The Physics Hypertextbook

Opus in profectus

Electromagnetic Waves

Discussion

maxwell's equations

Warning: big, fancy calculus derivation approaching. If you don't like or don't understand vector calculus, just skim through everything down to the paragraph before the last equation. The descriptive text is fairly easy to read.

Start with Maxwell's equations in derivative form for empty space.

$$\cdot \mathrm{E} = 0 \text{ (Gauss)} \qquad \times \mathrm{E} = \ -\frac{\partial \mathrm{B}}{\partial t} \text{ (Faraday)}$$

$$\cdot \mathrm{B} = 0 \text{ (no name)} \qquad \times \mathrm{B} = \mu_0\varepsilon_0\frac{\partial \mathrm{E}}{\partial t} \text{ (Ampère)}$$

These equations are first order, which usually means the mathematics should be easy (good!), but they're also coupled, which means it might be difficult (rats!). Let's separate them using this little trick. Take the curl of both sides of Faraday's and Ampère's laws. The left side of each equation is the curl of the curl, for which there is a special identity. The right side of each equation, on the other hand, is the curl of a time derivative. We'll switch it around into a time derivative of the curl.

$$\times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t} \qquad \times \mathrm{B} = \mu_0\varepsilon_0\frac{\partial \mathrm{E}}{\partial t}$$

$$\times(\ \times \mathrm{E}) = \ \times\ -\frac{\partial \mathrm{B}}{\partial t} \qquad \times(\ \times \mathrm{B}) = \ \times\ \mu_0\varepsilon_0\frac{\partial \mathrm{E}}{\partial t}$$

$$(\ \cdot \mathrm{E}) - \ {}^2\mathrm{E} = -\frac{\partial}{\partial t}(\ \times \mathrm{B}) \qquad (\ \cdot \mathrm{B}) - \ {}^2\mathrm{B} = \mu_0\varepsilon_0 \frac{\partial}{\partial t}(\ \times \mathrm{E}$$

Now if you look carefully, you'll see that one term in each equation equals zero and the other can be replaced with a time derivative.

$$0 - \ {}^2\mathrm{E} = -\frac{\partial}{\partial t}\ \mu_0\varepsilon_0\frac{\partial \mathrm{E}}{\partial t} \qquad 0 - \ {}^2\mathrm{B} = \mu_0\varepsilon_0\frac{\partial}{\partial t}\ -\frac{\partial \mathrm{B}}{\partial t}$$

Let's clean it up a bit and see what we get.

$$ {}^2\mathrm{E} = \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathrm{E} \qquad {}^2\mathrm{B} = \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathrm{B}$$

These equations are now decoupled (E and B have their own private equations), which certainly simplifies things, but in the process we've changed them from first to second order (notice all the squares). I know I said earlier that lower order implies easier to work with, but these second order equations aren't as difficult as they look. Raising the order has not made things more complicated, it's made things more interesting.

Here's one set of possible solutions.

$$\mathrm{E}\ (x,t) = E_0 \sin\left[2\pi\left(ft - \frac{x}{\lambda} + \varphi\right)\right]\ \mathrm{j} \qquad \mathrm{B}\ (x,t) = B_0 \sin\left[2\pi\left(ft - \frac{x}{\lambda} + \varphi\right)\right]\ \mathrm{k}$$

This particular example is one dimensional, but there are two dimensional solutions as well — many of them. The interesting ones have electric and magnetic fields that change in time. These changes then propagate away at a finite speed. Such a solution is an electromagnetic wave.

