Chapter 1 Special Relativity

And now in our time, there has been unloosed a cataclysm which has swept away space, time, and matter hitherto regarded as the firmest pillars of natural science, but only to make place for a view of things of wider scope, and entailing a deeper vision. This revolution was promoted essentially by the thought of one man, Albert Einstein.

Hermann Weyl - Space-Time-Matter

1.1 Introduction to Relative Motion

***Relativity** has as its basis the observation of the motion of a body by two different observers in relative motion to each other.* This observation, apparently innocuous when dealing with motions at low speeds has a revolutionary effect when the objects are moving at speeds near the velocity of light. At these high speeds, it becomes clear that the simple concepts of space and time studied in Newtonian physics no longer apply. Instead, *there becomes a fusion of space and time into one physical entity called spacetime. All physical events occur in the arena of spacetime. As we shall see, the normal Euclidean geometry, studied in high school, that applies to everyday objects in space does not apply to spacetime. That is, spacetime is non-Euclidean. The apparently strange effects of relativity, such as length contraction and time dilation, come as a result of this non-Euclidean geometry of spacetime.*

The earliest description of relative motion started with Aristotle who said that the earth was at absolute rest in the center of the universe and everything else moved relative to the earth. As a proof that the earth was at absolute rest, he reasoned that if you throw a rock straight upward it will fall back to the same place from which it was thrown. If the earth moved, then the rock would be displaced on landing by the amount that the earth moved. This is shown in figures 1.1(a) and 1.1(b).

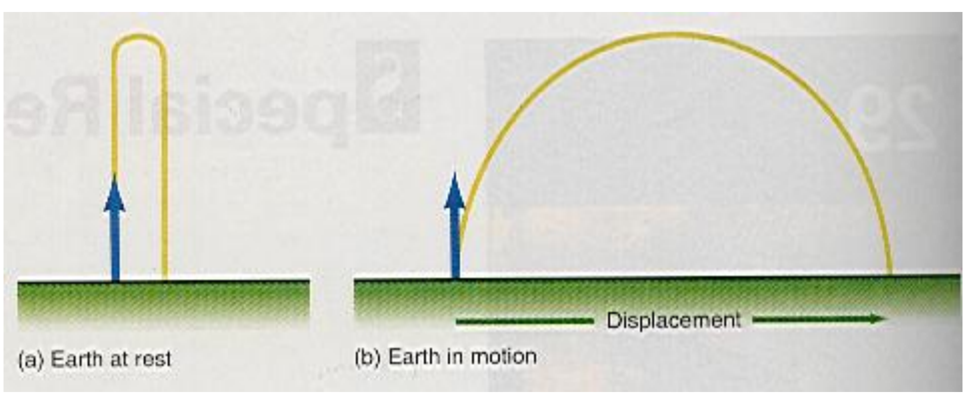

Figure 1.1 Aristotle's argument for the earth's being at rest.

Based on the prestige of Aristotle, the belief that the earth was at absolute rest was maintained until Galileo Galilee (1564-1642) pointed out the error in Aristotle's reasoning. Galileo suggested that if you throw a rock straight upward in a boat that is moving at constant velocity, then, as viewed from the boat, the rock goes straight up and straight down, as shown in figure 1.2(a). If the same

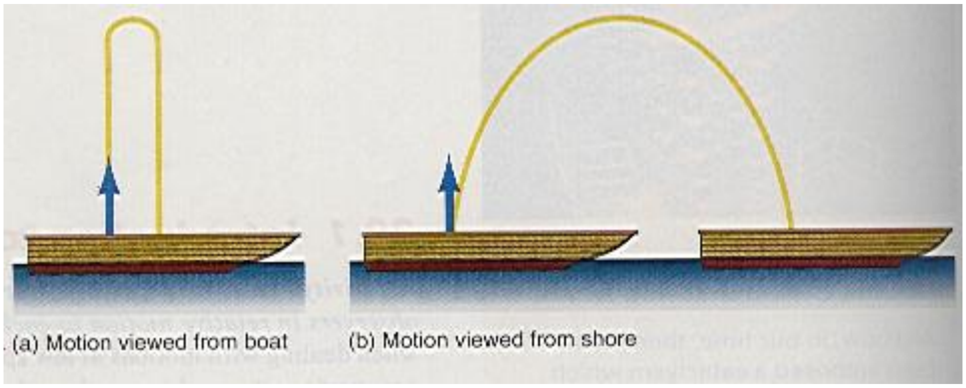

Figure 1.2 Galileo's rebuttal of Aristotle's argument of absolute rest.

projectile motion is observed from the shore, however, the rock is seen to be displaced to the right of the vertical path. The rock comes down to the same place on the boat only because the boat is also moving toward the right. Hence, to the observer on the boat, the rock went straight up and straight down and by Aristotle's reasoning the boat must be at rest. But as the observer on the shore will clearly state, the boat was not at rest but moving with a velocity **v.** Thus, Aristotle's argument is not valid. *The distinction between rest and motion at a constant velocity, is relative to the observer.* The observer on the boat says the boat is at rest while the observer on the shore says the boat is in motion. We then must ask, is there any way to distinguish between a state of rest and a state of motion at constant velocity?

Let us consider Newton's second law of motion as studied in general physics,

$$\mathbf{F} = m\mathbf{a}$$

If the unbalanced external force acting on the body is zero, then the acceleration is also zero. But since $\mathbf{a} = d\mathbf{v}/dt$, this implies that there is no change in velocity of the body, and the velocity is constant. *We are capable of feeling forces and accelerations but we do not feel motion at constant velocity, and rest is the special case of zero constant velocity.* Recall from general physics, concerning the weight of a person in an elevator, the scales read the same numerical value for the weight of the person when the elevator is either at rest or moving at a constant velocity. There is no way for the passenger to say he or she is at rest or moving at a constant velocity unless he or she can somehow look out of the elevator and see motion. When the elevator accelerates upward, on the other hand, the person experiences a greater force pushing upward on him. When the elevator accelerates downward, the person experiences a smaller force on him. Thus, accelerations are easily felt but not constant velocities. Only if the elevator accelerates can the passenger tell that he or she is in motion. While you sit there reading this sentence you are sitting on the earth, which is moving around the sun at about 30 km/s, yet you do not notice this motion.[1] When a person sits in a plane or a train moving at constant velocity, the motion is not sensed unless the person looks out the window. The person senses his or her motion only while the plane or train is accelerating.

Since relative motion depends on the observer, there are many different ways to observe the same motion. For example, figure 1.3(a) shows body 1 at rest while

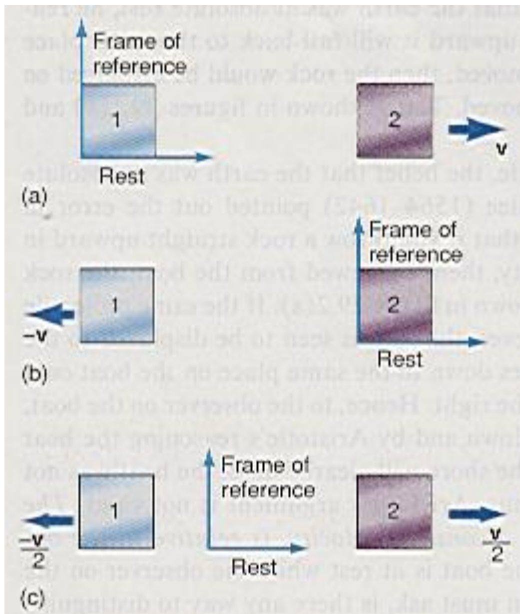

Figure 1.3 Relative motion.

body 2 moves to the right with a velocity **v.** But from the point of view of body 2, he can equally well say that it is he who is at rest and it is body 1 that is moving to the left with the velocity –**v,** figure 1.3(b). Or an arbitrary observer can be placed at rest between bodies 1 and 2, as shown in figure 1.3(c), and she will observe body 2 moving to the right with a velocity **v**/2 and body 1 moving to the left with a velocity of –**v**/2. We can also conceive of the case of body 1 moving to the right with a velocity **v** and body 2 moving to the right with a velocity 2**v,** the relative velocities between the two bodies still being **v** to the right. Obviously an infinite number of such possible cases can be thought out. Therefore, we must conclude that, *if a body in motion at constant velocity is indistinguishable from a body at rest, then there is no reason why a state of rest should be called a state of rest, or a state of motion a state of motion.* Either body can be considered to be at rest while the other body is moving in the opposite direction with the speed v.

To describe the motion, we place a coordinate system at some point, either in the body or outside of it, and call this coordinate system a frame of reference. The motion of any body is then made with respect to this frame of reference. *A frame of reference that is either at rest or moving at a constant velocity is called an inertial*

1[1]Actually the earth's motion around the sun constitutes an accelerated motion. The average centripetal acceleration is $a_c = v^2/r = (33.7 \times 10^3 \text{ m/s})^2/(1.5 \times 10^{11} \text{ m}) = 5.88 \times 10^{-3} \text{ m/s}^2 = 0.0059$ m/s^2. This orbital acceleration is so small compared to the acceleration of gravity, 9.80 m/s^2, that we do not feel it and it can be ignored. Hence, we feel as though we were moving at constant velocity.

frame of reference or an **inertial coordinate system.** *Newton's first law defines the inertial frame of reference.* That is, when $\mathbf{F} = 0$, and the body is either at rest or moving uniformly in a straight line, then the body is in an inertial frame. There are an infinite number of inertial frames and Newton's second law, in the form $\mathbf{F} = m\mathbf{a}$, holds in all these inertial frames.

An example of a noninertial frame is an accelerated frame, and one is shown in figure 1.4. A rock is thrown straight up in a boat that is accelerating to the

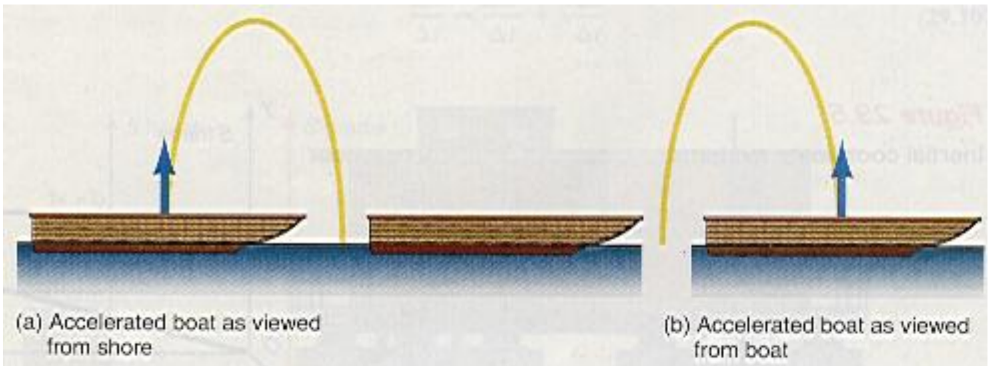

Figure 1.4 A linearly accelerated frame of reference.

right. An observer on the shore sees the projectile motion as in figure 1.4(a). The observed motion of the projectile is the same as in figure 1.2(b), but now the observer on the shore sees the rock fall into the water behind the boat rather than back onto the same point on the boat from which the rock was launched. Because the boat has accelerated while the rock is in the air, the boat has a constantly increasing velocity while the horizontal component of the rock remains a constant. Thus the boat moves out from beneath the rock and when the rock returns to where the boat should be, the boat is no longer there. When the same motion is observed from the boat, the rock does not go straight up and straight down as in figure 1.2(a), but instead the rock appears to move backward toward the end of the boat as though there was a force pushing it backward. The boat observer sees the rock fall into the water behind the boat, figure 1.4(b). In this accelerated reference frame of the boat, there seems to be a force acting on the rock pushing it backward. Hence, Newton's second law, in the form $\mathbf{F} = m\mathbf{a}$, does not work on this accelerated boat. Instead a fictitious force must be introduced to account for the backward motion of the projectile.

For the moment, we will restrict ourselves to motion as observed from inertial frames of reference, the subject matter of the special or restricted theory of relativity. In chapter 34, we will discuss accelerated frames of reference, the subject matter of general relativity.

1.2 The Galilean Transformations of Classical Physics

The description of any type of motion in classical mechanics starts with an inertial coordinate system *S*, which is considered to be at rest. Let us consider the occurrence of some "event" that is observed in the *S* frame of reference, as shown in figure 1.5. The event might be the explosion of a firecracker or the lighting of a

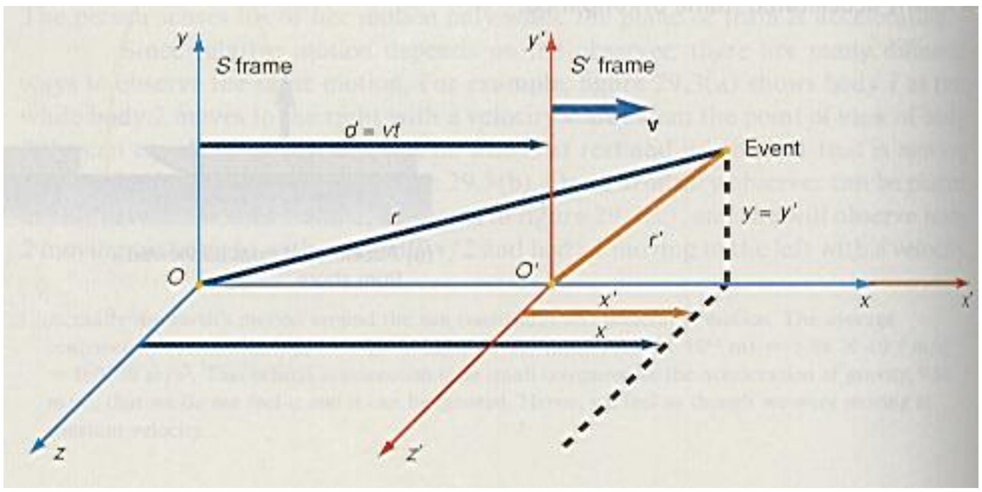

Figure 1.5 Inertial coordinate systems.

match, or just the location of a body at a particular instance of time. For simplicity, we will assume that the event occurs in the *x,y* plane. The event is located a distance *r* from the origin *O* of the *S* frame. The coordinates of the point in the *S* frame, are *x* and *y*. A second coordinate system *S'*, moving at the constant velocity **v** in the positive *x*-direction, is also introduced. The same event can also be described in terms of this frame of reference. The event is located at a distance *r'* from the origin *O'* of the *S'* frame of reference and has coordinates *x'* and *y'*, as shown in the figure. We assume that the two coordinate systems had their origins at the same place at the time, $t = 0$. At a later time *t*, the *S'* frame will have moved a distance, $d = vt$, along the *x*-axis. The *x*-component of the event in the *S* frame is related to the *x'*-component of the same event in the *S'* frame by

$$x = x' + vt \tag{1.1}$$

which can be easily seen in figure 1.5, and the *y*- and *y'*-components are seen to be

$$y = y' \tag{1.2}$$

Notice that because of the initial assumption, *z* and *z'* are also equal, that is

$$z = z' \tag{1.3}$$

It is also assumed, but usually never stated, that the time is the same in both frames of reference, that is,

$$t = t' \quad (1.4)$$

*These equations, that describe the event from either inertial coordinate system, are called the **Galilean transformations** of classical mechanics and they are summarized as*

$$x = x' + vt \quad (1.1)$$
$$y = y' \quad (1.2)$$
$$z = z' \quad (1.3)$$
$$t = t' \quad (1.4)$$

The inverse transformations from the S frame to the S' frame are

$$x' = x - vt \quad (1.5)$$
$$y' = y \quad (1.6)$$
$$z' = z \quad (1.7)$$
$$t' = t \quad (1.8)$$

Example 1.1

The Galilean transformation of distances. A student is sitting on a train 10.0 m from the rear of the car. The train is moving to the right at a speed of 4.00 m/s. If the rear of the car passes the end of the platform at $t = 0$, how far away from the platform is the student at 5.00 s?

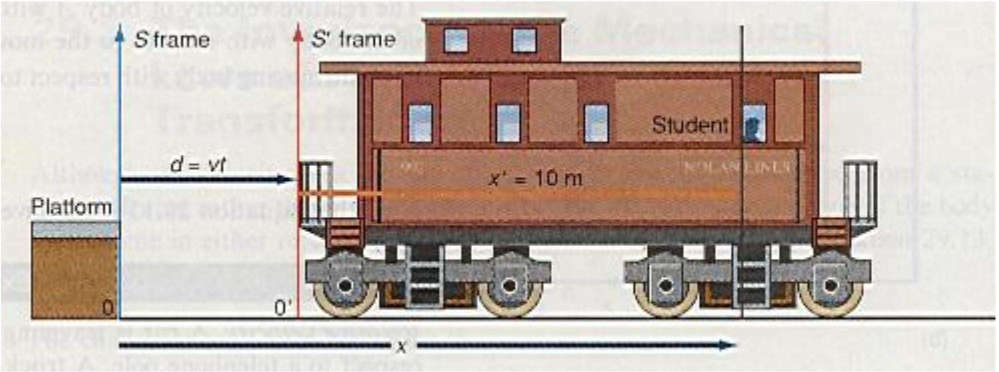

Figure 1.6 An example of the Galilean transformation.

Solution

The picture of the student, the train, and the platform is shown in figure 1.6. The platform represents the stationary S frame, whereas the train represents the moving S' frame. The location of the student, as observed from the platform, found from equation 1.1, is

$$\begin{aligned} x &= x' + vt \\ &= 10.0 \text{ m} + (4.00 \text{ m/s})(5.00 \text{ s}) \\ &= 30 \text{ m} \end{aligned}$$

To go to this Interactive Example click on this sentence.

The speed of an object in either frame can be easily found by differentiating the Galilean transformation equations with respect to t. That is, for the x-component of the transformation we have

$$x = x' + vt$$

Upon differentiating

$$\frac{dx}{dt} = \frac{dx'}{dt} + v\frac{dt}{dt} \tag{1.9}$$

But $dx/dt = v_x$, the x-component of the velocity of the body in the stationary frame S, and $dx'/dt = v'_x$, the x-component of the velocity in the moving frame S'. Thus equation 1.9 becomes

$$v_x = v'_x + v \tag{1.10}$$

Equation 1.10 is a statement of the Galilean addition of velocities.

