, is the cross sectional area of a plate of the capacitor and  $d$  is the separation between the two plates. Because all these terms are constant for a particular capacitor, they are set equal to a new constant  $C$ , called the capacitance of the capacitor. Therefore, *the capacitance of a parallel plate capacitor* is

$$C = \frac{\epsilon_o A}{d} \quad (6.14)$$

The capacitance is thus a function of the geometry of the capacitor itself. The larger the area  $A$  of the plates, the greater will be the value of the capacitance  $C$ . The greater the plate separation  $d$  the smaller will be the capacitance  $C$ . A parallel plate capacitor of any value can be obtained by proper selection of area, plate separation, and the medium between the plates. So far, our discussion has been limited to a ca-

## Chapter 6: Capacitance

capacitor with air between the plates. In chapter 7 an insulating material will be placed between the plates.

The introduction of the concept of the capacitance allows us to write equation 6.13 in the more general form

$$q = CV \quad (6.15)$$

*The charge  $q$  on a capacitor is directly proportional to the potential  $V$  between the plates, and the constant of proportionality is the capacitance  $C$  of the particular capacitor. The capacitance can then be defined in general, from equation 6.15, to be*

$$C = \frac{q}{V} \quad (6.16)$$

The SI unit for capacitance is defined from equation 6.16 to be a farad F where

$$1 \text{ farad} = 1 \text{ coulomb/volt}$$

This is abbreviated as

$$1 \text{ F} = 1 \text{ C/V}$$

This unit is named in honor of Michael Faraday (1791-1867), an English physicist. If a charge of one coulomb is placed on the plates and the potential difference between the plates is one volt, the capacitance is then defined to be one farad. A capacitance of one farad is extremely large, and the smaller units of microfarads,  $\mu\text{F}$ , or picofarads,  $\text{pF}$ , are usually used.

$$1 \mu\text{F} = 10^{-6} \text{ F}$$

$$1 \text{ pF} = 10^{-12} \text{ F}$$

### ***Example 6.1***

*Capacitance of a parallel plate capacitor.* A parallel plate capacitor consists of two metal disks, 5.00 cm in radius. The disks are separated by air and are a distance of 4.00 mm apart. A potential of 50.0 V is applied across the plates by a battery. Find (a) the capacitance  $C$  of the capacitor, and (b) the charge  $q$  on the plate.

### ***Solution***

(a) The area of the plate is

$$A = \pi r^2 = \pi(0.0500 \text{ m})^2 = 7.85 \times 10^{-3} \text{ m}^2$$

The capacitance is found from equation 6.14 to be

$$C = \frac{\epsilon_0 A}{d} = \frac{(8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2)(7.85 \times 10^{-3} \text{ m}^2)}{(4.00 \times 10^{-3} \text{ m})}$$

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$$\begin{aligned}C &= 17.4 \times 10^{-12} \frac{\text{C}^2}{\text{N m}} \left( \frac{\text{N m}}{\text{J}} \right) \\C &= 17.4 \times 10^{-12} \frac{\text{C}}{\text{J/C}} \left( \frac{\text{J/C}}{\text{V}} \right) \left( \frac{\text{F}}{\text{C/V}} \right) \\C &= 17.4 \times 10^{-12} \text{ F} \\C &= 17.4 \text{ pF}\end{aligned}$$

Note how the conversion factors have been carried through in the example to show that the capacitance does indeed come out to have the unit of farads.

(b) The charge on the plate is determined from equation 6.15 as

$$\begin{aligned}q &= CV \\q &= (17.4 \times 10^{-12} \text{ F})(50.0 \text{ V}) \\q &= 8.70 \times 10^{-10} \text{ C}\end{aligned}$$

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### 6.3 The Cylindrical Capacitor

