General Physics: Mechanics
Physics 211
Lecture Notes

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Contents

Introduction

The following lecture notes are prepared based on the textbook "Physics for Scientists and Engineers," Fourth Edition, by Giancoli for use in PHY211, General Physics: Mechanics at Penn State University, Hazleton Campus. They are only a guide and should not be used as a substitute for the text or for attending the lecture.

If you choose to print these lectures, I have double-spaced the text for you to insert notes. Feel free to send me any comments or corrections you may have.

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This course is a calculus-based introduction to classical mechanics, including topics ranging from measurement to gravitational forces and oscillations. This course is designed to provide students with a working knowledge of these elementary physics principles, as well as their applications, and to enhance their conceptual understanding of physical laws. The introduction of data acquisition and analysis methods will be stressed in a laboratory setting. The syllabus can be downloaded at the course website located at `www.david-starling.com`.

We will use the text **"Fundamentals of Physics," Volume 1, Ninth Edition by Halliday and Resnick**. If you plan to take PHY 212, you may consider getting the Extended edition, which includes both volumes.

Chapter 1

Measurement

1.1 Opening Question 1

How is physics different from other sciences (e.g., chemistry, biology, mathematics)?

1.2 Opening Question 2

Which of the following are equivalent to 8.0 m^2?

(a) 8.0×10^{-4} cm^2

(b) 8.0×10^{2} cm^2

(c) 8.0×10^{-2} cm^2

(d) 8.0×10^{4} cm^2

(e) 8.0×10^{3} cm^2

Answer: (d) since $(100\text{cm}/1\text{m})^2 = 10^4\ \text{cm}^2/\text{m}^2$.

1.3 Model vs. Theory vs. Law

- Model - a useful "picture" of how something behaves (e.g., atomic models)
- Theory - an explanation of why something happens, often with testable predictions
- Law - a concise but general statement about how nature behaves (like $F = ma$; is prescriptive vs. descriptive)

1.4 Measurement Concepts

When a scientist makes a measurement, there is always some uncertainty. Consider a ruler. The result of a measurement: e.g., 8.8 ± 0.1 cm. The percent uncertainty is given by

$$\frac{0.1}{8.8} \times 100\% \approx 1\%. \tag{1.1}$$

If unspecified, we assume one or two units of the last digit, so 8.8 cm $\rightarrow$ 8.8 ± 0.1 or 8.8 ± 0.2 cm. What if it is written as 8.80? Must consider:

Significant Figures

number	sig figs
8.8	2
8.80	3
0.8	1
0.80	2
8.0008	5
80	1 or 2
80.	2
80.00	4

Rounding to the proper number of significant figures is important. For example,

$$A = lw = 11.3 \text{ cm} \times 6.8 \text{ cm} = 76.84 \text{ cm}^2 = 77 \text{ cm}^2. \tag{1.2}$$

Why? Well...

$$A_{min} = 11.2 \text{ cm} \times 6.7 \text{ cm} = 75.04 \text{ cm}^2 \tag{1.3}$$

$$A_{max} = 11.4 \text{ cm} \times 6.9 \text{ cm} = 78.66 \text{ cm}^2 \tag{1.4}$$

It therefore makes sense to say 77 cm^2.

When adding, subtracting, multiplying or dividing, keep as many significant figures as the number with the least number of significant figures. Also, be careful with calculators. They give too many and too few significant figures. Consider: $2.5 \times 3.2 = 8.0$. A calculator would return "8."

Sometimes, a number is too large or small to be written easily. In this case, we use Scientific Notation. Consider the speed of light, defined to be 299,792,458 m/s. We will typically write this as 3×10^8 m/s. We move the decimal point as may times as the exponent says:

number	order of magnitude
$3 \times 10^0 = 3 \times 1 = 3$	1
$3 \times 10^1 = 3 \times 10 = 30$	10
$3 \times 10^2 = 3 \times 100 = 300$	100
$3 \times 10^5 = 3 \times 100000 = 300000$	100000
$3 \times 10^{-2} = 3 \times 0.01 = 0.03$	10^{-2}

Accuracy vs. Precision

- Precision - how well can I repeat a measurement with a given apparatus?
- Accuracy - how close is my measurement result to the actual value?
- Uncertainty includes both accuracy and precision

1.5 Units

Every measurable quantity must have a unit or multiple units associated with it. Consider the seven base units in the S.I. (Systeme International):

quantity	unit	abbreviation
Length	meter	m
Time	second	s
Mass	kilogram	kg
Electric Current	ampere	A
Temp	kelvin	K
Amount of Substance	mole	mol
Luminous intensity	candela	cd

Each of these is defined based on some physical property (a meter is how far light travels in 1/299792458 of a second, and 1 second is defined to be the time that it takes a cesium atom to oscillate 9192631770 times between its ground states).

Derived quantities are things like velocity (m/s) and force (kg m/s^2).

Unit conversion is how we switch from S.I. to the British system, for example 1 in = 2.54 cm, so 1 = 2.54 cm/in. We can insert "1" into an equation to convert things, like height: 69 in = 69 in $\times$ 1 = 69 in $\times$ 2.54 cm/in = 175.26 cm = 180 cm.

1.6 Dimensional Analysis

The dimensions of a quantity are the base units that make it up. In this class, we will focus on

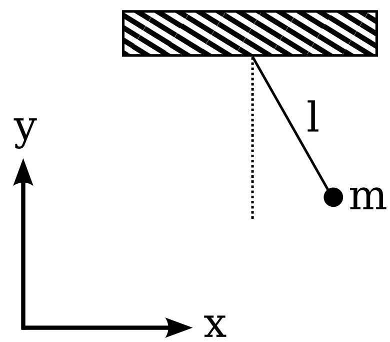
(a) Length [L]

(b) Time [T]

(c) Mass [M]

For example, area → [L^2] and velocity → [L]/[T]. We can use dimensional analysis to check our solutions.

Consider a pendulum:



We want to determine the period of the pendulum. We suspect it depends on the lenght l, the mass m and the acceleration due to gravity g. Therefore, we let

$$T \propto l^a g^b m^c, \tag{1.5}$$

where a, b and c are constants. To find them, we look at the units:

$$[\mathrm{T}] = [\mathrm{L}]^a([\mathrm{L}]/[\mathrm{T}^2])^b[\mathrm{M}]^c \tag{1.6}$$

We immediately see that $c = 0$ and find that $a + b = 0$ and $-2b = 1$. This gives $a = 1/2$ and $b = -1/2$. Therefore,

$$T \propto \sqrt{l/g}. \tag{1.7}$$

We will later find that this is true, and that the proportionality constant is simply 2π.

Chapter 2

Motion in 1D

2.1 Opening Question

A race car, traveling at constant speed, makes one lap around a circular track of radius r in a time t. Which of the following statements concerning this car is true?

(a) The displacement of the car does not change with time.

(b) The instantaneous velocity of the car is constant.

(c) The average speed of the car is the same over any time interval.

(d) The average velocity of the car is the same over any time interval.

(e) The average speed of the car over any time interval is equal to the magnitude of the average velocity over the same time interval.

2.2 Mechanics

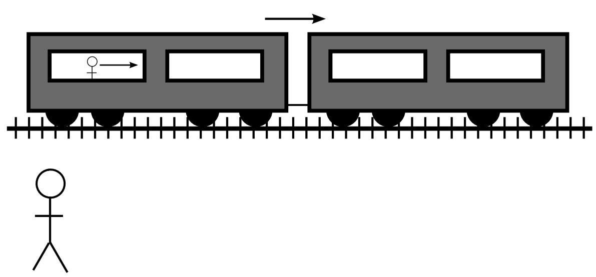
"Mechanics" is split into two parts:

- 1) kinematics, the motion of objects
- 2) dynamics, why objects move (forces)

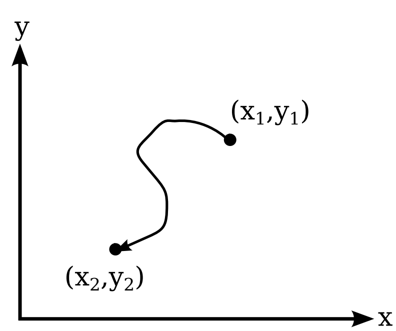
We begin with translation (no rotation) in one dimension.

Reference Frames

Position, distance and speed are quantities that require a reference frame; we measure them with respect to (w.r.t.) a particular frame of reference.

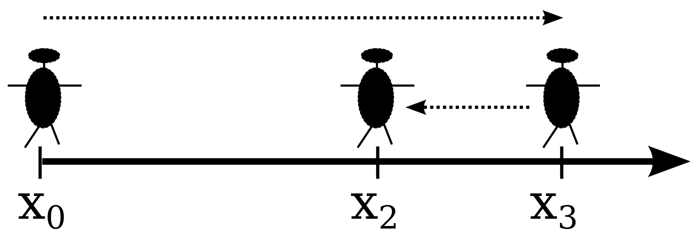


Coordinate axes - always write down for every problem in kinematics.



Displacement is how far an object has moved from its initial position, and the *direction* it moved.

Bob walks down the street, 10 m to the east, then 4 m to the west.



We write

- $x_0 = 0$ m
- $x_1 = 10$ m
- $x_2 = 10 - 4 = 6$ m

Bob has displaced 6 m to the east (magnitude, and direction). The **distance** Bob has traveled is $10 + 4 = 14$ m. In general,

$$\Delta x = x_f - x_i \tag{2.1}$$

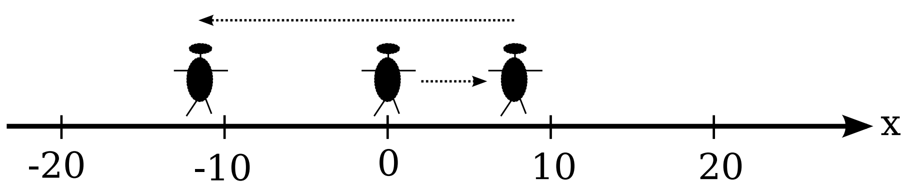
is the displacement = (final position - initial position). Positive/negative determines direction relative to the axis we have chosen.

2.3 Motion

- **speed** - the rate at which an object moves
- **average speed** $= \frac{\text{distance traveled}}{\text{time elapsed}} > 0$

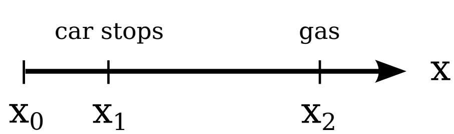
- **velocity** - includes the magnitude and direction of the motion
- **average velocity** $= \frac{\text{displacement}}{\text{time elapsed}} = \frac{\Delta x}{\Delta t}$

example: Alice walks 7.0 ft to the right, then 18 ft to the left, in 10 seconds.



- distance: $|7-0|+|-11-7|=7+18=25$ ft
- displacement: $-11-0=-11$ ft (notice the negative sign)
- time: 10 s
- average speed: $s_{avg}=25/10=2.5$ ft/s
- average velocity: $v_{avg}=-11/10=-1.1$ ft/s

example: You drive a car along a straight road for 8.4 km at 70 km/hr where is runs out of gas. Over the next 30 minutes, you walk another 2.0 km farther along the road to a gas station.

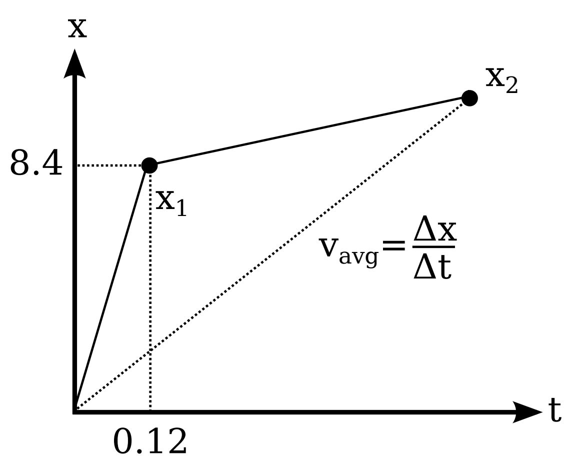


(a) What is your total displacement? $\Delta x = x_2 - x_0 = (8.4+2.0) - 0 = 10.4$ km

(b) How long does this take? Total time is $t = t_{drive} + t_{walk}$. We know t_{walk}. t_{drive} is given by $t_{drive} = \Delta x_{drive}/v_{avg} = 0.12$ hr. $t = 0.62$ hr.

(c) What is your average velocity for this trip? $v_{avg} = \Delta x/\Delta t = (8.4 + 2 - 0)/0.62 = 16.8 = 17$ km/hr.

(d) Graph the position as a function of time



Problem Solving

(a) Read and understand the problem - draw a picture!

(b) Identify and record known and unknown data

(c) Identify appropriate principles and equations

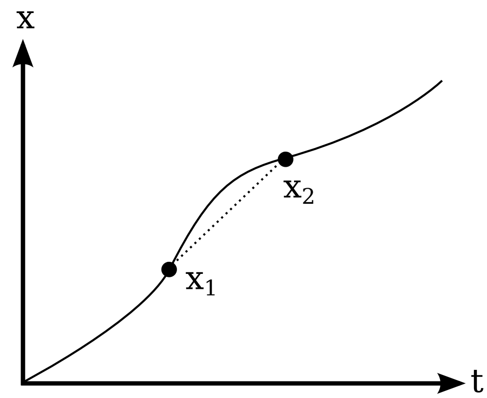
(d) Using symbols (variables), work the problem

(e) Calculate the result and ask: is the answer reasonable?

2.4 Instantaneous Velocity and Speed

We are often interested in the velocity or speed at a specific moment in time, not just the average.

$$v_{avg} = \frac{\Delta x}{\Delta t} \tag{2.2}$$

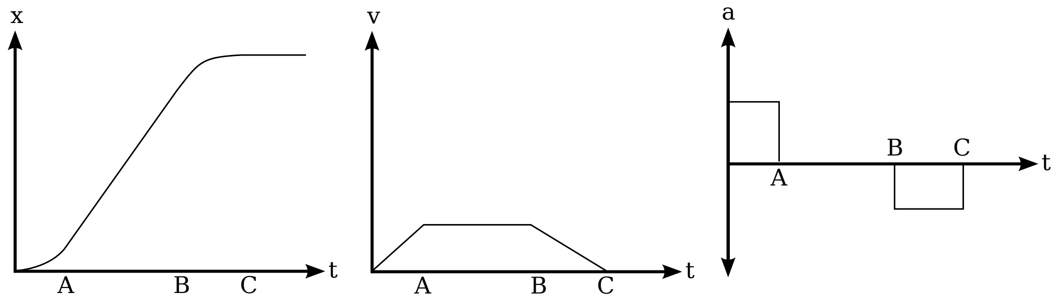


But now, let's shrink Δt to zero. This gives us the derivative:

$$v = \frac{dx}{dt}, \tag{2.3}$$

which is related to the slope of the line. Remember, velocity can be negative.

example: Consider a car moving in one dimension along a straight road. Its motion as a function of time is described by the following graphs:



Derivatives give you the slope of a curve: "rise over run" (v is distance over time, etc). In the case of time, it's a *rate of change*.

$$v = \frac{dx}{dt} \text{ rate of change of } x \tag{2.4}$$

$$a = \frac{dv}{dt} \text{ rate of change of } v. \tag{2.5}$$

Integrals are derivatives in reverse and give you the area under a curve, which is related to a summation.

$$x_1 - x_0 = \int_{t_0}^{t_1} v(t)dt \tag{2.6}$$

$$v_1 - v_0 = \int_{t_0}^{t_1} a(t)dt \tag{2.7}$$

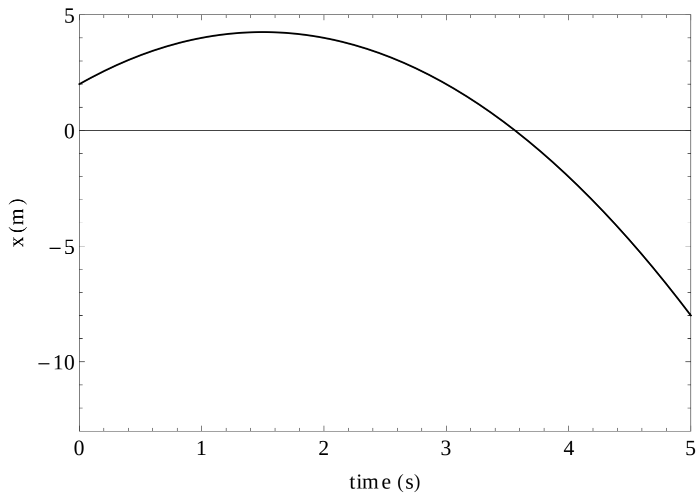
example: The position of a particle in one dimension is given by $x(t) = 2 + 3t - t^2$ m, where t is in seconds.

(a) find $v(t)$ (instantaneous velocity)

(b) find v at $t = 3/2$ s

(c) find $a(t)$

(d) Try to find $x(t)$ from $v(t)$



answer:

(a) $v(t) = \frac{dx}{dt} = 3 - 2t$

(b) $v(3/2) = 0$

(c) $a = dv/dt = -2$ (constant acceleration)

(d) $x = \int v dt = \int (3 - 2t) dt = 3t - t^2 + C$ or...

$x(t) - x(0) = \int_0^t v dt' = \int_0^t (3 - 2t') dt' = 3t - t^2$

Apparently we need some more information, namely $x(0)$.

2.5 Acceleration

When the velocity of a particle changes, it undergoes acceleration.

$$a_{avg} = \frac{v_1 - v_0}{t_1 - t_0} = \frac{\Delta v}{\Delta t} \tag{2.8}$$

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \tag{2.9}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{2.10}$$

Acceleration is the slope of the v vs. t graph, and is the rate of change of v. If $v(t)$ increases in time, $a > 0$. If $v(t)$ decreases in time, $a < 0$. The units of $a(t)$ are $[L]/[T^2]$, e.g., m/s^2.

example: The position of a particle is given by $x(t) = 4 - 27t + t^3$.

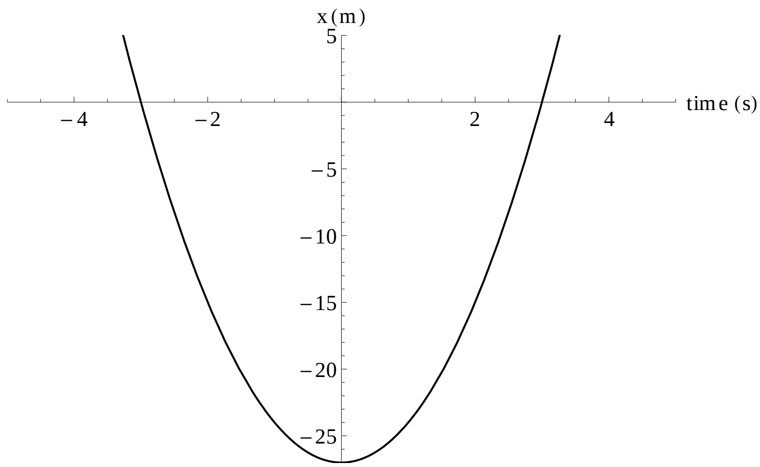
(a) find $v(t)$ and $a(t)$

(b) find the time when $v = 0$

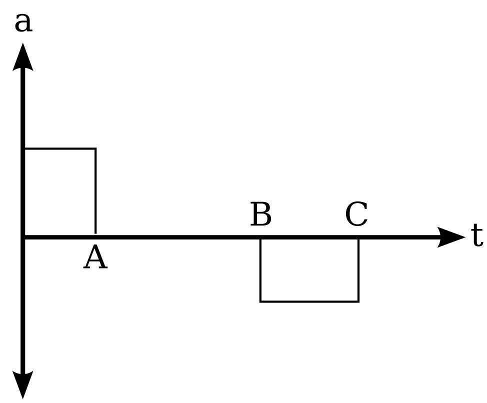
answer:

(a) $v(t) = \frac{dx}{dt} = -27 + 3t^2$ m/s and $a(t) = \frac{dv}{dt} = 6t$ m/s^2

(b) $v(t) = 0 = -27 + 3t^2 \rightarrow t = \pm 3$ s. Two solutions?



Consider the car from before, which starts and then stops. Note that $a = dv/dt$ implies $v_1 - v_0 = \int_{t_0}^{t_1} a dt$ = area under the curve. Since $v_1 = v_0 = 0$, the area must be zero!



2.6 Constant Acceleration

Three important equations:

$$a_{avg} = a = \frac{\Delta v}{\Delta t} = \frac{v - v_0}{t - t_0} \rightarrow a(t - t_0) = v - v_0 \tag{2.11}$$

This gives us our first equation:

$$v(t) = v_0 + a(t - t_0) = v_0 + at \tag{2.12}$$

which is true only for constant acceleration and with $v = v_0$ at the initial time $t = 0$. Also, since $v = dx/dt$, we can integrate to get our second equation:

$$\begin{aligned}
\int_{x_0}^{x} dx' &= \int_{t_0}^{t} v(t')dt' = \int_{t_0}^{t} (v_0 + at')dt' && (2.13)\\
x - x_0 &= v_0 t + \frac{1}{2}at^2, && (2.14)
\end{aligned}$$

assuming $t_0 = 0$. I often write this as $x(t) = x_0 + v_0 t + \frac{1}{2}at^2$.

What if we don't know t? We can eliminate it by solving the first equation for t and putting it into the second equation:

$$\begin{aligned}
t &= \frac{v - v_0}{a} && (2.15)\\
x - x_0 &= v_0\left(\frac{v - v_0}{a}\right) + \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 && (2.16)\\
2a(x - x_0) &= 2v_0(v - v_0) + (v - v_0)^2. && (2.17)
\end{aligned}$$

Changing the order of the terms gives us

$$v(t)^2 = v_0^2 + 2a(x - x_0). \tag{2.18}$$

To use these equations, we first identify the known and unknown quantities.

example: Alice applies the brake in a car, starting at 100 km/hr and slowing to 80 km/hr in 88 m at a constant acceleration.

(a) What is a?

(b) How long did this take?

answer: First, identify knowns and unknowns and convert to S.I. units.

knowns	unknowns
$v_0 = 100$ km/hr $= 27.8$ m/s	$t_f = ???$
$v_f = 80$ km/hr $= 22.2$ m/s	$a = \text{constant} = ???$
$t_0 = 0$	
$x_f - x_0 = 88$ m	

(a) $v_f^2 = v_0^2 + 2a(x_f - x_i) \rightarrow a = \frac{v_f^2 - v_0^2}{2(x_f - x_i)} = -1.6$ m/s^2 (seems reasonable)

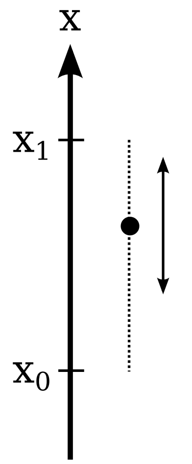
(b) $v_f = v_0 + at \rightarrow t = \frac{v_f - v_0}{a} = 3.5$ s

2.7 "Freely" Falling Objects

example: If a ball is projected vertically upward w/ a velocity of v_0, what is

(a) the maximum height is goes?

(b) and how long does this take?



answer: First, is this a constant acceleration problem? Yes!

At a given location on Earth and in the absence of air resistance, all objects fall with the same acceleration g.

So lets identify our knowns and unknowns:

<u>knowns</u>	<u>unknowns</u>
v_0	$\Delta x = ???$
$v_f = 0$	$t = ???$
$a = -g$	

(a) $v_f^2 = v_0^2 + 2a\Delta x \rightarrow \Delta x = h = \frac{v_f^2 - v_0^2}{-2g} = \frac{v_0^2}{2g}$

(b) $v_f = v_0 + at \rightarrow 0 = v_0 - gt \rightarrow t = v_0/g > 0.$

The ball's position is just $x(t) = x_0 + v_0 t + \frac{1}{2}at^2 = v_0 t - \frac{g}{2}t^2$. This is a downward facing parabola with a max of height of $v_0^2/2g$ at time v_0/g.

Chapter 3

Vectors

3.1 Opening Question

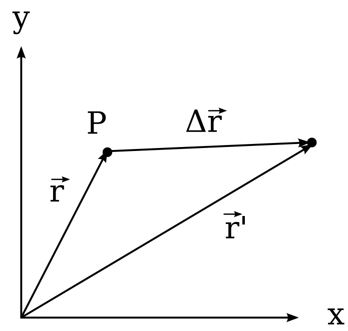
What are the differences between scalars and vectors?

Scalars are physical quantities with magnitude and no direction, e.g., distance and speed.

Vectors are physical quantities with magnitude *and* direction, e.g., displacement and velocity.

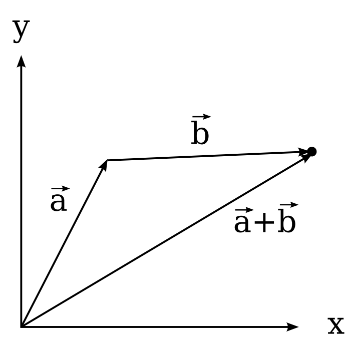
3.2 Vectors

example: position and displacement vectors



- $\vec{r}$ is in the initial position of particle P
- $\vec{r'}$ is the final position
- $\Delta\vec{r} = \vec{r'} - \vec{r}$ is the displacement

Adding vectors $\vec{a}$ and $\vec{b}$ looks like:



with properties

(a) Commutative: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(b) Associative: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

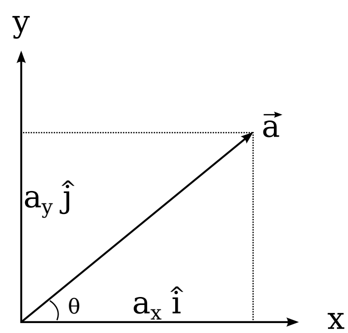
(c) Additive inverse exists: $\vec{a} + (-\vec{a}) = \vec{0}$

(d) Distributive: $m \times (\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

Unit vectors (length one)

- $\hat{i}$ is a vector of unit length along x-axis
- $\hat{j}$ is a vector of unit length along y-axis
- $\hat{k}$ is a vector of unit length along z-axis
- $\hat{a}$ is a vector of unit length along the vector $\vec{a}$

Components of a vector



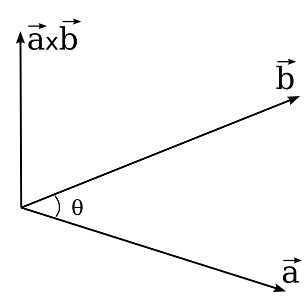
- a_x = projection of $\vec{a}$ on x-axis
- a_y = projection of $\vec{a}$ on y-axis
- a_z = projection of $\vec{a}$ on z-axis
- $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$
- $a = |a| = \sqrt{a_x^2 + a_y^2}$ length/magnitude of $\vec{a}$
- $a_x = a\cos(\theta)$ (SOH CAH TOA)
- $a_y = a\sin(\theta)$

- $\tan(\theta) = a_y/a_x$ is angle with x-axis

Scalar Product (dot product) - useful for work calculations

- $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$
- Commutative: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- If $\vec{a} = a_x\hat{i}+a_y\hat{j}+a_z\hat{k}$ and $\vec{b} = b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$, then $\vec{a}\cdot\vec{b} = a_xb_x+a_yb_y+a_zb_z$
- Dot product of basis vectors: $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$
- To find θ, just solve for it: $\cos(\theta) = \vec{a} \cdot \vec{b}/ab$

Vector Product (cross product) - useful for torque calculations



- $|\vec{a} \times \vec{b}|$ = Area of a parallelogram
- $|\vec{a} \times \vec{b}| = ab\sin(\theta)$
- Right Hand Rule to determine direction of $\vec{a} \times \vec{b}$
- Direction of $\vec{a} \times \vec{b}$ is perpendicular to both vectors

- Anti-commutative: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

- To compute components, use determinants: $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} =$
  $\hat{i}(a_y b_z - a_z b_y) - \hat{j}(a_x b_z - a_z b_x) + \hat{k}(a_x b_y - a_y b_x)$

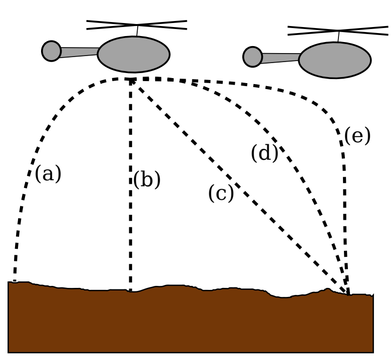
- Alternatively, we can note that $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$ and, using the right-hand-rule, $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$ and $\hat{k} \times \hat{i} = \hat{j}$.