Example 1.2

The Galilean transformation of velocities. The student on the train of example 1.1, gets up and starts to walk. What is the student's speed relative to the platform if (a) the student walks toward the front of the train at a speed of 2.00 m/s and (b) the student walks toward the back of the train at a speed of 2.00 m/s?

Solution

a. The speed of the student relative to the stationary platform, found from equation 1.10, is

$$v_x = v'_x + v = 2.00 \text{ m/s} + 4.00 \text{ m/s}$$
$$= 6.00 \text{ m/s}$$

b. If the student walks toward the back of the train $\Delta x' = x'_2 - x'_1$ is negative because x'_1 is greater than x'_2, and hence, v'_x is a negative quantity. Therefore,

$$v_x = v'_x + v$$
$$= -2.00 \text{ m/s} + 4.00 \text{ m/s}$$
$$= 2.00 \text{ m/s}$$

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If there is more than one body in motion with respect to the stationary frame, the relative velocity between the two bodies is found by placing the S' frame on one of the bodies in motion. That is, if body A is moving with a velocity $\mathbf{v}_{AS}$ with respect to the stationary frame S, and body B is moving with a velocity $\mathbf{v}_{BS}$, also with respect to the stationary frame S, the velocity of A as observed from B, $\mathbf{v}_{AB}$, is simply

$$\mathbf{v}_{AB} = \mathbf{v}_{AS} - \mathbf{v}_{BS} \tag{1.11}$$

as seen in figure 1.7(a).

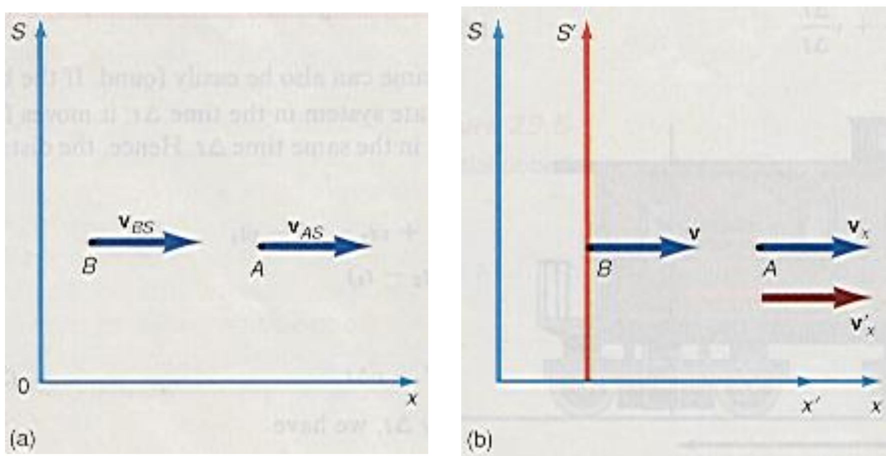

Figure 1.7 Relative velocities.

If we place the moving frame of reference S' on body B, as in figure 1.7(b), then $\mathbf{v}_{BS} = \mathbf{v}$, the velocity of the S' frame. The velocity of the body A with respect to S, $\mathbf{v}_{AS}$, is now set equal to $\mathbf{v}_x$, the velocity of the body with respect to the S frame. The relative velocity of body A with respect to body B, $\mathbf{v}_{AB}$, is now $\mathbf{v}'_x$, the velocity of the body with respect to the moving frame of reference S'. Hence the velocity $\mathbf{v}'_x$ of the moving body with respect to the moving frame is determined from equation 1.11 as

$$\mathbf{v}'_x = \mathbf{v}_x - \mathbf{v} \tag{1.12}$$

Note that equation 1.12 is the inverse of equation 1.10.

Example 1.3

Relative velocity. A car is traveling at a velocity of 95.0 km/hr to the right, with respect to a telephone pole. A truck, which is behind the car, is also moving to the right at 65.0 km/hr with respect to the same telephone pole. Find the relative velocity of the car with respect to the truck.

Solution

We represent the telephone pole as the stationary frame of reference S, while we place the moving frame of reference S' on the truck that is moving at a speed v = 65.0 km/hr. The auto is moving at the speed v = 95.0 km/hr with respect to S. The velocity of the auto with respect to the truck (or S' frame) is v'_x and is found from equation 1.12 as

$$\begin{aligned} v'_x &= v_x - v \\ &= 95.0 \text{ km/hr} - 65.0 \text{ km/hr} \\ &= 30.0 \text{ km/hr} \end{aligned}$$

The relative velocity is +30.0 km/hr. This means that the auto is pulling away or separating from the truck at the rate of 30.0 km/hr. If the auto were moving toward the S observer instead of away, then the auto's velocity with respect to S' would have been

$$\begin{aligned} v'_x = v_x - v &= -95.0 \text{ km/hr} - 65.0 \text{ km/hr} \\ &= -160.0 \text{ km/hr} \end{aligned}$$

That is, the truck would then observe the auto approaching at a closing speed of –160 km/hr. Note that when the relative velocity v'_x is positive the two moving objects are separating, whereas when v'_x is negative the two objects are closing or coming toward each other.

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To complete the velocity transformation equations, we use the fact that $y = y'$ and $z = z'$, thereby giving us

$$v'_y = \frac{dy'}{dt} = \frac{dy}{dt} = v_y \tag{1.13}$$

and

$$v'_z = \frac{dz'}{dt} = \frac{dz}{dt} = v_z \tag{1.14}$$

The Galilean transformations of velocities can be summarized as:

$$v_x = v'_x + v \tag{1.10}$$
$$v'_x = v_x - v \tag{1.12}$$
$$v'_y = v_y \tag{1.13}$$
$$v'_z = v_z \tag{1.14}$$

1.3 The Invariance of the Mechanical Laws of Physics under a Galilean Transformation

Although the velocity of a moving object is different when observed from a stationary frame rather than a moving frame of reference, the acceleration of the body is the same in either reference frame. To see this, let us start with equation 1.12,

$$v'_x = v_x - v$$

The change in each term with time is

$$\frac{dv'_x}{dt} = \frac{dv_x}{dt} - \frac{dv}{dt} \tag{1.15}$$

But v is the speed of the moving frame, which is a constant and does not change with time. Hence, $dv/dt = 0$. The term $dv'_x/dt = a'_x$ is the acceleration of the body with respect to the moving frame, whereas $dv_x/dt = a_x$ is the acceleration of the body with respect to the stationary frame. Therefore, equation 1.15 becomes

$$a'_x = a_x \tag{1.16}$$

Equation 1.16 says that the acceleration of a moving body is invariant under a Galilean transformation. *The word invariant when applied to a physical quantity means that the quantity remains a constant. We say that the acceleration is an* ***invariant quantity.*** *This means that either the moving or stationary observer would measure the same numerical value for the acceleration of the body.*

If we multiply both sides of equation 1.16 by m, we get

$$ma'_x = ma_x \tag{1.17}$$

But the product of the mass and the acceleration is equal to the force F, by Newton's second law. Hence,

$$F' = F \tag{1.18}$$

Thus, Newton's second law is also invariant to a Galilean transformation and applies to all inertial observers.

The laws of conservation of momentum and conservation of energy are also invariant under a Galilean transformation. We can see this for the case of the perfectly elastic collision illustrated in figure 1.8. We can write the law of conservation of momentum for the collision, as observed in the S frame, as

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{V}_1 + m_2\mathbf{V}_2 \tag{1.19}$$

where v_1 is the velocity of ball 1 before the collision, v_2 is the velocity of ball 2 before the collision, V_1 is the velocity of ball 1 after the collision, and V_2 is the velocity of ball 2 after the collision. But the relation between the velocity in the S and S' frames, found from equation 1.11 and figure 1.8, is

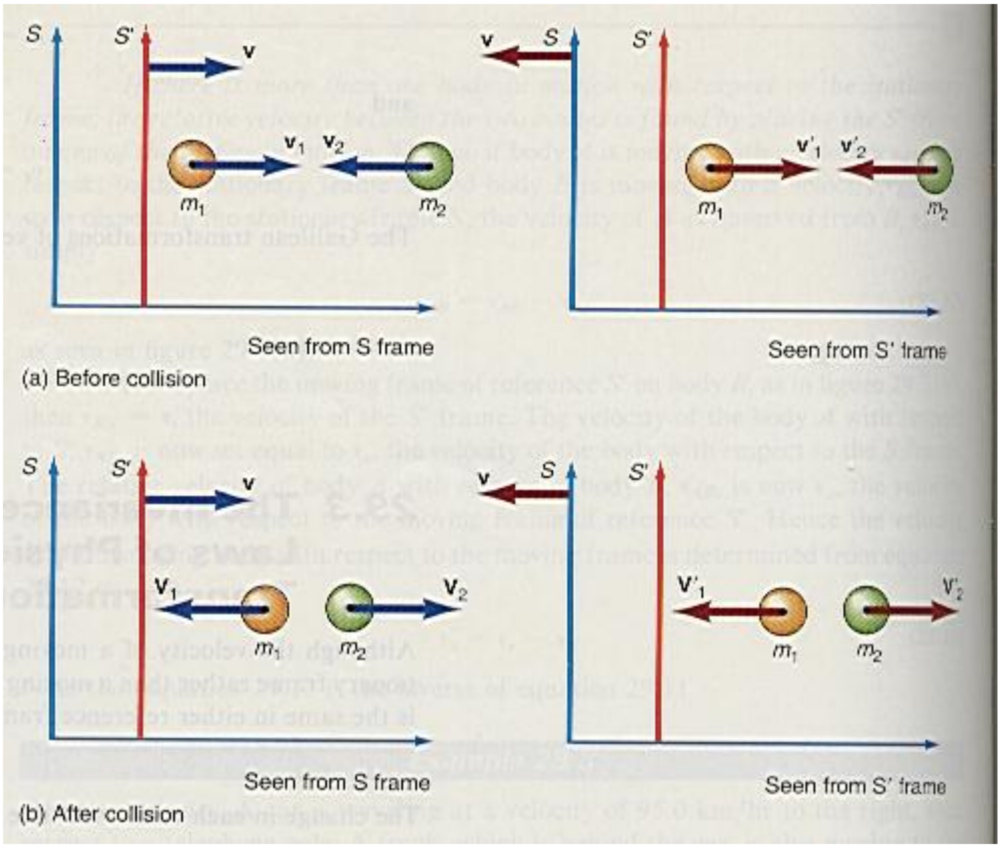


Figure 1.8 A perfectly elastic collision as seen from two inertial frames.

$$\left.\begin{aligned} \mathbf{v}_1 &= \mathbf{v}'_1 + \mathbf{v} \\ \mathbf{v}_2 &= \mathbf{v}'_2 + \mathbf{v} \\ \mathbf{V}_1 &= \mathbf{V}'_1 + \mathbf{v} \\ \mathbf{V}_2 &= \mathbf{V}'_2 + \mathbf{v} \end{aligned}\right\} \tag{1.20}$$

Substituting equations 1.20 into equation 1.19 for the law of conservation of momentum yields

$$m_1\mathbf{v}'_1 + m_2\mathbf{v}'_2 = m_1\mathbf{V}'_1 + m_2\mathbf{V}'_2 \tag{1.21}$$

Equation 1.21 is the law of conservation of momentum as observed from the moving S' frame. Note that it is of the same form as the law of conservation of momentum as observed from the S or stationary frame of reference. Thus, *the law of conservation of momentum is invariant to a Galilean transformation.*

The law of conservation of energy for the perfectly elastic collision of figure 1.8 as viewed from the S frame is

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 \tag{1.22}$$

By replacing the velocities in equation 1.22 by their Galilean counterparts, equation 1.20, and after much algebra we find that

$$\frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}m_2V_2'^2 \tag{1.23}$$

Equation 1.23 is the law of conservation of energy as observed by an observer in the moving S' frame of reference. Note again that the form of the equation is the same as in the stationary frame, and hence, *the law of conservation of energy is invariant to a Galilean transformation.* If we continued in this manner we would prove that all the laws of mechanics are invariant to a Galilean transformation.

1.4 Electromagnetism and the Ether

We have just seen that the laws of mechanics are invariant to a Galilean transformation. Are the laws of electromagnetism also invariant?

Consider a spherical electromagnetic wave propagating with a speed c with respect to a stationary frame of reference, as shown in figure 1.9. The speed of this

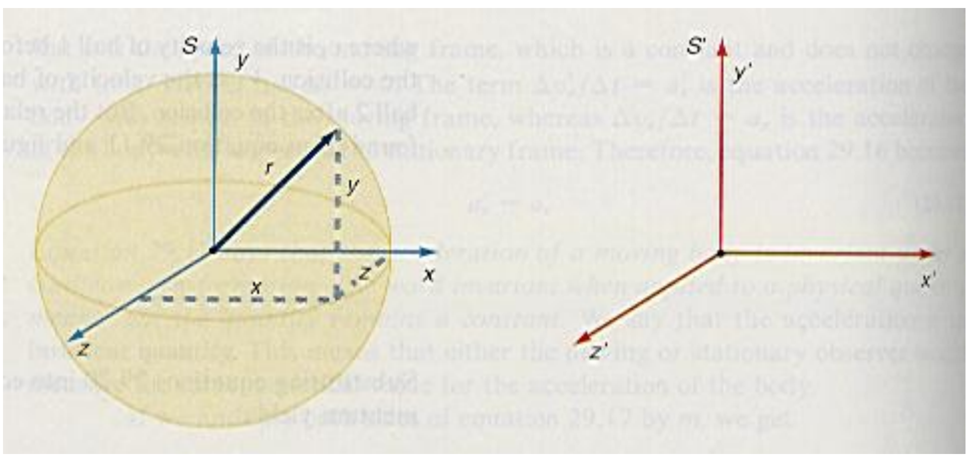


Figure 1.9 A spherical electromagnetic wave.

electromagnetic wave is

$$c = \frac{r}{t}$$

where r is the distance from the source of the wave to the spherical wave front. We can rewrite this as

$$r = ct$$

or

$$r^2 = c^2t^2$$

or

$$r^2 - c^2t^2 = 0 \tag{1.24}$$

The radius r of the spherical wave is

$$r^2 = x^2 + y^2 + z^2$$

Substituting this into equation 1.24, gives

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{1.25}$$

for the light wave as observed in the S frame of reference. Let us now assume that another observer, moving at the speed v in a moving frame of reference S' also observes this same light wave. The S' observer observes the coordinates x' and t', which are related to the x and t coordinates by the Galilean transformation equations as

$$x = x' + vt \tag{1.1}$$
$$y = y' \tag{1.2}$$
$$z = z' \tag{1.3}$$
$$t = t' \tag{1.4}$$

Substituting these Galilean transformations into equation 1.25 gives

$$(x' + vt)^2 + y'^2 + z'^2 - c^2t'^2 = 0$$
$$x'^2 + 2x'vt + v^2t^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

or

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = -2x'vt - v^2t^2 \tag{1.26}$$

Notice that the form of the equation is not invariant to a Galilean transformation. That is, equation 1.26, the velocity of the light wave as observed in the S' frame, has a different form than equation 1.25, the velocity of light in the S frame. Something is very wrong either with the equations of electromagnetism or with the Galilean transformations. Einstein was so filled with the beauty of the unifying effects of Maxwell's equations of electromagnetism that he felt that there must be something wrong with the Galilean transformation and hence, a new transformation law was required.

A further difficulty associated with the electromagnetic waves of Maxwell was the medium in which these waves propagated. Recall from your general physics course, that a wave is a disturbance that propagates through a medium. When a rock, the disturbance, is dropped into a pond, a wave propagates through the water, the medium. Associated with a transverse wave on a string is the motion of the particles of the string executing simple harmonic motion perpendicular to the direction of the wave propagation. In this case, the medium is the particles of the string. A sound wave in air is a disturbance propagated through the medium air. In fact, when we say that a sound wave propagates through the air with a velocity of 330 m/s at 0 ^{0}C, we mean that the wave is moving at 330 m/s with respect to the air.

A sound wave in water propagates through the water while a sound wave in a solid propagates through the solid. The one thing that all of these waves have in common is that they are all propagated through some medium. Classical physicists then naturally asked, "Through what medium does light propagate?" According to everything that was known in the field of physics in the nineteenth century, a wave must propagate through some medium. Therefore, it was reasonable to expect that a light wave, like any other wave, must propagate through some medium. This medium was called the luminiferous ether or just **ether** for short. It was assumed that this ether filled all of space, the inside of all material bodies, and was responsible for the transmission of all electromagnetic vibrations. Maxwell assumed that his electromagnetic waves propagated through this ether at the speed $c = 3 \times 10^8$ m/s.

An additional reason for the assumption of the existence of the ether was the phenomena of interference and diffraction of light that implied that light must be a wave. If light is an electromagnetic wave, then it is waving through the medium called ether.

There are, however, two disturbing characteristics of this ether. First, the ether had to have some very strange properties. The ether had to be very strong or rigid in order to support the extremely large speed of light. Recall from your general physics course that the speed of sound at 0 ^{0}C is 330 m/s in air, 1520 m/s in water, and 3420 m/s in iron. Thus, the more rigid the medium the higher the velocity of the wave. Similarly, for a transverse wave on a taut string the speed of propagation is

$$v = \sqrt{\frac{T}{m/l}}$$

where T is the tension in the string. The greater the value of T, the greater the value of the speed of propagation. Greater tension in the string implies a more rigid string. Although a light wave is neither a sound wave nor a wave on a string, it is reasonable to assume that the ether, being a medium for propagation of an electromagnetic wave, should also be quite rigid in order to support the enormous speed of 3×10^8 m/s. Yet the earth moves through this rigid medium at an orbital speed of 3×10^4 m/s and its motion is not impeded one iota by this rigid medium. This is very strange indeed.