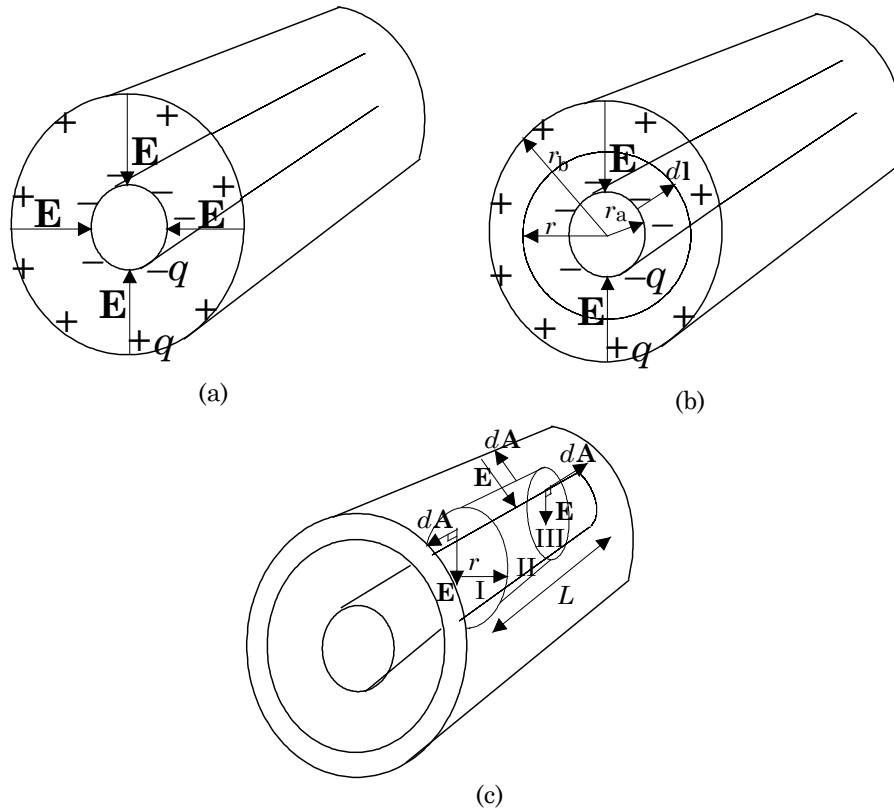
A cylindrical capacitor, consisting of two coaxial cylinders of radii  $r_a$  and  $r_b$ , is shown in figure 6.3(a). The capacitor has a length  $l$ . A charge  $-q$  is placed on the inner cylinder and  $+q$  on the outer cylinder. Hence, an electric field  $\mathbf{E}$  emanates from the outer cylinder, points inward, and terminates on the inner cylinder as shown. The capacitance  $C$  of the cylindrical capacitor is found from equation 6.16 as

$$C = \frac{q}{V} \quad (6.16)$$

In order to determine the capacitance from equation 6.16 it is first necessary to determine the potential  $V$  between the two cylinders. The potential difference between the two cylinders is found from equation 5.25 as

$$V_B - V_A = -\int \mathbf{E} \cdot d\mathbf{l} \quad (5.25)$$

Remember that equation 5.25 was derived as the work done per unit charge as a charge was moved up a potential hill from a point  $A$  at a lower potential to a point  $B$ , at a higher potential. In figure 6.3(b) the work is done as we move along a path from the lower potential at the negative inner cylinder to the higher potential at the outer positive cylinder. Hence the path  $d\mathbf{l}$  points outward while the direction of the electric field vector  $\mathbf{E}$  points into the inner cylinder. Therefore the angle  $\theta$  between  $\mathbf{E}$  and  $d\mathbf{l}$  is  $180^\circ$ , and equation 5.25 becomes



**Figure 6.3** A cylindrical capacitor.

$$V_B - V_A = -\int \mathbf{E} \cdot d\mathbf{l} = -\int E dl \cos 180^\circ = -\int E dl(-1) = \int E dl$$

We will assume that the negative plate is at the ground potential and hence we will take  $V_A = 0$ . The potential difference between the plates is then  $V$  and is given by

$$V = \int E dl \quad (6.17)$$

But the element of path  $dl$  is equal to the element of radius  $dr$ , and hence

$$V = \int_{r_a}^{r_b} E dr \quad (6.18)$$

In order to evaluate equation 6.18 it is necessary to know the electric field  $\mathbf{E}$  between the cylinders. To do this we use Gauss's law, equation 4-14, modified to take into account that *the charge enclosed by the Gaussian surface is negative, that is, the charge on the inner cylinder is  $-q$* . That is,

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{-q}{\epsilon_0} \quad (6.19)$$



## Chapter 6: Capacitance

To determine the electric field between the two conducting cylinders,  $r_a < r < r_b$ , we draw a cylindrical Gaussian surface of radius  $r$  and length  $L$  between the inner cylinder and the outer cylinder as shown in figure 6.3(c). We assume that the charge is uniformly distributed and has a surface charge density  $\sigma$ . The integral in Gauss's law is over the entire Gaussian surface. As we did in chapter 4, we break the entire cylindrical Gaussian surface into three surfaces. Surface I is the end cap on the left-hand side of the cylinder, surface II is the main cylindrical surface, and surface III is the end cap on the right-hand side of the cylinder as shown in figure 6.3(c). The total flux  $\Phi$  through the entire Gaussian surface is the sum of the flux through each individual surface. That is,

$$\Phi = \Phi_I + \Phi_{II} + \Phi_{III} \quad (6.20)$$

where

$\Phi_I$  is the electric flux through surface I  
 $\Phi_{II}$  is the electric flux through surface II  
 $\Phi_{III}$  is the electric flux through surface III

Hence, Gauss's law becomes

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \int_I \mathbf{E} \cdot d\mathbf{A} + \int_{II} \mathbf{E} \cdot d\mathbf{A} + \int_{III} \mathbf{E} \cdot d\mathbf{A} = \frac{-q}{\epsilon_0} \quad (6.21)$$