Chapter 4

Motion in Higher Dimensions

4.1 Opening Question

A small heavy box of emergency supplies is dropped from a moving helicopter at a point A as it flies along in a horizontal direction. Which path in the drawing below best describes the path of the box?



4.2 Important Quantities

Position and Displacement

Particle P is at position 1, then moves to position 2. We write this as:

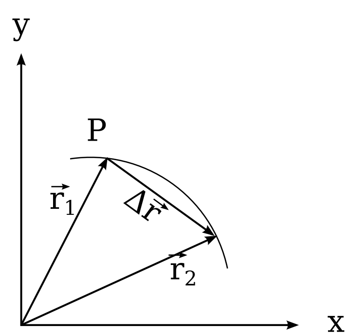
$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \tag{4.1}$$

$$\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}, \tag{4.2}$$

Before, the change in position (displacement) was just $\Delta x = x_2 - x_1$. In 3D, we have

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \tag{4.3}$$

$$= \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}. \tag{4.4}$$



Average Velocity

Before, we had $v_{avg} = \Delta x/\Delta t$. Now, we replace Δx with $\Delta\vec{r}$:

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k} \tag{4.5}$$

Instantaneous Velocity

Before, we had $v = \lim_{\Delta t \to 0} \Delta x/\Delta t = dx/dt$. Now, we replace Δx with $\Delta\vec{r}$:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{x}}{dt}\hat{i} + \frac{d\vec{y}}{dt}\hat{j} + \frac{d\vec{z}}{dt}\hat{k} = \frac{d\vec{r}}{dt} \tag{4.6}$$

So, as expected, $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$. Its direction is given by $\Delta\vec{r}$ in the limit of $\Delta t \to 0$.

Average Acceleration

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta\vec{v}_x}{\Delta t}\hat{i} + \frac{\Delta\vec{v}_y}{\Delta t}\hat{j} + \frac{\Delta\vec{v}_z}{\Delta t}\hat{k} \tag{4.7}$$

Instantaneous Acceleration

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}_x}{dt}\hat{i} + \frac{d\vec{v}_y}{dt}\hat{j} + \frac{d\vec{v}_z}{dt}\hat{k} = \frac{d\vec{v}}{dt} \tag{4.8}$$

The acceleration vector $\vec{a}$ depends on the rate of change of $\vec{v}$, both in magnitude *and* direction.

First consider a change in the magnitude of $\vec{v}$, so that $\vec{v} = v_x(t)\hat{i}$. Then $\vec{a} = d\vec{v}/dt = \hat{i}dv_x/dt$. That is, if a particle speeds up, its acceleration is in the same direction as the velocity vector.

Now consider a change in direction of $\vec{v}$, with $\vec{v} = v_0\hat{v}$. The unit vector $\hat{v}$ is a function of time. When we take the derivative, we find

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(v_0\hat{v})}{dt} = v_0\frac{d\hat{v}}{dt}. \tag{4.9}$$

What is the direction of $d\hat{v}/dt$?

example: The position of a particle is given by $\vec{r}(t) = (7-3t^3)\hat{\imath}+4\hat{\jmath}+\ln(x)\hat{k}$ m. Find $\vec{v}(t)$ and $\vec{a}(t)$.

Constant Acceleration Equations in 3D

(a) $\vec{v}(t) = \vec{v}_0 + \vec{a}t$

(b) $\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$

example: Two particles 1 and 2 move with constant velocities $\vec{v}_1$ and $\vec{v}_2$. At the initial moment their position vectors are $\vec{r}_1$ and $\vec{r}_2$. What is the condition required for a collision?

Answer: At some time t later, they must have the same position, i.e. $\vec{r}_3 = \vec{r}_1 + \vec{v}_1 t = \vec{r}_2 + \vec{v}_2 t$. Solve for t to get

$$t = \frac{|\vec{r}_1 - \vec{r}_2|}{|\vec{v}_1 - \vec{v}_2|}. \tag{4.10}$$

example: You toss a sharp knife to your friend on the ground from the roof of a house. He stands 5 meters below and 1 meter away from the house. What velocity is required to make sure the knife is caught 2 seconds later?

Knowns: $\vec{r}_0 = 1\hat{\imath} + 5\hat{\jmath}$ m, $\vec{a} = -9.8\hat{\jmath}$ m/s^2, $\vec{r}(t=2) = 0\hat{\imath} + 0\hat{\jmath}$ m, $t = 2$. The unknown is just $\vec{v}_0$.

This gives:

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \tag{4.11}$$

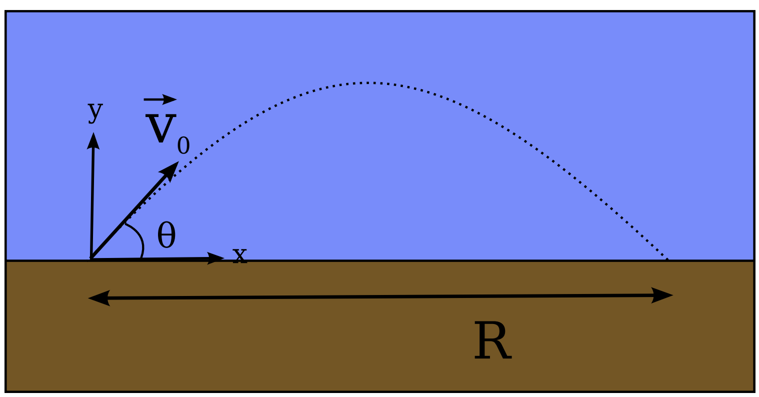
$$0\hat{\imath} + 0\hat{\jmath} = (1\hat{\imath} + 5\hat{\jmath}) + \vec{v}_0(2) + \frac{1}{2}(-9.8)\hat{\jmath}2^2 \tag{4.12}$$

$$\vec{v}_0 = [(-\hat{\imath} - 5\hat{\jmath}) + 4.5 \times 4\hat{\jmath}]/2 \tag{4.13}$$

$$= -0.5\hat{\imath} + 6.5\hat{\jmath} \text{ m/s} \tag{4.14}$$

4.3 Projectile Motion

In projectile problems, the horizontal and vertical motions are independent of each other. The observed motion is a combination of both.



The horizontal motion is given by $(x - x_0) = v_{0x}t$ (constant velocity). The vertical motion is given by $(y - y_0) = v_{0y}t - \frac{1}{2}gt^2$, $v_y = v_{0y} - gt$ and $v_y^2 = v_{0y}^2 - 2g\Delta y$. Note the negative sign, since positive y is typically taken to be upward, and $g > 0$.

Let us consider an object with an initial position of $(x_0, y_0) = (0, 0)$, the components of $\vec{v}_0$ to be $v_{0x} = v_0\cos(\theta)$ and $v_{0y} = v_0\sin(\theta)$, and solve for y in terms of x by eliminating t.

$$\begin{aligned} y &= v_{0y}t - \frac{1}{2}gt^2 && (4.15) \\ &= v_0\sin(\theta)\frac{x}{v_0\cos(\theta)} - \frac{1}{2}g\left(\frac{x}{v_0\cos(\theta)}\right)^2 && (4.16) \\ &= \tan(\theta)x - \frac{g}{2(v_0\cos(\theta))^2}x^2. && (4.17) \end{aligned}$$

This is an equation of a parabola $y = ax + bx^2$. This is what we see when an object flies through the air.

Horizontal Range R can be calculated by finding the position x at which y has become 0, i.e.,

$$\begin{aligned} R &= v_0\cos(\theta)t \text{ (horizontal)} && (4.18) \\ 0 &= v_0\sin(\theta)t - \frac{1}{2}gt^2 \text{ (vertical)} && (4.19) \end{aligned}$$

We can get rid of $t(= R/v_0\cos(\theta))$ and solve for R, giving

$$R = 2\sin(\theta)\cos(\theta)v_0^2/g = \frac{v_0^2}{g}\sin(2\theta). \qquad (4.20)$$

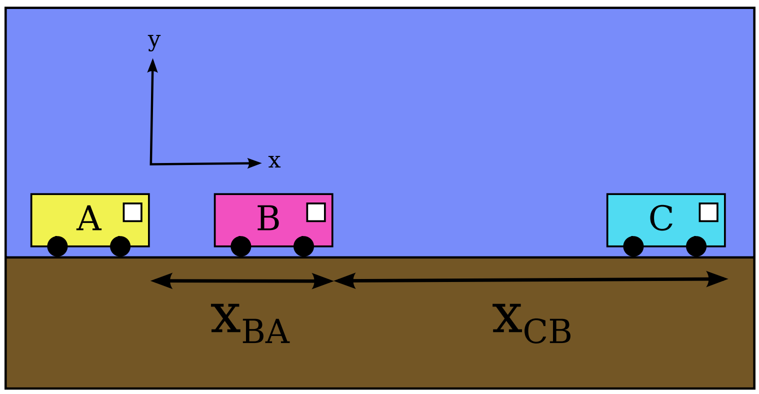
Horizontal range is therefore maximum when $\theta = 45°$.

example: A boy on a small hill aims his water-balloon slingshot horizontally, straight at a second boy hanging from a tree branch a distance d away. At the second the water-balloon is released, the second boy lets go and falls from the tree, hoping to avoid being hit. Was this a good strategy?

No! Both the balloon and the boy always have the same y-position since they have the same acceleration and initial v_{0y} (zero).

4.4 Relative Motion

In one dimension, consider three cars A, B and C.



- Car A is at rest
- Car B is at a distance of x_{BA} from A
- Car C is at a distance of x_{CB} from B

Therefore, assuming space isn't "warped," we know that $x_{CA} = x_{BA} + x_{CB}$. Taking the derivative with respect to time, we get $v_{CA} = v_{BA} + v_{CB}$. Taking the derivative again, we get $a_{CA} = a_{BA} + a_{CB}$. If Car B is not accelerating with respect to A, then $a_{CA} = a_{CB}$.

In higher dimensions, everything is the same.

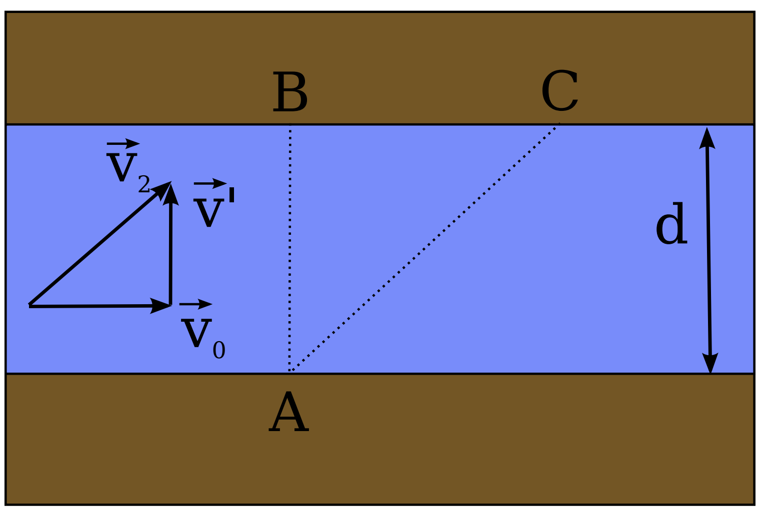
- $\vec{r}_{CA} = \vec{r}_{BA} + \vec{r}_{CB}$

- $\vec{v}_{CA} = \vec{v}_{BA} + \vec{v}_{CB}$

- $\vec{a}_{CA} = \vec{a}_{BA} + \vec{a}_{CB}$

We usually think of car B as the "moving reference frame," and car C as the object we want to study.

example: Two swimmers leave point A to meet at point B, directly across the river d meters away. Swimmer 1 swims directly there, despite a strong current of 2.0 km/hr. Swimmer 2 lets the current carry him and puts all of his effort into crossing the river, arriving at point C. He then must walk the distance back to point B. Both swimmers have a maximum swimming speed of 2.5 km/hr (with respect to the water). If they arrive at point B at the same time, what must his walking speed u be?



Answer: This question relies on the concept of *relative* velocity. We must add the velocity components of each swimmer in the correct way to get the resultant velocity. For the swimmers, call $\vec{v}_{1,2}$ the velocity of swimmer 1 (2) with respect to the ground, $\vec{v}'$ the velocity of the swimmers with with respect to the water, and $\vec{v}_0$ the velocity of the current with respect to the ground.

We find that, in either case, $\vec{v}' + \vec{v}_0 = \vec{v}_{1,2}$, but the direction of $\vec{v}'$ is different for each swimmer. For swimmer 1, the total time to reach B is just $t = d/\sqrt{v'^2 - v_0^2}$.

For swimmer 2, his trip is composed of two times, $t = t_1 + t_2$. The first part of the trip takes $t_1 = d/v'$. The second part of his trip takes $t_2 = L/u$, where L is how far down stream he was carried, which is $L = v_0 t_1 = v_0 d/v'$. Therefore,

$$
\begin{aligned}
t &= t_1 + t_2 && (4.21)\\
\frac{d}{\sqrt{v'^2 - v_0^2}} &= \frac{d}{v'} + \frac{v_0 d}{v' u} && (4.22)\\
u &= \frac{v_0}{\sqrt{\frac{1 - v_0^2}{v'^2}} - 1}. && (4.23)
\end{aligned}
$$

Relative motion part: if we observe them from a boat, floating along with the current, what would we see? $\vec{v}_1 = \vec{v}_0 + \vec{v}_s$, where

- $\vec{v}_1$ is w.r.t. the ground
- $\vec{v}_s$ is w.r.t. the boat
- $\vec{v}_0$ is boat w.r.t. the ground

So $\vec{v}_s = \vec{v}_1 - \vec{v}_0 = (\vec{v}' + \vec{v}_0) - \vec{v}_0 = \vec{v}'$. That is, from the boat, we only see the velocity of the swimmer, ignoring the current. Both boat and swimmer are in the same reference frame.

Chapter 5

Dynamics: Newton's Laws

5.1 Opening Question

A 150 kg football player collides head-on with a 75 kg running back. During the collision, the heavier player exerts a force of magnitude F_A on the smaller player. If the smaller player exerts a force F_B back on the heavier player, which response is most accurate?

(a) $F_B = F_A$

(b) $F_B < F_A$

(c) $F_B > F_A$

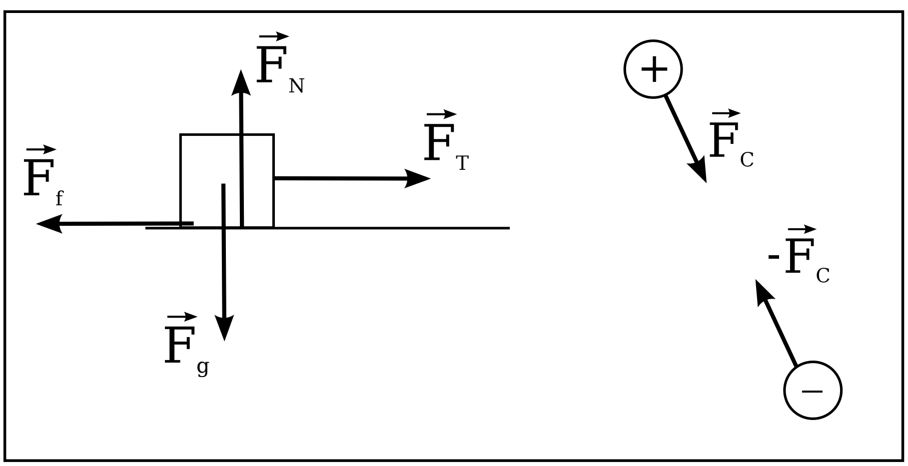
(d) $F_B = 0$

(e) We need more information.

5.2 Forces

Forces are vectors (magnitude and direction) and they come in all types; here are a few:

- Push (via contact)
- Pull (e.g., tension on a rope)
- Friction
- Gravity
- Electric
- Magnetic



Why are forces important? What do they do?

5.3 Newton's First Law

Dynamics is the study of what causes acceleration. Newton was the first person to really nail down the relationship between forces and acceleration, and this was possible because he could imagine a frictionless world.

Newton's First Law: Every body continues to remain in its state of rest or uniform motion in a straight line, unless a net external force is applied. Also known as *the Law of Inertia.* This Law is more conceptual in nature, and will help guide you when dealing with problems of uniform motion.

Sometimes we will say "inertial reference frame" to indicate a reference frame which is in uniform motion, i.e., $a = 0$. Newton's laws are only valid in inertial reference frames.

5.4 Newton's Second Law

Newton's Second Law: A specification of the first Law which states that the net force applied to an object is proportional to its acceleration, where the constant is given by its mass m:

$$\sum_i \vec{F}_i = m\vec{a}. \tag{5.1}$$

Things to note:

- This is a vector equation, i.e., three equations in one!
- $\vec{F}_1 = F_{1x}\hat{\imath} + F_{1y}\hat{\jmath} + F_{1z}\hat{k}$

- Each component satisfies $F = ma$ separately
- There is only one mass and one acceleration for an object
- We sum all the forces
- "Weight" is the force due to gravity, equal to $W = mg$

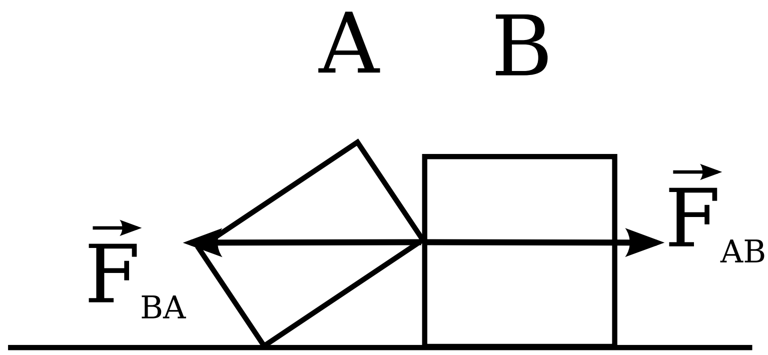
A **normal force** is a force applied by a surface to an object to keep the object in place; this force is perpendicular (normal) to the surface. For instance, when the earth pulls you down toward the ground, the ground pushed up on your feet. This "ground to foot" force is a normal force.

It's very important to separate motion from acceleration in your mind. Examples:

- Lifting a box from the ground
- Riding and elevator
- Sliding a canoe along a sandy shore

5.5 Newton's Third Law

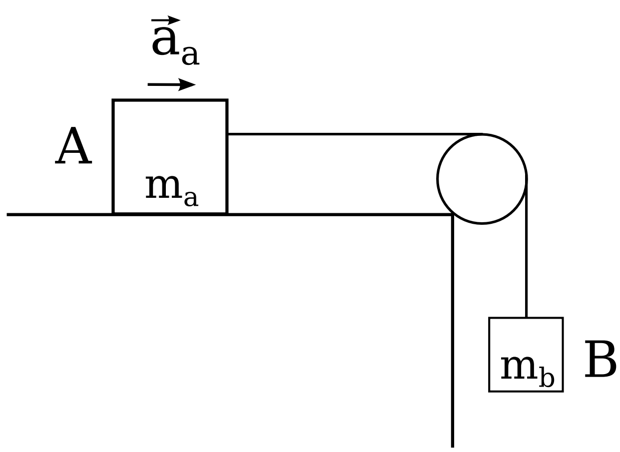
Every action has an equal and opposite reaction. Said another way, every force is associated with an equal and opposite force.



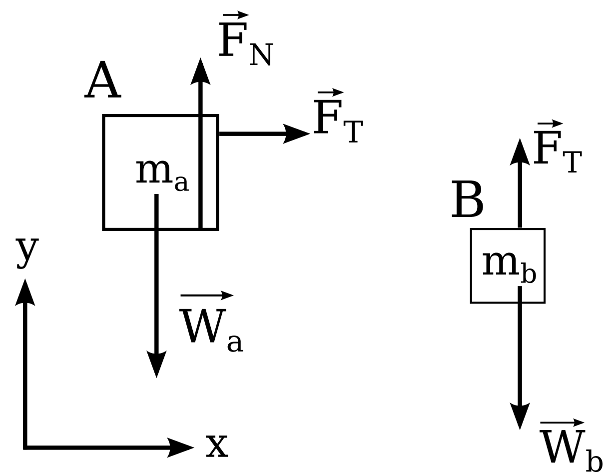
Here, $\vec{F}_{AB} = -\vec{F}_{BA}$. This is a normal force.

5.6 Applying Newton's Laws

example 1: Find the (i) acceleration of the sliding block, (ii) acceleration of the hanging block and (iii) the tension in the cord. Assume that the string does not stretch, and that the pulley is massless.



Let us first isolate each block and draw the applied forces with a coordinate axis. This is known as a "free body force diagram" and it simplifies the problem.



We then write down these forces mathematically and set them equal to $m_a a_a$ or $m_b a_b$. We have to use some logic to determine some parameters. For instance, what is the y-component of the acceleration of block A ($a_y = 0$)? Similarly, the x-component of the acceleration of block B is 0. Since there are no angles, we can easily write for block A

$$\sum_i F_{xi} = F_T = m_a a_a \tag{5.2}$$

$$\sum_i F_{yi} = F_N - W_a = 0. \tag{5.3}$$

And, for block B,

$$\sum_i F_{xi} = 0 \tag{5.4}$$

$$\sum_i F_{yi} = F_T - W_b = m_b a_b. \tag{5.5}$$

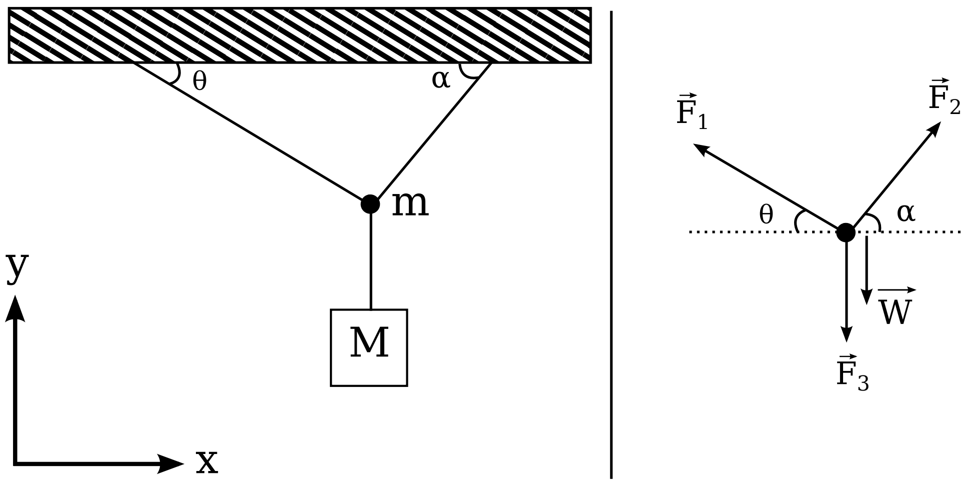
We can conclude that, since the rope does not stretch, that $a_a = -a_b = a$ (positive acceleration for each block is defined oppositely). We also know that $W_a = m_a g$ and $W_b = m_b g$. We have two unknowns (F_T and a) and two equations. Easy! We find,

$$a = \frac{m_b g}{m_a + m_b}. \tag{5.6}$$

Both blocks have same acceleration. Since they both start from rest, they also have the same velocity. Their displacement is also equal.

To find the tension F_T, we plug in our expression for a into the first equation and find $F_T = m_a m_b g/(m_a + m_b)$.

example 2: Find the tension in all three ropes.



First, note that the block is stationary, that is, $\vec{a} = \vec{0}$. Let us now separate horizontal and vertical force components to obtain two equations, then evaluate our situation.

Vertical forces: $F_1 \sin(\theta) + F_2 \sin(\alpha) - F_3 - mg = 0$ (1).

Horizontal forces: $F_2 \cos(\alpha) - F_1 \cos(\theta) = 0$ (2).

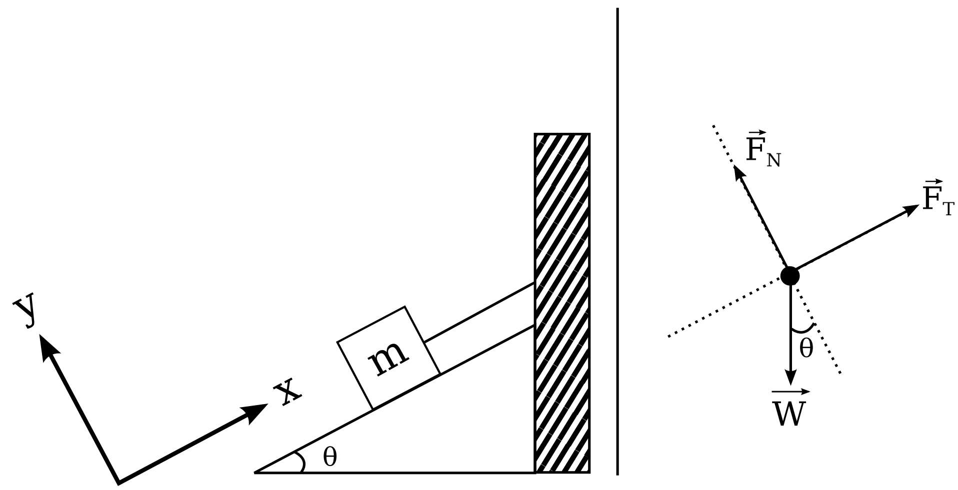
We also know that $F_3 = Mg$ is the weight of the block, from Newton's third law. We now have three equations and three unknowns (F_1, F_2 and F_3). We can use the matrix method to solve this, or by "brute force," as follows.

If we multiple Eq. 1 by $\cos(\theta)$ and Eq. 2 by $\sin(\theta)$ and then add them, we find

$$F_2(\sin(\alpha)\cos(\theta) + \cos(\alpha)\sin(\theta)) = (M + m)g\cos(\theta) \tag{5.7}$$

Therefore, $F_2 = (M + m)g\cos(\theta)/\sin(\theta + \alpha)$. We easily find that $F_3 = (M + m)g\cos(\alpha)/\sin(\theta + \alpha)$.

example 3: Find the tension in the string and the normal force on the block.

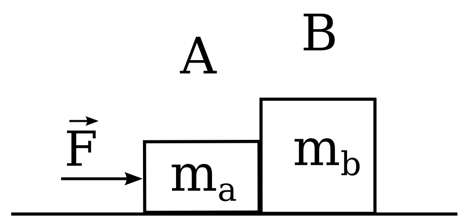


We set up the axes along the surface of the inclined plane, and normal to it. We then decompose the forces in these directions to find (noting that $\vec{a} = \vec{0}$),

$$N - mg\cos(\theta) = 0 \tag{5.8}$$

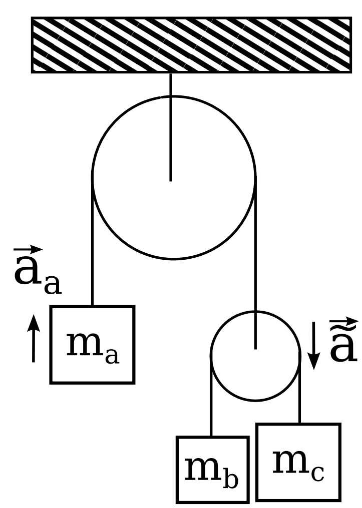
$$T - mg\sin(\theta) = 0. \tag{5.9}$$

example 4: Find the acceleration of the blocks and the force of block A on B.