The second disturbing characteristic of this ether hypothesis is that if electromagnetic waves always move at a speed c with respect to the ether, then maybe the ether constitutes an absolute frame of reference that we have not been able to find up to now. *Newton postulated an absolute space and an absolute time in his Principia: "Absolute space, in its own nature without regard to anything external remains always similar and immovable." And, "Absolute, true, and mathematical time, of itself and from its own nature flows equally without regard to anything external."* Could the ether be the framework of absolute space? In order to settle these apparent inconsistencies, it became necessary to detect this medium, called the ether, and thus verify its very existence. Maxwell suggested a crucial experiment to detect this ether. The experiment was performed by A. A. Michelson and E. E. Morley and is described in section 1.5.

1.5 The Michelson-Morley Experiment

If there is a medium called the ether that pervades all of space then the earth must be moving through this ether as it moves in its orbital motion about the sun. From the point of view of an observer on the earth the ether must flow past the earth, that is, it must appear that the earth is afloat in an ether current. The ether current concept allows us to consider an analogy of a boat in a river current.

Consider a boat in a river, L meters wide, where the speed of the river current is some unknown quantity v, as shown in figure 1.10. The boat is capable

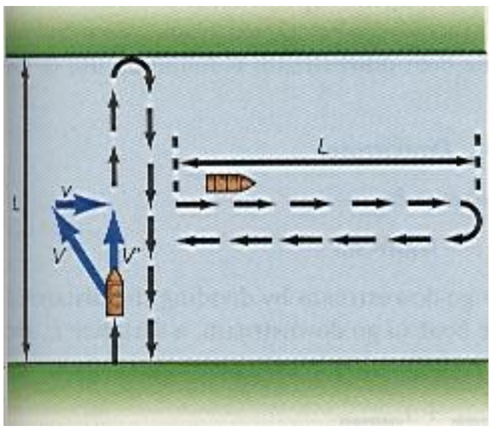


Figure 1.10 Current flowing in a river.

of moving at a speed V with respect to the water. The captain would like to measure the river current v, using only his stopwatch and the speed of his boat with respect to the water. After some thought the captain proceeds as follows. He can measure the time it takes for his boat to go straight across the river and return. But if he heads straight across the river, the current pushes the boat downstream. Therefore, he heads the boat upstream at an angle such that one component of the boat's velocity with respect to the water is equal and opposite to the velocity of the current downstream. Hence, the boat moves directly across the river at a velocity **V'**, as shown in the figure. The speed V' can be found from the application of the Pythagorean theorem to the velocity triangle of figure 1.10, namely

$$V^2 = V'^2 + v^2$$

Solving for V', we get

$$V' = \sqrt{V^2 - v^2}$$

Factoring out a V, we obtain, for the speed of the boat across the river,

$$V' = V\sqrt{1 - v^2/V^2} \tag{1.27}$$

We find the time to cross the river by dividing the distance traveled by the boat by the boat's speed, that is,

$$t_{\text{across}} = \frac{L}{V'}$$

The time to return is the same, that is,

$$t_{\text{return}} = \frac{L}{V'}$$

Hence, the total time to cross the river and return is

$$t_1 = t_{\text{across}} + t_{\text{return}} = \frac{L}{V'} + \frac{L}{V'} = \frac{2L}{V'}$$

Substituting V' from equation 1.27, the time becomes

$$t_1 = \frac{2L}{V\sqrt{1 - v^2/V^2}}$$

Hence, the time for the boat to cross the river and return is

$$t_1 = \frac{2L/V}{\sqrt{1 - v^2/V^2}} \tag{1.28}$$

The captain now tries another motion. He takes the boat out to the middle of the river and starts the boat downstream at the same speed V with respect to the water. After traveling a distance L downstream, the captain turns the boat around and travels the same distance L upstream to where he started from, as we can see in figure 1.10. The actual velocity of the boat downstream is found by use of the Galilean transformation as

$$V' = V + v \quad \text{Downstream}$$

while the actual velocity of the boat upstream is

$$V' = V - v \quad \text{Upstream}$$

We find the time for the boat to go downstream by dividing the distance L by the velocity V'. Thus the time for the boat to go downstream, a distance L, and to return is

$$t_2 = t_{\text{downstream}} + t_{\text{upstream}}$$

$$= \frac{L}{V + v} + \frac{L}{V - v}$$

Finding a common denominator and simplifying,

$$t_2 = \frac{L(V - v) + L(V + v)}{(V + v)(V - v)}$$
$$= \frac{LV - Lv + \mathrm{LV} + \mathrm{L}v}{V^2 + vV - vV - v^2}$$
$$= \frac{2LV}{V^2 - v^2}$$
$$= \frac{2LV/V^2}{V^2/V^2 - v^2/V^2}$$
$$t_2 = \frac{2L/V}{1 - v^2/V^2} \tag{1.29}$$

Hence, t_2 in equation 1.29 is the time for the boat to go downstream and return. Note from equations 1.28 and 1.29 that the two travel times are not equal.

The ratio of t_1, the time for the boat to cross the river and return, to t_2, the time for the boat to go downstream and return, found from equations 1.28 and 1.29, is

$$\frac{t_1}{t_2} = \frac{(2L/V)/\sqrt{1 - v^2/V^2}}{(2L/V)/(1 - v^2/V^2)}$$
$$= \frac{(1 - v^2/V^2)}{\sqrt{1 - v^2/V^2}}$$
$$\frac{t_1}{t_2} = \sqrt{1 - v^2/V^2} \tag{1.30}$$

Equation 1.30 says that if the speed v of the river current is known, then a relation between the times for the two different paths can be determined. On the other hand, if t_1 and t_2 are measured and the speed of the boat with respect to the water V is known, then the speed of the river current v can be determined. Thus, squaring equation 1.30,

$$\frac{t_1^2}{t_2^2} = 1 - \frac{v^2}{V^2}$$
$$\frac{v^2}{V^2} = 1 - \frac{t_1^2}{t_2^2}$$

or

$$v = V\sqrt{1 - \frac{t_1^2}{t_2^2}} \tag{1.31}$$

Thus, by knowing the times for the boat to travel the two paths the speed of the river current v can be determined.

Using the above analogy can help us to understand the experiment performed by Michelson and Morley to detect the ether current. The equipment used to measure the ether current was the Michelson interferometer and is sketched in figure 1.11. The interferometer sits in a laboratory on the earth. Because the earth moves through the ether, the observer in the laboratory sees an ether current moving past him with a speed of approximately $v = 3.00 \times 10^4$ m/s,

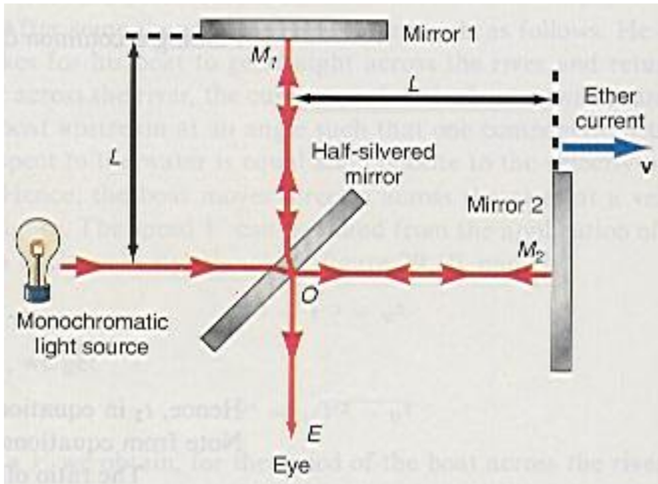


Figure 1.11 The Michelson-Morley experiment.

the orbital velocity of the earth about the sun. The motion of the light throughout the interferometer is the same as the motion of the boat in the river current. Light from the extended source is split by the half-silvered mirror. Half the light follows the path OM_1OE, which is perpendicular to the ether current. The rest follows the path OM_2OE, which is first in the direction of the ether current until it is reflected from mirror M_2, and is then in the direction that is opposite to the ether current. *The time for the light to cross the ether current is found from equation 1.28, but with V the speed of the boat replaced by c, the speed of light.* Thus,

$$t_1 = \frac{2L/c}{\sqrt{1-v^2/c^2}}$$

The time for the light to go downstream and upstream in the ether current is found from equation 1.29 but with V replaced by c. Thus,

$$t_2 = \frac{2L/c}{1-v^2/c^2}$$

The time difference between the two optical paths because of the ether current is

$$\Delta t = t_2 - t_1$$

$$\Delta t = \frac{2L/c}{1-v^2/c^2} - \frac{2L/c}{\sqrt{1-v^2/c^2}} \tag{1.32}$$

To simplify this equation, we use the *binomial theorem.* That is,

$$(1 - x)^n = 1 - nx + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n-2)x^3}{3!} + \ldots \tag{1.33}$$

This is a valid series expansion for $(1 - x)^n$ as long as x is less than 1. In this particular case,

$$x = \frac{v^2}{c^2} = \frac{(3.00 \times 10^4 \text{ m/s})^2}{(3.00 \times 10^8 \text{ m/s})^2} = 10^{-8}$$

which is much less than 1. In fact, since $x = 10^{-8}$, which is very small, it is possible to simplify the binomial theorem to

$$(1 - x)^n = 1 - nx \tag{1.34}$$

That is, since $x = 10^{-8}$, $x^2 = 10^{-16}$, and $x^3 = 10^{-24}$, the terms in x^2 and x^3 are negligible when compared to the value of x, and can be set equal to zero. Therefore, we can write the denominator of the first term in equation 1.32 as

$$\frac{1}{1 - v^2/c^2} = \left(1 - \frac{v^2}{c^2}\right)^{-1} = 1 - (-1)\frac{v^2}{c^2} = 1 + \frac{v^2}{c^2} \tag{1.35}$$

The denominator of the second term can be expressed as

$$\frac{1}{\sqrt{1 - v^2/c^2}} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 - \left(-\frac{1}{2}\right)\frac{v^2}{c^2}$$

$$\frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{1v^2}{2c^2} \tag{1.36}$$

Substituting equations 1.35 and 1.36 into equation 1.32, yields

$$\begin{aligned}\Delta t &= \frac{2L}{c}\left(1 + \frac{v^2}{c^2}\right) - \frac{2L}{c}\left(1 + \frac{1v^2}{2c^2}\right)\\ &= \frac{2L}{c}\left(1 + \frac{v^2}{c^2} - 1 - \frac{1v^2}{2c^2}\right)\\ &= \frac{2L}{c}\left(\frac{1v^2}{2c^2}\right)\end{aligned}$$

The path difference d between rays OM_1OE and OM_2OE, corresponding to this time difference Δt, is

$$d = c\Delta t = c\left[\frac{2L}{c}\left(\frac{1v^2}{2c^2}\right)\right]$$

or

$$d = \frac{Lv^2}{c^2} \tag{1.37}$$

Equation 1.37 gives the path difference between the two light rays and would cause the rays of light to be out of phase with each other and should cause an interference fringe. However, as explained in your optics course, the mirrors M_1 and M_2 of the Michelson interferometer are not quite perpendicular to each other and we always get interference fringes. However, if the interferometer is rotated through 90^0, then the optical paths are interchanged. That is, the path that originally required a time t_1 for the light to pass through, now requires a time t_2 and vice versa. The new time difference between the paths, analogous to equation 1.32, becomes

$$\Delta t' = \frac{2L/c}{\sqrt{1 - v^2/c^2}} - \frac{2L/c}{1 - v^2/c^2}$$

Using the binomial theorem again, we get

$$\begin{aligned}\Delta t' &= \frac{2L}{c}\left(1 + \frac{1v^2}{2c^2}\right) - \frac{2L}{c}\left(1 + \frac{v^2}{c^2}\right)\\ &= \frac{2L}{c}\left(1 + \frac{1v^2}{2c^2} - 1 - \frac{v^2}{c^2}\right)\\ &= \frac{2L}{c}\left(-\frac{1v^2}{2c^2}\right)\\ &= -\frac{Lv^2}{cc^2}\end{aligned}$$

The difference in path corresponding to this time difference is

$$d' = c\Delta t' = c\left(-\frac{Lv^2}{cc^2}\right)$$

or

$$d' = -\frac{Lv^2}{c^2}$$

By rotating the interferometer, the optical path has changed by

$$\Delta d = d - d' = \frac{Lv^2}{c^2} - \left(-\frac{Lv^2}{c^2}\right)$$

$$\Delta d = \frac{2Lv^2}{c^2} \tag{1.38}$$

This change in the optical paths corresponds to a shifting of the interference fringes. That is,

$$\Delta d = \Delta n\ \lambda$$

or

$$\Delta n = \frac{\Delta d}{\lambda} \tag{1.39}$$

Using equations 1.38 for Δd*, the number of fringes,* Δn*, that should move across the screen when the interferometer is rotated is*

$$\Delta n = \frac{2Lv^2}{\lambda\ c^2} \tag{1.40}$$

In the actual experimental set-up, the light path L was increased to 10.0 m by multiple reflections. The wavelength of light used was 500.0 nm. The ether current was assumed to be 3.00×10^4 m/s, the orbital speed of the earth around the sun. When all these values are placed into equation 1.40, the expected fringe shift is

$$\Delta n = \frac{2(10.0\text{ m})(3.00 \times 10^4\text{ m/s})^2}{(5.000 \times 10^{-7}\text{ m})(3.00 \times 10^8\text{ m/s})^2}$$
$$= 0.400\text{ fringes}$$

That is, if there is an ether that pervades all space, the earth must be moving through it. ***This ether current should cause a fringe shift of 0.400 fringes in the rotated interferometer, however, no fringe shift whatsoever was found.*** It should be noted that the interferometer was capable of reading a shift much smaller than the 0.400 fringe expected.

On the rare possibility that the earth was moving at the same speed as the ether, the experiment was repeated six months later when the motion of the earth was in the opposite direction. Again, no fringe shift was observed. ***The ether cannot be detected. But if it cannot be detected there is no reason to even assume that it exists.*** *Hence, the* ***Michelson-Morley experiment's*** *null result implies that the all pervading medium called the ether simply does not exist.* Therefore light, and all electromagnetic waves, are capable of propagating without the use of any medium. If there is no ether then the speed of the ether wind v is equal to zero. The null result of the experiment follows directly from equation 1.40 with $v = 0$.

The negative result also suggested a new physical principle. Even if there is no ether, when the light moves along path OM_2 the Galilean transformation equations with respect to the "fixed stars" still imply that the velocity of light along OM_2 should be $c + v$, where v is the earth's orbital velocity, with respect to the fixed stars, and c is the velocity of light with respect to the source on the interferometer. Similarly, it should be $c - v$ along path M_2O. But the negative result of the experiment requires the light to move at the same speed c whether the light was moving with the earth or against it. Hence, ***the negative result implies that the speed of light in free space is the same everywhere regardless of the motion of the source or the observer.*** *This also implies that there is something wrong with the Galilean transformation, which gives us the $c + v$ and $c - v$ velocities. Thus, it would appear that a new transformation equation other than the Galilean transformation is necessary.*

1.6 The Postulates of the Special Theory of Relativity

In 1905, Albert Einstein (1879-1955) formulated his **Special or Restricted Theory of Relativity** in terms of two postulates.

> *Postulate 1: The laws of physics have the same form in all frames of reference moving at a constant velocity with respect to one another.* This first postulate is sometimes also stated in the more succinct form: The laws of physics are invariant to a transformation between all inertial frames.
> *Postulate 2: The speed of light in free space has the same value for all observers, regardless of their state of motion.*

Postulate 1 is, in a sense, a consequence of the fact that all inertial frames are equivalent. If the laws of physics were different in different frames of reference, then we could tell from the form of the equation used which frame we were in. In particular, we could tell whether we were at rest or moving. But the difference between rest and motion at a constant velocity cannot be detected. Therefore, the laws of physics must be the same in all inertial frames.

Postulate 2 says that the velocity of light is always the same independent of the velocity of the source or of the observer. This can be taken as an experimental fact deduced from the Michelson-Morley experiment. However, Einstein, when asked years later if he had been aware of the results of the Michelson-Morley experiment, replied that he was not sure if he had been. Einstein came on the second postulate from a different viewpoint. According to his first postulate, the laws of physics must be the same for all inertial observers. If the velocity of light is different for different observers, then the observer could tell whether he was at rest or in motion at some constant velocity, simply by determining the velocity of light in his frame of reference. If the observed velocity of light c' were equal to c then the observer would be in the frame of reference that is at rest. If the observed velocity of light were $c' = c - v$, then the observer was in a frame of reference that was receding from the rest frame. Finally, if the observed velocity $c' = c + v$, then the observer would be in a frame of reference that was approaching the rest frame. Obviously these various values of c' would be a violation of the first postulate, since we could now define an absolute rest frame ($c' = c$), which would be different than all the other inertial frames.

The second postulate has revolutionary consequences. Recall that a velocity is equal to a distance in space divided by an interval of time. *In order for the velocity of light to remain a constant independent of the motion of the source or observer, space and time itself must change.* This is a revolutionary concept, indeed, because as already pointed out, Newton had assumed that space and time were absolute. A length of 1 m was considered to be a length of 1 m anywhere, and a time interval of 1 hr was considered to be a time interval of 1 hr anywhere. However, *if space and time change, then these concepts of absolute space and absolute time can no longer be part of the picture of the physical universe.*

The negative results of the Michelson-Morley experiment can also be explained by the second postulate. The velocity of light must always be c, never the $c + v$, $c - v$, or $\sqrt{c^2 - v^2}$ that were used in the original derivation. Thus, there would be no difference in time for either optical path of the interferometer and no fringe shift.

The Galilean equations for the transformation of velocity, which gave us the velocities of light as $c' = c + v$ and $c' = c - v$, must be replaced by some new transformation that always gives the velocity of light as c regardless of the velocity of the source or the observer. In section 1.7 we will derive such a transformation.

1.7 The Lorentz Transformation

Because the Galilean transformations violate the postulates of relativity, we must derive a new set of equations that relate the position and velocity of an object in one inertial frame to its position and velocity in another inertial frame. And we must derive the new transformation equations directly from the postulates of special relativity.