Along cylindrical surface I,  $d\mathbf{A}$  is everywhere perpendicular to the surface and points toward the left as shown in figure 6.3(c). The electric field intensity vector  $\mathbf{E}$  lies in the plane of the end cylinder cap and is everywhere perpendicular to the surface vector  $d\mathbf{A}$  of surface I and hence  $\theta = 90^\circ$ . Therefore the electric flux through surface I is

$$\Phi_I = \int_I \mathbf{E} \cdot d\mathbf{A} = \int_I E dA \cos \theta = \int_I E dA \cos 90^\circ = 0 \quad (6.22)$$

Surface II is the cylindrical surface itself. As can be seen in figure 6.3(c),  $\mathbf{E}$  is everywhere perpendicular to the cylindrical surface pointing inward, and the area vector  $d\mathbf{A}$  is also perpendicular to the surface and points outward. Hence  $\mathbf{E}$  and  $d\mathbf{A}$  are antiparallel to each other and the angle  $\theta$  between  $\mathbf{E}$  and  $d\mathbf{A}$  is  $180^\circ$ . Hence, the flux through surface II is

$$\Phi_{II} = \int_{II} \mathbf{E} \cdot d\mathbf{A} = \int_{II} E dA \cos \theta = \int_{II} E dA \cos 180^\circ = \int_{II} E dA (-1) = - \int_{II} E dA \quad (6.23)$$

Along cylindrical surface III,  $d\mathbf{A}$  is everywhere perpendicular to the surface and points toward the right as shown in figure 6.3(c). The electric field intensity vector  $\mathbf{E}$  lies in the plane of the end cylinder cap and is everywhere perpendicular to the surface vector  $d\mathbf{A}$  of surface III and therefore  $\theta = 90^\circ$ . Hence the electric flux through surface III is

$$\Phi_{III} = \int_{III} \mathbf{E} \cdot d\mathbf{A} = \int_{III} E dA \cos \theta = \int_{III} E dA \cos 90^\circ = 0 \quad (6.24)$$

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Combining the flux through each portion of the cylindrical surfaces we get

$$\begin{aligned}\Phi_E &= \oint_I \mathbf{E} \cdot d\mathbf{A} + \oint_{II} \mathbf{E} \cdot d\mathbf{A} + \oint_{III} \mathbf{E} \cdot d\mathbf{A} = \frac{-q}{\epsilon_o} \\ \Phi_E &= 0 - \oint_{II} E dA + 0 = \frac{-q}{\epsilon_o}\end{aligned}\tag{6.25}$$

From the symmetry of the problem, the magnitude of the electric field intensity  $E$  is a constant for a fixed distance  $r$  from the axis of the inner cylinder. Hence,  $E$  can be taken outside the integral sign to yield

$$-\oint_{II} E dA = -E \oint_{II} dA = \frac{-q}{\epsilon_o}\tag{6.26}$$

The integral  $\oint dA$  represents the sum of all the elements of area  $dA$ , and that sum is just equal to the total area of the cylindrical surface. As we showed in section 4.6, the total area of the cylindrical surface can be found by unfolding the cylindrical surface. One length of the surface is  $L$ , the length of the cylinder, while the other length is the unfolded circumference  $2\pi r$  of the end of the cylindrical surface. The total area  $A$  is just the product of the length times the width of the rectangle formed by unfolding the cylindrical surface, that is,  $A = (L)(2\pi r)$ . Hence, the integral of  $dA$  is

$$\oint dA = A = (L)(2\pi r)$$

Thus, Gauss's law, equation 6.26, becomes

$$E \oint_{II} dA = E(L)(2\pi r) = \frac{q}{\epsilon_o}\tag{6.27}$$

Notice that the minus signs were on both side of the equation in Gauss's law, equation 6.26, and now have been canceled out. Solving for the magnitude of the electric field intensity  $E$  we get

$$E = \frac{q}{2\pi\epsilon_o r L}\tag{6.28}$$

Equation 6.28, as it now stands, describes the electric field in terms of the charge enclosed by Gaussian surface II and the area of Gaussian surface II. *If the surface charge density on the inner cylinder is uniform, the surface charge density enclosed by Gaussian surface II is the same as the surface charge density of the entire inner cylinder. Hence the ratio of  $q/A$  for the Gaussian surface is the same as the ratio  $q/A$  for the entire cylinder.* Thus  $q$  in equation 6.28 can also be interpreted as the total charge  $q$  on the cylinder, and  $A$  can be interpreted as the total area of the inner conducting cylinder. Therefore, *equation 6.28, the electric field between the cylinders, is rewritten as*