Let's examine our possible solution in more detail. Find the second space and time derivatives of the electric field…

$${}^2\mathrm{E} = -\frac{4\pi^2}{\lambda^2} E_0 \sin\left[2\pi\left(ft - \frac{x}{\lambda} + \varphi\right)\right]\ \mathrm{j}$$

$$\frac{\partial^2}{\partial t^2}\mathrm{E} = -\ 4\pi^2 f^2 E_0 \sin\left[2\pi\left(ft - \frac{x}{\lambda} + \varphi\right)\right]\ \mathrm{j}$$

and substitute them back into the second order partial differential equation.

$${}^2\mathrm{E} = \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}\ \mathrm{E}$$

Work on the left side first.

$$^2\mathrm{E} = -\frac{4\pi^2}{\lambda^2} E_0 \sin[2\pi(ft - \frac{x}{\lambda} + \varphi)]\ \mathbf{j}$$

Work on the right side second.

$$\mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\,\mathrm{E} = \mu_0\varepsilon_0\{-\,4\pi^2 f^2 E_0 \sin[2\pi(ft - \frac{x}{\lambda} + \varphi)]\}\ \mathbf{j}$$

All kinds of stuff cancels.

$$\frac{1}{\lambda^2} = \mu_0\varepsilon_0 f^2$$

Rearrange a bit.

$$f^2\lambda^2 = \frac{1}{\mu_0\varepsilon_0}$$

I see a wave speed in there ($f\lambda$). We'll use c for this one since it's the first letter in the Latin word for swiftness — *celeritas*.

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

Very interesting.

Given Maxwell's four equations (which are based on observation) we have shown that electromagnetic waves must exist as a consequence. They can have any amplitude E_0 (with B_0 depending on E_0 as will be shown later), any wavelength λ, and be retarded or advanced by any phase φ, but they can only travel through empty space at one wave speed c.

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

$$c = \frac{1}{\sqrt{[(4\pi \times 10^{-7}\text{ Tm/A})(8.85418782 \times 10^{-12}\text{ C}^2\text{/Nm}^2)]}}$$

$$c = 299{,}792{,}458\text{ m/s}$$

In the words of Maxwell…

> This velocity is so nearly that of light, that it seems we have

> strong reasons to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws.
>
> James Clerk Maxwell, 1865

This is the speed of light in a vacuum, which means that…

1. electromagnetic waves propagate at the speed of light,
2. light is an electromagnetic wave, and
3. there are other forms of electromagnetic radiation.

Those are the three important conclusions from this mathematical excursion.

history

Let's recall the steps that led to the formulation of Maxwell's four laws.

1. Gauss's law is an extension of Coulomb's law and has its origins in

the study of charged objects and the forces of attraction and repulsion between them. In everyday terms, the study of static cling, which has its roots in ancient times when it was noticed that amber rubbed with animal fur attracted bits of cloth and paper. The Greek word for amber, ηλεκτρον (elektron), is the root of the English words electric, electrical, electricity, electrician, and so on.

2. No one's law comes from the observational fact that every magnet has both a north and a south pole. No one has ever seen a magnetic monopole. Whenever a magnet is broken it always has a north and a south pole. This is true down to the subatomic level. From this observation we can deduce that magnetic field lines must form continuous loops. The study of magnetism goes back to the time when magnetic rocks were first found by peoples around the world — most notably outside the ancient Greek city of Magnesia, which is the root of the English word magnetism.
3. Faraday's law deals with induced electric currents. Given a loop of wire and a magnet, one can induce current to flow through the loop by moving the loop or moving the magnet. The static charges studied by Gauss, Coulomb, and Franklin can be made to move by the unusual rocks found lying around in the lands of the old Greek Empire.
4. Ampère's law originally dealt with the magnetism that arose from moving charges. Run charges through a wire and you've made a

> magnet — an electromagnet. Maxwell's key insight was that the space between two parallel metal plates in the process of being charged will behave in a manner similar to the space around a current-carrying wire. There's the magnetism that comes from electric currents (like the current through a working electromagnet) and the magnetism that comes from displacement currents (like the changing electric field in a capacitor that's just been switched on or off).

It is the last law in the list — Ampère's law as modified by Maxwell — that is the key. A changing electric field can produce a magnetic field in much the same way as an electric current can produce a magnetic field. Thus, electric charges did not have to flow or even to exist. A changing electric field will generate a changing magnetic field all on its own. This would result in a changing electric field, which would result in a a changing magnetic field, and so on — the whole thing flying away out into empty space at the speed of light.