Since the Galilean transformations are correct when dealing with the motion of a body at low speeds, the new equations should reduce to the Galilean equations at low speeds. Therefore, the new transformation should have the form

$$x' = k(x - vt) \tag{1.41}$$

where x is the position of the body in the "rest" frame, t is the time of its observation, x' is the position of the body in the moving frame of reference, and finally k is some function or constant to be determined. For the classical case of low speeds, k should reduce to the value 1, and the new transformation equation would then reduce to the Galilean transformation, equation 1.5. This equation says that if the position x and velocity v of a body are measured in the stationary frame, then its position x' in the moving frame is determined by equation 1.41. Using the first postulate of relativity, this equation must have the same form in the frame of reference at rest. Therefore,

$$x = k(x' + vt') \tag{1.42}$$

where x' is the position of the body in the moving frame at the time t'. The sign of v has been changed to a positive quantity because, as shown in figure 1.3, a frame 2 moving to the right with a velocity v as observed from a frame 1 at rest, is equivalent to frame 2 at rest with frame 1 moving to the left with a velocity $-v$. This equation says that if the position x' and time t' of a body are measured in a moving frame, then its position x in the stationary frame is determined by equation 1.42. The position of the y- and z-coordinates are still the same, namely,

$$\begin{aligned} y' &= y \\ z' &= z \end{aligned} \tag{1.43}$$

The time of the observation of the event in the moving frame is denoted by t'. We deliberately depart from our common experiences by arranging for the possibility of a different time t' for the event in the moving frame compared to the time t for the same event in the stationary frame. In fact, t' can be determined by substituting equation 1.41 into equation 1.42. That is,

$$\begin{aligned} x = k(x' + vt') &= k[k(x - vt) + vt'] \\ &= k^2x - k^2vt + kvt' \\ kvt' &= x - k^2x + k^2vt \end{aligned}$$

and

$$t' = kt + \left(\frac{1-k^2}{kv}\right)x \tag{1.44}$$

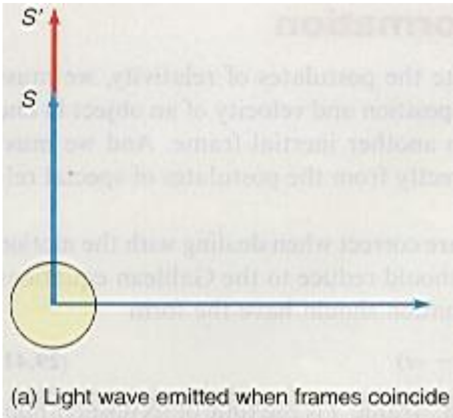
Thus, according to the results of the first postulate of relativity, the time t' in the moving coordinate system is not equal to the time t in the stationary coordinate system. The exact relation between these times is still unknown, however, because we still have to determine the value of k.

To determine k, we use the second postulate of relativity. Imagine a light wave emanating from a source that is located at the origin of the S and S' frame of reference, which momentarily coincide for $t = 0$ and $t' = 0$, figure 1.12. By the second postulate both the stationary and moving observer must observe the same velocity c of the light wave. The distance the wave moves in the x-direction in the S frame is

$$x = ct \tag{1.45}$$

whereas the distance the same wave moves in the x'-direction in the S' frame is

$$x' = ct' \tag{1.46}$$



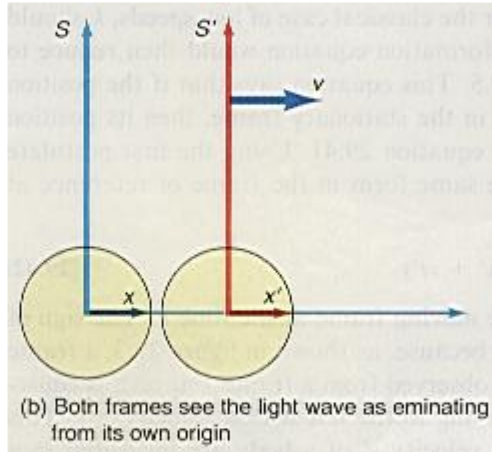


Figure 1.12 The same light wave observed from two inertial frames.

Substituting for x' from equation 1.41, and for t' from equation 1.44, into equation 1.46, yields

$$k(x - vt) = c\left[kt + \left(\frac{1-k^2}{kv}\right)x\right]$$

Performing the following algebraic steps, we solve for x

$$kx - kvt = ckt + \frac{c(1-k^2)}{kv}x$$

$$kx - \frac{c(1-k^2)}{kv}x = ckt + kvt$$

$$x\left[k - \frac{c(1-k^2)}{kv}\right] = ct\left(k + \frac{kv}{c}\right)$$

$$x = ct\left[\frac{k + kv/c}{k - c\left[(1-k^2)/kv\right]}\right] \tag{1.47}$$

But as already seen in equation 1.45, $x = ct$. Therefore, the term in braces in equation 1.47 must be equal to 1. Thus,

$$\frac{k + kv/c}{k - [c(1-k^2)/kv]} = 1$$

$$\frac{k(1 + v/c)}{k\{1 - [c(1-k^2)/k^2v]\}} = 1$$

$$1 + \frac{v}{c} = 1 - \frac{c}{v}\left(\frac{1}{k^2} - 1\right) = 1 - \frac{c}{vk^2} + \frac{c}{v}$$

$$1+\frac{v}{c}-1-\frac{c}{v}=-\frac{c}{vk^2}$$
$$k^2\left(\frac{v}{c}-\frac{c}{v}\right)=-\frac{c}{v}$$
$$k^2=\frac{-c/v}{v/c-c/v}=\frac{c/v}{c/v-v/c}=\frac{1}{1-\left[(v/c)/(c/v)\right]}=\frac{1}{1-v^2/c^2}$$

Thus, the function k becomes

$$k=\frac{1}{\sqrt{1-v^2/c^2}} \tag{1.48}$$

Substituting this value of k into equation 1.41 gives the first of the new transformation equations, namely

$$x'=\frac{x-vt}{\sqrt{1-v^2/c^2}} \tag{1.49}$$

Equation 1.49 gives the position x' of the body in the moving coordinate system in terms of its position x and velocity v at the time t in the stationary coordinate system. Before discussing its physical significance let us also substitute k into the time equation 1.44, that is,

$$t'=\frac{t}{\sqrt{1-v^2/c^2}}+\left(\frac{1-(1)^2/\left(\sqrt{1-v^2/c^2}\right)^2}{v/\sqrt{1-v^2/c^2}}\right)x$$

Simplifying,

$$\begin{aligned} t'&=\frac{t}{\sqrt{1-v^2/c^2}}+\left(1-\frac{1}{1-v^2/c^2}\right)\left(\frac{x}{v}\right)\sqrt{1-v^2/c^2} \\ &=\frac{t}{\sqrt{1-v^2/c^2}}+\frac{\left(1-v^2/c^2-1\right)x}{\left(1-v^2/c^2\right)v}\sqrt{1-v^2/c^2} \\ &=\frac{t}{\sqrt{1-v^2/c^2}}-\frac{v^2/c^2}{\sqrt{1-v^2/c^2}}\frac{x}{v} \end{aligned}$$

and the second transformation equation becomes

$$t'=\frac{t-xv/c^2}{\sqrt{1-v^2/c^2}} \tag{1.50}$$

These new transformation equations are called the **Lorentz transformations.**[2] The Lorentz transformation equations are summarized as

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{1.49}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} \tag{1.50}$$

Now that we have obtained the new transformation equations, we must ask what they mean. First of all, note that the coordinate equation for the position does look like the Galilean transformation for position except for the term $\sqrt{1 - v^2/c^2}$ in the denominator of the x'-term. If the velocity v of the reference frame is small compared to c, then $v^2/c^2 \approx 0$, and hence,

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} = \frac{x - vt}{\sqrt{1 - 0}} = x - vt$$

Similarly, for the time equation, if v is much less than c then $v^2/c^2 \approx 0$ and also $xv/c^2 \approx 0$. Therefore, the time equation becomes

$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} = \frac{t - 0}{\sqrt{1 - 0}} = t$$

Thus, the Lorentz transformation equations reduce to the classical Galilean transformation equations when the relative speed between the observers is small as compared to the speed of light. This reduction of a new theory to an old theory is called the correspondence principle and was first enunciated as a principle by Niels Bohr in 1923. It states that any new theory in physics must reduce to the well-established corresponding classical theory when the new theory is applied to the special situation in which the less general theory is known to be valid.

Because of this reduction to the old theory, the consequences of special relativity are not apparent unless dealing with enormous speeds such as those comparable to the speed of light.

2 [2]These equations were named for H. A. Lorentz because he derived them before Einstein's theory of special relativity. However, Lorentz derived these equations to explain the negative result of the Michelson-Morley experiment. They were essentially empirical equations because they could not be justified on general grounds as they were by Einstein.

Example 1.4

The value of $1/\sqrt{1-v^2/c^2}$ *for various values of* v. What is the value of $1/\sqrt{1-v^2/c^2}$ for (a) v = 1610 km/hr = 1000 mph, (b) v = 1610 km/s = 1000 mi/s, and (c) $v = 0.8c$. Take $c = 3.00 \times 10^8$ m/s in SI units. It will be assumed in all the examples of relativity that the initial data are known to whatever number of significant figures necessary to demonstrate the principles of relativity in the calculations.

Solution

a. The speed v = (1610 km/hr)(1 hr/3600 s) = 0.447 km/s = 447 m/s. Hence,

$$\frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\left(447\text{ m/s}\right)^2/\left(3.00\times10^8\text{ m/s}\right)^2}}$$
$$= \frac{1}{\sqrt{1-2.22\times10^{-12}}}$$

This can be further simplified by the binomial expansion as

$$(1-x)^n = 1 - nx$$

and hence,

$$\frac{1}{\sqrt{1-v^2/c^2}} = 1-\left(\frac{1}{2}\right)2.22\times10^{-12} = 1.00000000000111 = 1$$

That is, the value is so close to 1 that we cannot determine the difference.

b. The velocity v = 1610 km/s, gives a value of

$$\frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-(1.61\times10^6\text{ m/s})^2/(3.00\times10^8\text{ m/s})^2}}$$
$$= \frac{1}{\sqrt{1-2.88\times10^{-5}}}$$
$$= \frac{1}{\sqrt{0.99997}} = \frac{1}{0.99999}$$
$$= 1.00001$$

Now 1610 km/s is equal to 3,600,000 mph. Even though this is considered to be an enormous speed, far greater than anything people are now capable of moving at (for example, a satellite in a low earth orbit moves at about 18,000 mph, and the velocity of the earth around the sun is about 68,000 mph), the effect is still so small that it can still be considered to be negligible.

c. For a velocity of 0.8c the value becomes

$$\frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-(0.8c)^2/c^2}} = \frac{1}{\sqrt{1-0.64}}$$
$$= \frac{1}{\sqrt{0.36}} = \frac{1}{0.600}$$
$$= 1.67$$

Thus, at the speed of eight-tenths of the speed of light the factor becomes quite significant.

To go to this Interactive Example click on this sentence.

The Lorentz transformation equations point out that space and time are intimately connected. Notice that the position x' not only depends on the position x but also depends on the time t, whereas the time t' not only depends on the time t but also depends on the position x. *We can no longer consider such a thing as absolute time, because time now depends on the position of the observer. That is, all time must be considered relative. Thus, we can no longer consider space and time as separate entities. Instead there is a union or fusion of space and time into the single reality called spacetime.* That is, space by itself has no meaning; time by itself has no meaning; only spacetime exists. The coordinates of an event in four-dimensional spacetime are (x, y, z, t). We will say more about spacetime in chapter 2.

An interesting consequence of this result of special relativity is its effects on the fundamental quantities of physics. In general physics we saw that the world could be described in terms of three fundamental quantities--space, time, and matter. It is now obvious that there are even fewer fundamental quantities. *Because space and time are fused into spacetime, the fundamental quantities are now only two, spacetime and matter.*

It is important to notice that the Lorentz transformation equations for special relativity put a limit on the maximum value of v that is attainable by a body, because if $v = c$,

$$x' = \frac{x - vt}{\sqrt{1-v^2/c^2}} = \frac{x - vt}{\sqrt{1-c^2/c^2}}$$
$$= \frac{x - vt}{0}$$

Since division by zero is undefined, we must take the limit as v approaches c. That is,

$$x' = \lim_{v \to c} \frac{x - vt}{\sqrt{1-v^2/c^2}} = \infty$$

and similarly

$$t' = \lim_{v \to c} \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} = \infty$$

That is, for $v = c$, the coordinates x' and t' are infinite, or at least undefinable. If $v > c$ then $v^2/c^2 > 1$ and $1 - v^2/c^2 < 1$. This means that the number under the square root sign is negative and the square root of a negative quantity is imaginary. Thus x' and t' become imaginary quantities. Hence, according to the theory of special relativity, no object can move at a speed equal to or greater than the speed of light.

Example 1.5

Lorentz transformation of coordinates. A man on the earth measures an event at a point 5.00 m from him at a time of 3.00 s. If a rocket ship flies over the man at a speed of $0.800c$, what coordinates does the astronaut in the rocket ship attribute to this event?

Solution

The location of the event, as observed in the moving rocket ship, found from equation 1.49, is

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$x' = \frac{5.00 \text{ m} - (0.800)(3.00 \times 10^8 \text{ m/s})(3.00 \text{ s})}{\sqrt{1 - (0.800c)^2/c^2}}$$

$$= -1.20 \times 10^9 \text{ m}$$

This distance is quite large because the astronaut is moving at such high speed. The event occurs on the astronaut's clock at a time

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{3.00 \text{ s} - (0.800)(3.00 \times 10^8 \text{ m/s})(5.00 \text{ m})/(3.00 \times 10^8 \text{ m/s})^2}{\sqrt{1 - (0.800c)^2/c^2}}$$

$$= 5.00 \text{ s}$$

To go to this Interactive Example click on this sentence.

The inverse Lorentz transformation equations from the moving system to the stationary system can be written down immediately by the use of the first postulate. That is, their form must be the same, but $-v$ is replaced by $+v$ and primes and unprimes are interchanged. Therefore, the inverse Lorentz transformation equations are

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \tag{1.51}$$

$$y = y' \tag{1.52}$$

$$z = z' \tag{1.53}$$

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \tag{1.54}$$

1.8 The Lorentz-Fitzgerald Contraction

Consider a rod at rest in a stationary coordinate system S on the earth, as in figure 1.13(a). What is the length of this rod when it is observed by an astronaut in the S' frame of reference, a rocket ship traveling at a speed v? One end of the rod is located at the point x_1, while the other end is located at the point x_2. *The length of this stationary rod, measured in the frame where it is at rest, is called its* ***proper length*** *and is denoted by* L_0*, where*

$$L_0 = x_2 - x_1 \tag{1.55}$$

What is the length of this rod as observed in the rocket ship? The astronaut must measure the coordinates x_1' and x_2' for the ends of the rod at the same time t' in his frame S'.

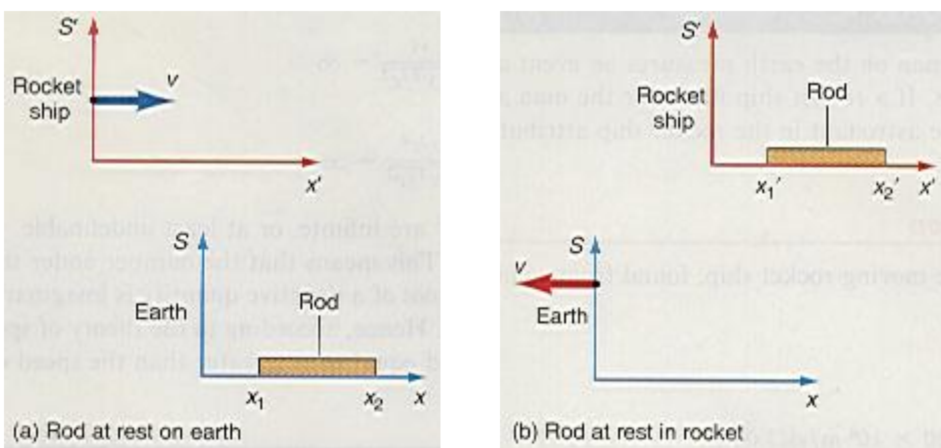


Figure 1.13 The Lorentz-Fitzgerald contraction.

The measurement of the length of any rod in a moving coordinate system must always be measured simultaneously in that coordinate system or else the ends of the rod will have moved during the measurement process and we will not be measuring the true length of the object. An often quoted example for the need of simultaneous measurements of length is the measurement of a fish in a tank. If the tail of the fish is measured first, and the head some time later, the fish has moved to the left and we have measured a much longer fish than the one in the tank, figure 1.14(a). If the head of the fish is measured first, and then the tail, the fish appears smaller than it is, figure 1.14(b). If, on the other hand, the head and tail are measured simultaneously we get the actual length of the fish, figure 1.14(c).