This is a one dimension problem. When pushed by an amount F, the blocks will accelerate together as one object. Therefore, $F = ma = (m_a + m_b)a$, so $a = F/(m_a + m_b)$. To determine the force on block B, let's consider the only force on B, which is the normal force F_N pushing to the right. Therefore, $F_N = m_b a = m_b(F/(m_a + m_b) = Fm_b/(m_b + m_a)$. The heavier the first block is, the less force is transfered to the second block.

example 5: Consider a double Atwood machine made of frictionless, massless pulleys with rope that does not stretch. Find the acceleration of each mass.



We note that the pulley is massless, so $T_a - 2T = 0$, so the tensions in all the ropes are easily related. How are the accelerations of the the blocks related? In one case, the accelerations add, in the other, the accelerations subtract. We therefore find

$$2T - m_a g = m_a a_a \tag{5.10}$$

$$T - m_b g = m_b a_b = m_b(\tilde{a} - a_a) \tag{5.11}$$

$$T - m_c g = m_c a_c = m_c(\tilde{a} + a_a). \tag{5.12}$$

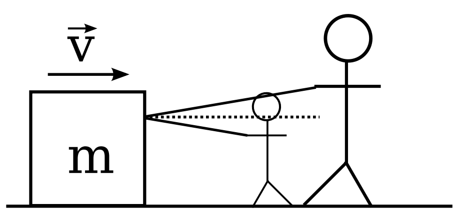
We now have three unknowns (a_a, $\tilde{a}$ and T) with three equations. The rest is math!

Chapter 6

Friction and Circular Motion

6.1 Opening Question

A short man and a tall man take turns pulling a large sleigh along a gravel road using a rope, as shown in the figure below (the angles are the same). Who must apply more force to maintain a constant speed?



(a) The tall man

(b) The short man

(c) The force is the same

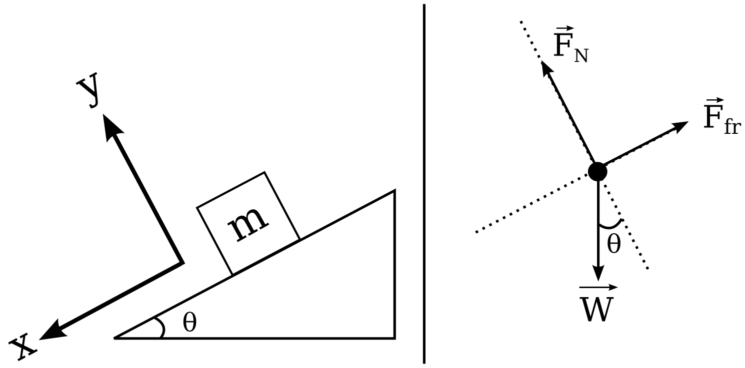
6.2 Friction

The frictional force opposes any motion and tendency of motion.

- Static friction: holds an object in place, $\vec{F}_{fr} = -\vec{F}_{applied}$
- Kinentic friction: retards motion, F_{fr}

The magnitude of the frictional force is μF_N, where μ is the coefficient of friction and F_N is the magnitude of the normal force (the force perpendicular to the surface). For stationary objects, we call μ_s the coefficient of static friction. For moving objects, μ_k is the coefficient of kinetic friction. The direction of $\vec{F}_{fr}$ is always opposite the motion.

example 1: Consider a block of mass m resting on an inclined plane with coefficients of friction μ_s and μ_k. What is the maximum angle θ before the block starts to fall? Once it begins to fall (at this angle), what is the acceleration of the block?



answer: when the mass is at rest, the equations are

$$Mg\sin(\theta) = F_{fr} \tag{6.1}$$

$$F_{fr} = \mu_s F_N \tag{6.2}$$

$$F_N - mg\cos(\theta) = 0. \tag{6.3}$$

Putting this together, we find $\tan(\theta) = \mu_s$. The angle does not depend on g or M!

If we nudge the block just a bit so it begins to fall, the equations change to

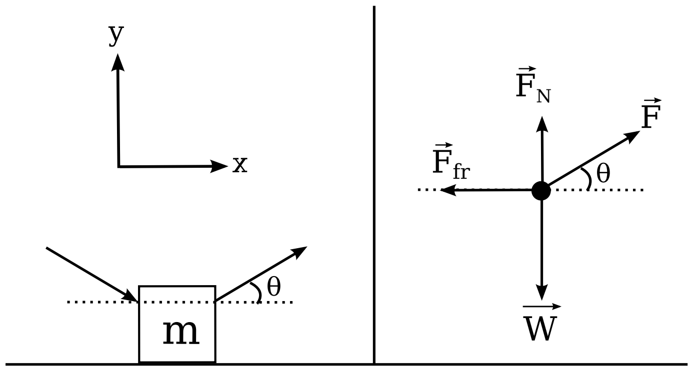
$$mg\sin(\theta) - F_{fr} = ma \tag{6.4}$$

$$F_{fr} = \mu_k F_N \tag{6.5}$$

$$F_N - mg\cos(\theta) = 0. \tag{6.6}$$

This gives us $a = g(\sin(\theta) - \mu_k\cos(\theta))$. What if the angle were smaller? What would F_{fr} be?

example 2: Consider a block of mass m resting on an a rough, horizontal surface with coefficient of friction μ_k. Is it easier to push or pull the block if you apply a force above the horizontal?



Push: the equations become

$$F\cos(\theta) - \mu_k F_N = ma \tag{6.7}$$

$$mg + F\sin(\theta) - F_N = 0. \tag{6.8}$$

Getting rid of the unknown N and solving for F, we find $F = (ma + \mu mg)/(\cos(\theta) - \mu\sin(\theta))$.

Pull: the equations become

$$F\cos(\theta) - \mu_k F_N = ma \tag{6.9}$$

$$mg - F\sin(\theta) - F_N = 0. \tag{6.10}$$

We now find $F = (ma + \mu mg)/(\cos(\theta) + \mu\sin(\theta))$. For $0 < \theta < 90°$, we find that it's easier to pull!

6.3 Opening Question 2

You are on a carnival ride called the Gravitron. It spins around so quickly that you are stuck to the wall. If the rotation of the ride increases to a new constant speed, does the new frictional force of the wall on your body

(a) increase,

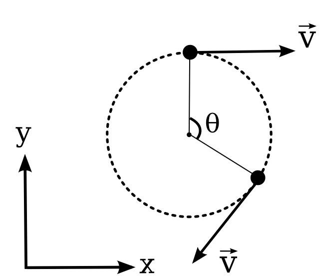
(b) decrease,

(c) or stay the same?

6.4 Uniform Circular Motion

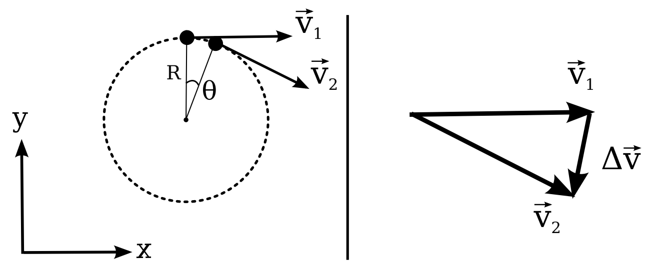
In the case of gravity, where the force pulls the object to the ground, we have seen the resulting curved trajectory (parabolic). However, the speed of objects in free-fall is not constant.

Consider instead an object spinning in a circle at a constant speed (Merry-Go-Round). The velocity changes direction at every instant, even if the magnitude is unchanged.



This is "uniform circular motion." The acceleration is always inward. Why?

To find the magnitude of the acceleration, let's look at two moments in time, separated by Δt.



The angle changed by an amount $\Delta\theta$. If we write the angle as a function of time, we can deduce that $\Delta\theta = (2\pi/\tau)\Delta t$, where τ is how long it takes for the ball to travel all 2π radians around. We also know that $v = C/\tau = 2\pi R/\tau$ is the constant velocity of the ball. This gives us $\Delta\theta = v\Delta t/R$, or $\Delta\theta/\Delta t = v/R$.

The acceleration is given by $\vec{a} = \Delta\vec{v}/\Delta t$. From the figure, we know that the direction of a is directly inward. We also see that the magnitude of $\Delta\vec{v}$ is just $v\Delta\theta$ (for small angles). Therefore,

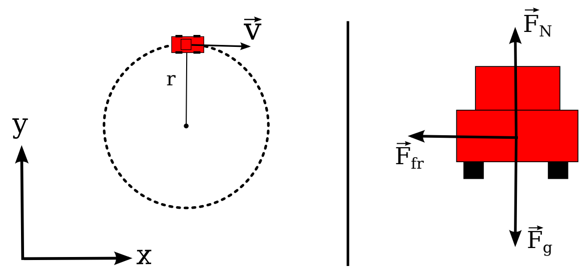
$$a = \frac{v\Delta\theta}{\Delta t} = v^2/R. \tag{6.11}$$

This is the cetripetal acceleration. The greater the speed, the faster the velocity changes direction. The larger the radius, the slower the velocity changes direction.

The moon has an acceleration toward the earth of about $0.0003g$.

Dynamical problems are solved in the same way, by applying $\sum_i \vec{F}_i = m\vec{a}$.

example 1: A car of mass m is traveling at a speed of v in a circle of radius r. What is the minimum coefficient of friction required to keep the car in this motion?



answer: There are three forces: gravity, the normal force and the frictional force. The acceleration of the car is inward, and so the frictional force must equal $F_{fr} = ma = mv^2/r$. The frictional force is $F_{fr} = \mu F_N = \mu mg$. Therefore, $\mu = v^2/rg$.

(a) Is this static or kinetic friction?

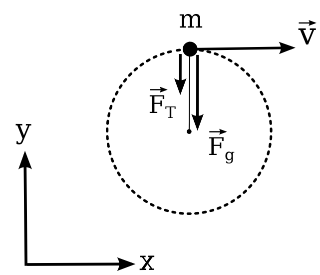
(b) What should the driver do to ensure no skidding?

(c) What if the road is banked at an angle α toward the center?

Centripetal acceleration is the acceleration toward the center during circular motion. During circular motion, you feel pulled toward the outside of the circle. Is this "pull" an actual force?

No. There is no "centrifugal force." This sensation comes from the non-inertial reference frame. Although Newton's Laws do not hold in accelerating reference frames there is a trick involving "fictitious forces."

example 2: A ball on string is swung in a vertical circle. Although the motion is not uniform, $a = v^2/r$ is still true at each point. Find the speed at which the ball must travel at its highest point to ensure the rope does not go slack.



answer: The ball has two forces acting upon it: gravity and F_T, the tension in the string. We set the limiting case to $F_T = 0$ and find $mg = mv^2/r$ which gives $v = \sqrt{rg}$.

Chapter 7

Work and Energy

7.1 Opening Question

Alice is moving into her new dorm room. Her father carries a box full of supplies up to the third floor. When he gets there, Alice tells him that she doesn't need that box. The father brings the box back down stairs an returns it to the car. If the work done on the box is W, which statement is true?

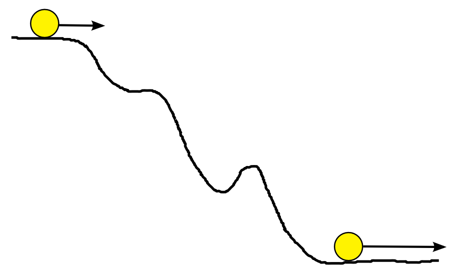
(a) $W > 0$

(b) $W < 0$

(c) $W = 0$

(d) Not enough information.

The motion of objects can be complicated; e.g., the slope may be nonuniform.



How do we find the velocity of the ball at the bottom? We can use energy concepts!

7.2 Work Done by a Constant Force

Unlike force and velocity, work and energy are scalars instead of vectors.

Work is a common word, but in physics, it means

$$W = \vec{F} \cdot \vec{d} = Fd\cos(\theta) \tag{7.1}$$

where $\vec{d} = d_x\hat{i} + d_y\hat{j} + d_z\hat{k}$ is the displacement vector, $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ and θ is the angle between the vectors. This is a *definition*. Remember: a dot product takes a projection of two vectors.

The SI unit of work is the joule defined to be 1 J = 1 N·m.

You need both movement d and force F for work to be done. Work can also be negative!

Positive Work:

- carrying a box up the stairs
- pushing a table along the floor

- hitting a baseball

Negative Work:

- carrying a box down the stairs
- lowering a barbell to the ground
- slowing down a car

Zero Work:

- pushing against the wall
- holding a bag of groceries
- sliding on ice at a constant velocity

7.3 Work Done by a Varying Force

If the force varies at each position, then we can't simply multiply $\vec{F}$ and $\vec{d}$ to get the total work. Instead, we must do a summation.

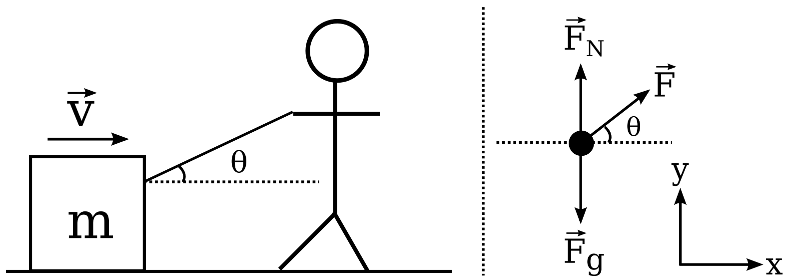
First, the small amount of of work done over a small distance $d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}$: $\vec{F} \cdot d\vec{l}$.

Next, we add up all of these small contributions. This is an integral!

$$W = \int_a^b \vec{F} \cdot d\vec{l} = \int_{x_a}^{x_b} F_x dx + \int_{y_a}^{y_b} F_y dy + \int_{z_a}^{z_b} F_z dz \tag{7.2}$$

In particular, this is a "line integral," since we add up the contributions of $\vec{F} \cdot d\vec{l}$ along a line from point a to b.

example: Bob pulls a 50 kg crate 40 m along a floor by applying a force of 100 N at 37°. There is no friction. Find the work done by each force on the crate, and the net work done on the crate.



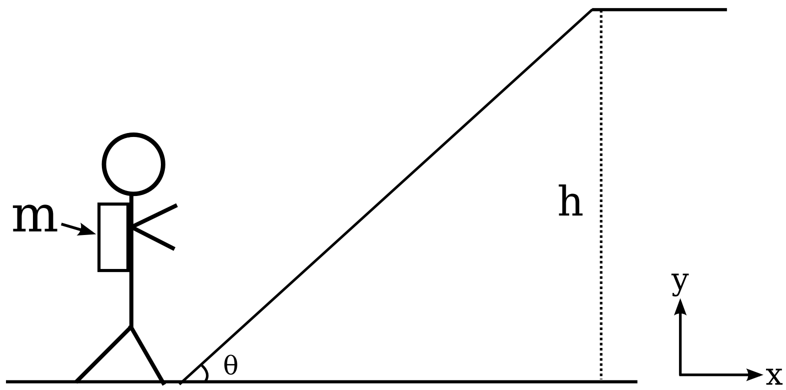
The free body force diagram shows three forces, $\vec{F}_g$, $\vec{F}_N$ and $\vec{F}$.

In the y direction, there is no movement, so $\vec{F}_N \cdot \Delta y\hat{j} = \vec{F}_g \cdot \Delta y\hat{j} = F\sin(\theta)\Delta y = 0$.

In the x direction, there is movement, so we have $W = (F\cos(\theta)\hat{i}) \cdot (\Delta x\hat{i}) = F\Delta x\cos(\theta) = 100 \times 40 \times \cos(37°) = 3200$ J.

The total work is just the sum of the individual work done by each force: $0 + 0 + 3200 = 3200$ J.

example: Bob carries a backpack up a mountain. The backpack is $m = 15.0$ kg and the height of the moutain is $h = 10.0$ m. What is the work done by gravity on the backpack?



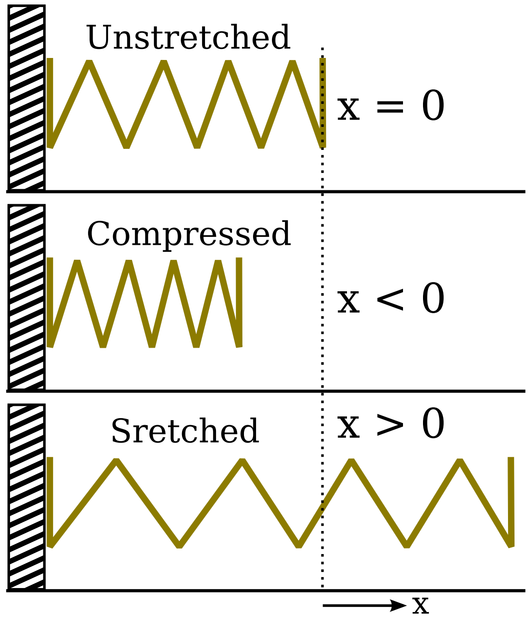
Consider the figure. The force due to gravity on the backpack is in the -y direction. The motion of the backpack is in the positive x and positive y direction. We can write $\vec{F}_g = -mg\hat{j}$ and $\vec{d} = x\hat{i} + h\hat{j}$.

Is this a constant force? Yes. The work done is therefore

$$W = \vec{F}_g \cdot \vec{d} = (-mg\hat{j}) \cdot (x\hat{i} + h\hat{j}) = -mgh = -1470 \text{ J} \tag{7.3}$$

since $\hat{i} \cdot \hat{j} = 0$.

example: Let's consider a varying force, such as a spring. If $F = -kx$, where $k = 2.5 \times 10^3$ N/m (Hooke's law) and $x = 0$ is when the spring is unstretched. The negative sign indicates a "restoring" force.



If you stretch the spring from equilibrium by 3.0 cm, how much work do you do?

$$W = \int_{x_0}^{x_1} F dx = \int_{0.0}^{0.03} kx dx = \frac{1}{2}kx^2\Big|_{0.0}^{0.03} = 1.1 \text{ J}. \tag{7.4}$$

What if you compress it instead, to $x = -0.03$ m? Then,

$$W = \int_{x_0}^{x_1} F dx = \int_{0.0}^{-0.03} kx dx = \frac{1}{2}kx^2\Big|_{0.0}^{-0.03} = 1.1 \text{ J}. \tag{7.5}$$

Since the distance is squared, the work is the same. Does this make sense?

In general, I can give you any force law, e.g., $F = F_0\left(1 + x^2/x_0^2\right)$. Can you find the work done by this force from $x = x_1$ to $x = x_2$?

Note that forces parallel to the motion do positive work, and force anti-parallel to the motion do negative work.

7.4 Kinetic Energy and Work Energy Theorem

Energy has many forms: motion, potential, heat. The sum of all the energy in a system is conserved: the energy before equals the energy after. We can loosely define energy as "the ability to do work." For example, a flying object does work on the object it strikes by moving it. This energy of motion is called kinetic energy.

Consider a car at a speed v_0. We apply a constant force F to accelerate it to v_f in a distance d.

- The total work done on the car is $W = Fd$.
- Using $F = ma$, we see $W = mad$.
- Using our kinematics equations, we know $v_f^2 = v_0^2 + 2ad$, so $a = \frac{v_f^2 - v_0^2}{2d}$
- We therefore find $W = m\frac{v_f^2 - v_0^2}{2d}d = m\frac{v_f^2 - v_0^2}{2} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2$

We can interpret this as final translational kinetic energy minus initial translational kinetic energy! The translational kinetic energy is therefore

$$K = \frac{1}{2}mv^2, \tag{7.6}$$

by definition. We see that $W = \Delta K = K_f - K_0$. That is, the work done results in a change in kinetic energy.

Work-Energy Theorem

> The net work done on an object is equal to the change in the object's kinetic energy.

Note that this is *net* work.

example: How much net work is required to accelerate a 1000 kg car from 20 m/s to 30 m/s?

$$W = \Delta K = K_f - K_0 = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_f^2 = 2.5 \times 10^5 \text{ J} \tag{7.7}$$

From 0 m/s to 20 m/s?

$$W = \Delta K = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_f^2 = \frac{1}{2}1000 \times (20^2 - 0^2) = 2 \times 10^5 \text{ J} \tag{7.8}$$

example: A car can apply a maximum breaking force F to stop from 60 km/h down to 0 in 20 m. If the car is going twice as fast, what is the new distance?

$$W_{net} = Fd\cos(\theta) = -Fd = \Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 = -\frac{1}{2}mv_0^2 \tag{7.9}$$

That is, the stopping distance goes as the square of the initial speed v_0. Twice as fast means $2^2 = 4$ times as far, so $4 \times 20 = 80$ m is the new stopping distance.

example: A spring ($k = 360$ N/m) pushes a block ($m = 1.85$ kg) on a frictionless, horizontal plane.

(a) How much work to compress from $x = 0$ to $x = 11$ cm?

(b) If we let it go with the block in place, what is the speed at $x = 0$ (when it releases)

(c) Do it again, with friction force $F_{fr} = 7.0$ N

Answer:

(a) We know the force varies as $F = kx$, pushing against us as we compress it. F and the displacement d are in the same direction. Therefore,

$$W = \int_{x_0}^{x_f} kx dx = \int_0^{0.11} kx dx = \frac{1}{2} 360 x^2 \Big|_0^{0.11} = 2.18 J. \tag{7.10}$$

(b) In order to find the speed, we would usually use our kinematics equations, say $v_f^2 = v_0^2 + 2a\Delta x$. Can we do that? No, because the acceleration is not constant. However, we can repeat the previoius calculation in reverse to find that the spring does 2.18 J of work on the block. We know this has to equal the kinetic energy, so

$$W_{net} = K = \frac{1}{2} m v^2 \rightarrow v = \sqrt{\frac{2K}{m}} = 1.54 \text{ m/s} \tag{7.11}$$

(c) To incorporate friction, we note that the work done by friction is still just $W = F_{fr} d \cos(\theta) = 7 \times 0.11 \times (-1) = -0.77$ J. That is, friction does negative work. Therefore,

$$W_{net} = K = \frac{1}{2}mv^2 \rightarrow v = \sqrt{\frac{2W_{net}}{m}} = 1.23 \text{ m/s} \tag{7.12}$$

Important Points to remember:

- If the force and displacement are in opposite directions, work is negative
- Work done by object A on object B is opposite the work done by object B on object A (Newton's Third Law)
- The net work done is equal to the *change* in kinetic energy of an object.
- If the force varies with position (spring), we have to integrate

Chapter 8

Conservation of Energy

8.1 Opening Question

In which case will the speed of a puck be greatest?

(a) A puck slides down an icy, frictionless surface of height h

(b) A puck is dropped from the same height h

(c) Speed in (a) and (b) are the same

(d) Not enough information

What if there is friction?

Sometimes a Newton's Laws analysis is too complicated or impossible; in these situations, an analysis of energy can be helpful. Here, we assume rigid bodies and no rotations.

8.2 Conservative vs. Non-conservative Forces

There are two types of forces: conservative and non-conservative.

> A force is conservative if the net work done by the force on an object moving around any closed path is zero.

This is the same as:

> The work done by a conservative force on an object moving from one point to another depends only on the initial and final positions of the object

Let's look at a few forces we know.

- Gravity $\vec{F}_g$
- Constant pushing/pulling force: $\vec{F}$
- Elastic force $\vec{F}_s$
- Tension $\vec{T}$
- Friction $\vec{F}_{fr}$

Which of these is conservative? Let's consider gravity $\vec{F}_g$ on a ball moving from point a to b: $a = (x_a, y_a)$ and $b = (x_b, y_b)$ along a curved path. To compute work, we need the force $\vec{F}_g = -mg\hat{j}$ and the displacement $d\vec{l} = dx\hat{i} + dy\hat{j}$. This gives,

$$W = \int_a^b \vec{F}_g \cdot d\vec{l} = \int_a^b (-mg)dy = -mg(y_b - y_a) = -mgh \tag{8.1}$$

The work done depends only on the height, not on the path at all. Gravity is a conservative force.

What if we push a box along a rough surface at a constant speed with a constant force F for a distance d? Although the direction of $\vec{F}$ changes, F does not, so $\vec{F} \cdot d\vec{l} = Fdl$. Therefore, the work done is just Fd. Is this conservative? Nope!

Conservative	Non-conservative
Gravitational	Friction
Elastic	Air resistance
Electric	Tension in cord
	Motor or rocket propulsion
	Push or pull by person

8.3 Potential Energy

Why are conservative forces interesting?

If a force is conservative, we can define a *potential energy* associated with it.

An object in motion has a kinetic energy $K = \frac{1}{2}mv^2$. But does a rock sitting at the top of a hill have energy? What about a compressed spring?

To lift a brick up to a height h above the ground, I have to do work on it.

The force: $\vec{F} = -m\vec{g} = mg\hat{\jmath}$.

The displacement: $\Delta\vec{y} = h\hat{j}$.

The work is $\vec{F} \cdot \Delta\vec{y} = mhg$ (constant force). If we then let it go and it falls to the ground, how fast will it be moving?

$$v_{fy}^2 = v_{0y}^2 + 2a\Delta y \rightarrow v_{fy}^2 = 2gh \tag{8.2}$$

Its kinetic energy is therefore $K = \frac{1}{2}mv^2 = mgh$.

The work we did to raise the brick was stored as gravitational potential energy. When released, it is converted into kinetic energy, which we can then use to do work equal to mgh.

By definition, for a conservative force $\vec{F}$,

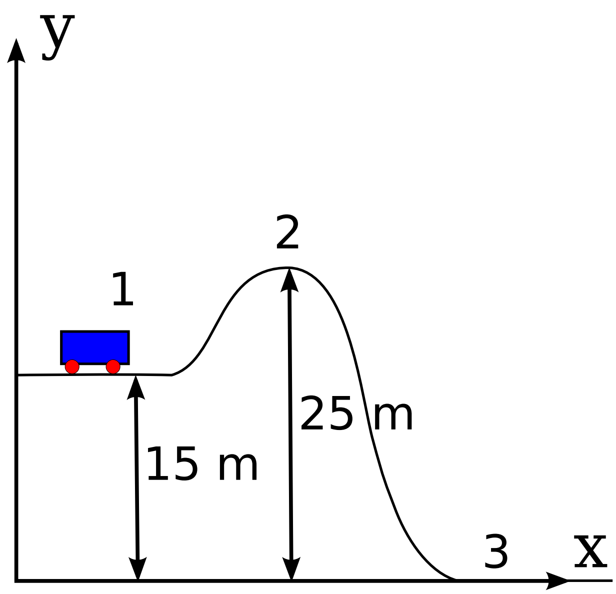
- Potential energy: U
- Change in potential energy: $\Delta U = U_2 - U_1 = -\int_1^2 \vec{F} \cdot d\vec{l} = -W$.

Note the negative sign.

- Gravity points down, but a brick gains potential energy when it is moved up.
- A spring fights against you, but gains potential energy when you stretch/compress it.
- $U = mgy$ is not physically meaningful! Only changes are meaningful.

example: Consider a roller coaster with $m = 1000$ kg.

- We can calculate U of the roller coaster at each point, relative to any height we like.
- Regardless of our choice, ΔU will always be the same
- Differences are meaningful



Take the bottom as $y = 0$. Then, $y_1 = 15$ m, $y_2 = 25$ m and $y_3 = 0$ m. Therefore, $U_1 = mgy_1 = 1000 \times 9.8 \times 15 = 1.5 \times 10^5$ J, $U_2 = 2.5 \times 10^5$ J and $U_3 = 0$ J.

Next, take point 1 as $y = 0$, so then $U_1 = 0$ J, $U_2 = 1000 \times 9.8 \times 10 = 9.8 \times 10^4$ J and $U_3 = -1.5 \times 10^5$ J.