The implications are huge. Perhaps there are other forms of electromagnetic waves that are invisible to the human eye. The equations impose no limits on wavelength or frequency. The only requirement is that they propagate with the speed of light in a vacuum.

These conclusions were made in 1864 before there was any experimental evidence for invisible electromagnetic waves. Before Maxwell there was light and nothing else. Now we have an unlimited electromagnetic spectrum that includes radio waves, microwaves, infrared, visible light, ultraviolet, x-rays, and gamma rays. Perhaps the most amazing thing about this story is not that Maxwell showed that light was an electromagnetic wave, but that he stumbled upon it. It wasn't his goal. It was an unintended consequence. To quote Maxwell once again…

> The value of [*c*] was determined by measuring the electromotive force with which a condenser of known capacity was charged, and then discharging the condenser through a galvanometer, so as to measure the quantity of electricity in it in electromagnetic measure. The only use made of light in the experiment was to see the instruments. The value of [*c*] found by M. Foucault was obtained by determining the angle through which a revolving mirror is turned, while the light reflected from it went and returned along a measured course. No use whatever was made of electricity or magnetism.
>
> The agreement of the results seem to show that light and magnetism are affections of the same substance, and that light is an electromagnetic disturbance propagated through the field

according to electromagnetic laws. [expand]

Amber, animal fur, rocks from Magnesia, loops of wire, and batteries connected to metal plates. What else have I missed? Dutch gentlemen wrapping glass jars with metal foil and shocking each other; Franklin flying a kite on a stormy, summer afternoon; and Chinese sailors navigating with compasses. You might not see it today, and you definitely wouldn't have seen it coming in the middle of the Nineteenth Century, but these seemingly disconnected events are all related by the speed of light. This means we must add to the list Newton, Snell, Fermat and all the rest watching light bend through glass; Young's double slit apparatus; and Galileo with his telescope to name but a few. Although none of them knew it at the time, they were all working on the same vast project — the study of electromagnetism.

The rest is history…

4. James Clerk Maxwell (1831–1879) Scotland–England
   Prediction of electromagnetic waves
7. Heinrich Hertz (1857–1894) Germany
   Experimental confirmation of radio waves (spark gap transmitter-

receiver)

1. Guglielmo Marconi (1874–1937) Italy
   First transatlantic Morse code transmission (England to Newfoundland)
6. Reginald Fessenden (1866–1932) Canada–Bermuda
   First amplitude modulation broadcast (AM)
9. Philo T. Farnsworth (1906–1971) USA
   First all-electronic television broadcast
3. Edwin Howard Armstrong (1890–1954) USA
   First frequency modulation broadcast (FM)

energy, power, and pressure

The electric field describes an electromagnetic wave completely in free space. The magnetic field is related to the electric field by a simple relationship. Start from Faraday's law.

$$\times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}$$

Work on the left side first. Substitute the one dimensional wave equation for electricity and find its curl.

$$\times \mathrm{E} = \qquad \times\{E_0\sin[2\pi(ft - \frac{x}{\lambda} + \varphi)]\ \hat{\mathbf{j}}\}$$

$$\times \mathrm{E} = -\frac{2\pi}{\lambda}\ E_0\cos[2\pi(ft - \frac{x}{\lambda} + \varphi)]\ \mathrm{k}$$

Work on the right side second. Substitute the one dimensional wave equation for magnetism and find its time derivative.

$$\frac{\partial}{\partial t}\mathrm{B} = -\frac{\partial}{\partial t}\{B_0\sin[2\pi(ft - \frac{x}{\lambda} + \varphi)]\ \mathrm{k}\}$$

$$\frac{\partial}{\partial t}\mathrm{B} = -\ 2\pi f B_0\cos[2\pi(ft - \frac{x}{\lambda} + \varphi)]\ \mathrm{k}$$

Set the two sides equal. Cancel the cosine terms and some other stuff.

$$\frac{1}{\lambda}E_0 = fB_0$$

Rearrange it to look nice…

$$\frac{E_0}{B_0} = f\lambda$$

and then recall that $f\lambda$ is the speed of light.

$$\frac{E_0}{B_0} = c$$

Well, we actually cancelled out too much stuff. This relationship holds true for all field values, not just the maximum. The ratio of the electric to magnetic fields in an electromagnetic wave in free space is always equal to the speed of light.

$$\frac{E}{B} = c$$

This knowledge can then be used to simplify the energy density situation a bit. Start with the magnetic energy density and replace it with an expression containing the electric field.

$$\eta_B = \frac{1}{2\mu_0} B^2 = \frac{1}{2\mu_0} \frac{E}{c^2}$$

Recall that the speed of light is related to the permeability and permittivity constants.