In a coordinate system where the rod or body is at rest, simultaneous measurements are not necessary because we can measure the ends at any time, since the rod is always there in that place and its ends never move. When the values of the coordinates of the end of the bar, x_1' and x_2', are measured at the time t', the values of x_1 and x_2 in the earth frame S are computed by the Lorentz transformation. Thus,

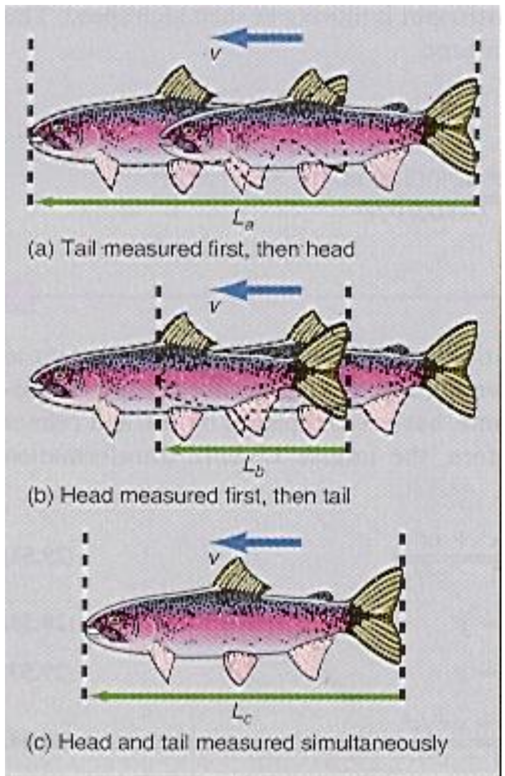


Figure 1.14 Measurement of the length of a moving fish.

$$x_1 = \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}} \tag{1.56}$$

while

$$x_2 = \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}} \tag{1.57}$$

The length of the rod L_0, found from equations 1.55, 1.56, and 1.57, is

$$L_0 = x_2 - x_1 = \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}} - \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{x_2' + vt' - x_1' - vt'}{\sqrt{1 - v^2/c^2}}$$

$$L_0 = \frac{x_2' - x_1'}{\sqrt{1 - v^2/c^2}} \tag{1.58}$$

Let us designate L as the length of the rod as measured in the moving rocket frame S', that is,

$$L = x_2' - x_1' \tag{1.59}$$

Then equation 1.58 becomes

$$L_0 = \frac{L}{\sqrt{1 - v^2/c^2}}$$

or

$$L = L_0\sqrt{1 - v^2/c^2} \tag{1.60}$$

Because v is less than c, the quantity $\sqrt{1 - v^2/c^2} < 1$, which means that $L < L_0$. *That is, the rod at rest in the earth frame would be measured in the rocket frame to be smaller than it is in the earth frame. From the point of view of the astronaut in the rocket, the rocket is at rest and the rod in the earth frame is moving toward him at a velocity v. Hence, the astronaut considers the rod to be at rest in a moving frame, and he then concludes that a moving rod contracts, as given by equation 1.60.* That is, if the rod is a meterstick, then its proper length in the frame where it is at rest is L_0 = 1.00 m = 100 cm. If the rocket is moving at a speed of 0.8c, then the length as observed from the rocket ship is

$$L = L_0\sqrt{1 - v^2/c^2} = 1.00 \text{ m}\sqrt{1 - (0.8c)^2/c^2} = 0.600 \text{ m}$$

Thus, the astronaut says that the meterstick is only 60.0 cm long. *This contraction of length is known as the* ***Lorentz-Fitzgerald contraction*** because it was derived earlier by Lorentz and Fitzgerald. However Lorentz and Fitzgerald attributed this effect to the ether. But since the ether does not exist, this effect cannot be attributed to it. It was Einstein's derivation of these same equations by the postulates of relativity that gave them real meaning.

This length contraction is a reciprocal effect. If there is a rod (a meterstick) at rest in the rocket S' frame, figure 1.13(b), then the astronaut measures the length of that rod by measuring the ends x_1' and x_2' at any time. The length of the rod as observed by the astronaut is

$$L_0 = x_2' - x_1' \tag{1.61}$$

This length is now the proper length L_0 because the rod is at rest in the astronaut's frame of reference. The observer on earth (S frame) measures the coordinates of the ends of the rod, x_1 and x_2, simultaneously at the time t. The ends of the rod x_2' and x_1' are computed by the earth observer by the Lorentz transformations:

$$x_2' = \frac{x_2 - vt}{\sqrt{1 - v^2/c^2}}$$

$$x_1' = \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}}$$

Thus, the length of the rod becomes

$$\begin{aligned} L_0 = x_2' - x_1' &= \frac{x_2 - vt}{\sqrt{1 - v^2/c^2}} - \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}} \\ &= \frac{x_2 - vt - x_1 + vt}{\sqrt{1 - v^2/c^2}} \\ &= \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

But

$$x_2 - x_1 = L$$

the length of the rod as observed by the earth man. Therefore,

$$L_0 = \frac{L}{\sqrt{1 - v^2/c^2}}$$

and

$$L = L_0\sqrt{1 - v^2/c^2} \tag{1.62}$$

But this is the identical equation that was found before (equation 1.60). If L_0 is again the meterstick and the rocket ship is moving at the speed 0.800*c,* then the length L as observed on earth is 60.0 cm. *Thus the length contraction effect is reciprocal.* When the meterstick is at rest on the earth the astronaut thinks it is only 60.0 cm long. When the meterstick is at rest in the rocket ship the earthbound observer thinks it is only 60.0 cm long. Thus each observer sees the other's length as contracted. This reciprocity is to be expected. If the two observers do not agree that the other's stick is contracted, they could use this information to tell which stick is at rest and which is in motion--a violation of the principle of relativity. One thing that is important to notice is that in equation 1.60, L_0 is the length of the rod at rest in the earth or S frame of reference, whereas in equation 1.62, L_0 is the length of the rod at rest in the moving rocket ship (S' frame). *L_0 is always the length of the rod in the frame of reference where it is at rest.* It does not matter if the frame of reference is at rest or moving so long as the rod is at rest in that frame. This is why L_0 is always called its proper length.

The Lorentz-Fitzgerald contraction can be summarized by saying that the length of a rod in motion with respect to an observer is less than its length when measured by an observer who is at rest with respect to the rod. This contraction occurs only in the direction of the relative motion. Let us consider the size of this contraction.

Example 1.6

Length contraction. What is the length of a meterstick when it is measured by an observer moving at (a) v = 1610 km/hr = 1000 mph, (b) v = 1610 km/s = 1000 miles/s, and (c) $v = 0.8c$. It is assumed in all these problems in relativity that the initial data are known to whatever number of significant figures necessary to demonstrate the principles of relativity in the calculations.

Solution

a. The speed v = (1610 km/hr)(1 hr/3600 s) = 0.447 km/s = 447 m/s. Take $c = 3.00 \times 10^8$ m/s in SI units. The length contraction, found from either equation 1.60 or equation 1.62, is

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$
$$= (1.00 \text{ m})\sqrt{1-\frac{(447 \text{ m/s})^2}{(3.00\times 10^8 \text{ m/s})^2}}$$
$$= (1.00 \text{ m})\sqrt{1-2.22\times 10^{-12}}$$

This can be further simplified by the binomial expansion as

$$(1 - x)^n = 1 - nx$$

and hence

$$\sqrt{1-\frac{v^2}{c^2}} = 1-\left(\frac{1}{2}\right)2.22\times 10^{-12} = 1-0.00000000000111$$
$$= 0.99999999999888$$

and

$$L = 1.00 \text{ m}$$

Thus, at what might be considered the reasonably fast speed of 1000 mph, the contraction is so small that it is less than the width of one atom, and is negligible.

b. The contraction for a speed of 1610 km/s is

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$
$$= (1.00 \text{ m})\sqrt{1-\frac{(1.610\times 10^6 \text{ m/s})^2}{(3.00\times 10^8 \text{ m/s})^2}}$$
$$= (1.00 \text{ m})\sqrt{0.99997}$$
$$= 0.99997 \text{ m}$$

A speed of 1610 km/s is equivalent to a speed of 3,600,000 mph, which is an enormous speed, one man cannot even attain at this particular time. Yet the associated contraction is very small.

c. For a speed of $v = 0.8c$ the contraction is

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$
$$= (1.00\text{ m})\sqrt{1-\frac{(0.8c)^2}{c^2}} = (1.00\text{ m})\sqrt{0.360}$$
$$= 0.600\text{ m}$$

At speeds approaching the speed of light the contraction is quite significant. Table 1.1 gives the Lorentz contraction for a range of values of speed approaching the speed of light. Notice that as v increases, the contraction becomes greater and greater, until at a speed of $0.999999c$ the meterstick has contracted to a thousandth of a meter or 1 mm. Therefore, the effects of relativity do not manifest themselves unless very great speeds are involved. This is why these effects had never been seen or even anticipated when Newton was formulating his laws of physics. However, in the present day it is possible to accelerate charged particles, such as electrons and protons, to speeds very near the speed of light, and the relativistic effects are observed with such particles.

Table 1.1
The Lorentz Contraction and Time Dilation

Speed	$\boxed{L = L_0\sqrt{1-\frac{v^2}{c^2}}}$	$\boxed{\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}}$
$0.1c$	$0.995L_0$	$1.01\Delta t_0$
$0.2c$	$0.980L_0$	$1.02\Delta t_0$
$0.4c$	$0.917L_0$	$1.09\Delta t_0$
$0.6c$	$0.800L_0$	$1.25\Delta t_0$
$0.8c$	$0.602L_0$	$1.66\Delta t_0$
$0.9c$	$0.437L_0$	$2.29\Delta t_0$
$0.99c$	$0.141L_0$	$7.08\Delta t_0$
$0.999c$	$0.045L_0$	$22.4\Delta t_0$
$0.9999c$	$0.014L_0$	$70.7\Delta t_0$
$0.99999c$	$0.005L_0$	$224\Delta t_0$
$0.999999c$	$0.001L_0$	$707\Delta t_0$

To go to this Interactive Example click on this sentence.

1.9 Time Dilation

Consider a clock at rest at the position x' in a moving coordinate system S' attached to a rocket ship. The astronaut sneezes and notes that he did so when his clock, located at x', reads a time t_1'. Shortly thereafter he sneezes again, and now notes that his clock indicates the time t_2'. The time interval between the two sneezes is

$$\Delta t' = t_2' - t_1' = \Delta t_0 \tag{1.63}$$

*This interval $\Delta t'$ is set equal to Δt_0, and is called the **proper time** because this is the time interval on a clock that is at rest relative to the observer.* The observer on earth in the S frame finds the time for the two sneezes to be

$$t_1 = \frac{t_1' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$
$$t_2 = \frac{t_2' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

Thus, the time interval between the sneezes Δt, as observed by the earth man, becomes

$$\Delta t = t_2 - t_1 = \frac{t_2' + vx'/c^2 - t_1' - vx'/c^2}{\sqrt{1 - v^2/c^2}}$$
$$= \frac{t_2' - t_1'}{\sqrt{1 - v^2/c^2}}$$

But $t_2' - t_1' = \Delta t_0$, by equation 1.63, therefore,

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} \tag{1.64}$$

Notice that because $v < c$, $v^2/c^2 < 1$ and thus $\sqrt{1 - v^2/c^2} < 1$. Therefore,

$$\Delta t > \Delta t_0 \tag{1.65}$$

*Equation 1.64 is the **time dilation formula** and equation 1.65 says that the clock on earth reads a longer time interval Δt than the clock in the rocket ship Δt_0. Or as is sometimes said, moving clocks slow down.* Thus, if the moving clock slows down, a smaller time duration is indicated on the moving clock than on a stationary clock. *Hence, the astronaut ages at a slower rate than a person on earth.* The amount of this slowing down of time is relatively small as seen in example 1.7.

Example 1.7

Time dilation. A clock on a rocket ship ticks off a time interval of 1 hr. What time elapses on earth if the rocket ship is moving at a speed of (a) 1610 km/hr = 1000 mph, (b) 1610 km/s = 1000 mi/s, and (c) 0.8*c*?

Solution

a. The time elapsed on earth, found from equation 1.64, is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{1 \text{ hr}}{\sqrt{1 - (447 \text{ m/s})^2/(3.00 \times 10^8 \text{ m/s})^2}}$$

$$= 1 \text{ hr}$$

The difference between the time interval on the astronaut's clock and the time interval on the earthman's clock is actually about 4 ns. This is such a small quantity that it is effectively zero and the difference between the two clocks can be considered to be zero for a speed of 1600 km/s = 1000 mph.

b. The time elapsed for a speed of 1610 km/s is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} = \frac{1 \text{ hr}}{\sqrt{1 - (1.61 \times 10^6 \text{ m/s})^2/(3.00 \times 10^8 \text{ m/s})^2}}$$

$$= 1.0000144 \text{ hr}$$

Even at the relatively large speed of 1610 km/s = 3,600,000 mph, the difference in the clocks is practically negligible, that is a difference of 0.05 s in a time interval of 1 hr.

c. The time elapsed for a speed of 0.8*c* is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} = \frac{1 \text{ hr}}{\sqrt{1 - (0.8c)^2/(c)^2}}$$

$$= 1.66 \text{ hr}$$

Therefore, at very high speeds the time dilation effect is quite significant. Table 1.1 shows the time dilation for various values of v. As we can see, the time dilation effect becomes quite pronounced for very large values of v.

To go to this Interactive Example click on this sentence.

It should be noted that the time dilation effect, like the Lorentz contraction, is also reciprocal. That is, a clock on the surface of the earth reads the proper time interval Δt_0 to an observer on the earth. An astronaut observing this earth clock assumes that he is at rest, but the earth is moving away from him at the velocity $-v$. Thus, he considers the earth clock to be the moving clock, and he finds that time on earth moves slower than the time on his rocket ship. This reciprocity of time dilation has led to the most famous paradox of relativity, called the twin paradox. (A paradox is an apparent contradiction.) The reciprocity of time dilation seems to be a contradiction when applied to the twins.

As an example, an astronaut leaves his twin sister on the earth as he travels, at a speed approaching the speed of light, to a distant star and then returns. According to the formula for time dilation, time has slowed down for the astronaut and when he returns to earth he should find his twin sister to be much older than he is. But by the first postulate of relativity, the laws of physics must be the same in all inertial coordinate systems. Therefore, the astronaut says that it is he who is the one at rest and the earth is moving away from him in the opposite direction. Thus, the astronaut says that it is the clock on earth that is moving and hence slowing down. He then concludes that his twin sister on earth will be younger than he is, when he returns. Both twins say that the other twin should be younger after the journey, and hence there seems to be a contradiction. How can we resolve this paradox?

With a little thought we can see that there is no contradiction here. The Lorentz transformations apply to inertial coordinate systems, that is, coordinates that are moving at a constant velocity with respect to each other. The twin on earth is in fact in an inertial coordinate system and can use the time dilation equation. The astronaut who returns home, however, is not in an inertial coordinate system. If the astronaut is originally moving at a velocity v, then in order for him to return home, he has to decelerate his spaceship to zero velocity and then accelerate to the velocity $-v$ to travel homeward. *During the deceleration and acceleration process the spaceship is not an inertial coordinate system, and we cannot justify using the time dilation formula that was derived on the basis of inertial coordinate systems.* Hence there is a very significant difference between the twin that stays home on the earth and the astronaut. Here again is that same conflict that occurs when we try to use an equation that was derived by using certain assumptions. When the assumptions hold, the equation is correct. When the assumptions do not hold, the equation no longer applies. In this example, the Lorentz transformation equations were derived on the assumption that two coordinate systems were moving with respect to each other at constant velocity. The astronaut is in an accelerated coordinate system when he turns around to come home. Hence, he is not in an inertial coordinate system and is not entitled to use the time dilation formula.[3] However, as correctly predicted by the earth twin, time has slowed down for the astronaut and when he returns to earth he should find his twin sister to be much older than he is.

We will consider a deeper insight into the slowing down of time in chapter 2 when we draw spacetime diagrams and discuss the general theory of relativity, and again in chapter 2 when we examine the gravitational red shift by the theory of the quanta.

1.10 Transformation of Velocities

We have seen that the Galilean transformation of velocities is incorrect when dealing with speeds at or near the speed of light. That is, velocities such as $V = c + v$ or $V = c - v$ are incorrect. Therefore, new transformation equations are needed for velocities. The necessary equations are found by the Lorentz equations. The components of the velocity of an object in a stationary coordinate system S are

$$V_x = \frac{dx}{dt} \tag{1.66}$$

$$V_y = \frac{dy}{dt} \tag{1.67}$$

$$V_z = \frac{dz}{dt} \tag{1.68}$$

whereas the components of the velocity of that same body, as observed in the moving coordinate system, S', are

$$V_x' = \frac{dx'}{dt'} \tag{1.69}$$

$$V_y' = \frac{dy'}{dt'} \tag{1.70}$$

$$V_z' = \frac{dz'}{dt'} \tag{1.71}$$

The transformation of the x-component of velocity is obtained as follows. The Lorentz transformation for the x' coordinate, equation 1.49, is

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

The differential dx' becomes

3 [3]This also points out a flaw in the derivation of the Lorentz transformation equations. Starting with inertial coordinate systems, if there is any time dilation caused by the acceleration of the coordinate system to the velocity v, it cannot be determined in this way.

$$dx' = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}} \tag{1.72}$$

The time interval dt' is found from the Lorentz transformation equation 1.50, as

$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}}$$

Taking the time differential dt we get

$$dt' = \frac{dt - dx(v/c^2)}{\sqrt{1 - v^2/c^2}} \tag{1.73}$$

The transformation for V'_x, found from equations 1.69, 1.72, and 1.73, is

$$V'_x = \frac{dx'}{dt'} = \frac{(dx - vdt)/\sqrt{1 - v^2/c^2}}{[dt - dx(v/c^2)]/\sqrt{1 - v^2/c^2}} \tag{1.74}$$

Canceling out the square root term in both numerator and denominator, gives

$$V'_x = \frac{dx - vdt}{dt - dx(v/c^2)}$$

Dividing both numerator and denominator by dt, gives

$$V'_x = \frac{dx/dt - vdt/dt}{dt/dt - (dx/dt)(v/c^2)}$$

But $dx/dt = V_x$ from equation 1.66, and $dt/dt = 1$. Hence,

$$V'_x = \frac{V_x - v}{1 - (v/c^2)V_x} \tag{1.75}$$

Equation 1.75 is the Lorentz transformation for the x-component of velocity. Notice that if v is very small, compared to c, then the term $(v/c^2)V_x$ approaches zero, and this equation reduces to the Galilean transformation equation 1.12 as would be expected for low velocities.