## Chapter 6: Capacitance

$$E = \frac{q}{2\pi\epsilon_0 r l} \quad (6.29)$$

Placing equation 6.29 back into equation 6.18 we get

$$\begin{aligned} V &= \int_{r_a}^{r_b} E dr = \int_{r_a}^{r_b} \left( \frac{q}{2\pi\epsilon_0 r l} \right) dr = \int_{r_a}^{r_b} \frac{q}{2\pi\epsilon_0 l} \frac{dr}{r} \\ V &= \frac{q}{2\pi\epsilon_0 l} \ln r \Big|_{r_a}^{r_b} = \frac{q}{2\pi\epsilon_0 l} (\ln r_b - \ln r_a) \end{aligned}$$

and the potential between the cylinders becomes

$$V = \frac{q}{2\pi\epsilon_0 l} \ln \frac{r_b}{r_a} \quad (6.30)$$

The capacitance is now found, by placing equation 6.30 into equation 6.16, as

$$C = \frac{q}{V} = \frac{q}{\frac{q}{2\pi\epsilon_0 l} \ln \frac{r_b}{r_a}} \quad (6.31)$$

Simplifying, the capacitance of the coaxial cylinders becomes

$$C = \frac{2\pi\epsilon_0 l}{\ln(r_b/r_a)} \quad (6.32)$$

### Example 6.2

*The capacitance of a cylindrical capacitor.* Find the capacitance of a cylindrical capacitor of radii  $r_a = 2.00$  cm and  $r_b = 2.50$  cm. The length of the capacitor is 15.0 cm.

### Solution

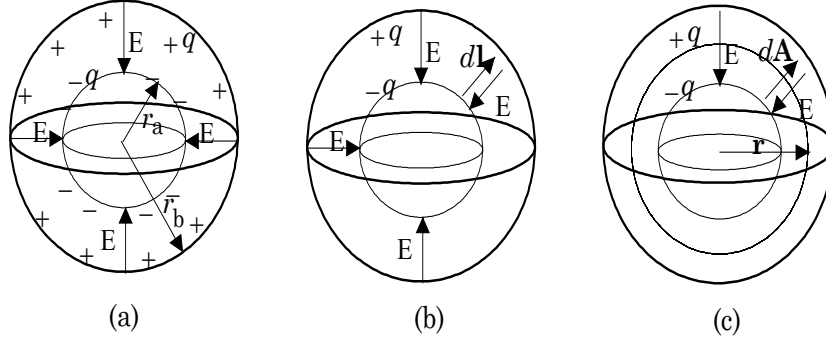
The capacitance  $C$  of the cylindrical capacitor is found from equation 6.32 as

$$\begin{aligned} C &= \frac{2\pi\epsilon_0 l}{\ln(r_b/r_a)} \\ C &= \frac{2\pi(8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2)(0.15 \text{ m})}{\ln[(0.025 \text{ m})/(0.020 \text{ m})]} \\ C &= 3.74 \times 10^{-11} \text{ F} = 37.4 \text{ pF} \end{aligned}$$

**To go to this Interactive Example click on this sentence.**

## 6.4 The Spherical Capacitor

A spherical capacitor, consisting of two concentric spheres of radii  $r_a$  and  $r_b$ , is shown in figure 6.4(a). A charge  $-q$  is placed on the inner sphere and  $+q$  on the



**Figure 6.4** A spherical capacitor.

outer sphere. Hence, an electric field  $\mathbf{E}$  emanates from the outer sphere, points inward, and terminates on the inner sphere as shown. The capacitance  $C$  of the spherical capacitor is found from equation 6.16 as

$$C = \frac{q}{V} \quad (6.16)$$

In order to determine the capacitance from equation 6.16 it is first necessary to determine the potential  $V$  between the two spheres. The potential difference between the two spheres is found from equation 5.25 as

$$V_B - V_A = -\int \mathbf{E} \cdot d\mathbf{l} \quad (5.25)$$

Remember that equation 5.25 was derived as the work done per unit charge as a charge was moved up a potential hill from a point  $A$  at a lower potential to a point  $B$ , at a higher potential. In figure 6.4(b) the work is done as we move along a path from the lower potential at the negative inner sphere to the higher potential at the outer positive sphere. Hence the path  $d\mathbf{l}$  points outward while the direction of the electric field vector  $\mathbf{E}$  points into the inner sphere. Therefore the angle  $\theta$  between  $\mathbf{E}$  and  $d\mathbf{l}$  is  $180^\circ$ , and equation 5.25 becomes

$$V_B - V_A = -\int \mathbf{E} \cdot d\mathbf{l} = -\int E dl \cos 180^\circ = -\int E dl(-1) = \int E dl$$

We will assume that the inner negative sphere is at the ground potential and hence we will take  $V_A = 0$ . The potential difference between the two spheres is then  $V$  and is given by

$$V = \int E dl \quad (6.33)$$

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But the element of path  $dl$  is equal to the element of radius  $dr$ , and hence

$$V = \int_{r_a}^{r_b} E dr \quad (6.34)$$

In order to evaluate equation 6.34 it is necessary to know the electric field  $\mathbf{E}$  between the spheres. To do this we use Gauss's law, equation 4-14, modified to take into account that *the charge enclosed by the Gaussian surface is negative, that is, the charge on the inner sphere is  $-q$* . Thus,