In both cases, $U_3 - U_2 = -2.5 \times 10^5$ J (etc).

example: Consider a spring that applies a force $F_S = -kx$ (force in direction opposite displacement). What is the potential energy stored in the spring?

$$\Delta U = U(x) - U(0) = -\int_{x'=0}^{x'=x} \vec{F}_s \cdot d\vec{l} = -\int_0^x (-kx)dx = \frac{1}{2}kx^2. \qquad (8.3)$$

This is relative to equilibrium ($x = 0$).

In fact, we can go in reverse if we want:

$$U(x) = -\int F(x)dx + C \rightarrow \frac{d}{dx}U(x) = -\frac{d}{dx}\int F(x)dx = -F(x) \qquad (8.4)$$

That is, $F(x) = -dU/dx$. In 3D, we get

$$\vec{F}(x, y, z) = -\hat{i}\frac{\partial U}{\partial x} - \hat{j}\frac{\partial U}{\partial y} - \hat{k}\frac{\partial U}{\partial z} \qquad (8.5)$$

example: Find the force that results in a potential energy of $U(x, y) = -ax/(b^2 + y^2)$.

Potential and Kinetic energy are called "mechanical energy," unlike e.g., heat.

8.4 Conservation of Energy

If we consider only conservative forces on an object, then

$$\Delta U = -\int_1^2 \vec{F} \cdot d\vec{l} = -W = -\Delta K \qquad (8.6)$$

That is, a change in potential energy means an opposite change in kinetic energy. A brick at a large height moves faster the farther it falls. Written another way,

$$\Delta U + \Delta K = \Delta(U + K) = \Delta E = 0 \tag{8.7}$$

There is no change in energy, assuming conservative forces!

However...

> The total energy is neither increased nor decreased in any process. Energy can be transferred or transformed, but never destroyed or created.

Important notes:

- This is true, even if there are non-conservative forces!
- Work and energy are equivalent

Let's do some examples!

example: A rock, 3 m above the ground, is dropped. After it falls 2 m, what is the rock's speed?

First, choose the ground as the reference $y = 0$. Then, we know the *mechanical* energy does not change (conservative forces). We immediately see

$$K_1 + U_1 = K_2 + U_2 \tag{8.8}$$

$$\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv_2^2 + mgy_2 \tag{8.9}$$

where we choose, for example, the top to be y_1 and 1 m up to be y_2. The initial velocity $v_1 = 0$, and the m's cancel out. Therefore,

$$v_2 = \sqrt{2g(y_1 - y_2)} = 6.3 \text{ m/s}. \tag{8.10}$$

Notice, only the *difference* in height mattered.

This same method can be applied to any object subject to gravitational forces only.

example: Consider a roller coaster at an initial height of 40 m. How fast will it be traveling at the bottom of the hill? At what height will the coaster have half this speed?

$$\begin{aligned} K_1 + U_1 &= K_2 + U_2 && (8.11) \\ mgy_1 &= \frac{1}{2} m v_2^2 && (8.12) \\ v_2 &= \sqrt{2gy_1} = 28 \text{ m/s} && (8.13) \end{aligned}$$

To find the new height when $v = 28/2 = 14$ m/s, we solve for y_2, where

$$\begin{aligned} K_1 + U_1 &= K_2 + U_2 && (8.14) \\ \frac{1}{2} m v_1^2 + mgy_1 &= \frac{1}{2} m v_2^2 + mgy_2 && (8.15) \\ y_2 &= y_1 - v_2^2/2g = 30 \text{ m} && (8.16) \end{aligned}$$

Imagine trying to do this 2D problem with Newton's Laws.

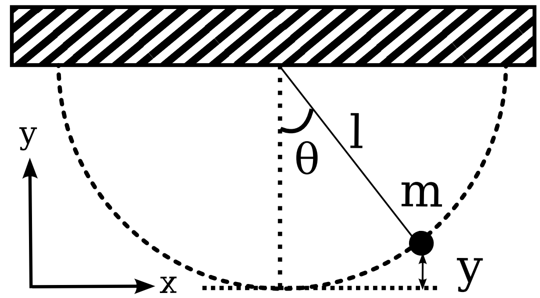
example: Consider a dart of mass 0.100 kg pushed against a spring of stiffness k in a toy gun. If the dart is compressed against the spring by 6.0 cm and released, what is the maximum speed of the dart? (assume the dart separates from the spring at $x = 0$, the "natural length")

We know $U_1 + K_1 = U_2 + K_2$, where $U = \frac{1}{2}kx^2$ is the potential energy stored in the spring. Initially (point 1), the spring is compressed and the dart is at rest. Then, the dart is moving, but the spring is at its natural length. Therefore,

$$U_1 = K_2 \tag{8.17}$$
$$\frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 \tag{8.18}$$
$$v_2^2 = kx_1^2/m \tag{8.19}$$
$$v_2 = 3.0 \text{ m/s} \tag{8.20}$$

We can even mix potential energies, like a spring hanging vertically. In that case, $U_1 + K_1 = U_2 + K_2$, just like before, but U_1 and U_2 have two terms (e.g., $U_1 = mgy_1 + \frac{1}{2}kx_1^2$).

example: A swinging pendulum, ball of mass m, rope of length l, max angle θ_0. Find the speed of the bob as a function of θ and the tension $\vec{F}_T$ in the cord.



The total mechanical energy of the system is $E = K + U = \frac{1}{2}mv^2 + mgy$, with $y = 0$ at the lowest point of swing. The initial energy E_0 is just mgy_0, where $y_0 = l - l\cos\theta_0$ (see figure).

As the bob swings down, $E = mgy_0 = \frac{1}{2}mv^2 + mgy$, where v and y change accordingly. Solving for v, we find $v = \sqrt{2g(y_0 - y)}$. Changing to θ, $y_0 - y = (l - l\cos\theta_0) - (l - l\cos\theta) = l(\cos\theta - \cos\theta_0)$. So

$$v = \sqrt{2gl(\cos\theta - \cos\theta_0)} \tag{8.21}$$

The tension is given by Newton's second law, so

$$F_T - mg\cos\theta = ma = m\frac{v^2}{l} \rightarrow F_t = m\left(\frac{v^2}{l} + g\cos\theta\right) = (3\cos\theta - 2\cos\theta_0)mg \tag{8.22}$$

8.5 Energy Conservation with Dissipative Forces

When a ball is kicked on the soccer field, it slows down. The kinetic energy of the ball is dissipated. This is due to the work done by friction. The grass pulls on the ball, slowing it down, and produces heat.

In general,

$$\Delta K + \Delta U + F_{fr} l = 0 \rightarrow \frac{1}{2} m v_1^2 + m g y_1 = \frac{1}{2} m v_2^2 + m g y_2 + F_{fr} l \quad (8.23)$$

That is Initial equals Final plus any Thermal or Internal energy created in the process.

example: Friction with a spring. A block, mass m, slides on a rough surface at a speed of v_0 when it strikes a spring of constant k. The spring compresses a distance d. Determine μ_k.

The block initially has $K = \frac{1}{2} m v_0^2$ with $U_{spring} = 0$. At max compression, $K_f = 0$ and $U_f = \frac{1}{2} k d^2$. We lost some energy due to friction. Therefore,

$$\frac{1}{2} m v_0^2 = \frac{1}{2} k d^2 + F_{fr} d \quad (8.24)$$

$$= \frac{1}{2} k d^2 + \mu_k m g d \quad (8.25)$$

$$\mu_k = \frac{1}{2g} \left(\frac{v_0^2}{d} - \frac{kd}{m} \right) \quad (8.26)$$

8.6 Gravitational Potential Energy

We know that g is not a constant for different heights. We have to use the more general form of $F_g = GMm/r^2$. Therefore, the potential energy is not simply mgh when talking about planets and satellites.

The work done by the gravitation force on an object near Earth is just

$$W = \int_1^2 \vec{F} \cdot d\vec{l} = -Gm_e m \int_1^2 \frac{\hat{r} \cdot d\vec{l}}{r^2} = -Gm_e m \int_{r_1}^{r_2} \frac{dr}{r^2}. \tag{8.27}$$

Even if the object moves to the side, and not just up, gravity always points down. Therefore, the work only depends on the distance between the two objects. This gives $W = Gm_e m/r_2 - Gm_e m/r_1 = -\Delta U$. That is,

$$U(r) = -\frac{Gm_e m}{r} \tag{8.28}$$

is the gravitational potential energy. So we can often write

$$E_1 = E_2 \tag{8.29}$$

$$K_1 + U_1 = K_2 + U_2 \tag{8.30}$$

$$\frac{1}{2}mv_1^2 - \frac{Gm_e m}{r_1} = \frac{1}{2}mv_2^2 - \frac{Gm_e m}{r_2} \tag{8.31}$$

If we want an object to escape the gravitational pull of earth, we simply find v_1 that allows $r_2 \to \infty$ and $v_2 \approx 0$. Therefore, $v_1 = \sqrt{2Gm_e}r_1$ is the "escape velocity."

8.7 Power

Power P is the rate at which work is done, or the rate of change of work, or the rate at which energy is transformed. We can write it as

$$P_{avg} = \frac{W}{t} \quad (8.32)$$

$$P = \frac{dW}{dt} \quad (8.33)$$

$$P = \vec{F} \cdot \vec{v} \quad (8.34)$$

The last line can be seen from the second line and the definition of W. The units of power are J/s, also known as the watt (W). Some people use horsepower (hp), which is the British unit.

example: Calculate the power required by a 1400 kg car climbing a 10° hill at a constant speed of 22 m/s. There is a constant retarding force of $F_R = 700N$.

The required force provided by the car to climb the hill is $F = mg\sin\theta + F_R = 3100$ N. Therefore, $P = Fv = 6.8 \times 10^4$ W = 91 hp. Note that the force and the velocity are in the same direction.

The efficiency of an engine (or other work producing object) is simply $e = P_{out}/P_{in}$.

Chapter 9

Momentum

9.1 Opening Question

A railroad car loaded with rocks coasts on a level track without friction. A worker throws rocks out of the back.

(a) The car slows down.

(b) The car speeds up.

(c) First the car speeds up and then it slows down.

(d) The car's speed remains constant.

(e) None of these.

9.2 Momentum and Force

Linear momentum of an object is defined as the product:

$$\vec{p} = m\vec{v}. \tag{9.1}$$

It is a vector. Heavier objects → more momentum. Faster object → more momentum.

What about the derivative $d\vec{p}/dt = d(m\vec{v})/dt = md\vec{v}/dt = m\vec{a}$ (constant mass). But we know that $m\vec{a}$ is net force!

$$\sum_i \vec{F}_i = \frac{d\vec{p}}{dt} \tag{9.2}$$

So, if mass changes, then this covers that scenario!

On average,

$$F_{avg} = \frac{\Delta p}{\Delta t}. \tag{9.3}$$

example: A tennis serve accelerates a ball from 0 to 55 m/s in about 4 ms. If the ball is 0.060 kg, what is the average force on the ball?

$$F_{avg} = \frac{p_f - p_0}{\Delta t} = \frac{mv_f - mv_0}{\Delta t} = 0.060 \times 55/0.004 = 800N. \tag{9.4}$$

People are about 600 N.

example: Water from a hose leaves at a rate of 1.5 kg/s at a speed of 20 m/s and hits a car, which stops it. What is the force exerted by the car on the water to stop it?

$$F_{avg} = \frac{p_f - p_0}{\Delta t} = \frac{mv_f - mv_0}{\Delta t} = \frac{0 - 1.5 \times 20}{1} = -30N. \tag{9.5}$$

We used the idea that in 1 second, 1.5 kg of water strikes the car.

9.3 Conservation of Momentum

Consider two objects (billiard balls) of mass m_A and m_B. Their momenta are $\vec{p}_A$ and $\vec{p}_B$. If they collide, they may have new momenta: $\vec{p}'_A$ and $\vec{p}'_B$.

The force exerted by A on B is $\vec{F}$. Therefore, the force exerted by B on A is $-\vec{F}$. No other forces are acting on the balls. Therefore,

$$\vec{F} = \frac{d\vec{p}_B}{dt} \tag{9.6}$$

$$-\vec{F} = \frac{d\vec{p}_A}{dt} \tag{9.7}$$

$$\vec{F} - \vec{F} = \frac{d\vec{p}_B}{dt} + \frac{d\vec{p}_A}{dt} \tag{9.8}$$

$$0 = \frac{d}{dt}(\vec{p}_B + \vec{p}_A) \tag{9.9}$$

Thus, $\vec{p}_B + \vec{p}_A$ is a constant: momentum is conserved.

$$\text{momentum before} = \text{momentum after} \tag{9.10}$$

$$m_a\vec{v}_A + m_B\vec{v}_B = m_a\vec{v}'_A + m_B\vec{v}'_B. \tag{9.11}$$

This is just for two objects, but it is true for any number of objects:

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + ... = \sum_i m_i\vec{v}_i = \sum_i \vec{p}_i \tag{9.12}$$

$$\sum_i \vec{F}_i = \frac{d\vec{P}}{dt} \tag{9.13}$$

We have

> When the net external force on a system of objects is zero, the total momentum of the system remains constant.

or

> The total momentum of an isolated system of objects remains constant.

example: Two objects collide in one dimension (think: trains on a track). Object B is at rest, and object A strikes it and they stick together. What is their new velocity v'?

$$\vec{P}_0 = \vec{P}_f \tag{9.14}$$

$$P_{x0} = P_{xf} \tag{9.15}$$

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B \tag{9.16}$$

$$m_A v_A = m_A v' + m_B v' \tag{9.17}$$

$$v' = \frac{m_A}{m_A + m_b} v_A \tag{9.18}$$

example: Rifle recoil: what is the velocity of a rifle that shoots a 0.020 kg bullet at a speed of 620 m/s?

$$\text{momentum before} = \text{momentum after} \tag{9.19}$$

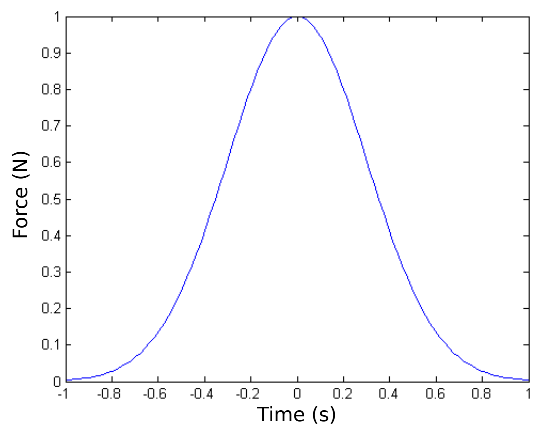
$$m_B v_B + m_R v_R = m_B v'_B + m_B v'_R \tag{9.20}$$

$$0 + 0 = m_B v'_B + m_B v'_R \tag{9.21}$$

$$v'_R = -\frac{m_B v'_B}{m_R} = -2.5 \text{ m/s} \tag{9.22}$$

9.4 Collisions and Impulse

During a collision, the force between two objects abruptly increases, then drops to zero. This is called an impulsive force:



From Newton's 2nd Law, we find

$$\vec{F} = \frac{d\vec{p}}{dt} \rightarrow d\vec{p} = \vec{F}dt \rightarrow \int_0^f d\vec{p} = \int_{t_0}^{t_f} \vec{F}dt \tag{9.23}$$

$$\vec{p}_f - \vec{p}_0 = \int_{t_0}^{t_f} \vec{F}dt = \vec{J} \tag{9.24}$$

$\vec{J}$ is the impulse: the net force times the time it acts, which has units of momentum (kg-m/s). This is equivalent to

$$\vec{J} = \vec{F}_{avg}\Delta t \tag{9.25}$$

where $\Delta t = t_f - t_i$.

9.5 Conservation of Energy and Momentum in Collisions

During a collision, energy can be lost (heat, sound, vibrations, rotations). If no energy is lost, then the collision is said to be *elastic*. In this case, we have

$$\begin{aligned} \vec{P}_0 &= \vec{P}_f & (9.26) \\ K_A + K_B &= K'_A + K'_B. & (9.27) \end{aligned}$$

If energy is lost during the collision, then the collision is *inelastic*. We then have

$$\begin{aligned} \vec{P}_0 &= \vec{P}_f & (9.28) \\ K_A + K_B &= K'_A + K'_B + E_L. & (9.29) \end{aligned}$$

where E_L is energy lost.

In general, we have two equations:

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B \tag{9.30}$$
$$\frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 = \frac{1}{2} m_A v'^2_A + \frac{1}{2} m_B v'^2_B \tag{9.31}$$

(often, masses do not change during a collision.) We can simplify this result by grouping like masses together and getting rid of common terms:

$$m_A(v_A - v'_A) = m_B(v_B - v'_B) \tag{9.32}$$
$$m_A(v_A^2 - v'^2_A) = m_B(v_B^2 - v'^2_B) \tag{9.33}$$
$$m_A(v_A - v'_A)(v_A + v'_A) = m_B(v_B - v'_B)(v_B + v'_B) \tag{9.34}$$
$$v_A + v'_A = v_B + v'_B \tag{9.35}$$
$$v_A - v_B = -(v'_A - v'_B) \tag{9.36}$$
$$\tag{9.37}$$

A head-on (1D) collision implies that the difference of the velocities is opposite its initial value.

example: Unequal masses, target at rest. Mass A is moving at v_A, and mass B is at rest. Assume elastic and head-on. What are v'_B and v'_A?

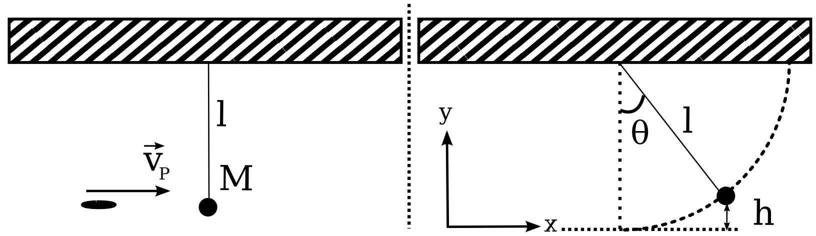
$$
\begin{aligned}
m_b v'_B &= m_A(v_A - v'_A) \text{ (zero initial speed of B)} & (9.38)\\
v'_A &= v'_B - v_A \text{ (from before)} & (9.39)\\
v'_B &= v_A\left(\frac{2m_A}{m_A + m_B}\right) & (9.40)\\
v'_A &= v_A\left(\frac{m_A - m_B}{m_A + m_B}\right). & (9.41)
\end{aligned}
$$

If $m_A \gg m_B$, then $v'_B = 2v_A$ and $v'_A = v_A$. If $m_A \ll m_B$, then $v'_B = 0$ and $v'_A = -v_A$. Does this make sense?

example: Railroad cars again. For a completely inelastic collision (the objects stick together), how much of the initial kinetic energy is transformed to thermal or other forms of energy? ($m_A = 10000$ kg and $v_A = 24.0$ m/s)

Initially, we had $K_0 = \frac{1}{2}m_A v_A^2 = 2.88 \times 10^6$ J. After the collision, we found the speed was reduced by half (if the cars had the same mass): $K_f = \frac{1}{2}(m_A + m_B)v'^2 = 1.44 \times 10^6$ J. The energy was reduced by half.

example: Ballistic pendulum. A projectile of mass m strikes a block of mass M attached to a string hung from the ceiling (like a pendulum). If the block swings up to a height h, what is the velocity of the projectile?



First, we know momentum is conserved: $mv_P = (m+M)v'$. v' is the speed of the block/projectile combo after they collide. During this collision, energy is not conserved.

After they are together, energy is conserved, and so $K_i \rightarrow U_f$ gives $\frac{1}{2}(m+M)v'^2 = (m+M)gh$ (initial kinetic energy turns into final potential energy). Therefore,

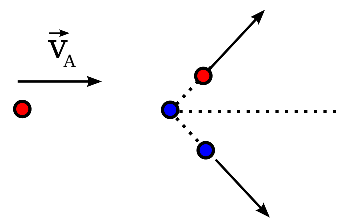
$$v' = \sqrt{2gh} \tag{9.42}$$

$$v_P = \frac{m+M}{m}v' = \frac{m+M}{m}\sqrt{2gh} \tag{9.43}$$

9.6 Collisions in higher dimensions

Momentum conservation is a vector statement: $\vec{P}_0 = \vec{P}_f$. That means, the momentum in each direction is conserved! Look at each direction separately. In two dimensions, we get two equations: we can solve for two unknowns!

example: Billiard ball collision (2D). Ball A moves with speed $v_A = 3.0$ m/s in the x-direction. It strikes ball B, initially at rest. The move off at 45° above (A) and below (B) the x-axis. What are the speeds of the two balls after collision?



x-direction:

$$mv_A = mv'_A \cos(45°) + mv'_B \cos(-45°) \tag{9.44}$$

y-direction

$$0 = mv'_A \sin(45°) + mv'_B \sin(-45°) \tag{9.45}$$

The m's cancel out and the second equation tells us $v'_B = v'_A$. The first equation tells us

$$v'_A = v'_B = v_A / \cos(45°) = 2.1 \text{ m/s}. \tag{9.46}$$

For elastic collisions, we have a third equation ($K_0 = K_f$). In this case, we can solve for three unknowns. This is how particle accelerator detectors work!

9.7 Center of Mass

If an object rotates or vibrates, does conservation of momentum still hold? Yes. We apply our equations to the *center of mass* (CM) of the object or objects.

> The motion of an extended object (or system of objects) can be considered as the sum of the translational motion of the CM, plus rotational or vibrational motion about the CM.

Spinning wrench/knife/diver all follow projectile motion if you look only at the CM. The center of mass is defined as

$$
\begin{aligned}
x_{CM} &= \frac{\sum_i m_i x_i}{M} \text{ (1D)} && (9.47)\\
\vec{r}_{CM} &= \frac{\sum_i m_i \vec{r}_i}{M} = x_{CM}\hat{i} + y_{CM}\hat{j} + z_{CM}\hat{k} \text{ (3D)} && (9.48)\\
\vec{r}_{CM} &= \frac{\int \vec{r} dm}{M} \text{ (continuous 3D)} && (9.49)
\end{aligned}
$$

example: CM in 1D: three guys on a raft. $x_A = 1.0$ m, $x_B = 5.0$ m and $x_C = 6.0$ m, and equal masses. Then,

$$x_{CM} = \frac{mx_A + mx_b + mx_c}{m + m + m} = 4.0 \text{ m.} \tag{9.50}$$

example: CM in 2D: three marbles on a table. $\vec{r}_A = (0,0)$ m, $\vec{r}_B = (2,0)$ m and $\vec{r}_C = (2,1.5)$ m; equal masses $m = 2.5$ kg. Then,

$$\vec{r}_{CM} = \frac{m\vec{r}_A + m\vec{r}_b + m\vec{r}_c}{m+m+m} = \frac{1}{3}[(0\hat{\imath}+0\hat{\jmath})+(2\hat{\imath}+0\hat{\jmath})+(2\hat{\imath}+1.5\hat{\jmath}) = 1.33\hat{\imath}+0.50\hat{\jmath} \text{ m.} \tag{9.51}$$

example: CM of an thin rod. The rod is uniform with density $\lambda = M/l$ (mass M per length l). This is one dimensional. The CM is

$$x_{CM} = \frac{1}{M}\int_0^l x dm = \frac{1}{M}\int_0^l \lambda x dx = l/2 \tag{9.52}$$

If the mass density is instead $\lambda = \lambda_0(1 + x/l)$, we find

$$x_{CM} = \frac{1}{M}\int_0^l \lambda_0(1 + x/l)x dx = \frac{5}{6}\frac{\lambda_0}{M}l^2 \tag{9.53}$$

We can find λ_0 in terms of M by integrating up all the mass:

$$M = \int_0^l dm = \int_0^l \lambda_0(1 + x/l)dx = \frac{3}{2}\lambda_0 l. \tag{9.54}$$

This gives $x_{CM} = \frac{5}{9}l$, more than half way as expected.

For a system of particles (or an extended rigid object), we have

$$\sum_i \vec{F}_{ext,i} = M\vec{a}_{CM} = \frac{d\vec{P}}{dt}. \tag{9.55}$$

That is, Newton's Second Law applies to systems of particles, where the force it the external force on the system and $\vec{a}$ is the acceleration of the CM.

Chapter 10

Rotational Motion

10.1 Opening Question

If a marble moves down an inclined plane, in which case does it have the largest acceleration?

(a) It rolls down

(b) It rolls down, but is hollow

(c) It slides down with no friction

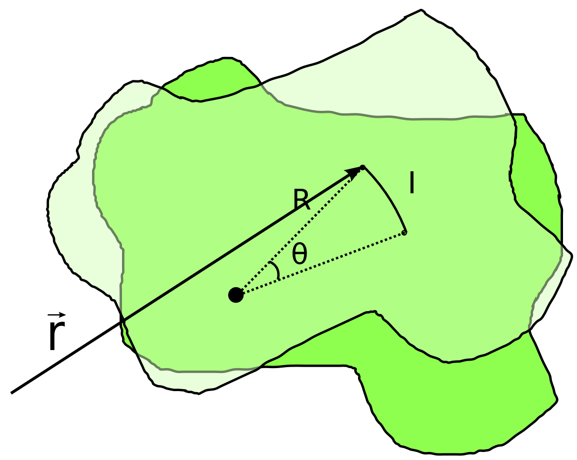
(d) Need more information

10.2 Angular Quantities

We consider only rigid objects: particles within an object have a fixed distance from each other.

Consider an object that rotates about a fixed point. This is called the **axis of rotation**. All points around it trace out circles.

A fixed point on a rotating object travels a distance $l = R\theta$, where R is the distance from the axis of rotation and θ is the angle through which the object has rotated. θ is measured in radians.



Note that $\vec{r}$ is still the position of the particle w.r.t. the origin.

Objects "subtend" an angle θ if they are l wide a distance R away.

Angular displacement may look familiar:

$$\Delta\theta = \theta_f - \theta_0 \tag{10.1}$$

What might the angular velocity be?

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t} \tag{10.2}$$

$$\omega = \frac{d\theta}{dt} \tag{10.3}$$

This is how fast the angle of an object changes in some time interval, or the rate of change of the angle. The angular acceleration is an extension of this:

$$\alpha_{avg} = \frac{\Delta\omega}{\Delta t} \tag{10.4}$$

$$\alpha = \frac{d\omega}{dt} \tag{10.5}$$

All points on the rotating object, if measured from the axis of rotation, have the same angular velocity and angular acceleration. This is because all points move through the same angle in the same time interval.

How are these concepts related to linear displacement, velocity and acceleration? We know the position of a point on a rigid body is just $l = R\theta$. Therefore,

$$v = \frac{dl}{dt} \tag{10.6}$$

$$= \frac{d}{dt}(R\theta) \tag{10.7}$$

$$= R\frac{d}{dt}(\theta) \tag{10.8}$$

$$= R\frac{d\theta}{dt} \tag{10.9}$$

$$v = R\omega \tag{10.10}$$

$$\tag{10.11}$$

We see that, since ω is the same for all points on a rigid body, v is greater for points farther from the axis of rotation.

Consider a merry-go-round. Do children near the edge travel faster or slower than children near the center? Consider the distance and time they travel one complete rotation.

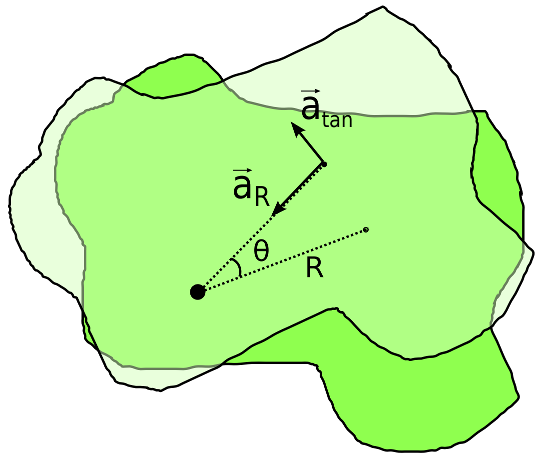
We can repeat the previous calculation, but for acceleration, and use $v = R\omega$ to obtain:

$$\begin{aligned} a_{tan} &= \frac{dv}{dt} && (10.12) \\ &= \frac{d}{dt}(R\omega) && (10.13) \\ &= R\frac{d}{dt}(\omega) && (10.14) \\ &= R\frac{d\omega}{dt} && (10.15) \\ a_{tan} &= R\alpha && (10.16) \\ & && (10.17) \end{aligned}$$

This is the linear acceleration tangent to the object's motion around the circle. There is also a component of the acceleration pointing inward. What is its magnitude? ($a_R = v^2/R$).