The y-component of the velocity transformation is obtained similarly. Thus,

$$V'_y = \frac{dy'}{dt'} = \frac{dy}{[dt - (v/c^2)dx]/\sqrt{1 - v^2/c^2}}$$

$$= \frac{dy\sqrt{1-v^2/c^2}}{[dt-(v/c^2)dx]}$$

Dividing numerator and denominator by dt gives

$$V_y' = \frac{(dy/dt)\sqrt{1-v^2/c^2}}{dt/dt-(v/c^2)dx/dt}$$

and therefore

$$V_y' = \frac{V_y\sqrt{1-v^2/c^2}}{1-(v/c^2)V_x} \tag{1.76}$$

A similar analysis for the z-component of the velocity gives

$$V_z' = \frac{V_z\sqrt{1-v^2/c^2}}{1-(v/c^2)V_x} \tag{1.77}$$

Note, that for v very much less than c, these equations reduce to the Galilean equations, $V_y' = V_y$ and $V_z' = V_z$, as expected.

The Lorentz velocity transformation equations are summarized as

$$V_x' = \frac{V_x - v}{1-(v/c^2)V_x} \tag{1.75}$$

$$V_y' = \frac{V_y\sqrt{1-v^2/c^2}}{1-(v/c^2)V_x} \tag{1.76}$$

$$V_z' = \frac{V_z\sqrt{1-v^2/c^2}}{1-(v/c^2)V_x} \tag{1.77}$$

The inverse transformations from the S' frame to the S frame can be written down immediately by changing primes for nonprimes and replacing $-v$ by $+v$. Thus,

$$V_x = \frac{V_x' + v}{1+(v/c^2)V_x'} \tag{1.78}$$

$$V_y = \frac{V_y'\sqrt{1-v^2/c^2}}{1+(v/c^2)V_x'} \tag{1.79}$$

$$V_z = \frac{V_z'\sqrt{1-v^2/c^2}}{1+(v/c^2)V_x'} \tag{1.80}$$

Example 1.8

Galilean transformation of velocities versus the Lorentz transformation of velocities. Two rocket ships are approaching a space station, each at a speed of $0.9c$, with respect to the station, as shown in figure 1.15. What is their relative speed according to (a) the Galilean transformation and (b) the Lorentz transformation?

Solution

a. According to the space station observer, the space station is at rest and the two spaceships are closing on him, as shown in figure 1.15. The spaceship to the right is approaching at a speed $V_x = -0.9c$, in the space station coordinate system. The spaceship to the left is considered to be a moving coordinate system approaching

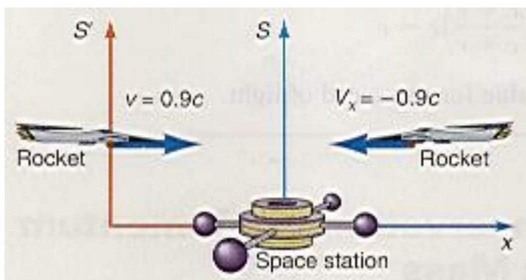


Figure 1.15 Galilean and Lorentz transformations of velocities.

with the speed $v = 0.9c$. The relative velocity according to the Galilean transformation, as observed in the moving spaceship to the left, is

$$\begin{aligned} V_x' &= V_x - v \\ &= -0.9c - 0.9c \\ &= -1.8c \end{aligned}$$

That is, the spaceship to the left sees the spaceship to the right approaching at a speed of $1.8c$. The minus sign means the velocity is toward the left in the S' frame of reference. Obviously this result is incorrect because the relative velocity is greater than c, which is impossible.

b. According to the Lorentz transformation the relative velocity of approach as observed by the S' spaceship, given by equation 1.75, is

$$V_x' = \frac{V_x - v}{1 - (v/c^2)V_x} = \frac{-0.9c - 0.9c}{1 - (0.9c/c^2)(-0.9c)} = \frac{-1.8c}{1 + 0.81c^2/c^2}$$

$$= \frac{-1.8c}{1.81} = -0.994c$$

Thus, the observer in the left-hand spaceship sees the right-hand spaceship approaching at the speed of 0.994c. The minus sign means that the speed is toward the left in the diagram. Notice that the relative speed is less than c as it must be.

To go to this Interactive Example click on this sentence.

Example 1.9

Transformation of the speed of light. If a ray of light is emitted from a rocket ship moving at a speed v, what speed will be observed for that light on earth?

Solution

The speed of light from the rocket ship is $V'_x = c$. The speed observed on earth, found from equation 1.78, is

$$V_x = \frac{V_x' + v}{1 + (v/c^2)V'_x} = \frac{c + v}{1 + vc/c^2} = \frac{c + v}{1 + \left(\frac{v}{c}\right)}$$
$$= \frac{c+v}{(c+v)/c} = \left(\frac{c+v}{(c+v)}\right)c = c$$

Thus, all observers observe the same value for the speed of light.

1.11 The Law of Conservation of Momentum and Relativistic Mass

In section 1.3 we saw that momentum was conserved under a Galilean transformation. Does the law of conservation of momentum also hold in relativistic mechanics? Let us first consider the following perfectly elastic collision between two balls that are identical when observed in a stationary rest frame S in figure 1.16. The first ball m_A is thrown upward with the velocity V_y, whereas the second ball is thrown straight downward with the velocity $-V_y$. Thus the speed of each ball is the same. We assume that the original distance separating the two balls is small and the velocity V_y is relatively large, so that the effect of the acceleration due to gravity can be ignored. Applying the law of conservation of momentum to the collision, we obtain

momentum before collision = momentum after collision

$$m_A V_y - m_B V_y = m_B V_y - m_A V_y$$

Simplifying,

$$2m_A V_y = 2m_B V_y$$

or

$$m_A V_y = m_B V_y \tag{1.81}$$

Equation 1.81 also indicates that $m_A = m_B$, as originally stated.

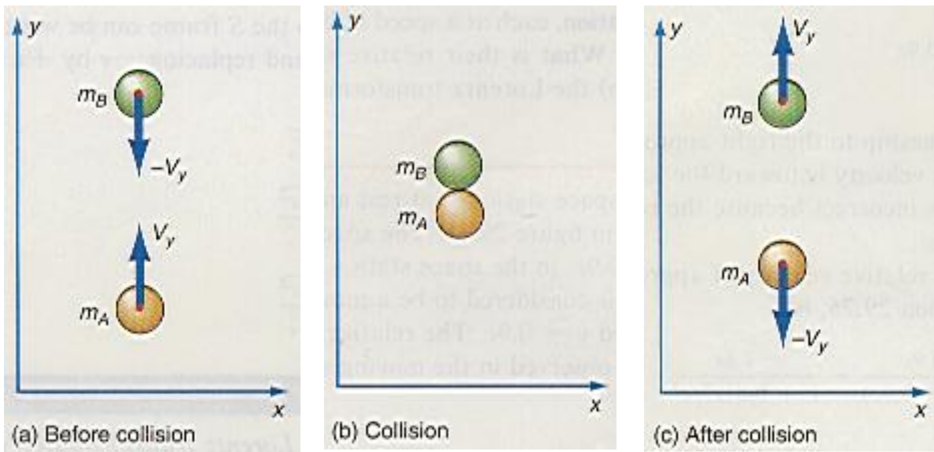


Figure 1.16 A perfectly elastic collision in a stationary frame of reference.

Let us now consider a similar perfectly elastic collision only now the ball B is thrown by a moving observer. The stationary observer is in the frame S and the moving observer is in the moving frame S', moving toward the left at the velocity $-v$, as shown in figure 1.17. In the stationary frame S, the observer throws a ball

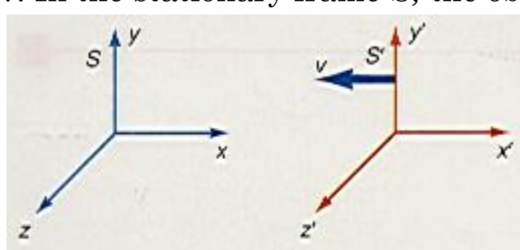


Figure 1.17 One observer is in rest frame and one in moving frame.

straight upward in the positive y-direction with the velocity V_y. The moving observer S' is on a truck moving to the left with the velocity $-v$. The moving observer throws an identical ball straight downward in the negative y-direction with the velocity $-u'$ in the moving frame of reference. In the stationary frame this velocity is observed as $-u$. We assume that both observers throw the ball with the same speed in their frames of reference. That is, the magnitude of the velocity V_y in the S frame is identical to the magnitude of the velocity u' in the S' frame. (As an example, let us assume that observer S throws the ball upward at a speed of 20 m/s and observer S' throws his ball downward at a speed of 20 m/s.) The two balls are exactly alike in that they have the same mass and size when they are at rest before the experiment starts.

After some practice, the experimenters are able to throw the balls such that a collision occurs. As observed from the S frame of reference on the ground, the collision appears as in figure 1.18(a). The mass m_A goes straight up, collides in a perfectly elastic collision with mass m_B, and is reflected with the velocity $-V_y$, since no energy, and hence speed, was lost in the collision. Ball B has a velocity component $-u$ straight downward (as seen by the S observer) but it is also moving in the negative x-direction with the velocity $-v$, the velocity of the truck and hence the velocity of the S' frame.

The y-component of the law of conservation of momentum, as observed in the rest frame S, can be written as

momentum before collision = momentum after collision

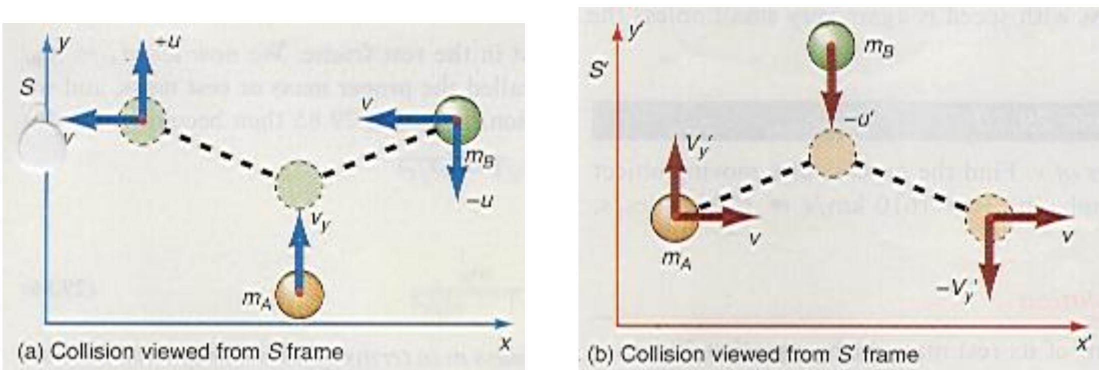


Figure 1.18 A perfectly elastic collision viewed from different frames of reference.

$$m_A V_y - m_B u = -m_A V_y + m_B u$$

Simplifying,

$$2m_A V_y = 2m_B u$$

or

$$m_A V_y = m_B u \tag{1.82}$$

But the velocity u in the stationary frame S is related to the velocity u' in the moving frame of reference S' through the velocity transformation, equation 1.79, as

$$u = \frac{u'\sqrt{1 - v^2/c^2}}{1 + (v/c^2)V_x'}$$

However, $V'x = 0$ in this experiment because m_B is thrown only in the y-direction. Hence, u becomes

$$u = u'\sqrt{1 - v^2/c^2} \tag{1.83}$$

Substituting u from equation 1.83 back into the law of conservation of momentum, equation 1.82, we obtain

$$m_A V_y = m_B u' \sqrt{1 - v^2 / c^2}$$

But recall that the initial speed of each ball was the same in each reference frame, that is, $V_y = u'$. Hence,

$$m_A V_y = m_B V_y \sqrt{1 - v^2 / c^2} \tag{1.84}$$

If we compare equation 1.84, for the conservation of momentum when one of the frames is in motion, with equation 1.81, for the conservation of momentum in a stationary frame, we see that the form of the equation is very different. Thus, in the form of equation 1.84, the law of conservation of momentum does not seem to hold. But the law of conservation of momentum is such a fundamental concept in physics that we certainly do not want to lose it in the description of relativistic mechanics. The law of conservation of momentum can be retained if we allow for the possibility that the moving mass changes its value because of that motion. That is, if both sides of equation 1.84 are divided by V_y we get

$$m_A = m_B \sqrt{1 - v^2 / c^2} \tag{1.85}$$

Now m_A is the mass of the ball in the stationary frame and m_B is the mass of the ball in the moving frame. If we consider the very special case where V_y is zero in the S frame, then the mass m_A is at rest in the rest frame. We now let $m_A = m_0$, the mass when it is at rest, henceforth called the **proper mass or rest mass,** and we let $m_B = m$, the mass when it is in motion. Equation 1.85 then becomes

$$m_0 = m \sqrt{1 - v^2 / c^2}$$

or, solving for m,

$$m = \frac{m_0}{\sqrt{1 - v^2 / c^2}} \tag{1.86}$$

Equation 1.86 defines the ***relativistic mass*** *m in terms of its rest mass m_0.* Because the term $\sqrt{1 - v^2 / c^2}$ is always less than one, *the relativistic mass m, the mass of a body in motion at the speed v, is always greater than m_0, the mass of the body when it is at rest.* The variation of mass with speed is again very small unless the speed is very great.

Example 1.10

The relativistic mass for various values of v. Find the mass m of a moving object when (a) v = 1610 km/hr = 1000 mph, (b) v = 1610 km/s = 1000 miles/s, (c) $v = 0.8c$, and (d) $v = c$.

Solution

The relativistic mass m is found in terms of its rest mass m_0 by equation 1.86.
a. For v = 1610 km/hr = 447 m/s, we obtain

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$
$$m = \frac{m_0}{\sqrt{1 - (447\ \text{m/s})^2/(3.00 \times 10^8\ \text{m/s})^2}}$$
$$= m_0$$

Thus, at this reasonably high speed there is no measurable difference in the mass of the body.

b. For v = 1610 km/s,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$
$$m = \frac{m_0}{\sqrt{1 - (1.610 \times 10^6\ \text{m/s})^2/(3.00 \times 10^8\ \text{m/s})^2}}$$
$$= \frac{m_0}{\sqrt{0.99997}}$$
$$= \frac{m_0}{0.99999}$$
$$= 1.00001 m_0$$

Thus, for a speed of 1610 km/s = 3,600,000 mph, a speed so great that macroscopic objects cannot yet attain it, the relativistic increase in mass is still practically negligible.

c. For $v = 0.8c$,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$
$$m = \frac{m_0}{\sqrt{1 - (0.8c)^2/c^2}}$$
$$= 1.67 m_0$$

For the rather large velocity of $0.8c$, the increase in mass is very significant. We should note that it is almost routine today to accelerate elementary particles to speeds approaching the speed of light and in all such cases this variation of mass with speed is observed.

d. For $v = c$,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = \frac{m_0}{\sqrt{1 - c^2/c^2}} = \frac{m_0}{0}$$
$$= \infty$$

Thus, as a particle approaches the speed of light c, the mass of the particle approaches infinity. Since an infinite force and infinite energy would be required to move an infinite mass it is obvious that a particle of a finite rest mass m_0 can never be accelerated to the speed of light.

To go to this Interactive Example click on this sentence.

The first of many experiments to verify the change in mass with speed was performed by A. H. Bucherer in 1909. Electrons were first accelerated by a large potential difference until they were moving at high speeds. They then entered a velocity selector. By varying the electric and magnetic field of the velocity selector, electrons with any desired velocity can be obtained by the equation $v = E/B$ These electrons were then sent through a uniform magnetic field B where they were deflected into a circular path. The centripetal force was set equal to the magnetic force, and we obtain

$$\frac{mv^2}{r} = qvB$$

Simplifying,

$$mv = qBr \tag{1.87}$$

But now we must treat the mass in equation 1.87 as the relativistic mass in equation 1.86. Thus, equation 1.87 becomes

$$\frac{m_0 v}{\sqrt{1 - v^2/c^2}} = qBr \tag{1.88}$$

Because B, r, and v could be measured in the experiment, the ratio of the charge of the electron q to its rest mass m_0, found from equation 1.88, is

$$\frac{q}{m_0} = \frac{v}{Br\sqrt{1 - v^2/c^2}} \tag{1.89}$$

Bucherer's experiment confirmed equation 1.89 and hence the variation of mass with speed. Since 1909, thousands of experiments have been performed confirming the variation of mass with speed.

The variation of mass with speed truly points out the meaning of the concept of inertial mass as a measure of the resistance of matter to motion. As we can see with this relativistic mass, at higher and higher speeds there is a much greater resistance to motion and this is manifested as the increase in the mass of the body. The rest mass m_0 should probably be called the "quantity of matter" of a body since it is truly a measure of how much matter is present in the body, whereas the relativistic mass is the measure of the resistance of that quantity of matter to being put into motion.

*With this new definition of relativistic mass the **relativistic linear momentum** can now be defined as*

$$\mathbf{p} = m\mathbf{v} = \frac{m_0\mathbf{v}}{\sqrt{1 - v^2/c^2}} \tag{1.90}$$

The law of conservation of momentum now holds for relativistic mechanics just as it did for Newtonian mechanics. In fact, we can rewrite equation 1.84 as

$$m_0 V_y = m V_y \sqrt{1 - v^2/c^2}$$

Substituting for m from equation 1.86,

$$m_0 V_y = \frac{m_0}{\sqrt{1 - v^2/c^2}} V_y \sqrt{1 - v^2/c^2}$$

Simplifying,

$$m_0 V_y = m_0 V_y \tag{1.91}$$

Hence, using the concept of relativistic mass, the same form of the equation for the law of conservation of momentum 1.91 is obtained as for the Newtonian case in equation 1.81. Thus, momentum is always conserved if the relativistic mass is used and the law of conservation of momentum is preserved for relativistic mechanics.