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{-q}{\epsilon_0} \quad (6.35)$$

To determine the electric field between the two conducting spheres,  $r_a < r < r_b$ , we draw a spherical Gaussian surface of radius  $r$  between the inner sphere and the outer sphere as shown in figure 6.4(c). We assume that the charge is uniformly distributed and has a surface charge density  $\sigma$ . The integral in Gauss's law is over the entire spherical Gaussian surface. As can be seen in figure 6.4(c),  $\mathbf{E}$  is everywhere perpendicular to the spherical surface pointing inward, and the area vector  $d\mathbf{A}$  is also perpendicular to the spherical surface and points outward. Hence  $\mathbf{E}$  and  $d\mathbf{A}$  are antiparallel to each other and the angle  $\theta$  between  $\mathbf{E}$  and  $d\mathbf{A}$  is  $180^\circ$ . Hence, the flux through the Gaussian surface is

$$\Phi = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E dA \cos \theta = \oint E dA \cos 180^\circ = \oint E dA (-1) = - \oint E dA = \frac{-q}{\epsilon_0} \quad (6.36)$$

From the symmetry of the problem, the magnitude of the electric field intensity  $E$  is a constant for a fixed distance  $r$  from the center of the inner sphere. Hence,  $E$  can be taken outside the integral sign to yield

$$- \oint E dA = -E \oint dA = \frac{-q}{\epsilon_0} \quad (6.37)$$

The integral  $\oint dA$  represents the sum of all the elements of area  $dA$ , and that sum is just equal to the total area of the spherical surface. But the area of a spherical surface is  $4\pi r^2$ . Hence, the integral of  $dA$  is

$$\oint dA = A = 4\pi r^2 \quad (6.38)$$

Thus, Gauss's law, equation 6.37, becomes

$$E \oint dA = E(4\pi r^2) = \frac{q}{\epsilon_0} \quad (6.39)$$

Notice that the minus signs were on both side of the equation in Gauss's law, equation 6.37, and now have been canceled out. Solving for the magnitude of the electric field intensity  $E$  we get

## Chapter 6: Capacitance

$$E = \frac{1}{4\pi\epsilon_o} \frac{q}{r^2} = \frac{kq}{r^2} \quad (6.40)$$

just as we would expect from our previous analyses in chapters 3 and 4. Placing equation 6.40 back into equation 6.34 we get

$$\begin{aligned} V &= \int_{r_a}^{r_b} E dr = \int_{r_a}^{r_b} \left( \frac{kq}{r^2} \right) dr = \int_{r_a}^{r_b} kqr^{-2} dr \\ V &= kq \frac{(r^{-2+1})}{(-2+1)} \Big|_{r_a}^{r_b} = kq \frac{(r^{-1})}{(-1)} \Big|_{r_a}^{r_b} = -kq \left( \frac{1}{r} \right) \Big|_{r_a}^{r_b} = -kq \left( \frac{1}{r_b} - \frac{1}{r_a} \right) = kq \left( \frac{1}{r_a} - \frac{1}{r_b} \right) \end{aligned}$$

and the potential between the two conducting spheres becomes

$$V = kq \left( \frac{r_b - r_a}{r_a r_b} \right) \quad (6.41)$$

The capacitance is now found from equations 6.16 and 6.41 as

$$C = \frac{q}{V} = \frac{q}{kq \left( \frac{r_b - r_a}{r_a r_b} \right)} = \frac{r_a r_b}{k(r_b - r_a)} \quad (6.42)$$

Simplifying, the capacitance of the concentric spheres becomes

$$C = \frac{4\pi\epsilon_o r_a r_b}{(r_b - r_a)} \quad (6.43)$$

### Example 6.3

*Capacitance of a spherical capacitor.* Find the capacitance of a spherical capacitor of radii  $r_a = 20.0$  cm and  $r_b = 25.0$  cm.

### Solution

The capacitance  $C$  of the spherical capacitor is found from equation 6.43 as

$$\begin{aligned} C &= \frac{4\pi\epsilon_o r_a r_b}{(r_b - r_a)} \\ C &= \frac{4\pi(8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2)(0.20 \text{ m})(0.25 \text{ m})}{(0.25 \text{ m}) - (0.20 \text{ m})} \\ C &= 1.11 \times 10^{-10} \text{ F} = 111 \text{ pF} \end{aligned}$$

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**Example 6.4**

*Capacitance of an isolated sphere.* Find the capacitance of a charged isolated sphere of 35.0 cm radius.