We combine these two components of acceleration using vector addition: $\vec{a} = \vec{a}_R + \vec{a}_{tan}$. We can rewrite the centripetal acceleration in terms of ω, so

$$a_R = v^2/R = (R\omega)^2/R = \omega^2 R. \tag{10.18}$$



example: A carousel is initially at rest and then has an angular acceleration for 8 s. At this moment, what is:

(a) The angular velocity. Since the acceleration is constant, we have $\alpha = \Delta\omega/\Delta t = (\omega_f - \omega_0)/(t_f - t_0) = \omega_f/t_f$. Therefore, $\omega_f = \alpha t_f = 0.48$ rad/s.

(b) The linear velocity of a child $R = 2.5$ m from the center: $v = R\omega = 1.2$ m/s.

(c) The tangential acceleration of the child: $a_{tan} = R\alpha = 0.15$ m/s^2.

(d) The centripetal acceleration of the child: $a_R = \omega^2 R = 0.58$ m/s^2

(e) The total acceleration of the child: $a = \sqrt{a_{tan}^2 + a_R^2} = 0.60$ m/s^2. The angle is $\theta = \tan^{-1}(a_{tan}/a_R) = 0.25$ rad.

We can relate the angular velocity ω to the frequency by noting that 1 full revolution is the same as 2π radians. Therfore,

$$f = \frac{\omega}{2\pi} \tag{10.19}$$

This tells us how many revolutions/sec, instead of radians/sec. The unit for f is the Hertz (Hz). The time required for a complete revolution is the inverse:

$$T = \frac{1}{f}. \tag{10.20}$$

This is the period, sometimes denoted as τ.

example: A disk of radius $R = 3.0$ m rotates at an angular velocity $\omega = 1.6 + 1.2t$ rad/s. At $t = 2.0$ s, what is

(a) The angular acceleration: $\alpha = d\omega/dt = 1.2$ rad/s^2.

(b) The speed v at a point near the edge of the disk: $v = R\omega = 12.0$ m/s

(c) The components of the linear acceleration: $a_{tan} = R\alpha = 3.6$ m/s^2 and $a_R = \omega^2 R = 48$ m/s^2.

We use the **Right Hand Rule** to determine the directions of these quantities. That is, $\vec{\omega}$ and $\vec{\alpha}$ are vectors.

10.3 Constant Angular Acceleration

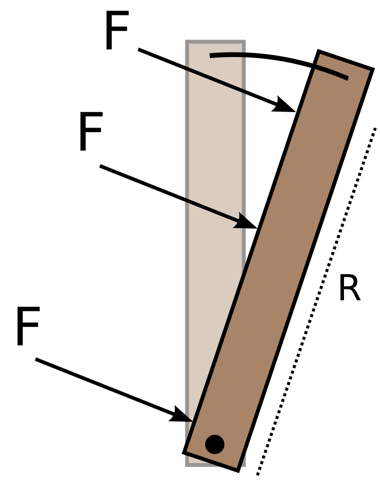
For constant acceleration, the derivation of the relationship between t, x, v and a was a matter of integrating $v = dx/dt$. For angular quantities, the same ideas hold, we just replace $x \to \theta$, $v \to \omega$ and $a \to \alpha$. We therefore find

Angular	Linear
$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x_f = x_0 + v_0 t + \frac{1}{2}a t^2$
$\omega_f = \omega_0 + \alpha t$	$v_f = v_0 + at$
$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta)$	$v_f^2 = v_0^2 + 2a(\Delta x)$

Remember to convert quantities into rad/s and rad/s^2 and not revolutions.

10.4 Torque

Just like forces generate linear acceleration in a system, forces can also create angular acceleration. The location and direction of the force are important. Consider a door on a hinge. Is it easier to open if you push near the hinge, or far from the hinge? The **lever arm** or **moment arm** $\vec{R}$ is the distance from the hinge. It is a vector, pointing from the hinge to where the force is applied.

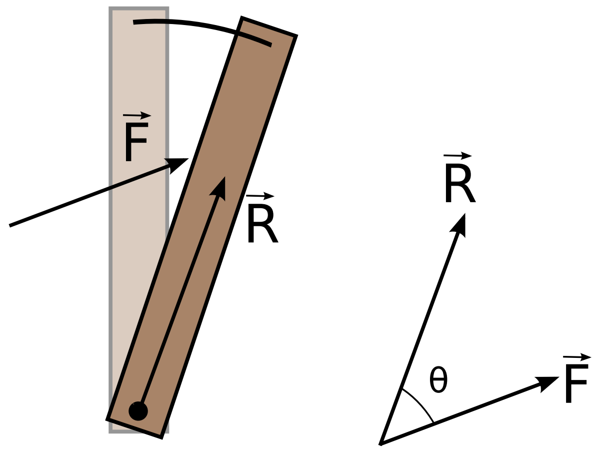


We notice two things: the longer the moment arm and the harder we push, the faster the door accelerates. That is, the angular acceleration is proportional to both the moment arm and the force. We therefore define a new quantity called torque:

$$\tau = RF \text{ (roughly)} \tag{10.21}$$

This is proportional to the angular acceleration: $\tau \propto \alpha$.

Clearly, the angle of the force w.r.t. the axis of rotation is very important. If we draw a vector from the axis of rotation to the location where the force is applied and call that $\vec{R}$, we want the force $\vec{F}$ to be perpendicular to this vector for maximum angular acceleration.



The component of $\vec{F}$ parallel to $\vec{R}$ is wasted and provides no angular acceleration about this hinge. We therefore have

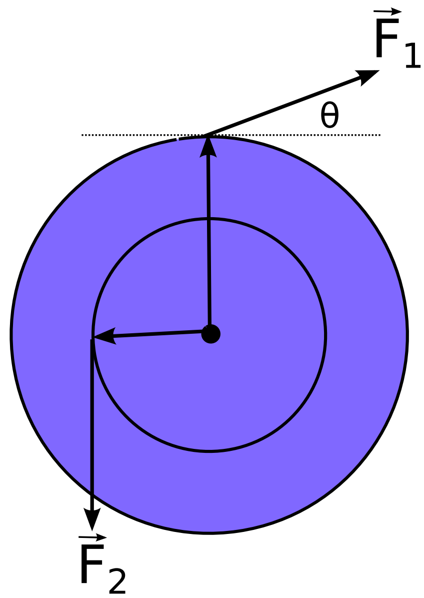
$$\tau = RF_{perp} = RF\sin(\theta) = |\vec{R} \times \vec{F}|. \tag{10.22}$$

Remembering that the cross product has direction according to the right hand rule, we note that

$$\vec{\tau} = \vec{R} \times \vec{F}. \tag{10.23}$$

Is this consistent with the way the door rotates? Yes!

example: Consider a wheel with two 50 N applied forces, one at a radius of 50 cm, 30° above the x-axis and another at a radius of 30 cm in the -y-direction. What is the net torque?



We note that one force tends to rotate the wheel clockwise, and the other force counterclockwise. Let's choose counterclockwise to be positive θ. For one force, the moment arm and the force are 60° apart. For the other force, they are 90° apart. Calling the first force negative and the second positive, we find

$$\tau = -.5 \times 50 \times \sin(60^\circ) + .3 \times 50 \times \sin(90^\circ) \tag{10.24}$$

$$= -6.7\text{N m} \tag{10.25}$$

Notice that the unit for Torque is the same as for energy/work. However, we should **never** call the units of torque Joules.

10.5 Rotational Dynamics

How does Torque generate motion? If we look at a single particle, we know

$$F = ma. \tag{10.26}$$

Let's multiply both side by the distance R of the particle from some imaginary point, then replace a with the angular acceleration about this point:

$$\begin{aligned} F &= ma & (10.27)\\ FR &= maR & (10.28)\\ FR &= mR^2\alpha & (10.29)\\ \tau &= mR^2\alpha & (10.30) \end{aligned}$$

It seems like the torque about some aribtrary point is equal to the angular acceleration about this point, multiplied by this quantity mR^2. For a system of particles, we get

$$\sum_i \tau_i = \left(\sum_i m_i R_i^2\right)\alpha \tag{10.31}$$

(alpha is the same for all particles).

The pre-factor is called the moment of inertia, and is the equivalent of mass for rotational motion:

$$I = \sum_i m_i R_i^2 = m_1 R_1^2 + m_2 R_2^2 + ... \tag{10.32}$$

We can then write:

$$\sum_i \tau_i = I\alpha \tag{10.33}$$

Here, the axis is fixed in an inertial reference frame.

example: Two weights on a bar: $m_1 = 5$ kg and $m_2 = 7$ kg separated by 4 m. What is I if the bar rotates about:

(a) The center between the two weights. In this case, $R_1 = R_2 = 2$ m. We therefore get $I = 5 \times 2^2 + 7 \times 2^2 = 48$ kg m^2.

(b) 0.5 m left of one of the weights. In this case, $R_1 = 0.5$ m and $R_2 = 4.5$ m. We therefore get $I = 5 \times 0.5^2 + 7 \times 4.5^2 = 143$ kg m^2.

This shows the that moment of inertia depends on the axis of rotation. Also, the objects close to the axis contribute less to I than objects farther away.

There is a table in your text with the moment of inertia of common shapes and axes.

In general, we can find the momentum of inertia for any object by computing:

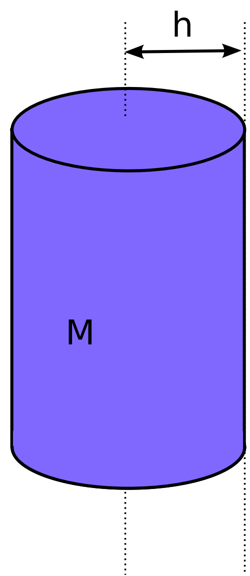
$$I = \int R^2 dm, \tag{10.34}$$

very similar to the center of mass.

If we know the moment of inertia about the center of mass of some object of mass M, we can find the moment of inertia about some other parallel axis a distance h away:

$$I = I_{CM} + Mh^2 \tag{10.35}$$

This is the **Parallel-Axis Theorem**. Consider a cylinder.



example: Consider a heavy pulley of mass 4 kg and radius 33 cm. A rope pulls tangential to the pulley with 15 N of force, and a frictional force opposes the motion at the axle with a torque of 1.10 N m. If the pulley accelerates up to a speed of 30 rad/s in 3.00 s, what is the moment of inertia?

We have two torques opposing each other, and their vector sum must equal $I\vec{\alpha}$. We can find α from the information given. Since this is a 1D problem, drop the vectors, and we find $I = (\tau_r - \tau_f)/\alpha$, where $\alpha = (\omega_f - \omega_i)/\Delta t = 10$ rad/s^2. Therefore, $I = (0.33 \times 15 - 1.10)/10 = 0.385$ kg m^2.

example: Consider a bucket hanging from a rope tied to the pulley from before (radius R_0). The bucket is 15 N heavy, and the inertia of the pulley is from I from before.

(a) What is α of the pulley and a of the bucket? We have two equations ($\tau = I\alpha$ and $F = ma$). Let's see what they tell us.

$$
\begin{aligned}
I\alpha &= \sum \tau = R_0 F_T - \tau_{fr} \text{ (pulley)} && (10.36)\\
mg - F_T &= ma \text{ (bucket)} && (10.37)
\end{aligned}
$$

We know that $R_0\alpha = a$. Why? We can therefore replace a above to find

$$
\begin{aligned}
\alpha &= \sum \tau / I && (10.38)\\
&= \frac{R_0 F_T - \tau_{fr}}{I} && (10.39)\\
&= \frac{R_0(mg - mR_0\alpha) - \tau_{fr}}{I} && (10.40)\\
\alpha &= \frac{mgR_0 - \tau_{fr}}{I + mR_0^2} && (10.41)\\
&= 6.98 \text{ rad/s}^2 && (10.42)
\end{aligned}
$$

And

$$a = R_0\alpha = 2.30 \text{ m/s}^2. \qquad (10.43)$$

(b) What is ω of the pulley and v of the bucket at $t = 3$ if at $t = 0$, $\omega = 0$. For constant acceleration, $\omega_f = \omega_0 + \alpha t_f = 20.9$ rad/sec, and

$v = R_0\omega = 6.91$ m/s.

example: Rotating Rod: A uniform rod of mass M and length l can pivot freely about a hinge at its end. You release the rod from horizontal, what is:

(a) The angular acceleration of the rod? There are two forces on the rod, the hinge and the force due to gravity. If we use the hinge as the pivot, the hinge exerts no torque on the rod (why?). Therefore, we focus on the force due to gravity. It acts on the center of mass, so

$$\alpha = \tau/I = \frac{Mgl/2}{\frac{1}{3}Ml^2} = \frac{3g}{2l} \tag{10.44}$$

The angle between the force and the moment arm is 90°. As the rod moves, this angle changes, giving an non-constant α.

(b) The linear acceleration of the tip of the rod? $a_{tan} = l\alpha = \frac{3}{2}g$. Craziness!!

10.6 Energy and Work

Rotating objects have rotational kinetic energy. Think about all those little particles moving around.

$$\begin{aligned} K &= \sum(\frac{1}{2}m_i v_i^2) && (10.45) \\ &= \sum(\frac{1}{2}m_i(R_i\omega)^2) && (10.46) \\ &= \frac{1}{2}(\sum m_i R_i^2)\omega^2 && (10.47) \\ &= \frac{1}{2}I\omega^2 && (10.48) \end{aligned}$$

in perfect analogy to linear kinetic energy. Not surprisingly, we have

$$W = \int \vec{F} \cdot d\vec{l} = \int_{\theta_1}^{\theta_2} \tau d\theta \qquad (10.49)$$

and

$$W = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2. \qquad (10.50)$$

The power is just

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega. \qquad (10.51)$$

example: We drop the hinged rod from before from rest. What is the angular velocity of the rod when it reaches the vertical position?

Gravity does positive work on the rod as it swings down. The center of mass drops by $l/2$, so the work done is $W = Mgl/2 = \Delta K$. Since the rod starts at rest, $K_0 = 0$, so

$$K_f = \frac{1}{2} I \omega_f^2 = Mgl/2 \rightarrow \omega = \sqrt{3g/l}. \tag{10.52}$$

10.7 Rolling without Slipping

If an object rolls, it has translational and rotational motion. If the object does not slip on the surface, then the velocity of the axis is $v = R\omega$. To see this, put yourself in the reference frame of the wheel, where there is no translational motion.

In the situation of rolling, we simply add the kinetic energies from both motions:

$$K_{tot} = K_{CM} + K_{rot} \tag{10.53}$$

where CM stands for the kinetic energy of the center of mass.

Important Note: We can only use $\tau = I\alpha$ when the axis is (1) fixed in an inertial reference fram or (2) fixed in direction but passes through the CM of the object.

example: Sphere of mass M and radius R_0 rolling down an incline of vertical height h. What will be its speed at the bottom?

Assume the sphere does not slip, so friction does no work. Therefore,

$$U_0 = U_f + K_{CM,f} + K_{rot,f} \tag{10.54}$$
$$Mgh = 0 + \frac{1}{2}Mv^2 + \frac{1}{2}I_{CM}\omega^2 \tag{10.55}$$

A sphere has $I = \frac{2}{5}MR_0^2$ and $\omega = v/R_0$. Therefore,

$$Mgh = 0 + \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{2}{5}MR_0^2\right)(v/R_0)^2 \tag{10.56}$$
$$v = \sqrt{\frac{10}{7}gh} \tag{10.57}$$

If the sphere just slid down the incline, $Mgh = 0.5mv^2 \rightarrow v = \sqrt{2gh}$.

example: If we roll a variety of round objects down the incline, which is fastest? In general, we have

$$U_0 = K_{CM,f} + K_{rot,f} \tag{10.58}$$
$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I_{CM}(v/R_0)^2 \tag{10.59}$$
$$v = \sqrt{\frac{2gh}{1 + I_{CM}/(MR_0^2)}} \tag{10.60}$$

For most round objects, $I_{CM} \propto MR_0^2$, so all that matters is the pre-factor f. We find

$$v = \sqrt{\frac{2gh}{1+f}}. \tag{10.61}$$

Larger inertia (factors) means lower velocity. So hoop (1) < hollow cylinder

$<$ solid cylinder (1/2) $<$ sphere (2/5).

example: Falling yo-yo (string wrapped around a solid cylinder) of mass M and radius R. As it falls from rest, what is its acceleration and the tension in the string?

Linearly, we have (positive down)

$$\sum_i F_i = Ma \rightarrow Mg - F_T = Ma \tag{10.62}$$

We need F_T to find a. Rotationally, we have

$$\sum_i \tau_i = I_{CM}\alpha_{CM} \rightarrow F_T R = \frac{1}{2}MR^2\alpha \tag{10.63}$$

If there is no slipping of the rope, then $a = R\alpha$. We can therefore solve for F_T in terms of a:

$$F_T R = \frac{1}{2}MR^2(a/R) \rightarrow F_T = \frac{1}{2}Ma. \tag{10.64}$$

Putting this into the linear motion, we find

$$Ma = Mg - \frac{1}{2}Ma \rightarrow a = \frac{2}{3}g. \tag{10.65}$$

The string slows down the yoyo. $F_T = Mg - Ma = M(g - \frac{2}{3}g) = Mg/3$.

Chapter 11

Angular Momentum

11.1 Opening Question

You spin on a desk chair with your arms pointing out. You bring your arms inward toward your body. What happens?

(a) Nothing

(b) You spin slower

(c) You spin faster

(d) You come to a stop

11.2 Angular Momentum

When an object moves, it has momentum equal to $\vec{p} = m\vec{v}$. When a rigid object spins, all of the small parts of the object have momentum given by this equation. However, for a symmetrical object, we can define the total angular momentum about the symmetry axis of this object as

$$\vec{L} = I\vec{\omega}. \tag{11.1}$$

The direction of the angular momentum is the same as the direction of the angular velocity (given by the right hand rule). We also know that $\sum F = dp/dt$. The equivalent rotational equation is

$$\sum_i \vec{\tau}_i = \frac{d\vec{L}}{dt}. \tag{11.2}$$

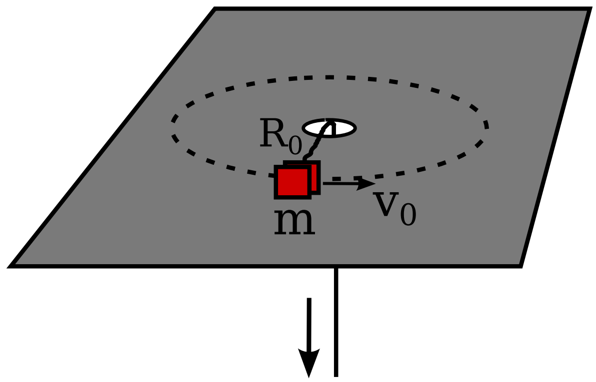
Just like before, I can change. We see that if $\sum \tau = 0$ (no net torque), then

$$\frac{d\vec{L}}{dt} = 0. \tag{11.3}$$

Angular momentum is conserved! $L = I\omega$ is a constant.

> The total angular momentum of a rotating object remains constant if the net external torque acting on it is zero.

example: Consider a rotating object moving in a circle and connected to a string, where the length of the string can change. Initially, the tangential speed is $v_0 = 2.4$ m/s and the radius of its circular motion is 0.80 m. If the string length is reduced to 0.48 m, what is v_f?



There is no net torque, the tension force is in the direction of the moment arm. Therefore, angular momentum is conserved!

$$
\begin{aligned}
I_0\omega_0 &= I_f\omega_f & (11.4)\\
(mR_0^2)(v_0/R_0) &= (mR_f^2)(v_f/R_f) & (11.5)\\
v_f &= v_0R_0/R_f = 4.0 \text{ m/s} & (11.6)
\end{aligned}
$$

As it moves in, the speed increases to maintain the same angular momentum. Compare this to spinning in a chair with your arms out-stretched!

example: Two metal cylindrical plates (radii 0.60 m) sit motionless near each other. Plate A ($m_A = 6.0$ kg) is accelerated to $\omega_0 = 7.2$ rad/s in 2.0 s. Then plate B ($m_B = 9.0$ kg) is pressed against A so that they move together at ω_f.

(a) What is the angular momentum of plate A? $L_A = I_A\omega_0 = \frac{1}{2}M_AR_0^2\omega_0 = 7.8 \text{ kg m}^2\text{/s}$.

(b) How much torque was needed to accelerate plate A to this initial speed of ω_0 from rest? $\tau = \Delta L/\Delta t = (7.8 - 0)/2 = 3.9$ m N.

(c) The two plates move together where $\omega_f < \omega_0$. Why? Conservation of angular momentum.

(d) What is ω_f?

$$\begin{aligned} I_A\omega_0 &= (I_A + I_B)\omega_f && (11.7) \\ \omega_f &= \frac{I_A}{I_A + I_B}\omega_0 && (11.8) \\ &= \frac{M_A}{M_A + M_B}\omega_0 && (11.9) \\ &= 2.9 \text{ rad/s.} && (11.10) \end{aligned}$$

example: A star of radius $r_0 = 7 \times 10^8$ m collapses to a neutron star of radius $r_f = 10^3$ m. If the original star rotated 1 revolution in 100 days, what is the new rotation frequency of the neutron star?

Assume the star is a sphere, and that angular momentum is conserved. Therefore,

$$
\begin{aligned}
I_0\omega_0 &= I_f\omega_f && (11.11)\\
\omega_f &= \frac{I_0}{I_f}\omega_0 && (11.12)\\
&= \frac{\frac{2}{5}mr_0^2}{\frac{2}{5}mr_f^2}\omega_0 && (11.13)\\
&= \frac{r_0^2}{r_f^2}\omega_0 && (11.14)
\end{aligned}
$$

Since $\omega = 2\pi f$, we have

$$
f = \frac{r_0^2}{r_f^2}\omega_0 = 600 \text{ rev/s.} \tag{11.15}
$$

example: A 60 kg person stands on a 6-m diameter circular platform at rest. The moment of inertia is 1800 kg m^2. When the person runs at a speed of 4.2 m/s (w.r.t. earth), the platform rotates. What is the angular velocity of the platform?

The angular momentum is zero to start, and zero always since there is no net torque on the person/platform system. Therefore,

$$
\begin{aligned}
L &= L_{per} + L_{plat} && (11.16)\\
0 &= mR^2\left(\frac{v}{R}\right) - I\omega && (11.17)\\
\omega &= \frac{mRv}{I} = 0.42 \text{ rad/s.} && (11.18)
\end{aligned}
$$

example: You are standing on a stationary turntable, holding a spinning bicycle wheel so that $\vec{\omega}$ points up. If you turn the wheel around so $\vec{\omega}$ points down, what happens?

Since there is no net torque, the angular momentum (which is pointing up along $\vec{\omega}$) must remain constant. Therefore, after the wheel is turned over, its angular momentum is pointing down ($-\vec{L}$) and so the rest of the system must have angular momentum $2\vec{L}$ (upward). Therefore, you and the turntable start to spin in the same direction that the wheel was spinning before you turned it over.

Question: If you walk toward the center of a large, rotating turn table, what happens?

example: A little review. Calculate the torque for a $\vec{r} = 1.2\hat{\imath} + 1.2\hat{k}$ and $\vec{F} = 150\hat{\imath}$.

$$
\begin{aligned}
\vec{\tau} &= \vec{r} \times \vec{F} && (11.19)\\
&= (1.2\hat{\imath} + 1.2\hat{k}) \times (150\hat{\imath}) && (11.20)\\
&= 1.2 \cdot 150\hat{\imath} \times \hat{\imath} + 1.2 \cdot 150\hat{k} \times \hat{\imath} && (11.21)\\
&= 180\hat{\jmath} \text{ m N.} && (11.22)
\end{aligned}
$$

Note: $\hat{k} \times \hat{\imath} = \hat{\jmath}$.

We know that the angular momentum of a symmetric, rigid body is given by $\vec{L} = I\vec{\omega}$. However, what is $I\omega$ for a single particle of mass m a distance r from the axis?

$$I\omega = (mr^2)(v/r) = rmv = rp, \tag{11.23}$$

where v is the velocity perpendicular to the radius. That is, there is a more general relationship:

$$\vec{L} = \vec{r} \times \vec{p}. \tag{11.24}$$

We have a number of useful relations:

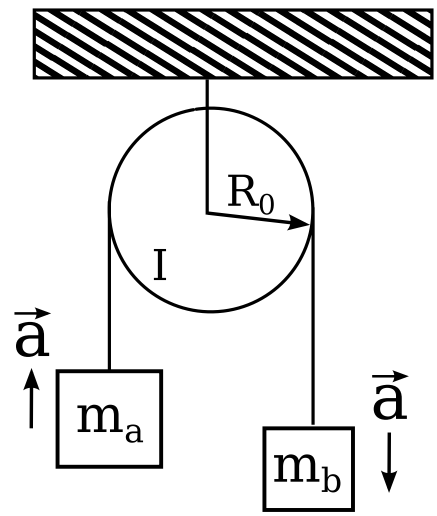
$$\vec{L} = I\vec{\omega} \tag{11.25}$$

$$\vec{L} = \vec{r} \times \vec{p} \tag{11.26}$$

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt} \tag{11.27}$$

The first equation is only true for symmetric objects about the symmetry axis. The latter equation is true for $\vec{\tau}_{ext}$ and $\vec{L}$ calculated w.r.t. either the origin of an inertial reference frame or the center of mass of a system of particles or a rigid object. (see the derivation in the text)

example: A simple Atwood machine consists of two masses m_A and m_B connected by a cord over a pulley of radius R_0 and moment of inertia I. What is the acceleration of the masses?



The total momentum is $L = L_A + L_B + L_P$. For the masses, the angular momentum is just mvR_0, both positive. Why? Draw a picture! Therefore,

$$L = (m_A + m_B)vR_0 + I\frac{v}{R_0} \tag{11.28}$$

The torque about the pulley is found to be

$$\tau = m_b g R_0 - m_A g R_0 \tag{11.29}$$

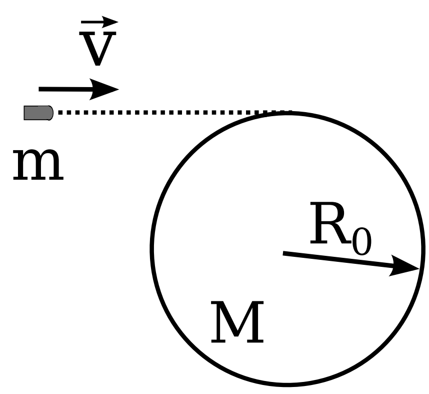
This is NOT the tension in the rope! This torque must equal dL/dr, so

$$\begin{aligned} \tau &= \frac{dL}{dt} &(11.30)\\ (m_B - m_A)gR_0 &= (m_A + m_B)R_0\frac{dv}{dt} + \frac{I}{R_0}\frac{dv}{dt} &(11.31)\\ a &= \frac{dv}{dt} = \frac{(m_B - m_A)g}{(m_A + m_B) + I/R_0^2}. &(11.32)\end{aligned}$$

example: Consider a bicycle wheel spinning as if you are moving forward on a bike (clockwise from the right side). Its angular momentum vector $\vec{L}$ points to the left. If you push on the handles to turn the bike left (so that the torque vector is upward), which way does the bike lean?