Newton's second law is still valid for relativistic mechanics, but only in the form

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) \tag{1.92}$$

1.12 The Law of Conservation of Mass-Energy

As we have just seen, because the mass of an object varies with its speed, Newton's second law also changes for relativistic motion. We now ask what effect does this changing mass have on the kinetic energy of a moving body? How do we determine the kinetic energy of a body when its mass is changing with time? First let us recall how we determined the kinetic energy of a body of constant mass. The kinetic energy of a moving body was equal to the work done to put the body into motion, i.e.,

$$\text{KE} = W = \int dW = \int \mathbf{F} \bullet d\mathbf{s} = \int ma\, dx$$

However, since the acceleration $a = dv/dt$ this became

$$\text{KE} = \int madx = \int m\frac{dv}{dt}dx = \int mdv\frac{dx}{dt} = \int m\frac{dx}{dt}dv$$

Since $dx/dt = v$ the velocity of the moving body, the kinetic energy became

$$\text{KE} = \int_0^v mvdv = \left[\frac{mv^2}{2}\right]_0^v = \tfrac{1}{2}mv^2$$

Let us now see how this changes when we compute the kinetic energy relativisticly. The kinetic energy is again equal to the work done to put the body into motion. That is,

$$\text{KE} = W = \int dW = \int F\, dx$$

But Newton's second law is now written in the form $F = dp/dt$ and the kinetic energy becomes

$$\text{KE} = \int \frac{dp}{dt}dx = \int dp\frac{dx}{dt} = \int vdp \tag{1.93}$$

We cannot integrate equation 1.93 directly because $p = mv$ and hence dp is a function of v. We solve equation 1.93 by the standard technique of integrating it by parts, that is $\int vdp$ has the form $\int udV$ which has the standard solution

$$\int udV = Vu - \int Vdu \tag{1.94}$$

We let $u = v$ and hence $du = dv$, and $dV = dp$ hence $V = p$ and the integration becomes

$$\text{KE} = \int vdp = [pv]_0^v - \int_0^v pdv$$
$$\text{KE} = [(mv)v]_0^v - \int_0^v mvdv$$
$$\text{KE} = mv^2 - \int_0^v \frac{m_0 vdv}{\sqrt{1 - v^2/c^2}} \tag{1.95}$$

We integrate the second term in equation 1.95 by making the following substitutions. Let $x = v^2/c^2$, and hence $dx = (2vdv)/c^2$. Then $vdv = c^2dx/2$ and the integral in equation 1.95 becomes

$$I = m_0 \int_0^v \frac{vdv}{\sqrt{1 - v^2/c^2}} = m_0 \int \frac{c^2 dx}{2\sqrt{1-x}} \tag{1.96}$$

We now let

$$y = \sqrt{1-x}$$

Therefore

$$dy = 1/2(1 - x)^{-1/2}(-dx)$$

and hence

$$dx = -2\sqrt{1-x}\,dy$$

and the second integral in equation 1.96 becomes

$$I = \frac{m_o c^2}{2}\int \frac{\left(-2\sqrt{1-x}\ \right)dy}{y} = \frac{m_o c^2}{2}\int \frac{(-2ydy)}{y}$$
$$= -m_0c^2 \int dy = -m_0c^2 y = -m_0c^2\sqrt{1-x} = \left[-m_0c^2\sqrt{1-v^2/c^2}\ \right]_0^v$$

and

$$I = \left[-m_0c^2\sqrt{1-v^2/c^2}\ \right]_0^v$$
$$I = m_0\left[-c^2\sqrt{1-v^2/c^2} + c^2\right]$$

Equation 1.95 therefore becomes

$$\text{KE} = mv^2 - m_0\left[-c^2\sqrt{1-v^2/c^2} + c^2\right]$$
$$\text{KE} = \frac{m_0}{\sqrt{1-v^2/c^2}}v^2 + m_0c^2\sqrt{1-v^2/c^2} - m_0c^2$$
$$\text{KE} = \frac{m_0v^2 + m_0c^2(1-v^2/c^2)}{\sqrt{1-v^2/c^2}} - m_0c^2$$
$$\text{KE} = \frac{m_0v^2 + m_0c^2 - m_0c^2v^2/c^2}{\sqrt{1-v^2/c^2}} - m_0c^2$$
$$\text{KE} = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2$$

and since $m_o/\sqrt{1-v^2/c^2} = m$, the relativistic kinetic energy becomes

$$\text{KE} = mc^2 - m_0c^2 \tag{1.97}$$

We can also write the relativistic mass m as

$$m = m_0 + \Delta m \tag{1.98}$$

That is, *the relativistic mass is equal to the rest mass plus the change in mass due to motion*. Substituting equation 1.98 back into equation 1.97, we have

$$\text{KE} = (m_0 + \Delta m)c^2 - m_0c^2$$

or

$$\text{KE} = (\Delta m)c^2 \tag{1.99}$$

Thus, relativistically, the kinetic energy of a body is equal to the change in mass of the body caused by the motion times the velocity of light squared.

Notice that the left-hand side of either equation 1.97 or 1.99 represents an energy. Since the left-hand side of the equation is equal to the right-hand side of the equation, the right-hand side must also represent an energy. That is, *the product of a mass times the square of the speed of light must equal an energy. The total* ***relativistic energy*** *of a body is, therefore, defined as*

$$E = mc^2 \tag{1.100}$$

We can rewrite equation 1.97 as

$$mc^2 = \text{KE} + m_0c^2$$

In view of the definition in equation 1.100, the total energy of a body is

$$E = \text{KE} + m_0c^2 \tag{1.101}$$

When a particle is at rest, its kinetic energy KE is equal to zero. Therefore, the total energy of the particle when it is at rest must be equal to m_0c^2. *The* ***rest mass energy*** *of a particle can then be defined as*

$$E_0 = m_0c^2 \tag{1.102}$$

Substituting equation 1.102 back into equation 1.101, we get

$$E = \text{KE} + E_0 \tag{1.103}$$

Equation 1.103 states that the total energy of a body is equal to its kinetic energy plus its rest mass energy. The result of these equations is that energy can manifest itself as mass, and mass can manifest itself as energy. In a sense, mass can be thought of as being frozen energy.

Example 1.11

Energy in a 1-kg mass. How much energy is stored in a 1.00-kg mass?

Solution

The rest mass energy of a 1.00-kg mass, given by equation 1.102, is

$$E_0 = m_0c^2$$
$$= (1.00 \text{ kg})(3.00 \times 10^8 \text{ m/s})^2$$
$$= 9.00 \times 10^{16} \text{ J}$$

This is an enormous amount of energy to be sure. It is about a thousand times greater than the energy released from the atomic bomb dropped on Hiroshima, and could supply 2.85 gigawatts of power for a period of one year.

To go to this Interactive Example click on this sentence.

We have found the total relativistic kinetic energy to be given by equation 1.97 as

$$\text{KE} = mc^2 - m_0c^2$$

What does this relation for the kinetic energy reduce to at low speeds? The term for the mass m can be written as

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = m_0(1 - v^2/c^2)^{-1/2}$$

For relatively small velocities the term $(1 - v^2/c^2)^{-1/2}$ can be expanded by the binomial theorem, as

$$(1 - x)^n = 1 - nx$$
$$\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2}$$

Substituting this back into the equation for the mass, we get

$$m = m_0\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)$$
$$= m_0 + \frac{1}{2}\frac{m_0v^2}{c^2}$$

Multiplying each term by c^2, gives

$$mc^2 = m_0c^2 + \frac{1}{2}m_0v^2$$

Replacing this into equation 1.97 for the kinetic energy gives

$$\text{KE} = mc^2 - m_0c^2 = m_0c^2 + \frac{1}{2}m_0v^2 - m_0c^2$$

and finally

$$\text{KE} = \frac{1}{2} m_0 v^2$$

Notice that the relativistic equation for the kinetic energy reduces to the classical form of the equation for the kinetic energy at low speeds as would be expected.

Example 1.12

Relativistic and classical kinetic energy. A 1.00-kg object is accelerated to a speed of 0.4c. Find its kinetic energy (a) relativistically and (b) classically.

Solution

a. The relativistic kinetic energy of the moving body, found from equation 1.97, is

$$\begin{aligned}\text{KE} &= mc^2 - m_0c^2 \\ &= \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2 \\ &= \frac{1.00\text{ kg}(3.0 \times 10^8\text{ m/s})^2}{\sqrt{1-(0.4c)^2/c^2}} - 1.00\text{ kg}(3.00 \times 10^8\text{ m/s})^2 \\ &= 8.20 \times 10^{15}\text{ J}\end{aligned}$$

b. The classical, and wrong, determination of the kinetic energy is

$$\begin{aligned}\text{KE} &= \frac{1}{2} mv^2 \\ &= \frac{1}{2}(1.00\text{ kg})[(0.4)(3.00 \times 10^8\text{ m/s})]^2 \\ &= 7.20 \times 10^{15}\text{ J}\end{aligned}$$

That is, if an experiment were performed to test these two results, the classical result would not agree with the experimental results, but the relativistic one would agree.

To go to this Interactive Example click on this sentence.

When dealing with charged elementary particles, the kinetic energy can be found as the work that you must do in order to accelerate the particle up to the speed v, and is given as

$$\text{KE} = \text{work done} = qV$$

Example 1.13

Kinetic energy of an electron. An electron is accelerated through a uniform potential difference of 2.00×10^6 V. What is its kinetic energy as it leaves the electric field?

Solution

The kinetic energy, found from equation 23.74, is

$$\begin{aligned} \text{KE} &= qV \\ &= (1.60 \times 10^{-19}\ \text{C})(2.00 \times 10^6\ \text{V}) \\ &= 3.20 \times 10^{-13}\ \text{J} \end{aligned}$$

To go to this Interactive Example click on this sentence.

It is customary in relativity and modern physics to express energies in terms of electron volts, abbreviated eV. The unit of energy called an electron volt is equal to the energy that an electron would acquire as it falls through a potential difference of 1 V. Hence,

$$\begin{aligned} \text{KE} &= qV \\ 1\ \text{eV} &= (1.60 \times 10^{-19}\ \text{C})(1.00\ \text{V}) \\ 1\ \text{eV} &= 1.60 \times 10^{-19}\ \text{J} \end{aligned} \tag{1.104}$$

Thus, the electron volt is also a unit of energy.

Now we can express the KE in example 1.13 in electron volts as

$$\begin{aligned} \text{KE} &= \left(3.20 \times 10^{-13}\ \text{J}\right)\left(\frac{1\ \text{eV}}{1.60 \times 10^{-19}\ \text{J}}\right) \\ &= 2.00 \times 10^6\ \text{eV} \end{aligned}$$

For larger quantities of energy the following units of energy are used:

$$\begin{aligned} 1\ \text{kilo electron volt} &= 1\ \text{keV} = 10^3\ \text{eV} \\ 1\ \text{mega electron volt} &= 1\ \text{MeV} = 10^6\ \text{eV} \\ 1\ \text{giga electron volt} &= 1\ \text{GeV} = 10^9\ \text{eV} \\ 1\ \text{tera electron volt} &= 1\ \text{TeV} = 10^{12}\ \text{eV} \end{aligned}$$

Hence, the energy in example 1.13 can be expressed as

$$\text{KE} = 2.00\ \text{MeV}$$

By far, the greatest implication of equations 1.100 and 1.102 is that mass and energy must be considered as a manifestation of the same thing. *Thus, mass and* *energy are not independent quantities, just as we found that space and time are no longer independent quantities. Just as space and time are fused into spacetime, we must now fuse the separate concepts of mass and energy into one concept called mass-energy. What was classically considered as two separate laws, namely the law of conservation of mass and the law of conservation of energy must now be considered as one single law --* ***the law of conservation of mass-energy.*** *That is, mass can be created or destroyed as long as an equal amount of energy vanishes or appears, respectively.*

Because mass and energy can be equated it is sometimes desirable to express the mass of a particle in terms of energy units. Let us start by *defining an atomic unit of mass, called the unified mass unit, and defined as one-twelfth of the mass of the carbon 12 atom.* Recall that the mass of a molecule is given by

$$m = \frac{M}{N_A}$$

where M is the molecular mass of the molecule and N_A is Avogadro's number. For a single atom the molecular mass is replaced by its atomic mass and the mass of a single atom is given by

$$m = \frac{\text{atomic mass}}{N_A}$$

Thus, we define the *unified mass unit,* u, as

$$1\text{ u} = \frac{1}{12} m_C = \frac{1}{12} \frac{12 \text{ kg/kilomole}}{(6.0221367 \times 10^{26} \text{ molecules/kilomole})}$$

$$1\text{ u} = 1.660540 \times 10^{-27} \text{ kg} \tag{1.105}$$

To express this mass unit in terms of energy, we use equation 1.102 as

$$E_0 = m_0 c^2$$
$$= (1\text{ u})(c^2)$$
$$= \frac{(1.660540 \times 10^{-27} \text{ kg})(2.997925 \times 10^8 \text{ m/s})^2}{\left(1.60219\times10^{-19} \ \frac{\text{J}}{\text{eV}}\right)\left(\frac{10^6 \text{ eV}}{\text{MeV}}\right)}$$
$$= 931.493 \text{ MeV}$$

More significant figures have been used in this calculation than has been customary in this book. The additional accuracy is necessary because of the small quantities that are dealt with. Hence, a unified mass unit u has an energy equivalent of 931.493 MeV, that is,

$$1\text{ u} = 931.493 \text{ MeV} \tag{1.106}$$

The masses of some of the elementary particles in terms of unified mass units and MeVs are given as

$$\text{rest mass of proton} = m_\mathbf{p} = 1.00726 \text{ u} = 938.256 \text{ MeV}$$
$$\text{rest mass of neutron} = m_\mathbf{n} = 1.00865 \text{ u} = 939.550 \text{ MeV}$$
$$\text{rest mass of electron} = m_\mathbf{e} = 0.00055 \text{ u} = 0.511006 \text{ MeV}$$
$$\text{rest mass of deuteron} = m_\mathbf{d} = 2.01410 \text{ u} = 1875.580 \text{ MeV}$$

Example 1.14

The energy of the deuteron. Deuterium is an isotope of hydrogen whose nucleus, called a *deuteron,* consists of a proton and a neutron. Find the sum of the rest mass energies of the proton and the neutron, and compare it with the rest mass energy of the deuteron.

Solution

The sum of the rest mass energy of the proton and neutron is

$$m_\mathbf{p} + m_\mathbf{n} = 938.26 \text{ MeV} + 939.55 \text{ MeV} = 1877.81 \text{ MeV}$$

The actual rest mass of the deuteron is $m_\mathbf{d}$ = 1875.58 MeV. Thus, the sum of the masses of the individual proton and neutron is greater than the mass of the deuteron itself. The difference in mass is

$$\begin{aligned} \Delta m &= (m_\mathbf{p} + m_\mathbf{n}) - m_\mathbf{d} \\ &= 1877.81 \text{ MeV} - 1875.58 \text{ MeV} \\ &= 2.23 \text{ MeV} \end{aligned}$$

That is, some mass Δm and hence energy is lost in combining the proton and the neutron. The lost energy that binds the proton and neutron together is called the binding energy of the system. This is the amount of energy that must be supplied to break up the deuteron.

To go to this Interactive Example click on this sentence.

A further and extremely important application of mass-energy conversions occurs in the fusion of light atoms into heavier atoms. The most famous of such fusion processes is the conversion of hydrogen to helium in the sun and in the hydrogen bomb. An extremely simplified version of the process can be obtained by considering the mass of helium as consisting of two protons, two neutrons, and two electrons. The atomic mass of helium, as determined by the rest masses of its constituents, is

$$\begin{aligned} m_{\mathrm{He}} &= 2m_{\mathrm{p}} + 2m_{\mathrm{n}} + 2m_{\mathrm{e}} \\ &= 2(938.256 \text{ MeV}) + 2(939.550 \text{ MeV}) + 2(0.511006 \text{ MeV}) \\ &= 1876.512 \text{ MeV} + 1879.100 \text{ MeV} + 1.0220 \text{ MeV} \\ &= 3756.634 \text{ MeV} \end{aligned}$$

If this value is compared to the atomic mass of helium from the table of elements we find

$$\begin{aligned} \text{Atomic mass of He} &= (4.002603\text{u})\left(\frac{931.493\,\text{MeV}}{\text{u}}\right) \\ &= 3728.397 \text{ MeV} \end{aligned}$$

Hence, helium is lighter than the sum of its constituent parts. The difference in mass between helium and its constituent parts is

$$\begin{aligned} \Delta m &= 3756.634 \text{ MeV} - 3728.397 \text{ MeV} \\ &= 28.237 \text{ MeV} \end{aligned}$$

Thus, 28.237 MeV of energy is given off for each atom of helium formed. For the formation of 1 mole of helium, there are 6.02×10^{23} atoms. Hence, the total energy released per mole of helium formed is

$$\begin{aligned} \frac{\text{Energy released}}{\text{mole}} &= \left(28.237\frac{\text{MeV}}{\text{atom}}\right)\left(6.02\times10^{23}\,\frac{\text{atoms}}{\text{mole}}\right) \\ &= \left(1.70\times10^{25}\text{ MeV}\right)\left(\frac{10^{6}\text{ eV}}{\text{MeV}}\right)\left(\frac{1.60\times10^{-19}\text{ J}}{\text{eV}}\right) \\ &= 2.73 \times 10^{12} \text{ J} \end{aligned}$$

Hence, in the formation of 1 mole of helium, a mass of only 4 g, 2,730,000,000,000 J of energy are released. This monumental amount of energy, which comes from the conversion of mass into energy, is continually being released by the sun. This fusion process is also the source of energy in the hydrogen bomb.

Example 1.15

A high-speed electron. An electron is accelerated from rest through a potential difference of 3.00×10^5 V. Find (a) the kinetic energy of the electron, (b) the total energy of the electron, (c) the speed of the electron, (d) the relativistic mass of the electron, and (e) the momentum of the electron.