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

Thus…

$$\eta_B = \frac{1}{2\mu_0}\mu_0\varepsilon_0 E^2 = \frac{\varepsilon_0}{2}E^2$$

Look familiar? It's the electric energy density. For an electromagnetic wave in free space, half the energy is in the electric field and half is in the magnetic field

$$\eta = \eta_E + \eta_B$$

$$\eta = \frac{\varepsilon_0}{2}E^2 + \frac{\varepsilon_0}{2}E^2$$

This gives us this compact equation for the total energy density of an electromagnetic wave…

$$\eta = \varepsilon_0 E^2$$

or this one, if you prefer to state things in terms of the magnetic field instead…

$$\eta = \frac{1}{\mu_0} B^2$$

This is an interesting and simple set of relations, but keep in mind that it only works for electromagnetic waves in free space. Things are different in a media and the electric and magnetic fields can have any values they want if they're static (meaning there's no accelerating charges).

Since waves are spread out in space and time, energy density is often a more useful concept than energy. By extension, the power of a wave should probably be replaced with the more useful concept of its power density. Since the energy content of a wave fills a volume of space it makes sense to define energy density as energy per volume.

$$\eta = \frac{U}{V}$$

Since power is energy on the move, the notion of power existing in a place doesn't make much sense. Instead we should speak of the power delivered to a place. The boundary between one place and another is described by an area. What's the difference between being inside a room and outside the room? The answer is what side of the doorway you're on. How is this opening described? By its area. The sensible definition of power density is then power per area.

$$S = \frac{P}{A}$$

This quantity is also known as irradiance, radiant flux, emissive power, energy flux or energy flux density. None of these words begin with "s" so why *S* was chosen as the symbol is unknown to me. Since I've also seen this quantity represented by the symbols *q*, *j*, and *E* maybe what I really should be saying here is I don't know why I chose *S*. My guess is that it's the way I learned it way back when and therefore it's the way you should learn it too.

The unit of this quantity is the watt per square meter, which has no special name.

$$\frac{\mathrm{W}}{\mathrm{m}^2} = \frac{\mathrm{W}}{\mathrm{m}^2}$$

We'll start the analysis of this quantity by recalling the definitions of power (the rate at which energy is transformed) and energy density (energy per volume).

$$S = \frac{P}{A} = \frac{1}{A}\frac{U}{t} = \frac{1}{A}\frac{\eta V}{t}$$

Now, imagine a beam of light or radio waves or any other kind of electromagnetic wave landing on a surface. The energy that falls on this surface in a given amount of time fills a column that travels through space at the speed of light. The volume of this column is the area of its base times its length. The area can be any arbitrary size, since we're dealing with a density here, and the length of this column is determined by the time it takes for the column to land on the surface while traveling at the speed of light. Let me show you what I'm talking about with mathematical symbols.

$$S = \frac{\eta V}{At} = \frac{\eta(A\ell)}{At} = \frac{\eta(Act)}{At} = \eta c$$

The next steps involve replacing η and *c* with the special relationships discussed earlier.

$$S = \eta c = \frac{1}{\mu_0} B^2 \frac{E}{B}$$

And here's what we end up with…

$$S = \frac{1}{\mu_0} EB$$

Certainly not what I expected, but this is the traditional way to write the power density of an electromagnetic wave. Well… almost. The real equation is written in vector form like this…

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

and is given the oddly appropriate name poynting vector, not because someone was making a joke about how vectors "poynt" but in honor of its discoverer, the English physicist John Poynting (1852–1914). Poynting's derivation involves vector mathematics that isn't appropriate for the level of this book. (Translation: I don't understand it.)