**Solution**

The capacitance of an isolated sphere can be found by equation 6.43 by assuming that the outer sphere is at infinity, i.e.,  $r_b = \infty$ . Hence,

$$C = \frac{4\pi\epsilon_0 r_a r_b}{(r_b - r_a)}$$

Dividing both numerator and denominator by  $r_b$  we get

$$C = \frac{4\pi\epsilon_0 r_a}{(\frac{r_b}{r_b} - \frac{r_a}{r_b})} = \frac{4\pi\epsilon_0 r_a}{(1 - \frac{r_a}{r_b})}$$

Letting  $r_b$  approach infinity, this becomes

$$C = \frac{4\pi\epsilon_0 r_a}{(1 - \frac{r_a}{\infty})}$$

The capacitance of an isolated sphere becomes

$$\begin{aligned} C &= 4\pi\epsilon_0 r_a \\ C &= 4\pi(8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2)(0.35 \text{ m}) \\ C &= 3.89 \times 10^{-11} \text{ F} = 38.9 \text{ pF} \end{aligned} \tag{6.44}$$

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**6.5 Energy Stored in a Capacitor**

As shown in figure 6.2(a), *using the concept of conventional current, the net effect of charging a capacitor by a battery is to take a positive charge  $q$  from the right plate, move it through the battery, and deposit it on the left plate. The work done by the battery in moving the charge from the negative plate, or ground plate, to the positive plate is converted to electrical potential energy of the charge.* The mechanical analogue is that of a person who does work by lifting a bowling ball from the floor to a table, where the ball now has potential energy with respect to the floor. *Since the charge creates the electric field between the plates, this energy associated with the*

## Chapter 6: Capacitance

*charge may be viewed as residing in the electric field between the plates. Thus, the energy stored in the capacitor is equal to the work done to charge the capacitor. The work done by the battery in moving a charge  $q$  through the battery is*

$$W = qV \quad (6.45)$$

But in charging the capacitor the rate at which work is done is not a constant. At the instant that the switch in figure 6.2(a) is closed, the initial potential  $V_i$  across the capacitor is zero. A small charge  $+q_i$  is then placed on the left hand plate, which then induces the charge  $-q_i$  on the right plate. An electric field  $E_1$  is established between the plates and a potential  $V_1$  appears across the plates. When the next charge  $q_1$  is brought to the left plate, an amount of work  $W_1 = q_1 V_1$  must be done by the battery. With the new charge on the left plate,  $q_i + q_1$ , a new electric field  $E_2$  is established between the plates, and a new potential  $V_2$ , which is greater than  $V_1$  appears across the plates. When the next charge  $q_2$  is brought to the left plate, the work done is  $W_2 = q_2 V_2$ . Since  $V_2$  is greater than  $V_1$ , the work  $W_2$ , must be greater than  $W_1$ . With the new charge on the left plate,  $q_i + q_1 + q_2$ , a new electric field  $E_3$  is established between the plates, and a new potential  $V_3$ , which is greater than  $V_2$ , appears across the plates. When the next charge  $q_3$  is brought to the left plate, the work done is  $W_3 = q_3 V_3$ . However, because  $V_3$  is greater than  $V_2$ , the work done,  $W_3$  is greater than the work  $W_2$ . It is obvious that a different amount of work must be done to move each charge to the plate, because as each charge is placed on the plate, a new potential appears across the plates and the product of  $qV$  will be different for each charge. Hence, the total work necessary to charge the capacitor is equal to the sum, and hence integral, of all the products of  $q_i V_i$  for an extremely large number of charges. The problem can be solved by noting that the small amount of work  $dW$  that is done in placing a small element of charge  $dq$  on the capacitor when there is a potential  $V$  across the plates is given by

$$dW = V dq \quad (6.46)$$

The work done in charging the capacitor is the sum or integral of all these elements of work and becomes

$$W = \int dW = \int V dq \quad (6.47)$$

but from equation 6.15,  $q = CV$ , and therefore  $V = q/C$ . Equation 6.47 then becomes

$$W = \int V dq = \int \frac{q}{C} dq$$

and upon integrating we get

$$W = \frac{q^2}{2C} \quad (6.48)$$

*Equation 6.48, gives the energy that is stored in the electric field of the capacitor. Because the energy stored in the capacitor is equal to the work done to charge the capa-*



## Chapter 6: Capacitance

capacitor, the letter  $W$  will now be used to designate the energy stored in a capacitor, since the letter  $E$ , usually associated with energy, is now being used for the electric field intensity. Using equation 6.15,  $q = CV$ , the energy stored in a capacitor can also be written as

$$W = \frac{q^2}{2C} = \frac{(CV)^2}{2C} = \frac{C^2 V^2}{2C} \quad (6.49)$$

$$W = \frac{1}{2} CV^2 \quad (6.50)$$

Rearranging equation 6.15 into the form,  $C = q/V$ , and substituting it into equation 6.50, the energy stored in the capacitor can also be written as