Solution

a. The kinetic energy of the electron, found from equation 23.74, is

$$\text{KE} = \text{work done} = qV$$
$$\text{KE} = (1.60 \times 10^{-19}\ \text{C})(3.00 \times 10^{5}\ \text{V})$$
$$\text{KE} = \left(4.80 \times 10^{-14}\ \text{J}\right)\left(\frac{1\ \text{eV}}{1.60 \times 10^{-19}\ \text{J}}\right)$$
$$\text{KE} = \left(3.00 \times 10^{5}\ \text{eV}\right)\left(\frac{1\ \text{MeV}}{10^{6}\ \text{eV}}\right)$$
$$= 0.300\ \text{MeV}$$

b. The rest mass energy of the electron is

$$E_0 = (m_0c^2)_{\text{electron}} = 0.511\ \text{MeV}$$

Thus, the total relativistic energy E, found from equation 1.101, is

$$E = \text{KE} + m_0c^2$$
$$= 0.300\ \text{MeV} + 0.511\ \text{MeV}$$
$$= 0.811\ \text{MeV}$$

c. To determine the speed of the electron, equation 1.97 is rearranged as

$$\text{KE} = mc^2 - m_0c^2$$
$$= \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2 = \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right)m_0c^2$$
$$\frac{1}{\sqrt{1 - v^2/c^2}} - 1 = \frac{\text{KE}}{m_0c^2}$$
$$\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{\text{KE}}{m_0c^2} + 1 = \frac{0.300\ \text{MeV}}{0.511\ \text{MeV}} + 1 = 1.587$$
$$\sqrt{1 - v^2/c^2} = \frac{1}{1.587} = 0.630$$
$$1 - v^2/c^2 = (0.630)^2 = 0.397$$
$$v^2/c^2 = 1 - 0.397 = 0.603$$
$$v = \sqrt{0.603c^2}$$
$$v = 0.776c$$

Hence, the speed of the electron is approximately seven-tenths the speed of light.

d. To determine the relativistic mass of the electron, we use equation 1.86:

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{9.11 \times 10^{-31}\ \text{kg}}{\sqrt{1-(0.776c)^2/c^2}}$$
$$= 14.4 \times 10^{-31}\ \text{kg}$$

The relativistic mass has increased by approximately 1.6 times the rest mass.

e. The momentum of the electron, found from equation 1.90, is

$$p = mv = \frac{m_0}{\sqrt{1-v^2/c^2}}v$$
$$= (14.4 \times 10^{-31}\ \text{kg})(0.776)(3.00 \times 10^8\ \text{m/s})$$
$$= 3.35 \times 10^{-22}\ \text{kg m/s}$$

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The Language of Physics

Relativity
The observation of the motion of a body by two different observers in relative motion to each other. At speeds approaching the speed of light, the length of a body contracts, its mass increases, and time slows down (p.).

Inertial coordinate system
A frame of reference that is either at rest or moving at a constant velocity (p.).

Galilean transformations
A set of classical equations that relate the motion of a body in one inertial coordinate system to that in a second inertial coordinate system. All the laws of classical mechanics are invariant under a Galilean transformation, but the laws of electromagnetism are not (p.).

Invariant quantity
A quantity that remains a constant whether it is observed from a system at rest or in motion (p.).

Ether
A medium that was assumed to pervade all space. This was the medium in which light was assumed to propagate (p.).

Michelson-Morley experiment
A crucial experiment that was performed to detect the presence of the ether. The

results of the experiment indicated that if the ether exists it cannot be detected. The assumption is then made that if it cannot be detected, it does not exist. Hence, light does not need a medium to propagate through. The experiment also implied that the speed of light in free space is the same everywhere regardless of the motion of the source or the observer (p.).

Special or Restricted Theory of Relativity
Einstein stated his special theory of relativity in terms of two postulates.

Postulate 1: The laws of physics have the same form in all inertial frames of reference.
Postulate 2: The speed of light in free space has the same value for all observers, regardless of their state of motion.

In order for the speed of light to be the same for all observers, space and time itself must change. The special theory is restricted to inertial systems and does not apply to accelerated systems (p.).

Lorentz transformations
A new set of transformation equations to replace the Galilean transformations. These new equations are derived by the two postulates of special relativity. These equations show that space and time are intimately connected. The effects of relativity only manifests itself when objects are moving at speeds approaching the speed of light (p.).

Proper length
The length of an object that is measured in a frame where the object is at rest (p.).

Lorentz-Fitzgerald contraction
The length of a rod in motion as measured by an observer at rest is less than its proper length (p.).

Proper time
The time interval measured on a clock that is at rest relative to the observer (p.).

Time dilation
The time interval measured on a moving clock is less than the proper time. Hence, moving clocks slow down (p.).

Proper mass or rest mass
The mass of a body that is at rest in a frame of reference (p.).

Relativistic mass
The mass of a body that is in motion. The relativistic mass is always greater than the rest mass of the object (p.).

Relativistic linear momentum
The product of the relativistic mass of a body and its velocity (p.).

Relativistic energy
The product of the relativistic mass of a body and the square of the speed of light. This total energy is equal to the sum of the kinetic energy of the body and its rest mass energy (p.).

Rest mass energy
The product of the rest mass and the square of the speed of light. Hence, mass can manifest itself as energy, and energy can manifest itself as mass (p.).

The law of conservation of mass-energy
Mass can be created or destroyed as long as an equal amount of energy vanishes or appears, respectively (p.).

Summary of Important Equations

Galilean transformation of coordinates

$$x = x' + vt \tag{1.1}$$
$$y = y' \tag{1.2}$$
$$z = z' \tag{1.3}$$
$$t = t' \tag{1.4}$$

Galilean transformation of velocities

$$v_x = v_x' + v \tag{1.11}$$
$$v_x' = v_x - v \tag{1.13}$$
$$v_y' = v_y \tag{1.14}$$
$$v_z' = v_z \tag{1.15}$$

Lorentz transformation equations of coordinates

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{1.49}$$
$$y' = y$$
$$z' = z$$
$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} \tag{1.50}$$

Inverse Lorentz transformation equations of coordinates

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \tag{1.51}$$

$$y = y' \tag{1.52}$$
$$z = z' \tag{1.53}$$
$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \tag{1.54}$$

Length contraction $$L = L_0\sqrt{1 - v^2/c^2} \tag{1.60}$$

Time dilation $$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} \tag{1.64}$$

Lorentz transformation of velocities

$$V'_x = \frac{V_x - v}{1 - (v/c^2)V_x} \tag{1.75}$$
$$V'_y = \frac{V_y\sqrt{1 - v^2/c^2}}{1 - (v/c^2)V_x} \tag{1.76}$$
$$V'_z = \frac{V_z\sqrt{1 - v^2/c^2}}{1 - (v/c^2)V_x} \tag{1.77}$$

Relativistic mass $$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \tag{1.86}$$

Linear momentum $$\mathbf{p} = m\mathbf{v} = \frac{m_0\mathbf{v}}{\sqrt{1 - v^2/c^2}} \tag{1.90}$$

Newton's second law $$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) \tag{1.92}$$

Relativistic kinetic energy $$\text{KE} = mc^2 - m_0c^2 \tag{1.97}$$
$$\text{KE} = (\Delta m)c^2 \tag{1.99}$$

Total relativistic energy $$E = mc^2 \tag{1.100}$$

Rest mass energy $$E_0 = m_0c^2 \tag{1.102}$$

Law of conservation of relativistic energy $$E = \text{KE} + E_0 \tag{1.103}$$

Electron volt $$1\ \text{eV} = 1.60 \times 10^{-19}\ \text{J} \tag{1.104}$$
$$\text{u} = 1.66 \times 10^{-27}\ \text{kg} \tag{1.105}$$

Unified mass unit $$\text{u} = 931.493\ \text{MeV} \tag{1.106}$$

Questions for Chapter 1

1. If you are in an enclosed truck and cannot see outside, how can you tell if you are at rest, in motion at a constant velocity, speeding up, slowing down, turning to the right, or turning to the left?

*2. Does a length contract perpendicular to its direction of motion?

*3. Lorentz explained the negative result of the Michelson-Morley experiment by saying that the ether caused the length of the telescope in the direction of motion to be contracted by an amount given by $L = L_0\sqrt{1 - v^2/c^2}$. Would this give a satisfactory explanation of the Michelson-Morley experiment?

4. If the speed of light in our world was only 100 km/hr, describe some of the characteristics of this world.

*5. Does time dilation affect the physiological aspects of the human body, such as aging? How does the body know what time is?

6. Are length contraction and time dilation real or apparent?

7. An elementary particle called a neutrino moves at the speed of light. Must it have an infinite mass? Explain.

*8. It has been suggested that particles might exist that are moving at speeds greater than *c*. These particles, which have never been found, are called tachyons. Describe how such particles might exist and what their characteristics would have to be.

9. In the equation for the total relativistic energy of a body, could there be another term for the potential energy of a body? Does a compressed spring, which has potential energy, have more mass than a spring that is not compressed?

*10. When helium is formed, the difference in the mass of helium and the mass of its constituents is given off as energy. When the deuteron is formed, the difference in mass is also given off as energy. Could the formation of deuterium be used as a source of commercial energy?

11. If the speed of light were infinite, what would the Lorentz transformation equations reduce to?

*12. Can you apply the Lorentz transformations to a reference frame that is moving in a circle?

Problems for Chapter 1

1.1 Introduction to Relative Motion

1. A projectile is thrown straight upward at an initial velocity of 25.0 m/s from an open truck at the same instant that the truck starts to accelerate forward at 5.00 m/s^2. If the truck is 4.00 m long, how far behind the truck will the projectile land?

2. A projectile is thrown straight up at an initial velocity of 25.0 m/s from an open truck that is moving at a constant speed of 10.0 m/s. Where does the projectile

land when (a) viewed from the ground (S frame) and (b) when viewed from the truck (S' frame)?

3. A truck moving east at a constant speed of 50.0 km/hr passes a traffic light where a car just starts to accelerate from rest at 2.00 m/s^2. At the end of 10.0 s, what is the velocity of the car with respect to (a) the traffic light and (b) with respect to the truck?

4. A woman is sitting on a bus 5.00 m from the end of the bus. If the bus is moving forward at a velocity of 7.00 m/s, how far away from the bus station is the woman after 10.0 s?

1.2 The Galilean Transformations of Classical Physics

5. The woman on the bus in problem 4 gets up and (a) walks toward the front of the bus at a velocity of 0.500 m/s. What is her velocity relative to the bus station? (b) The woman now walks toward the rear of the bus at a velocity of 0.500 m/s. What is her velocity relative to the bus station?

1.3 The Invariance of the Mechanical Laws of Physics under a Galilean Transformation

*6. Filling in the steps omitted in the derivation associated with figure 1.8, show that the law of conservation of momentum is invariant under a Galilean transformation.

*7. Show that the law of conservation of energy for a perfectly elastic collision is invariant under a Galilean transformation.

1.5 The Michelson-Morley Experiment

8. A boat travels at a speed V of 5.00 km/hr with respect to the water, as shown in figure 1.10. If it takes 90.0 s to cross the river and return and 95.0 s for the boat to go the same distance downstream and return, what is the speed of the river current?

1.7 The Lorentz Transformation

9. A woman on the earth observes a firecracker explode 10.0 m in front of her when her clock reads 5.00 s. An astronaut in a rocket ship who passes the woman on earth at $t = 0$, at a speed of $0.400c$ finds what coordinates for this event?

10. A clock in the moving coordinate system reads $t' = 0$ when the stationary clock reads $t = 0$. If the moving frame moves at a speed of $0.800c$, what time will the moving clock read when the stationary observer reads 15.0 hr on her clock?

*11. Use the Lorentz transformation to show that the equation for a light wave, equation 1.25, has the same form in a coordinate system moving at a constant velocity.

1.8 The Lorentz-Fitzgerald Contraction

12. The USS *Enterprise* approaches the planet Seti Alpha 5 at a speed of $0.800c$. Captain Kirk observes an airplane runway on the planet to be 2.00 km long. The air controller on the planet says that the runway on the planet is how long?

Diagram for problem 12.

13. The starship *Regulus* was measured to be 100 m long when in space dock. If it approaches a planet at a speed of $0.400c$, how long does it appear to an observer on the planet?

14. How fast must a 4.57 m car move in order to fit into a 30.5 cm garage? Could you park the car in this garage?

Diagram for problem 14.

15. A comet is observed to be 130 km long as it moves past an observer at a speed of $0.700c$. How long does the comet seem when it travels at a speed of $0.900c$ with respect to the observer?

16. A meterstick at rest makes an angle of 30.0^0 with the x-axis. Find the length of the meterstick and the angle it makes with the x'-axis for an observer moving parallel to the x-axis at a speed of $0.650c$.

1.9 Time Dilation

17. A particle is observed to have a lifetime of 1.50×10^{-6} s when it is at rest in the laboratory. (a) What is its lifetime when it is moving at $0.800c$? (b) How far will the particle move with respect to the moving frame of reference before it decays? (c) How far will the particle move with respect to the laboratory frame before it decays?

18. A stroboscope is flashing light signals at the rate of 2100 flashes/min. An observer in a rocket ship traveling toward the strobe light at $0.500c$ would see what flash rate?

19. A particle has a lifetime of 0.100 s when observed while it moves at a speed of $0.650c$ with respect to the laboratory. What is its lifetime in its rest frame?

1.10 Transformation of Velocities

20. A spaceship traveling at a speed of $0.600c$ relative to a planet launches a rocket backward at a speed of $0.500c$. What is the velocity of the rocket as observed from the planet?

21. The three electrons are moving at the velocities shown in the diagram. Find the relative velocities between (a) electrons 1 and 2, (b) electrons 2 and 3, and (c) electrons 1 and 3.

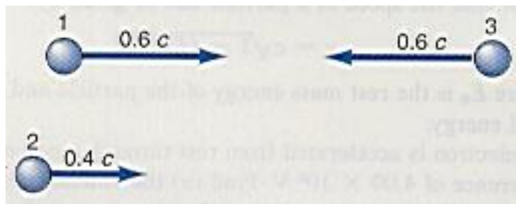

Diagram for problem 21.

1.11 The Law of Conservation of Momentum and Relativistic Mass

22. What is the mass of the following particles when traveling at a speed of $0.86c$: (a) electron, (b) proton, and (c) neutron?

23. Find the speed of a particle at which the mass m is equal to (a) 0.100 m_0, (b) 1.00 m_0, (c) 10.0 m_0, (d) 100 m_0, and (e) 1000 m_0.

24. Determine the linear momentum of an electron moving at a speed of $0.990c$.

25. How fast must a proton move so that its linear momentum is 8.08×10^{-19} kg m/s?

26. Compute the speed of a neutron whose total energy is 1.88×10^{-10} J.

1.12 The Law of Conservation of Mass-Energy

27. An isolated neutron is capable of decaying into a proton and an electron. How much energy is liberated in this process?

28. Since it takes 2.26×10^6 J to convert 1.00 kg of water to 1.00 kg of steam at 100 ^{0}C, what is the increase in mass of the steam?

29. What is the kinetic energy of a proton traveling at $0.800c$?

30. Through what potential difference must an electron be accelerated if it is to attain a speed of $0.800c$?

31. What is the total energy of a proton traveling at a speed of 2.50×10^8 m/s?

32. Calculate the speed of an electron whose kinetic energy is twice as large as its rest mass energy.

Additional Problems

33. If an ion-engine in a spacecraft can produce a continuous acceleration of 0.200 m/s^2, how long must the engine continue to accelerate if it is to reach the speed of 0.500*c*?

*34. The volume of a cube is V_0 in a frame of reference where it is at rest. Show that the volume observed in a moving frame of reference is given by

$$V = V_0\sqrt{1 - v^2/c^2}$$

35. The distance to Alpha Centari, the closest star, is about 4.00 light years as measured from earth. What would this distance be as observed from a spaceship leaving earth at a speed of 0.500*c*? How long would it take to get there according to a clock on the spaceship and a clock on earth?

36. A muon is an elementary particle that is observed to have a lifetime of 2.00×10^{-6} s before decaying. It has a typical speed of 2.994×10^8 m/s. (a) How far can the muon travel before it decays? (b) These particles are observed high in our atmosphere, but with such a short lifetime how do they manage to get to the surface of the earth?

*37. Show that the formula for the density of a cube of material moving at a speed v is given by

$$\rho = \frac{\rho_0}{\sqrt{1 - v^2/c^2}}$$

*38. A proton is accelerated to a speed of 0.500*c*. Find its (a) kinetic energy, (b) total energy, (c) relativistic mass, and (d) momentum.

*39. Show that the speed of a particle can be given by

$$v = c\sqrt{1 - \left(E_0/E\right)^2}$$

where E_0 is the rest mass energy of the particle and E is its total energy.

*40. An electron is accelerated from rest through a potential difference of 4.00×10^6 V. Find (a) the kinetic energy of the electron, (b) the total energy of the electron, (c) the velocity of the electron, (d) the relativistic mass, and (e) the momentum of the electron.

*41. From the solar constant, determine the total energy transmitted by the sun per second. How much mass is this equivalent to? If the mass of the sun is 1.99×10^{30} kg, approximately how long can the sun continue to radiate energy?

*42. A reference frame is accelerating away from a rest frame. Show that Newton's second law in the form $F = ma$ does not hold in the accelerated frame.

Interactive Tutorials

43. *Length contraction*. The length of a rod at rest is found to be $L_0 = 2.55$ m. Find the length L of the rod when observed by an observer in motion at a speed $v = 0.250c$.

44. *Time dilation*. A clock in a moving rocket ship reads a time duration $\Delta t_0 = 15.5$ hr. What time elapses, Δt, on earth if the rocket ship is moving at a speed $v = 0.355c$?

45. *Relative velocities*. Two spaceships are approaching a space station, as in figure 1.15. Spaceship 1 has a velocity of $0.55c$ to the left and spaceship 2 has a velocity of $0.75c$ to the right. Find the velocity of rocket ship 1 as observed by rocket ship 2.

46. *Relativistic mass*. A mass at rest has a value $m_0 = 2.55$ kg. Find the relativistic mass m when the object is moving at a speed $v = 0.355c$.

47. *Plot of length contraction and mass change versus speed*. The length of a rod at rest is $L_0 = 1.00$ m and its mass is $m_0 = 1.00$ kg. Find the length L and mass m of the rod as its speed v in the axial direction increases from $0.00c$ to $0.90c$, where c is the speed of light ($c = 3.00 \times 10^8$ m/s). Plot the results.

48. *An accelerated charged particle*. An electron is accelerated from rest through a potential difference $V = 4.55 \times 10^5$ V. Find (a) the kinetic energy of the electron, (b) the rest mass energy of the electron, (c) the total relativistic energy of the electron, (d) the speed of the electron, (e) the relativistic mass of the electron, and (f) the momentum of the electron.

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