The poynting vector is important because it aligns the three vectors of an electromagnetic wave: the electric field, the magnetic field, and the direction of propagation. These three vectors are mutually perpendicular; that is, each is perpendicular to the other two. Their relative arrangement is determined by the right hand rule of the cross product (that is; the × between E and B in the equation).

The example shown in the diagram below is consistent with this rule. Check it out for yourself. Mentally pick a pair of vectors coming out of the same point on the wave. Hold your right hand flat in front of your face with your thumb stuck out on the side at a right angle in the shape of an "L". Now rotate your hand until your fingers point in the direction of the electric field and your palm faces in the direction of the magnetic field. If your hand is aligned properly you should be able to fold your fingers so they point in the direction of the magnetic field. (Don't move your thumb.) This action imitates the "crossing" of the electric field into the magnetic field. The direction of this cross product is the poynting vector and is indicated by your thumb. If you've done this activity correctly, your thumb should be point out of the screen toward your face. The orientation of the rest of your hand depends on whether you aligned you fingers with an electric field vector pointing left or right. One of them is easy on the hand and the other makes you look like you're performing some odd form of yoga.

As we learned in an earlier section of this book, waves transfer both energy and momentum without transferring any mass. That might seem obvious for mechanical waves (especially if you've ever been bowled over by a strong ocean wave) but when's the last time you ever felt pressed by a radio wave or knocked down by a beam of light? We just don't experience radiation pressure. Still, it is something we can compute.

Begin with the definitions of pressure (force per area) and work (force times distance) and see what happens.

$$P = \frac{F}{A} = \frac{F\ell}{A\ell} = \frac{U}{V} = \eta$$

Well that's interesting. Pressure and energy density are the same thing. The only problem is that with waves there is no single value for the energy density. It's a quantity that fluctuates in time and space. What we really need here are time-averaged values. Such quantities are represented by the symbol between two angle brackets. Like this…

$$P = \frac{F}{A} = \frac{F \quad \ell}{A \quad \ell} = \frac{U}{V} = \eta$$

That's how you write it and here's how you do it for the case of a simple sine wave. Integrate the energy density equation over one period.

$$P = \eta$$

$$P = \frac{1}{T}\int \varepsilon_0 E^2 \, dt$$

$$P = \frac{1}{T}\int \varepsilon_0 \; E_0 \sin 2\pi \; ft - \frac{x}{\lambda} \;\; {}^2 dt$$

$$P = \frac{\varepsilon_0 E_0{}^2}{T}\int_0^T \; \sin 2\pi \; ft - \frac{x}{\lambda} \;\; {}^2 dt$$

That may look like a big mean integral, but it's not. Think of what the sine squared curve looks like. It's a wiggly line that goes up and down between 0 and 1. Over one complete cycle it splits a box 1 high by *T* wide in half. This gives us…

$$P = \frac{\varepsilon_0 E_0{}^2}{T}\frac{T}{2} = ½\varepsilon_0 E_0$$

which you might recognize as half the energy density.

$P = ½\eta$

The radiation pressure of an electromagnetic wave isn't equal to its energy density, it's equal to half its energy density. I believe this mathematics, but I think I still need to prove to myself that this equation is real. As I noted earlier, I've never felt pressed by a radio wave or been knocked down by a beam of light. It must be an exceptioanlly weak effect. We'll confirm this through computation in the practice problems that accompany this discussion.

miscellaneous

Do I need to discuss the impedance of free space here?

$$Z = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \mu_0 c$$

Show that this has ohm as the unit

Compute it.

$$Z = \mu_0 c$$
$$Z = (4\pi \times 10^{-7}\ \mathrm{Vs/Am})(299{,}792{,}458\ \mathrm{m/s})$$

$$Z = 376.730\ldots\ \Omega$$

Electromagnetic Waves

… maxwell | em-waves | em-spectrum …

mechanics

kinematics

motion
distance & displ
speed & velocity
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