$$W = \frac{1}{2} CV^2 = \frac{1}{2} (q/V) V^2 \quad (6.51)$$

$$W = \frac{1}{2} qV \quad (6.52)$$

Equations 6.48, 6.50, and 6.52 are different ways of expressing the energy that is stored in the capacitor. This stored energy can be related to the electric field intensity between the plates of a parallel plate capacitor by using equations 6.50, 6.14, and 6.3 as

$$W = \frac{1}{2} CV^2 = \frac{1}{2} \left( \frac{\epsilon_o A}{d} \right) (Ed)^2 \quad (6.53)$$

$$W = \frac{1}{2} \epsilon_o A d E^2 \quad (6.54)$$

The product of  $A$ , the cross sectional area of the plates, and  $d$ , the separation of the plates, is the volume between the plates that the electric field occupies, and as such is the volume of the electric field. *This allows us to define a quantity called the energy density  $U_E$  as the energy per unit volume, i.e.,*

$$U_E = \frac{\text{total energy}}{\text{volume}} \quad (6.55)$$

$$U_E = \frac{W}{Ad} = \frac{\frac{1}{2} \epsilon_o A d E^2}{Ad} \quad (6.56)$$

$$U_E = \frac{1}{2} \epsilon_o E^2 \quad (6.57)$$

The energy density of the electric field, equation 6.57, was derived for the parallel plate capacitor, but it holds true in general for any electric field. Notice that the energy density depends upon the electric field intensity. It is therefore certainly appropriate to think of the energy as residing in the electric field.

### **Example 6.5**

*Energy stored in a capacitor.* Find the energy stored in the capacitor of example 6.1.

### **Solution**

## Chapter 6: Capacitance

The energy stored in the capacitor is given by equation 6.50 as

$$\begin{aligned}W &= 1/2 CV^2 = 1/2 (17.4 \times 10^{-12} \text{ F})(50.0 \text{ V})^2 \\W &= 2.18 \times 10^{-8} \frac{\text{C V}^2}{\text{V}} \left( \frac{\text{J/C}}{\text{V}} \right) \\W &= 2.18 \times 10^{-8} \text{ J}\end{aligned}$$

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### Example 6.6

*Energy density in a capacitor.* Find the energy density in the electric field between the plates of the above parallel plate capacitor.

#### Solution

Since the energy stored in the capacitor,  $W$ , is already known, the energy density is found from equation 6.55 as

$$\begin{aligned}U_E &= \frac{W}{\text{volume}} = \frac{W}{Ad} = \frac{2.18 \times 10^{-8} \text{ J}}{(7.85 \times 10^{-3} \text{ m}^2)(4.00 \times 10^{-3} \text{ m})} \\U_E &= 6.94 \times 10^{-4} \text{ J/m}^3\end{aligned}$$

As a check, let us find the electric field between the plates of the capacitor and then use equation 6.57 to find the energy density. The electric field between the plates is found from equation 6.1 as

$$E = \frac{V}{d} = \frac{50.0 \text{ V}}{4.00 \times 10^{-3} \text{ m}} = 1.25 \times 10^4 \text{ V/m}$$

and the energy density from equation 6.57 as

$$\begin{aligned}U_E &= 1/2 \epsilon_0 E^2 \\U_E &= 1/2 (8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2)(1.25 \times 10^4 \text{ V/m})^2 \\U_E &= 6.94 \times 10^{-4} \text{ J/m}^3\end{aligned}$$

which is the same energy density determined in the previous calculation.

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## Summary of Important Concepts

**Capacitor** - Two conductors of any size or shape carrying equal and opposite charges is called a capacitor. The charge on the capacitor is directly proportional to the potential difference between the plates. The importance of the capacitor lies in the fact energy can be stored in the electric field between the two bodies.

**Energy Density** - The energy per unit volume that is stored in the electric field.

## Summary of Important Equations

Charge on a Capacitor	$q = CV$	(6.15)
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Definition of Capacitance	$C = q/V$	(6.16)
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### Parallel plate capacitor

Potential between the plates	$V = Ed$	(6.3)
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Electric field between the plates	$E = \frac{q}{\epsilon_o A}$	(6.11)
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Capacitance	$C = \frac{\epsilon_o A}{d}$	(6.14)
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### Coaxial cylindrical capacitor

Electric field between the cylinders	$E = \frac{q}{2\pi\epsilon_o r l}$	(6.29)
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Potential between the cylinders	$V = \frac{q}{2\pi\epsilon_o l} \ln \frac{r_b}{r_a}$	(6.30)
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Capacitance	$C = \frac{2\pi\epsilon_o l}{\ln(r_b/r_a)}$	(6.32)
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### Concentric spherical capacitor

Electric field between the spheres	$E = \frac{1}{4\pi\epsilon_o} \frac{q}{r^2} = \frac{kq}{r^2}$	(6.40)
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Potential between the spheres	$V = kq \left( \frac{r_b - r_a}{r_a r_b} \right)$	(6.41)
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Capacitance	$C = \frac{4\pi\epsilon_o r_a r_b}{r_b - r_a}$	(6.43)
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### Energy stored in a Capacitor