To the right! We can see this from the relation $\vec{\tau} = d\vec{L}/dt \rightarrow \Delta\vec{L} = \vec{\tau}\Delta t$. That is, the change in angular momentum is in the same direction as the torque. We know the torque is upward, so the change in angular momentum is also upward. Since the angular momentum was originally to the left, the wheel must tilt to the right to have the angular momentum start pointing upward.

example: A bullet of mass m and speed v embeds in a cylinder's (M) edge. The bullet was traveling tangential to the surface of the clinder. What is the angular velocity of the cylinder's rotation after the collision, assuming it rotates about its original CM?



Angular momentum is conserved, so $\vec{L}_0 = \vec{L}_f$. Initially, only the bullet is moving, so $L = |\vec{r} \times \vec{p}| = R_0 m v$.

After, we sum the angular momentum of each piece, or, more simply, calculate the momentum of inertial of the cylinder/bullet system since they both have the same ω.

$$L_f = I\omega = (I_{cyl} + mR_0^2)\omega = (\frac{1}{2}M + m)R_0^2\omega \tag{11.33}$$

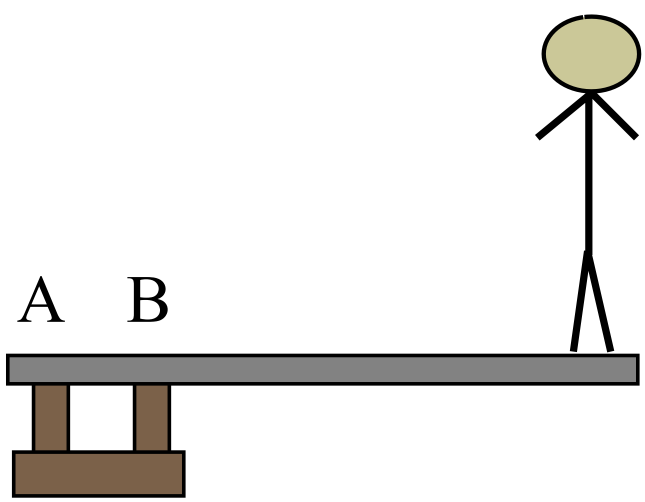
$$\omega = \frac{L}{(\frac{1}{2}M + m)R_0^2} = \frac{mvR_0}{(\frac{1}{2}M + m)R_0^2} = \frac{mv}{(\frac{1}{2}M + m)R_0} \tag{11.34}$$

Chapter 12

Equilibrium

12.1 Opening Question

The diving board is held by two supports at A and B. Which is true about the forces exerted on the diving board at A and B?



(a) $\vec{F}_A$ is down, $\vec{F}_B$ is up, and $F_A < F_B$

(b) $\vec{F}_A$ is down, $\vec{F}_B$ is up, and $F_A > F_B$

(c) $\vec{F}_B$ is down, $\vec{F}_A$ is up, and $F_A = F_B$

(d) Both force are up and $F_A < F_B$

(e) Both forces are down and $F_A = F_B$

12.2 Conditions for Equilibrium

This chapter looks at objects in equilibrium, i.e. objects that do not move or accelerate. This is the case for many things, like bridges and buildings. Later in this chapter we will discuss the ideas of stress, strain and fracture.

If an object does not move (linearly or rotationally), we know the forces and torques are completely balanced and so

$$\sum F_x = 0 \tag{12.1}$$

$$\sum F_y = 0 \text{ first condition} \tag{12.2}$$

$$\sum F_z = 0 \tag{12.3}$$

and

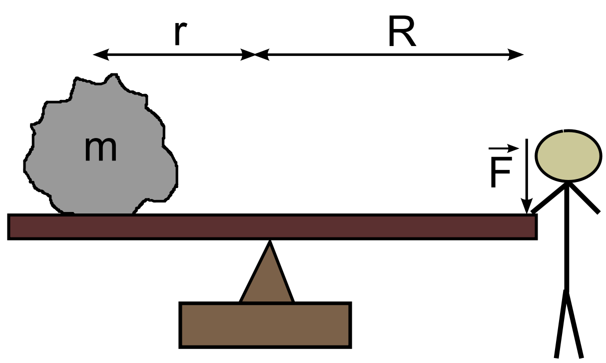
$$\sum \tau_x = 0 \tag{12.4}$$

$$\sum \tau_y = 0 \text{ second condition} \tag{12.5}$$

$$\sum \tau_z = 0. \tag{12.6}$$

Note that we have to choose an axis to calculate the torque. For equilibrium, the torques calculated w.r.t. *any* axis must sum to zero. Therefore, we typically choose an axis that makes our calculation simpler.

example: Consider a lever as shown below. Ignoring the weight of the lever itself, derive a relationship between the applied force and the weight of the rock. Assume the lever is in equilibrium.



The forces must all balance. The larger the applied force, the larger the fulcrum pushes up. If we choose the fulcrum as the pivot point and calculate torque about this point, then the torque due to $\vec{F}_{applied}$ is $-F_{applied}R$ and the torque due to the rock is mgr. We have chosen counterclockwise (out of the page) as positive. This is the standard choice. These torques must balance, so

$$-mgr + F_{applied}R = 0 \tag{12.7}$$
$$\frac{F_{applied}}{mg} = \frac{r}{R}. \tag{12.8}$$

In order to balance the weight of the rock, the applied force must be large, or the momentum arm must be large. Archimedes once said of the lever: "Give me a place to stand on, and I will move the Earth."

12.3 Solving Statics Problems

We typically work in a plane. Therefore, we have three equations:

$$\sum F_x = 0 \tag{12.9}$$

$$\sum F_y = 0 \tag{12.10}$$

$$\sum \tau = 0. \tag{12.11}$$

Often, gravity acts on our system. We assume its application is at the center of mass (CM). A standard method:

(a) Choose one object at a time and draw a F.B.F.D.

(b) Choose a convenient coordinate system and decompose all forces

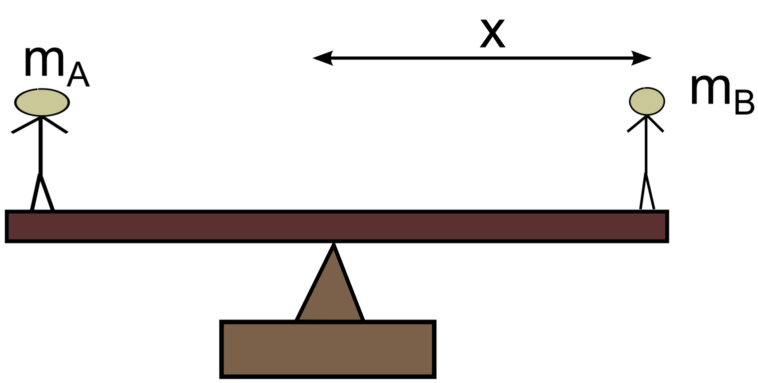
(c) Choose a simple axis of rotation, one that eliminates one or more forces

(d) Fill out the equilibrium conditions for forces and torques

(e) Solve the resulting system of equations

note: Sometimes we don't know the direction of a force, so just guess and the final result may have a negative sign.

example: A long board of mass $M = 2.0$ kg is balanced over a pivot. One child ($m_A = 30kg$) is to the left of the pivot by 2.5 m. The other child ($m_B = 25$ kg) is to the right of the pivot by a distance x. What is x such that the system is in equilibrium?



If the normal force exerted by the pivot is F_N, the torque equation about the pivot for equilibrium is:

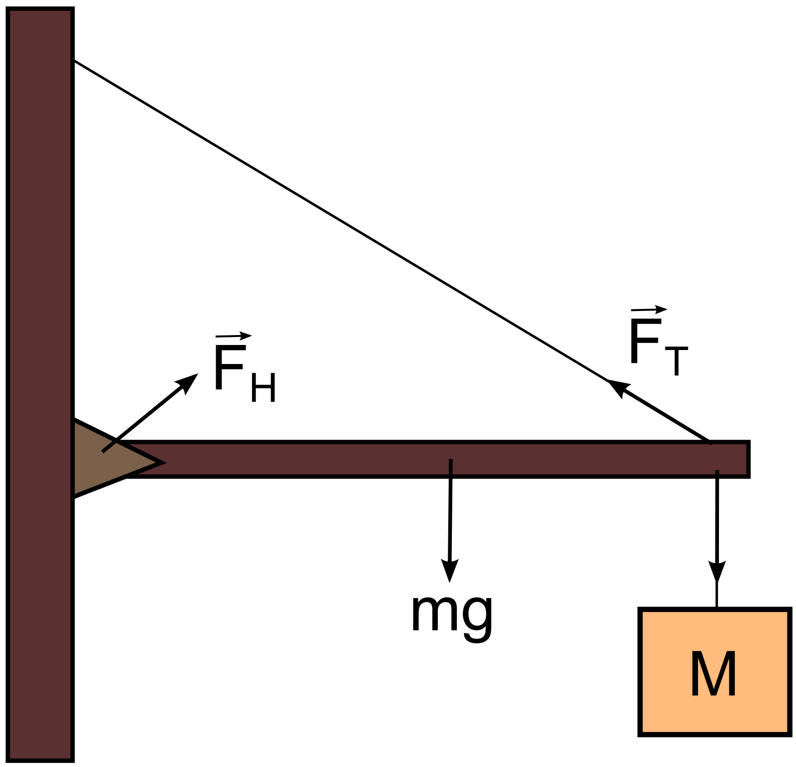
$$\sum \tau = 0 \tag{12.12}$$

$$m_A g 2.5 - m_B x + Mg(0) + F_N(0) = 0 \tag{12.13}$$

$$m_A g 2.5 - m_B x = 0 \tag{12.14}$$

$$x = \frac{m_A}{m_B} 2.5 = 3.0 \text{ m.} \tag{12.15}$$

example: Find the force of the hinge and the force of tension in the cable for a 2.2 m long beam of mass $m = 25$ kg that supports a sign of mass $M = 28$ kg. The angle of the cable is 30°.



We have three unknowns (F_{Hx}, F_{Hy} and F_T), so we will need all three equations. Let's choose the hinge as the pivot (although, we could choose the end as well!).

$$\sum F_y = 0 \;\rightarrow\; F_{Hy} + F_{Ty} - mg - Mg = 0 \tag{12.16}$$

$$\sum F_x = 0 \;\rightarrow\; F_{Hx} + F_{Tx} = 0 \tag{12.17}$$

$$\sum \tau = 0 \;\rightarrow\; -F_{Hy}(2.20) + mg(1.1) = 0 \tag{12.18}$$

$$\tag{12.19}$$

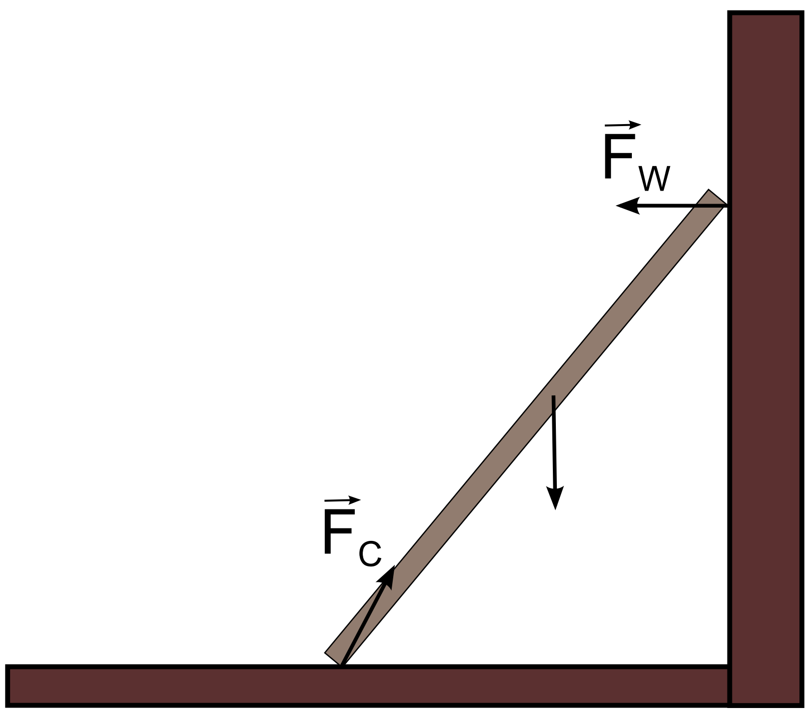
The torque equation gives $F_{Hy} = \frac{1.1}{2.2}mg = 123$ N. Since $\tan(30^\circ) =$

F_{Ty}/F_{Tx}, $F_{Ty} = 0.577 F_{Tx}$. From the first equation,

$$F_{Ty} = (m + M)g - F_{Hy} = 396 \text{ N} \tag{12.20}$$

So $F_{Tx} = 687$ N and $F_{Hx} = F_{Tx} = 687$ N. This gives $F_T = \sqrt{F_{Tx}^2 + F_{Ty}^2} =$ 793 N.

example: A 5.0 m long ladder leans against a wall at a point 4.0 m high. The mass of the ladder is uniform and is 12.0 kg. What are the forces on the ladder from the wall and the cement floor? (the wall is frictionless, the floor is not).



We have three unknowns: F_{Cx}, F_{Cy} and F_W. Therefore, we will need all three equations.

$$\sum F_y = 0 \;\rightarrow\; F_{Cy} - mg = 0 \tag{12.21}$$

$$\sum F_x = 0 \;\rightarrow\; F_{Cx} + F_W = 0 \tag{12.22}$$

$$\sum \tau = 0 \;\rightarrow\; F_W(4) - mg(1.5) = 0 \tag{12.23}$$

$$\tag{12.24}$$

Notice that the base is 3.0 m from the wall, so mg acts 1.5 m from the pivot (perpendicular distance). $F_W = 44$ N from the torque equation. $F_{Cx} = F_W = 44$ N, and $F_{Cy} = mg = 118$ N. The direction of $\vec{F}_C$ is $\theta = \tan^{-1}(118/44) = 70°$. This is not the angle of the ladder, unlike a tension force.

12.4 Stability and balance

There are three types of equilibrium:

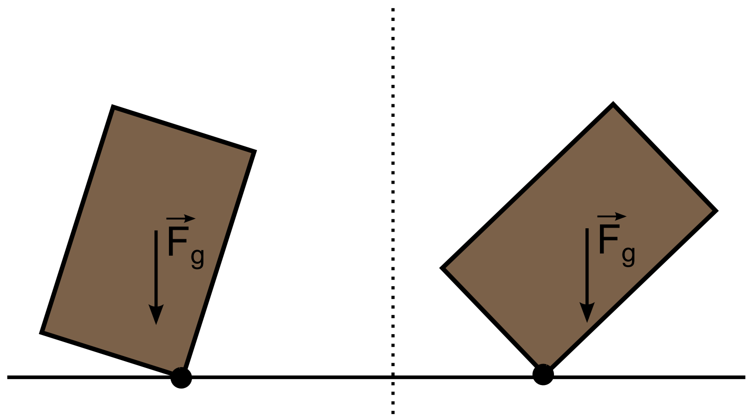
(a) Stable, like a ball on a string

(b) Unstable, like a ball at the top of a hill

(c) Neutral, like a ball on a flat plane

We can determine the type of equilibrium and look at the ensuing motion. Which way do the forces point?

Consider a brick: how far do I have to tilt it for the brick to fall over instead of falling back? When the center of mass passes over the point of contact! This is due to the torque due to gravity about the pivot.



12.5 Elasticity

Every solid object, from iron to bone, follows Hooke's Law:

$$F = k\Delta l \tag{12.25}$$

If we apply a force F to it, its length will change by an amount Δl. The value of k is different for each object, depending on its material, length and thickness. In general,

$$k = \frac{EA}{l_0} \tag{12.26}$$

where A is the cross-sectional area, l_0 is the original length of the object, and E is Young's Modulus. We can stretch a material up to the **proportional limit**. Past this, the behavior of the material is no longer linear. If we keep stretching the material past this limit, there is a "plastic region" and then a breaking point. Once in the plastic region, the object will no longer return to its original length.

example: A 1.60 m long steel piano wire stretches by 0.25 cm when tightened. If the diameter of the wire is 0.20 cm, what is the tension?

We know $F = k\Delta l = E\Delta l A/l_0$. We look up $E = 2 \times 10^11$ N/m^2 and calculate the area of the wire as $A = \pi r^2$ to find $F = 980$ N.

We can rearrange the equation above to find

$$F = k\Delta l = EA\frac{\Delta l}{l_0} \tag{12.27}$$

$$\frac{\Delta l}{l_0} = \frac{1}{E}\frac{F}{A} \tag{12.28}$$

Stress is what we call force per unit area F/A, and **strain** is the ratio of the change in length to its original length $\Delta l/l_0$. So E = stress/strain. There are three types of stress: tensile (pulling), compressive (pushing) and shear (transverse). Shear is slightly different:

$$\frac{\Delta}{l_0} = \frac{1}{G}\frac{F}{A}. \tag{12.29}$$

The constant G is called the shear modulus, and the the Δl here is how far one surface shears to the side.

We can compress a material from all sides by applying a uniform force over the outside area of a material (think: submerging an object in water).

In this case, the volume of the object may change according to

$$\frac{\Delta V}{V_0} = -\frac{1}{B}\Delta P \tag{12.30}$$

where B is the bulk modulus and ΔP is the change in pressure applied to the object (pressure, like strain, is just force/area).

There are three moduli: Young's (for tensile and compressive), Shear and Bulk. They all have units of N/m^2.

12.6 Fracture

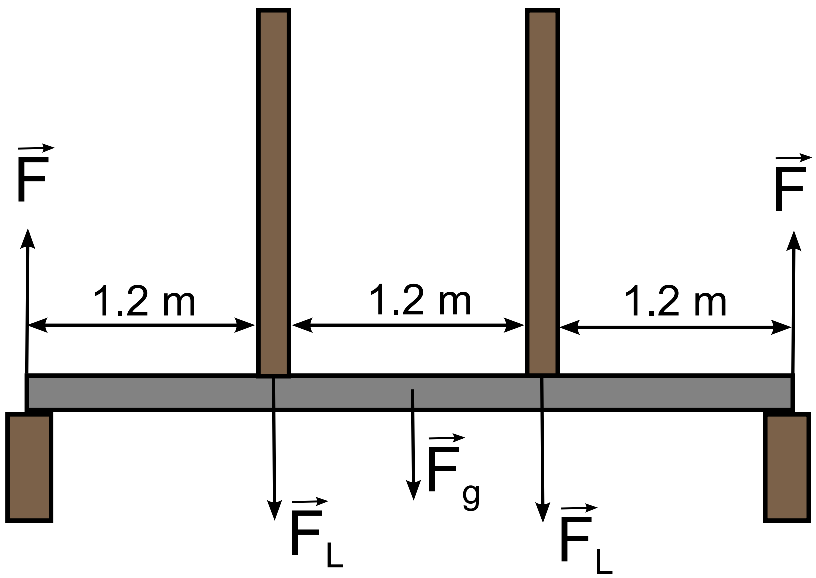
If we apply too much force to an object, it will fracture. The nature of the force (tensile, compressive or shear) determines how much stress an object and take. For instance, concrete can handle 20 MN/m^2 of compression, but only 2 MN/m^2 of tension.

example: What force do we apply to the piano wire to break it?

We look up the ultimate tensile strength of steel and find $F/A = 500 \times 10^6$ N/m^2. The area is still $\pi r^2 = \pi(10^{-3})^2$. This gives

$$F = A \times 500 \times 10^6 = 1600 \text{ N.} \tag{12.31}$$

example: What is the maximum load on a beam shown in the figure below such that the shearing forces at the end do not break the beam? The beam is pine, 25 kg, 3.6 m long with a cross-sectional area of $A = .013$ m^2.



First, let us find the maximum shearing force we can apply at the ends of the beam. The strength is listed as 5×10^6 N/m^2. To be safe, let's take 1/5 of that, so 10^6 N/m^2. Therefore, the maximum force is $F = 10^6 A = 13000$ N.

The torque about the left end is just

$$\sum \tau = -F_L 1.2 - 25 \times 9.81.8 - F_L 2.4 + F3.6 = 0 \tag{12.32}$$

$$F_L = 12900 \text{ N} \tag{12.33}$$

Chapter 13

Gravitation

13.1 Opening Question

A space station revolves around the Earth as a satellite, 100 km above the Earth's surface. What is the net force on an astronaut at rest inside the space station?

- Equal to her weight on Earth
- A little less than her weight on Earth
- Almost zero
- Zero (she is weightless)
- Somewhat larger than her weight on Earth

13.2 Newton's Law of Universal Gravitation

It was clear by Newton's time that large objects in the sky move in circular or elliptical orbits: but why? Newton sought to find the answer to this question.

What if large bodies pulled toward each other by some invisible force? By Newton's third law, if the earth pulls on the moon, the moon pulls on the earth. Think about the example of a ball on a string, where the string is replaced by "gravity."

First, we can measure the acceleration here on the surface of the earth and find it to be a nearly constant value of $g = 9.80$ m/s^2. Using our understanding of circular motion $a = v^2/R$, we can find how quickly the moon is accelerating toward the earth! A quick calculation shows that $a_m \approx \frac{1}{3600}g$. But we also know that the moon is 60 times farther than the surface of the Earth from center of the earth. Noting that $60^2 = 3600$, it seems to be that the acceleration (and thus the force) is related to the inverse square of the distance.

$$F \propto 1/R^2. \tag{13.1}$$

The farther away two objects are, the less they pull on each other. Also, we know that heavier objects are pulled more strongly toward the ground, so the force must be proportional to the mass as with $F = ma$. But remember, the force is symmetric (Newton's Third Law). Therefore, the force must be proportional to both masses:

$$F \propto \frac{m_1 m_2}{R^2} \tag{13.2}$$

The last piece of the equation comes for the idea that physics doesn't change from one part of the Universe to another. Therefore, the proportionality constant should not vary. This gives

$$F = \frac{G m_1 m_2}{R^2}, \tag{13.3}$$

where $G = 6.67 \times 10^{-}11$ N m^2/kg^2 is the experimentally/observationally determined universal gravitational constant.

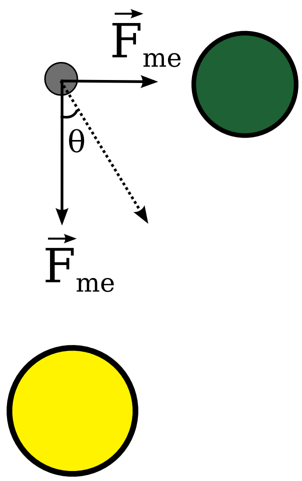
For *geophysical* applications, we consider an object on the surface of the earth with only the force due to gravity acting on it, and find

$$F = mg = G\frac{m m_e}{R_e^2} \rightarrow g = G\frac{m_e}{r_e^2}. \tag{13.4}$$

That is, we can determine g if we know the mass and radius of the earth.

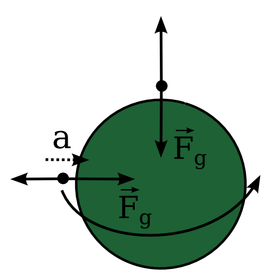
Important mathematical note: Using Gauss' Law, we can show that the force exerted on a particle due to a large spherical body of mass m is the same as the force exerted by a point mass m with no size. Therefore, we think of the Sun, Earth and Moon as points in space.

example: The gravitational forces from the sun and the earth act on the moon. Calculate the magnitude of each force. Assuming right angles, what is the resultant vector's angle?



For the earth, $F_{me} = Gm_m m_e/R_{me}^2 = 1.99 \times 10^{20}$ N. For the sun, $F_{ms} = Gm_m m_s/R_{ms}^2 = 4.34 \times 10^{20}$ N. They are the same order of magnitude! The angle is given by $\tan(\theta) = F_{me}/F_{ms} \rightarrow \theta = 24.6°$.

example: What is the effect of the earth's rotation on g? Compare an object of mass m at the pole and at the equator. Use a spring scale which measures force F_W.



At the pole, there is no acceleration, so $mg - F_W = 0 \rightarrow F_W = mg$ as expected.

At the equator, you are in circular motion, so now $mg - F_W' = mv^2/r_e$, where $v = 2\pi r_e/1$ day $= 464$ m/s. $F_W' = m(g - v^2/r_e)$. Therefore, the new g' is just $g' = F_W'/m$. The change is g is therefore $v^2/r_e = 0.0337$ m/s^2.

example: Consider a satellite in a circular orbit around the Earth. What keeps it from falling? Newton's second law states

$$F = ma \rightarrow G\frac{m_s m_e}{r^2} = m\frac{v^2}{r}. \tag{13.5}$$

The only force is due to gravity, and the acceleration is v^2/r. Therefore, its velocity must be $v = \sqrt{Gm_e/r}$ (depends on the mass of the earth and the radius of orbit).

13.3 Weightlessness

In an elevator, as it accelerates downward, the floor pushes on your feet less. If the elevator accelerated at g, you would feel weightless. Objects in the elevator would appear to float.

Has gravity disappeared? No. This is "apparent weightlessness," and it is what astronauts feel. Everything floats around them, but they are still accelerating toward the earth.

The only true weightlessness is found in the abyss of space.

13.4 Kepler's Laws

From detailed observations made by Tycho Brahe, Johannes Kepler (1571-1630) derived three laws of planetary motion. He did not know *why* the planets followed these trajectories, only how to describe the motion, and predict it.

- The path of each planet about the sun is an ellipse with the sun at one focus
- Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal time periods
- The periods and semi-major axis for any two planets has the following relationship: $s_1^3/T_1^2 = s_2^2/T_2^2$.

These laws are consistent with Newton's equations, assuming the planets do not interact with each other. Observed perturbations are due to planet-planet gravitational interaction.

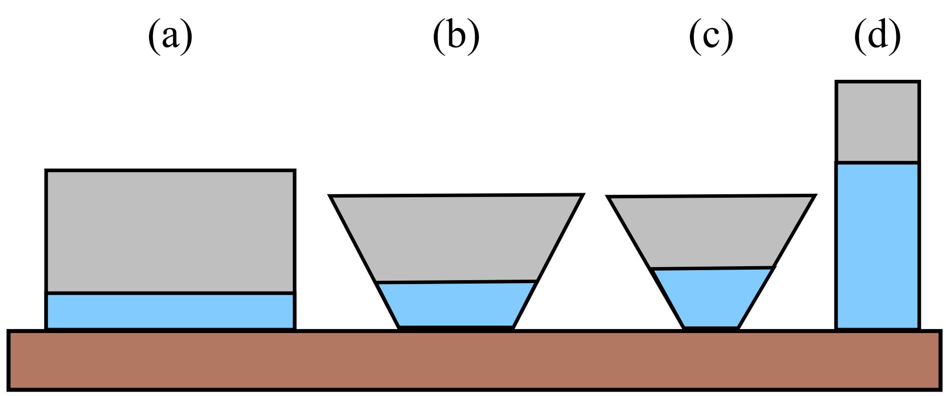
Discuss Lagrange points (if there is time).

Chapter 14

Fluids (not covered)

14.1 Opening Question

Which container has the largest pressure at the bottom? Each container has the same volume of water.



14.2 Phases of Matter

In this chapter, we look at non-rigid objects, like liquids and gases. The three phases of matter:

(a) Solid - fixed shape, fixed volume

(b) Liquid - take on shape of container, fixed volume

(c) Gas - fills the container, not a fixed volume

Liquids and gases are sometimes called "fluids" since they have the ability to flow. There are other more complex phases of matter: colloids, plasmas, liquid crystals.

14.3 Definition of Terms

Density ρ is the mass m of an object divided by its volume V:

$$\rho = \frac{m}{V}. \tag{14.1}$$

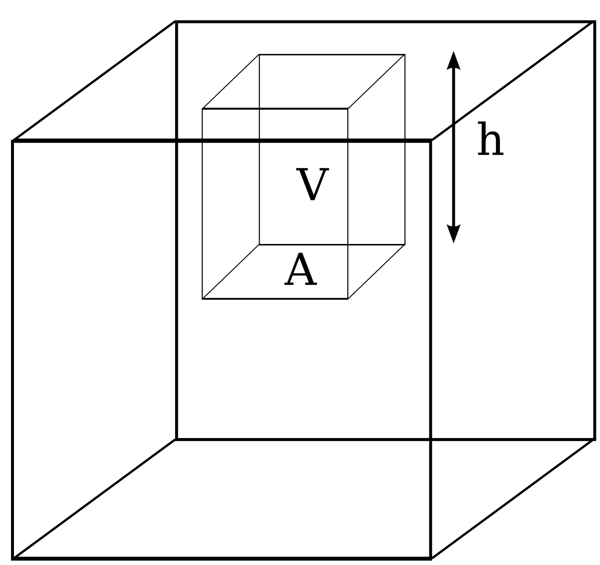
Any given element has a unique density associated with it. A block of gold, no matter how large, always has the same density. We can of course write: $m = \rho V$ to find an object's mass, or maybe its weight $mg = \rho V g$.

Specific gravity of a substance is the ratio of the density of that substance to the density of water at 4°C. This is convenient, because water has a density of 1 g/cm^3, or 10^3 kg/m^3.

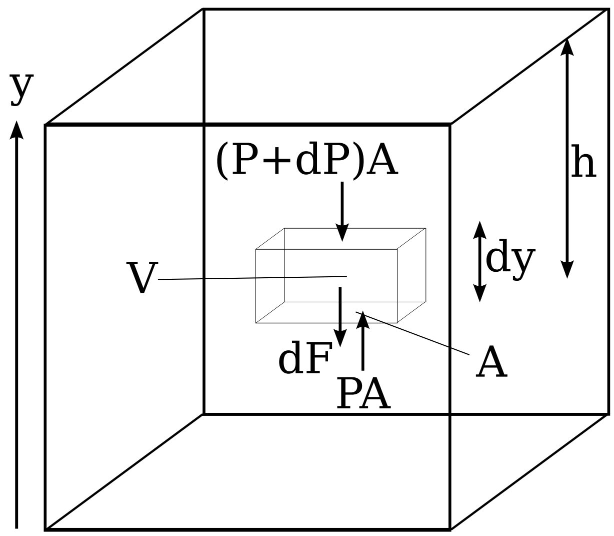
Pressure is the force per unit area. In particular, we only consider the force perpendicular to the surface. The S.I. unit of pressure is N/m^2, which is called the pascal (Pa).

14.4 Fluids

- A fluid exerts an equal pressure in all directions in a non-flowing (static) fluid at a given height. Otherwise, the fluid would flow.
- At the surface of a container, the fluid exerts only a perpendicular force on the surface. Otherwise, the fluid would flow!
- What is the pressure in a liquid of density ρ at a depth h? Consider a square area parallel with the surface. What is the force that the liquid exerts on this square? $F = mg = (\rho V)g = \rho Ahg$. So the pressure is just $P = F/A = \rho hg$. The pressure does not depend on the area.



There is a more general way to treat pressure when the density changes with height (such as in the atmosphere). Consider an infinitesimally narrow slab of fluid with an area A and a width dy.



There are three forces on the slab:

$$dF_G = -dmg = -\rho g dV = -\rho g A dy \tag{14.2}$$

$$F_{bottom} = PA \tag{14.3}$$

$$F_{top} = -(P + dP)A \tag{14.4}$$

We know these forces have to balance, so $0 = PA - (P - dP)A - \rho g A dy = AdP - \rho g A dy$. This results in

$$\frac{dP}{dy} = -\rho g. \tag{14.5}$$

This gives us the variation of the pressure as a function of height. It depends only on the density of the liquid and g. The negative sign indicates that the pressure increases as we move deeper. We can integrate to get:

$$\int_{P_1}^{P_2} dP = -\int_{y_1}^{y_2} \rho g dy \tag{14.6}$$

$$\Delta P = -\int_{y_1}^{y_2} \rho g dy \tag{14.7}$$

For a constant density fluid (such as water in a swimming pool), we can immediately integrate to obtain

$$\Delta P = -\rho g \Delta y \tag{14.8}$$

$$P_2 - P_1 = -\rho g \Delta(y_2 - y_1). \tag{14.9}$$

If we call $h = y_2 - y_1$ the depth of the liquid and let P_2 be the pressure at the surface, then the pressure below the surface is just

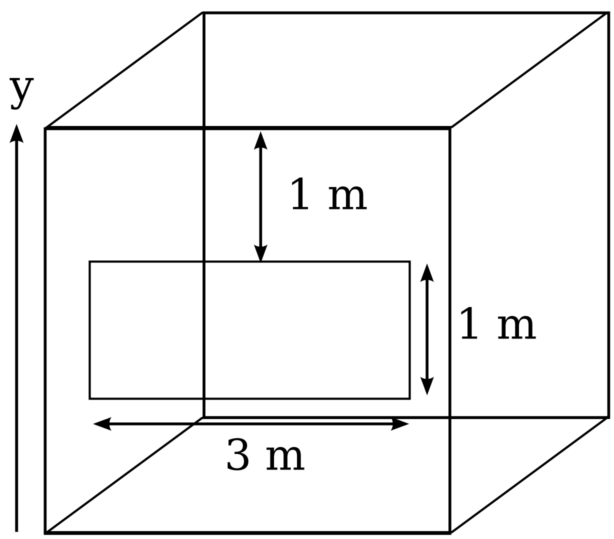
$$P_1 = P_2 + \rho g h. \tag{14.10}$$

Often, P_2 is just the atmospheric pressure (all that air pushing down on you!).

example: The surface of the water in a silo is 30 m above the faucet in your home. What is the difference in the water pressure between the surface and your faucet?

$$\Delta P = \rho g h = 10^3 \times 9.8 \times 30 = 2.9 \times 10^5 \text{N/m}^2. \tag{14.11}$$

example: What is the force due to water pressure on a 1.0×3.0 m^2 window whose top edge is 1.0 m below the water surface? The long edge $l = 3$ m is horizontal.



We divide the window up into horizontal strips of height dy. The small amount of force on this strip is just $dF = PdA = \rho g y l dy$. We integrate each bit of force, obtaining:

$$F = \int dF = \int_{y_1=1.0}^{y_2=2.0} \rho g y l dy = \frac{1}{2}\rho g l(y_2^2 - y_1^2) = 44000 \text{ N}. \tag{14.12}$$

Let us now consider a gas, where the density changes significantly as a function of height. Assume that the density is proportional to pressure, that is, $\rho = CP$, so that,

$$\frac{\rho}{\rho_0} = \frac{P}{P_0} \rightarrow \rho = \frac{\rho_0}{P_0}P, \tag{14.13}$$

where the 0 indicates atmospheric pressure and density. We can then use this in the relationship from before,

$$\frac{dP}{dy} = -\rho g = -P\frac{\rho_0}{P_0}g \rightarrow \frac{dP}{P} = -\frac{\rho_0}{P_0}g dy \tag{14.14}$$

We integrate this from $y = 0$ ($P = P_0$) at Earth's surface to some other height y (P). This gives

$$\int \frac{dP}{P} = -\frac{\rho_0}{P_0}g \int dy \tag{14.15}$$

$$\ln \frac{P}{P_0} = -\frac{\rho_0}{P_0}gy \tag{14.16}$$

$$P = P_0 e^{-(\rho_0 g/P_0)y} \tag{14.17}$$

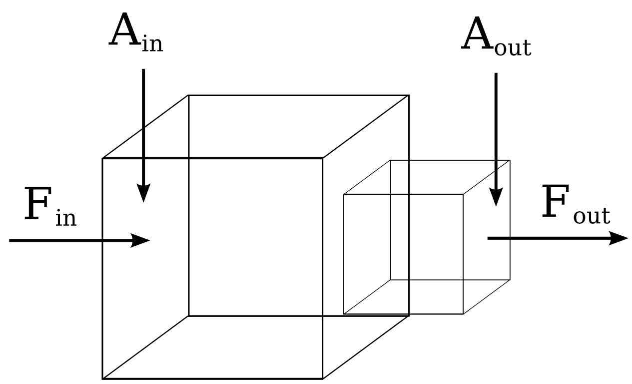
The pressure decreases exponentially with height!

The pressure at sea level on Earth averages about 1.013×10^5 N/m^2. This is given another name, called the atmosphere (atm). There is also a measure called the bar, where 1 bar = 1.000×10^5 N/m^2.

14.5 Pascal's Principle

If an external pressure is applied to a confined fluid, the pressure at every point within the fluid increases by that amount.

We can use this principle in applications such as a hydraulic lift. Consider a fluid confined to a vessel, with one piston on each end. If the input piston is smaller than the output piston, we can amplify the applied force.



We know that $P_{in} = P_{out}$, but $P = F/A$, so

$$\frac{F_{in}}{A_{in}} = \frac{F_{out}}{A_{out}} \tag{14.18}$$
$$\frac{A_{out}}{A_{in}} = \frac{F_{out}}{F_{in}} \tag{14.19}$$
$$\tag{14.20}$$

The ratio F_{out}/F_{in} is called the mechanical advantage. This is also how the brakes in a car work.

There are many types of pressure measuring devices. One such device is the open-tube manometer. One end connects to the atmosphere, and the other end connects to the pressure you want to measure. The difference in the heights of the two ends gives the pressure:

$$P = P_0 + \rho g \Delta h. \tag{14.21}$$

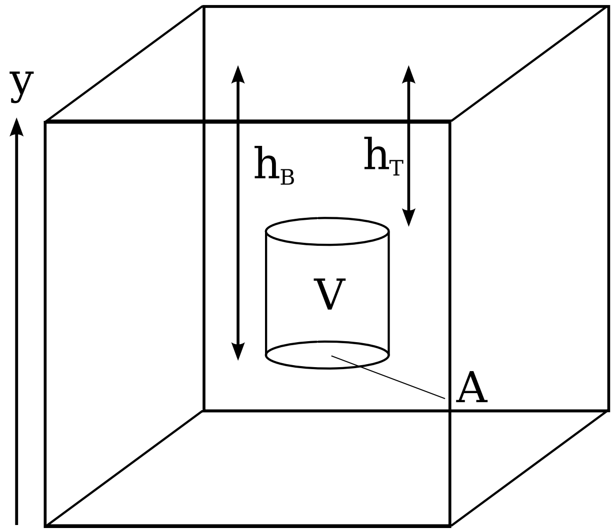
Suction is a funny thing: what is actually happening? We reduce the pressure on one end, and the atmosphere pushes the liquid up. But there is a limit to how far the atmosphere can push.

$$\begin{aligned} P &= \rho g \Delta h && (14.22)\\ \Delta h &= \frac{P_0}{\rho g} && (14.23)\\ &= \frac{1.013 \times 10^5}{10^3 \times 9.81} && (14.24)\\ &= 10.3 \text{ m} && (14.25) \end{aligned}$$

14.6 Buoyancy and Archimeedes' Principle

Why do things in water appear to be lighter than things in air? This is due to the pressure change as a function of height. Although gravity is pulling the object down, the fluid is pushing the object up because the pressure (force) near the bottom of the object is larger than the force near the top.

Think about a cylinder submerged in a fluid of density ρ_F.



The forces on the top and bottom of the cylinder are just

$$F_{top} = P_T A = \rho_F g h_T A \tag{14.26}$$

$$F_{bottom} = P_B A = \rho_F g h_B A \tag{14.27}$$

$$F_B = F_{bottom} - F_{top} = \rho_F g A(h_B - h_T) \tag{14.28}$$

$$= \rho_F g V \tag{14.29}$$

$$= m_F g \tag{14.30}$$

where F_B is the buoyant force and m_F is the mass of the displaced fluid.

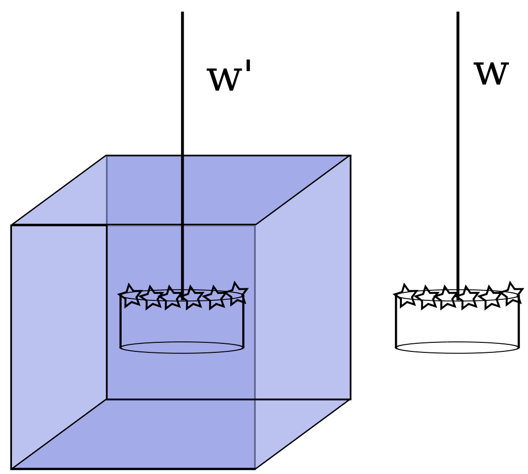
The shape of the object is irrelevant. Archimedes' principle states:

> The buoyant force on an object immersed in a fluid is equal to the weight of the fluid displaced by that object.

This is actually just a consequence of Newton's Laws. Consider an object submerged in liquid. Then, consider an object of the exact same dimensions, but made up by the liquid itself. Are the forces not balanced?

example: Two pails of water are filled to the brim, but one has a log of wood floating on top. Which pail weighs more? (they weigh the same)

example: Think about a crown made of some unknown material. A scale measures its weight to be 147 N, but when its submerged in water, its weight is measured to be 134 N. Is it made of gold?



Out of the water, the weight is simply $w = mg = \rho V g$, where V is the unknown volume and ρ is the unknown density.

In the water, the measured weight w' is given by $w - w' - F_B = 0 \rightarrow w - w' = F_B = \rho_F V g$, where ρ_F is the known density of water.

If we divide these two equations, we find

$$\frac{w}{w - w'} = \frac{\rho V g}{\rho_F V g} = \frac{\rho_0}{\rho_F} \tag{14.31}$$

The ratio of the densities is just the specific gravity! Therefore, we can compute that

$$\frac{w}{w - w'} = \frac{147}{147 - 134} = 11.3 \tag{14.32}$$

This is the specific gravity of lead, not gold.

The buoyant force ($\rho_F V g$)on a submerged object can be greater than the weight of the object ($\rho V g$) if its density is less than that of the fluid. If an object floats, the buoyant force is

$$F_B = mg \tag{14.33}$$

$$\rho_F V_{displaced} g = \rho_0 V_0 g \tag{14.34}$$

$$\frac{V_{displaced}}{V_0} = \frac{\rho_0}{\rho_F} \tag{14.35}$$

The fraction of the object submerged is given by the ratio of the object's density to that of the fluid.

example: What volume V of helium is needed if a balloon is to lift a load of 180 kg?

The buoyant force must have a value of $F_B = (m_{he} + 180)g = \rho_{air} V g$.

This gives a volume of

$$V = \frac{180}{\rho_{air} - \rho_{he}} = 160 \text{ m}^3. \tag{14.36}$$

This is only true near the surface of earth, where $\rho_{air} = 1.29$ kg/m^3.

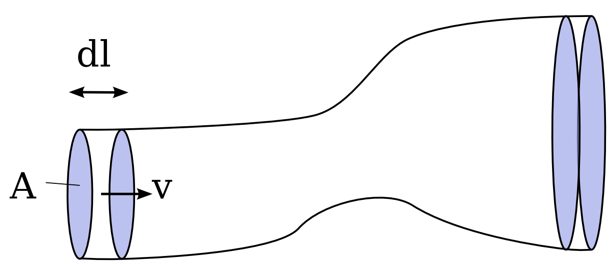
14.7 Fluids in Motion

Fluid dynamics (or hydrodynamics in the case of water). If the flow of the fluid is smooth, the flow is called **streamline** or **laminar**. Laminar means "in layers." Above a certain speed, the flow becomes turbulent, where eddy currents start to arise.

In laminar flow, we can define the mass flow rate:

$$\text{mass flow rate} = \frac{\Delta m}{\Delta t}, \tag{14.37}$$

which describes how much mass of a fluid passes a certain point along a tube.



Let's also define a few terms:

$$\begin{aligned}
A_1 & \quad \text{the cross-sectional area of the tube} & (14.38)\\
\Delta l_1 & \quad \text{the distance the fluid travels from point 1 in } \Delta t & (14.39)\\
v_1 &= \Delta l_1/\Delta t \text{ velocity of fluid at point 1} & (14.40)\\
\Delta V_1 &= A_1 \Delta l_1 \text{ volume of fluid passing point 1 in } \Delta t & (14.41)
\end{aligned}$$

This allows us to write

$$\frac{\Delta m_1}{\Delta t} = \frac{\rho_1 \Delta V_1}{\Delta t} = \frac{\rho_1 A_1 \Delta l_1}{\Delta t} = \rho_1 A_1 v_1 \tag{14.42}$$

Of course, this is true later down the tube, where the area, velocity and density may be different: $\rho_2 A_2 v_2$. Since no fluid flows out of the tube, the mass flow rate must be the same everywhere, giving

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \tag{14.43}$$

This is the **equation of continuity**. For incompressible fluids, we just have

$$A_1 v_1 = A_2 v_2 \tag{14.44}$$

Each side is called the volume rate of flow, with units m^3/s.

example: The radius of the aorta is 1.2 cm, and the radius of a capillary is just 4×10^{-4} cm. If the blood drops from 40 cm/s in the aorta to 5×10^{-4} m/s in a capillary, how many capillaries are there?

We know that all the blood in the aorta must flow through all of the capilaries, so we have something like:

$$A_a v_a = N A_c v_c \tag{14.45}$$

$$N = \frac{A_a v_a}{A_c v_c} \tag{14.46}$$

$$= \frac{r_a^2}{r_c^2}\frac{v_a}{v_c} \tag{14.47}$$

$$= 7 \times 10^9 \tag{14.48}$$

Just remember, all that mass has to go somewhere!

14.8 Bernoulli's Equation

How does and airplane fly? Where does the lift force come from?

Bernoulli's principle states:

> Where the velocity of a fluid is high, the pressure is low; where the velocity is low, the pressure is high.

Don't think about the pressure a fluid exerts on an object in its path: this would stop the fluid entirely, and would not be laminar! Instead, think of the pressure within the fluid at it flows.

Let's derive this relationship quantitatively. Consider a section of tube that gets narrorwer as the fluid flows to the right. The volume of fluid starts at the left and then moves an some amount to the right.

$$\Delta l_1 \quad \text{how far the left edge moves} \tag{14.49}$$

$$\Delta l_2 \quad \text{how far the right edge moves} \tag{14.50}$$

$$A_1 \quad \text{area of the left edge} \tag{14.51}$$

$$A_2 \quad \text{area of the right edge} \tag{14.52}$$

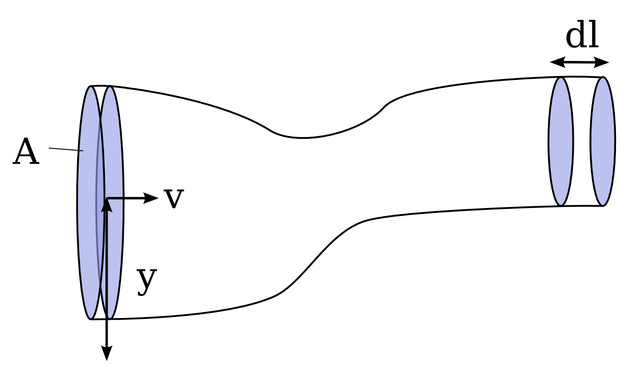
$$v_1 \quad \text{velocity at the left edge} \tag{14.53}$$

$$v_2 \quad \text{velocity at the right edge} \tag{14.54}$$

$$P_1 \quad \text{pressure exerted at the left edge} \tag{14.55}$$

$$P_2 \quad \text{pressure exerted at the right edge} \tag{14.56}$$

$$\tag{14.57}$$



How much work does the pressure at the left edge do? $W_1 = F_1 \Delta l_1 = P_1 A_1 \Delta l_1$.

Similarly, the work done by the pressure at the right edge is just $W_2 = -F_2 \Delta l_2 = -P_2 A_2 \Delta l_2$.

There is also work done by gravity to move the mass from point 1 to point 2. Therefore, $W_g = mg(y_2 - y_1)$, where m is the mass in the volume $A_1 \Delta y_1 = A_2 \Delta y_2$. The work-energy principle says this equals the change in kinetic energy!

$$\begin{aligned} W &= \Delta K && (14.58) \\ P_1 A_1 \Delta l_1 - P_2 A_2 \Delta l_2 - mgy_2 + mgy_1 &= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 && (14.59) \end{aligned}$$

If we replace $m = \rho A_1 \Delta l_1 = \rho A_2 \Delta l_2$, and then divide through by $A_1 \Delta l_1$ or $A_2 \Delta l_2$ as appropriate, we find

$$\begin{aligned} \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 &= P_1 - P_2 - \rho g y_2 + \rho g y_1 && (14.60) \\ P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 &= P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2 && (14.61) \\ P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 &= \text{constant} && (14.62) \end{aligned}$$

Since points 1 and 2 were chosen arbitrarily, we know this has to be a constant. What are the ramifications of this? This is **Bernoulli's equation**.

example: A hot water heater pumps water at 0.50 m/s through a 2.0 cm radius pipe in the basement under a pressure of 3.0 atm. What is the flow speed and the pressure through a 1.3 cm radius pipe on the second floor, 5.0 m above?

For an incompressible fluid, $A_1 v_1 = A_2 v_2$, so $v_2 = A_1 v_1 / A_2 = 1.2$ m/s.

To find the pressure, we use Bernoulli's equation, where 2 is the second floor, and 1 is the basement:

$$
\begin{aligned}
P_2 &= P_1 + \rho g(y_1 - y_2) + \frac{1}{2}(v_1^2 - v_2^2) && (14.63)\\
&= 2.5 \text{ atm} && (14.64)
\end{aligned}
$$

For an open container with a spigot, like a reservoir, the velocity of the liquid at the top is zero, and the pressures are just atmospheric, so Bernoulli's equation becomes

$$
\begin{aligned}
\frac{1}{2}\rho v_1^2 + \rho g y_1 &= \rho g y_2 && (14.65)\\
v_1 &= \sqrt{2g(y_2 - y1)} && (14.66)
\end{aligned}
$$

This is known as Torricellie's theorem, and was discovered 1 century prior to Bernoulli's more general form. Notice that this is the speed of an object dropped from the height of the reservoir! (conservation of energy)

Airplane wing revisited: the velocity of the air above the wing is higher than that below the wing (due to the different travel distance), so the pressure must be lower.

Bernoulli's equation is very important to understand sailboats, baseball motion and a venturi (velocity) meter as well.

We have ignored the effects of friction, called viscosity for fluids and mostly ignored the compressibility.

Chapter 15

Oscillations

15.1 Opening Question

A heavy ball of mass m hangs from the ceiling by a massless cord of length l, forming a pendulum. The pendulum is pulled by an angle of 5° and then released, and it oscillates at a frequency f. If we then pull the pendulum to 10°, its frequency would be

(a) twice as great.

(b) half as great.

(c) the same.

(d) not quite twice as great.

(e) a bit more than half as great

15.2 Springs and Simple Harmonic Motion

We know that the force from a spring on a mass is $F = -kx$. This force always makes the mass oscillate around the equilibrium point (where $\sum F = 0$). Note that sometimes the force is in the opposite direction of the motion of the mass. At the ends of its oscillation, the mass comes to a rest, then speeds up again.

A few terms:

(a) x, displacement: the distance the mass is from the equilibrium point

(b) A, amplitude: the maximum displacement of the mass from equilibrium

(c) cycle: the complete to-and-fro motion of the mass

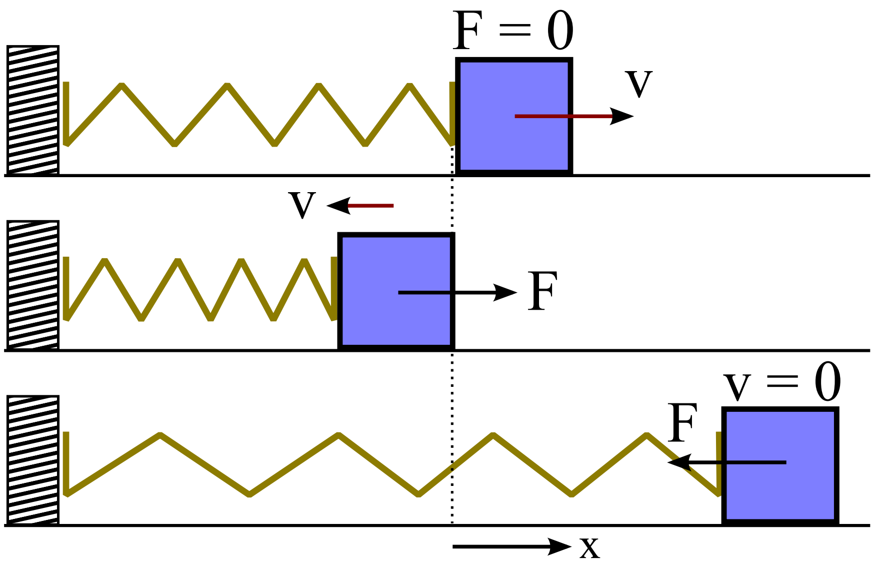
(d) T, period: the time it takes for one cycle

(e) f, frequency: the number of cycles in 1 second

Note that $f = 1/T$ and $T = 1/f$.

Even if a mass is hung vertically, we have the same behavior, but a new equilibrium point.

Simple harmonic motion is the motion of an object under the influence of a force like $F = -kx$. Examples include tuning forks, pendulums, vibrating rulers, guitar and piano strings, spider webs, buildings and bridges, cars, radio waves, light, atoms and molecules.



Let's consider Newton's Second Law with this force.

$$F = ma \tag{15.1}$$
$$-kx = m\frac{d^2x}{dt^2} \tag{15.2}$$
$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{15.3}$$

The last form is known as a differential equation and describes the motion of the mass subject to this force. We can try to solve this equation by making a smart guess:

$$x(t) = A\cos(\omega t + \phi) \tag{15.4}$$
$$\frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \tag{15.5}$$
$$\frac{dx^2}{dt^2} = -\omega^2 A\cos(\omega t + \phi). \tag{15.6}$$

Let's put this trial solution into the equation and see if it works:

$$\begin{aligned} \frac{d^2x}{dt^2} + \frac{k}{m}x &= 0 && (15.7) \\ -\omega^2 A\cos(\omega t+\phi) + A\frac{k}{m}\cos(\omega t+\phi) &= 0 && (15.8) \\ \left(\frac{k}{m} - \omega^2\right) A\cos(\omega t+\phi) &= 0 && (15.9) \end{aligned}$$

The left side equals zero if the cosine equals zero or if the part in parentheses equals zero. Since the cosine is not zero for all times t, we know that

$$\frac{k}{m} - \omega^2 = 0 \rightarrow \omega^2 = \frac{k}{m} \tag{15.10}$$

Using calculus, we can show that this is the general solution to the equation, and it has two constants: A and ϕ, which we would expect from the second derivative (two integrals). These two constants are determined by the **initial conditions**.

What if we stretch the mass on the spring, and then release it from rest at $t = 0$. In this case, $x = A\cos\omega t$. To confirm, we take a derivative:

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t+\phi) = 0 \tag{15.11}$$

$$\sin(\omega(0)+\phi) = 0 \tag{15.12}$$

$$\phi = 0, \pi, 2\pi, ... \tag{15.13}$$

Or we could have it resting at equilibrium and then give it a kick in the positive x-direction, so that

$$x = A\cos(\omega t + \phi) = A\cos(\phi) = 0 \rightarrow \phi = \pm\pi/2 \qquad (15.14)$$

In order to choose the correct sign, we look at the velocity:

$$v = -\omega A\sin(\omega t + \phi) = -\omega A\sin(\phi) > 0 \rightarrow \phi = -\pi/2 \qquad (15.15)$$

These are just two simple examples. The phase of the mass (determined by ϕ) tells you where the peak of the wave is at $t = 0$. The period of oscillation is given by ω, where $\omega = 2\pi f = 2\pi/T$. That is, the mass goes from positive to negative and back to positive in T seconds, so the cosine must change by 2π during this time. We therefore have

$$\begin{aligned} x(t) &= A\cos(2\pi t/T + \phi) && (15.16) \\ x(t) &= A\cos(2\pi f t + \phi) && (15.17) \end{aligned}$$

Remember that $\omega^2 = k/m$, so

$$\begin{aligned} f &= \frac{1}{2\pi}\sqrt{\frac{k}{m}} && (15.18) \\ T &= 2\pi\sqrt{\frac{m}{k}} && (15.19) \\ & && (15.20) \end{aligned}$$

The frequency and the period do not depend on the amplitude of the oscillations! How COOL.