$W = 1/2 qV$	(6.52)
--------------	--------

$W = 1/2 CV^2$	(6.50)
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$W = 1/2 q^2/C$	(6.48)
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Energy density in electric field	$U_E = 1/2 \epsilon_o E^2$	(6.57)
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## Questions for Chapter 6

1. Can you speak of the capacitance of an isolated conducting sphere?
2. How does the direction signal device on your car make use of a capacitor?
3. Discuss the statement, "If a capacitor contains equal amounts of positive and negative charge it should be neutral and have no effect whatsoever."
4. If the potential difference across a capacitor is doubled what does this do to the energy that the capacitor can store?

## Problems for Chapter 6

1. Find the charge on a  $4.00\text{-}\mu\text{F}$  capacitor if it is connected to a  $12.0\text{-V}$  battery.
2. If the charge on a  $9.00\text{-}\mu\text{F}$  capacitor is  $5.00 \times 10^{-4}\text{ C}$ , what is the potential across it?
3. How much charge must be removed from a capacitor such that the new potential across the plates is  $1/2$  of the original potential?
4. A charge of  $6.00 \times 10^{-4}\text{ C}$  is found on a capacitor when a potential of  $120\text{ V}$  is placed across it. What is the value of the capacitance of this capacitor?
5. A parallel plate capacitor has plates that are  $6.00\text{ cm}$  by  $4.00\text{ cm}$  and are separated by  $8.00\text{ mm}$ . Find the capacitance of the capacitor.
6. If a  $5.00\text{-}\mu\text{F}$  parallel plate capacitor has its plate separation doubled while its cross-sectional area is tripled, determine the new value of its capacitance.
7. You are asked to design your own parallel plate capacitor that is capable of holding a charge of  $3.60 \times 10^{-12}\text{ C}$  when placed across a potential difference of  $12.0\text{ V}$ . If the area of the plates is equal to  $100\text{ cm}^2$  what must the plate separation be? What is the value of the capacitance of this capacitor?
8. You are asked to design a parallel plate capacitor that can hold a charge of  $9.87\text{ pC}$  when a potential difference of  $24.0\text{ V}$  is applied across the plates. Find the ratio of the area of the plates to the plate separation, and then pick a reasonable set of values for them.
9. A cylindrical capacitor  $10.0\text{ cm}$  long has an inner radius of  $0.500\text{ mm}$  and an outside radius of  $5.00\text{ mm}$ . Find the capacitance of the capacitor.
10. A cylindrical capacitor  $1.00\text{ m}$  long has radii  $20.0\text{ cm}$  and  $50.0\text{ cm}$ . Find its capacitance.
11. A spherical capacitor has radii  $20.0\text{ cm}$  and  $50.0\text{ cm}$ . Find its capacitance.
12. A  $7.00\text{-}\mu\text{F}$  capacitor is connected to a  $400\text{-V}$  source. Find (a) the charge on the capacitor and (b) the energy stored in the capacitor.
13. How much energy can be stored in a capacitor of  $9.45\text{ }\mu\text{F}$  when it is placed across a potential difference of  $120\text{ V}$ ? How much charge will be on this capacitor?
14. What value of capacitance is necessary to store  $1.73 \times 10^{-3}\text{ J}$  of energy when it is placed across a potential difference of  $24.0\text{ V}$ ?

## Chapter 6: Capacitance

15. Show that if the radii of a spherical capacitor are very large, the equation for the capacitance of a spherical capacitor reduces to the equation for a parallel plate capacitor.

16. Consider the earth to be an isolated sphere. Find the capacitance of the earth. If the electric field of the earth is measured to be 100 V/m, what charge must be on the surface of the earth? (Note that the fair weather electric field points downward, indicating an effective negative charge on the solid sphere.)

17. A parallel plate capacitor with air between the plates has a plate separation  $d_0$  and area  $A$ . A sheet of aluminum of negligible thickness is now placed midway between the parallel plates. Find the capacitance (a) before and (b) after the aluminum sheet is placed between the plates.

18. A parallel plate capacitor with air between the plates has a plate separation  $d_0$  and area  $A$ . A half of a sheet of aluminum of negligible thickness is now placed midway between the parallel plates as shown. Show what this combination is equivalent to and find the equivalent capacitance.

19. A parallel plate capacitor has a moveable top plate that executes simple harmonic motion at a frequency  $f$ . The displacement of the top plate varies such that the plate separation varies from  $d$  to  $2d$ . Show that the voltage across  $MN$  varies as

$$V = V_0 \left( 1 + \frac{1}{2} \cos 2\pi ft \right)$$

where  $V_0$  is the voltage across the capacitor when the plates are at a constant separation  $d$ . Describe how you might use this as a sensor to convert a mechanical vibration to an AC signal that shows the vibration.

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