The S.H.O. moves as such:

$$x = A\cos(\omega t + \phi) \tag{15.21}$$

$$v = -\omega A\sin(\omega t + \phi) \tag{15.22}$$

$$a = -\omega^2 A\cos(\omega t + \phi) \tag{15.23}$$

The position and acceleration are *in phase*, but the velocity is *out of phase*. We also note that

$$x_{max} = A \tag{15.24}$$

$$v_{max} = \omega A = \sqrt{\frac{k}{m}}A \tag{15.25}$$

$$a_{max} = \omega^2 A = \frac{k}{m}A \tag{15.26}$$

For $\phi \neq 0$, we can find A and ϕ from,

$$x_0 = x(0) = A\cos(\phi) \tag{15.27}$$

$$v_0 = v(0) = -\omega A\sin(\phi) = -v_{max}\sin(\phi) \tag{15.28}$$

$$x_0 = x(0) = -\omega^2 A\cos(\phi) = -a_{max}\cos(\phi) \tag{15.29}$$

example: The cone of a speaker oscillates at 262 Hz. The amplitude is $A = 1.5 \times 10^{-4}$ m and at $t = 0$, $x = A$.

(a) What is the equation that describes the motion? $\omega = 2\pi f = 2\pi(262) = 1650$ rad/s. From our initial condition $x_0 = A\cos(\phi) = A$, we know that $\phi = 0$. Therefore, $x(t) = (1.5 \times 10^{-4})\cos(1650t)$ m.

(b) What about the velocity and acceleration?

$$v(t) = \frac{dx}{dt} = -0.25\sin(1650t) \tag{15.30}$$
$$a(t) = \frac{dv}{dt} = -410\cos(1650t) \tag{15.31}$$
$$\tag{15.32}$$

(c) What is the position of the cone at $t = 1$ ms? $x(0.001) = (1.5 \times 10^{-4})\cos(1650 \times 0.001) = -1.2 \times 10^{-5}$ m.

15.3 Energy in the Simple Harmonic Oscillator

We know that a spring stores potential energy equal to $U_s = \frac{1}{2}kx^2$, where x is the distance from equilibrium. Therefore, as a mass oscillates on a spring, its total energy is given by

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \tag{15.33}$$

When the mass is fully stretched and comes to a stop, then $E = 0 + \frac{1}{2}kA^2$. Since energy is conserved, this is the total amount of energy stored in the system. Therefore, at any time,

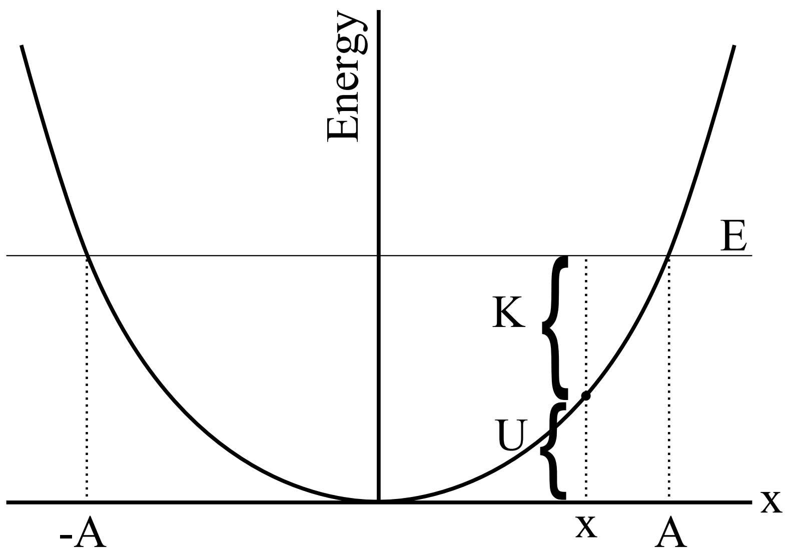
$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{15.34}$$

$$kA^2 = mv^2 + kx^2 \tag{15.35}$$

$$v = \pm\sqrt{\frac{k}{m}\left(A^2 - x^2\right)} \tag{15.36}$$

$$v = \pm v_{max}\sqrt{1 - \frac{x^2}{A^2}} \tag{15.37}$$

since $v_{max} = A\sqrt{k/m}$.



example: If a spring is stretched twice as far, how does its

(a) energy change? $E = \frac{1}{2}kA^2$, so doubling A would quadruple the energy.

(b) max velocity change? From above, we see that v_{max} is proportional to A, so doubling the amplitude would double the max velocity.

(c) max acceleration change? Again, $a_{max} = kA/m$, so doubling A doubles the acceleration.

Note that circular motion, viewed from the side, is exactly like simple harmonic motion (see your text).

15.4 The Simple Pendulum

We have seen the simple pendulum before and have used energy conservation to study its behavior. If we look at the displacement of the bob away from equilibrium, gravity tries to restore the bob back to equilibrium. The magnitude of this force is

$$F = -mg \sin\theta \approx= -mg\theta \tag{15.38}$$

where we assume that θ is small. For angles less ath 15°, the error in this approximation is less than 1%. Since the displacement along its path is just $a = l\theta$, we have

$$F = -\frac{mg}{l}x, \tag{15.39}$$

so $k = mg/l$ for our S.H.O. with $F = -kx$. We can then write $\theta(t) = \theta_{max}\cos(\omega t + \phi)$. This gives a period/frequency of

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}} \tag{15.40}$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \tag{15.41}$$

15.5 Damped Harmonic Motion

We now consider adding in a second force to our oscillator, one that damps, or slows the motion of the mass. Consider a force $F_{damping} = -bv = -bdx/dt$ which is proportional to the velocity of the mass, and always opposed the motion. Newton's Second Law becomes

$$\sum F = ma \quad (15.42)$$

$$-kx - bv = ma \quad (15.43)$$

$$ma + kx + bv = 0 \quad (15.44)$$

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + kx = = 0. \quad (15.45)$$

Again, we just guess a solution to this equation and see if it works. How about

$$x(t) = Ae^{-\gamma t}\cos(\omega' t) \quad (15.46)$$

where A, γ and ω' are constants and $x = A$ at $t = 0$. Note that $\omega' \neq \omega = \sqrt{k/m}$. We substitute this solution into the equation of motion above and find that

$$\gamma = \frac{b}{2m} \quad (15.47)$$

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \quad (15.48)$$

We can think of the decaying exponential as a just modifying the amplitude of the oscillations in time. The large b, the faster the decay.

Note that the term under the square root can go negative! This is bad news, and this solution is no longer valid.

$$\frac{k}{m} - \left(\frac{b}{2m}\right)^2 > 0 \tag{15.49}$$

$$b^2 < 4mk \tag{15.50}$$

describes a valid solution, where the mass oscillates many times before coming to equilibrium. Another option is of course $b^2 = 4mk$, in which case there are no oscillations ($\omega' = 0$) and the system comes to rest in the quickest time possible. The other option is if $b^2 > 4mk$, in which case the system comes to rest, but very slowly.

$$\begin{aligned} b^2 &< 4mk \text{ (underdamped)} && (15.51)\\ b^2 &= 4mk \text{ (critically damped)} && (15.52)\\ b^2 &> 4mk \text{ (overdamped)} && (15.53)\\ & && (15.54) \end{aligned}$$

example: A simple pendulum of length $l = 1.0$ m is swining with small-amplitude oscillations. After 5.0 minutes, the amplitude is only 50% of what it was initially. What is the value of γ?

We let $0.5 = e^{-\gamma t} = 0.5 = e^{-\gamma 300}$ gives $\gamma = \ln(2.0)/300 = 2.3 \times 10^3$ Hz $=$ 2.3 mHz.

Note that most oscillators have a natural frequency at which the want to oscillate. We can apply a sinusoidal force to the system (think: pushing a swing). If this force is applied at the natural frequency of the system, it is "on resonance" and adds to the amplitude of the system.

Chapter 16

Wave Motion

16.1 Opening Question

You drop a rock in a pond and water waves spread out.

(a) The waves carry water and energy out.

(b) The waves only move the water up and down, and no energy is transferred.

(c) The waves only move the water up and down, and energy is transferred.

16.2 Wave Motion

A mechanical wave is a wave that propagates (travels) as oscillations of matter (unlike electromagnetic waves).

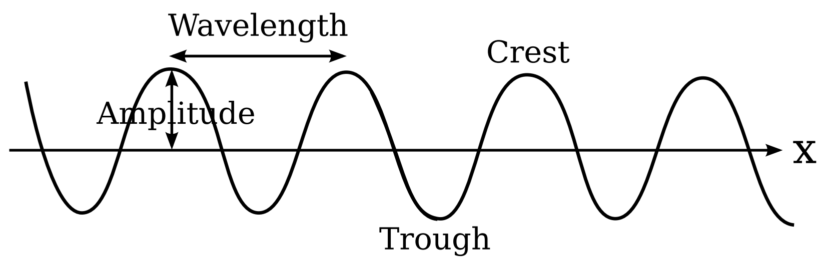
Reminder of terms:

(a) A, amplitude, the maximum height of a crest or depth of a trough

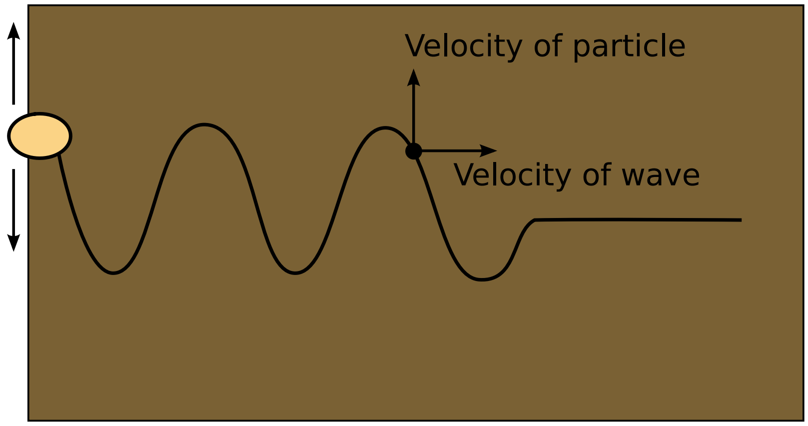
(b) λ, wavelength: the distance between two successive identical points on the wave

(c) T, period: the time elapsed between two successive crests passing by the same point in space

(d) f, frequency: the number of cycles that pass a given point per unit time



Unlike the motion of a mass on a spring, we know that waves travel. When I shake a rope, the waves travel down the rope, but the particles in the rope move up and down like harmonic oscillators. If I sing a note, the air particles vibrate back and forth at a particular frequency, transmitting a wave through the air.



There are “two” types of waves:

(a) Transverse: the particles/fields oscillate perpendicular to the motion (rope, light)

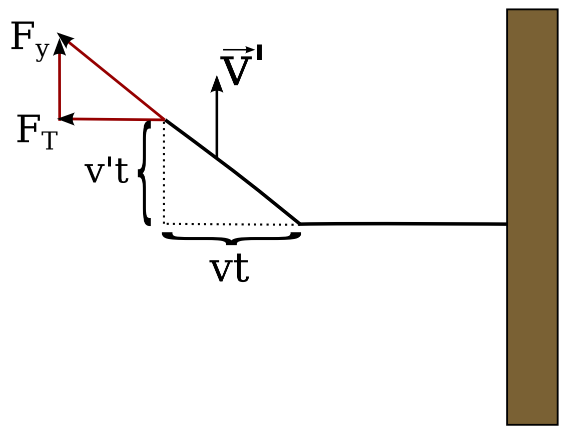
(b) Longitudinal: the particles/fields oscillate parallel to the motion (sound)

(c) Mixture of both: the particles/fields oscillate in all directions (earthquakes, surface waves in water)

There are two distinct velocities. We derived the velocity of the particles last chapter, by differentiating the displacement of an oscillating mass. The velocity of the wave is called the **wave velocity**, and is equal to

$$v = \frac{\Delta x}{\Delta t} = \frac{\lambda}{T} = \lambda f. \tag{16.1}$$

That is, the wave moves a distance of one wavelength in a time period T. We can predict this velocity based on the properties of the system.



Since our rope is flexible, the force on the particles in the rope follows the direction of travel, so that

$$\frac{F_T}{F_y} = \frac{vt}{v't} \rightarrow F_y t = \frac{v'}{v} F_T t \tag{16.2}$$

But $F_y t$ is just the impulse, equal to Δp. The change in momentum is equal to the mass of the cord segment times its velocity v'. For a cord with mass per unit length of μ, we find

$$F_y t = \frac{v'}{v} F_T t \tag{16.3}$$

$$\Delta p = \frac{v'}{v} F_T t \tag{16.4}$$

$$m v' = \frac{v'}{v} F_T t \tag{16.5}$$

$$(\mu v t) v' = \frac{v'}{v} F_T t \tag{16.6}$$

$$v = \sqrt{\frac{F_T}{\mu}}, \tag{16.7}$$

where we note that the mass is $m = \mu v' t$. This means that the velocity of a wave depends on the tension in the forces in and the density of the material the wave propagates through.

A longitudinal wave is one which oscillates along the direction of motion, like a slinky or sound waves. In this case, the velocity is very similar,

$$v = \sqrt{\frac{E}{\rho}} \text{ (in a rod)} \tag{16.8}$$

$$v = \sqrt{\frac{B}{\rho}} \text{ (in a fluid)} \tag{16.9}$$

where E is Young's (elastic) modulus and B is the bulk modulus.

example: A bird lands at the center of an 80 m long, 2.10-mm-diameter copper wire stretched between two poles, sending pulses out to each pole which reflect and arrive back at the bird 0.75 s later. What is the tension in the wire?

$$v = \sqrt{F_T/\mu} \tag{16.10}$$

$$F_T = \mu v^2 = (m/l)(d/t)^2 = (\rho\pi r^2 l/l)(d/t)^2 \tag{16.11}$$

$$= \rho\pi r^2 l^2/t^2 = 353 \text{ N} \tag{16.12}$$

example: A dolphin emits a 100,000 Hz wave pulse for location purposes.

(a) What is the wavelength of this wave in water? Since $v = \lambda f \rightarrow \lambda =$

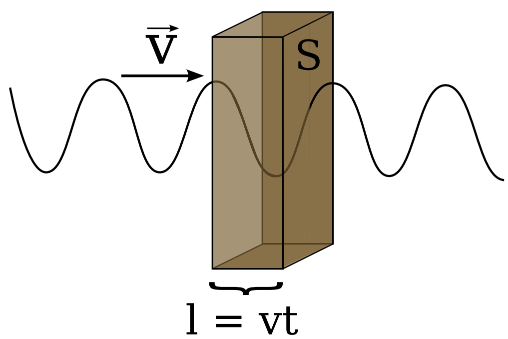
v/f, let's find the speed

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.0 \times 10^9}{1.025 \times 10^3}} = 1.4 \times 10^3 \text{ m/s} \tag{16.13}$$

Then, $\lambda = v/f = 14$ mm.

(b) How long does it take this pulse to hit a target 100 m and return to the dolphin? $t = \Delta x / v = 200/1.4 \times 10^3 = 0.14$ s.

16.3 Energy Transported by Waves



Let's consider a bulk medium of some density ρ. We know that the energy of each particle in the medium is given by $E = \frac{1}{2}kA^2$, either stored as potential or kinetic. The constant k we can rewrite, noting that

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \to k = 4\pi^2 m f^2. \tag{16.14}$$

In three dimensions, we know that $m = \rho V = \rho S l$, where S is the cross-sectional area that the wave travels through, and $l = vt$ is the distance the wave travels in a time t. This gives us

$$E = 2\pi^2 \rho S v t f^2 A^2 \tag{16.15}$$

There's a lot of stuff here, but note two important points: the energy transferred by the wave is proportional to the square of the amplitude and to the square of the frequency. The average rate of power transfer ($P = E/t$) is just

$$\bar{P} = 2\pi^2 \rho S v f^2 A^2 \tag{16.16}$$

And finally, the intensity of the wave I is the power transferred per unit cross sectional area S, so

$$I = 2\pi^2 \rho v f^2 A^2 \tag{16.17}$$

What if the power output is constant, and the wave spreads out evenly in all directions? This **spherical wave** has a surface area of $4\pi r^2$, so

$$I = \frac{\bar{P}}{S} = \frac{\bar{P}}{4\pi r^2} \tag{16.18}$$

$$I \propto \frac{1}{r^2} \tag{16.19}$$

So if the distance is doubled, the intensity of the wave drops by a factor of 1/4, etc. Since the intensity decreases, so must the amplitude,

$$A \propto \frac{1}{r}. \tag{16.20}$$

example: Then intensity of an earthquake P wave traveling through the Earth is 1.0×10^6 W/m^2 100 km from the source. What is the intensity 400 km from the source?

Assuming a spherical wave, we note that the distance is four times greater. Therefore, since $(1/4)^2 = 1/16$, we know that the wave's intensity is 1/16th of its value 100 km, or 6.3×10^4 W/m^2.

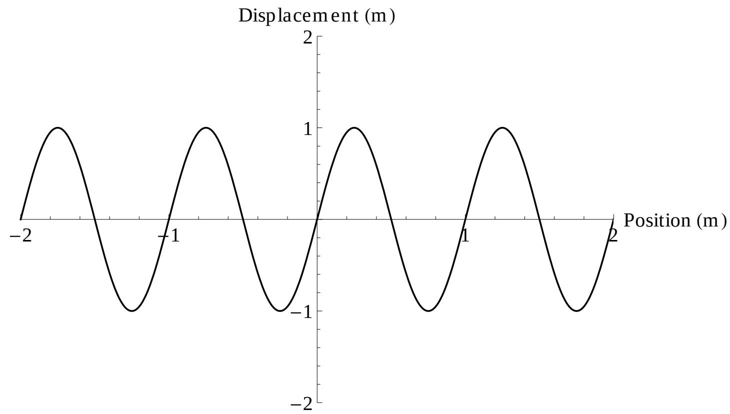
In one dimension, the area remains constant as the wave propagates, and therefore $A =$ constant and the intensity does not change. Of course, this is ignoring friction.

16.4 Mathematical Representation of a Traveling Wave

Let us think about the shape of a wave (ignoring ϕ). We can write it as

$$D(x) = A \sin\left(\frac{2\pi}{\lambda}x\right). \tag{16.21}$$

This wave looks something like this.



If this wave were to move to the right by an amount vt, where v is the wave velocity, this would be represented by the value of x being shifted, such that

$$D(x,t) = A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]. \tag{16.22}$$

Another way to think about this is, if you are riding on a crest of a wave, then the argument of the sine $\frac{2\pi}{\lambda}(x - vt)$ must remain the same. So if the time increases, then x must increase at the same rate so that $x - vt$ doesn't change. Since $v = \lambda f = \lambda/T$, we see that

$$D(x,t) = A \sin\left(\frac{2\pi x}{\lambda} - \frac{2\pi t}{T}\right). \tag{16.23}$$

Of course, $2\pi/T = 2\pi f = \omega$ and we define the wave number $k = 2\pi/\lambda$ (do not confuse with the spring constant k!). Then,

$$D(x,t) = A \sin\left(kx - \omega t\right). \tag{16.24}$$

All three of these forms are equivalent, but some are more convenient than others. The argument of the sine is called the **phase** of the wave, and the velocity is sometimes called the **phase velocity** $v = \lambda f = \omega/k$.

If the wave travels to the left, we let $v \to -v$, and we find that

$$D(x,t) = A\sin(kx + \omega t) \;\; \text{(to the left)}. \tag{16.25}$$

We can take a snapshot of our wave at a particular time (say $t = 0$) and we see that the wave varies sinusoidally:

$$D(x,0) = A\sin(kx)\,. \tag{16.26}$$

On the other hand, if we look at a particular spot, and ask how the particles move in time, we see

$$D(0,t) = A\sin(-\omega t)\,. \tag{16.27}$$

The wave varies sinusoidally in time and space. The most general form can have a phase ϕ, which alters the value of D at $t = 0$. Therefore, in general, we have

$$D(x,t) = A\sin(kx - \omega t + \phi)\,. \tag{16.28}$$

example: A rope oscillates at a frequency of 250 Hz with an amplitude of 2.6 cm. Its linear mass density is 0.12 kg/m and has a tension of 140 N.

(a) Was is the wavelength? Since $v = \lambda f$, we know

$$\lambda = \frac{v}{f} = \frac{1}{f}\sqrt{\frac{F_T}{\mu}} = 14 \text{ cm}. \tag{16.29}$$

(b) If at $t = 0$, the cord has a displacement of 1.6 cm and is falling toward zero, what is the equation for the wave? At $t = 0$ and $x = 0$, we know that the general form reduces to $D(0,0) = A\sin(\phi)$. Therefore, $\phi = \sin^{-1}(1.6/2.6) = 0.66$ rad (about 38 degrees). Is this the right angle? Or is it $38 + 90$ degrees?

$$\begin{aligned} D(x,t) &= A\sin\left(\frac{2\pi}{\lambda}x - 2\pi f t + \phi\right) && (16.30)\\ &= 0.026\sin\left(45x - 1570t + 0.66\right) && (16.31)\\ . & && (16.32) \end{aligned}$$

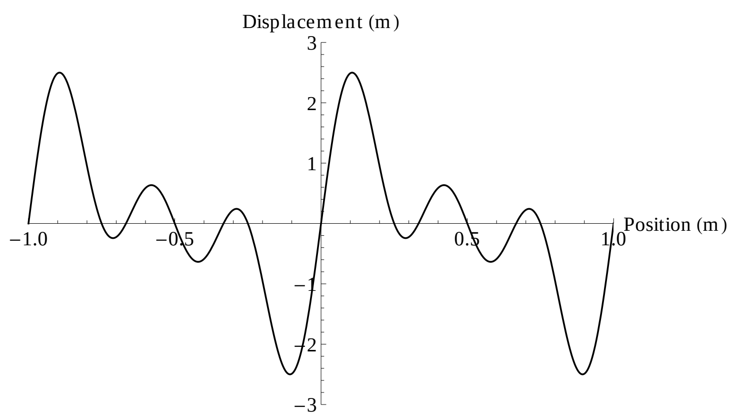
In many materials, the tension force or elastic/bulk modulus depends on the frequency of the wave. This means that the velocity of the wave depends on the frequency of oscillations. This phenomenon is known as **dispersion** and is responsible for the behavior of prisms (and my research!).

16.5 Superposition, Reflection and Standing Waves

When two waves pass over the same area, what happens? **The principle of superposition**:

> The displacement of a particle in a wave is the sum of the displacement caused by each wave passing through that particle.

What happens when we add two or more sine waves together? We start to get some complex shapes. If we add three sine waves of frequency f, $2f$ and $3f$, we start to produce a triangle wave!



In fact, any complex or composite wave (meaning: not sinusoidal) can be constructed from a sum of sine waves. This is Fourier's theorem.

When a wave strikes a barrier, some part of the wave is reflected. A common example is the echo of sound off of the walls in a large room. There are three common rope examples:

(a) Fixed end: pulse is inverted upon reflection

(b) Free end: pulse is right-side-up upon reflection

(c) Small to large rope: part of pulse is reflect, inverted, and the rest is transmitted, right-side-up.

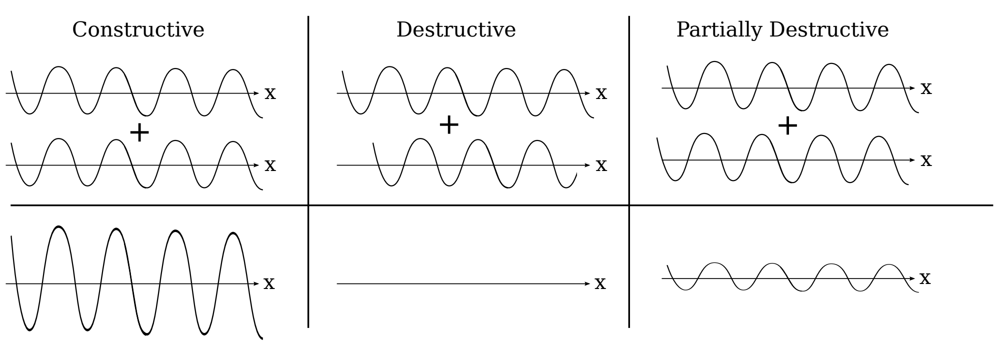
If you consider the forces involved, do these results make sense?

A wave front of a 2D or 3D wave is the line drawn through space that corresponds to a constant phase of the wave. Often, we take the crest of the wave and use this. A plane wave is a wave whose wave fronts are straight lines (like ocean waves crashing on the shore). When a plane wave strikes a barrier, it is reflected at the same angle of its incidence. That is,

the angle of reflection equals the angle of incidence.

The angle of incidence/reflection is measured from the perpendicular to the surface.

Since waves add together, we can get some interesting behavior. What if we add two waves of the same frequency and amplitude? If their phases are the same, then they will add **constructively**. However, if they are precisely π (180°) out of phase, then they will add **destructively**. We call these two extremes **in phase** and **out of phase**.



You can witness destructive and constructive interference by standing at certain locations in a large room with a sound source.

Note that if the frequency of the waves are not the same, or the amplitudes differ, we may get beat notes and partially destructive interference.

If you vibrate a rope attached to a wall at both ends, the waves will interfere with after reflection off of the walls. Although many frequencies are created, only the "natural" frequencies persist for very long. These waves will constructively interfere to produce a **standing wave**.

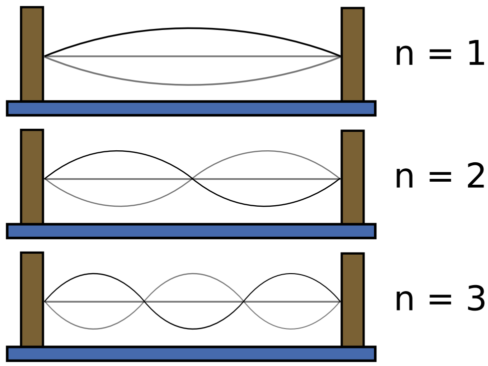
Let's add two waves traveling in opposite directions to see what this looks like:

$$\begin{aligned}
D_1(x,t) &= A\sin(kx-\omega t) && (16.33)\\
D_2(x,t) &= A\sin(kx+\omega t) && (16.34)\\
D(x,t) &= D_1+D_2 && (16.35)\\
&= A[\sin(kx-\omega t)+\sin(kx+\omega t)] && (16.36)\\
&= 2A\sin(kx)\cos(\omega t), && (16.37)
\end{aligned}$$

where we used $[\sin A+\sin B]/2=\sin[(A+B)/2]\cos[(A-B)/2]$. If $x=0$ is the left end, and $x=l$ is the right end (length of the string), then we know that $D=0$ at these two locations. $D(0,t)=0$ already, and $D(l,t)=0=2A\sin(kl)\cos(\omega t)$ means that

$$kl=\pi,2\pi,...,n\pi,... \qquad (16.38)$$

where n is any integer. Therefore, $\lambda = 2l/n$. That is, the wavelength of the standing wave is just some fraction of the total length of the string. The number n is called the harmonic, with $n = 1$ being the fundamental and $n = 2$ being the second harmonic.



These wavelengths (and corresponding frequencies) are the natural frequencies or resonant frequencies of the string. There are locations along the string where $D = 0$ for all t. These are called **nodes**, and are at $x = n\lambda/2$. The locations where the oscillations are maximum are called the **antinodes**, and are at $x = (2n - 1)\lambda/4$.