MAXWELL EQUATIONS AND FRESNEL COEFFICIENTS
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Contents

1. Maxwell's equations

The *electromagnetic field* (EM) is a physical field produced by electrically charged objects. It extends indefinitely throughout space and describes the electromagnetic interaction. It is one of the four fundamental forces of nature (the others are gravitation, the weak interaction, and the strong interaction). The field propagates by electromagnetic radiation; in order of increasing energy (decreasing wavelength) electromagnetic radiation comprises: radio waves, microwaves, infrared, visible light, ultraviolet, X-rays, and gamma rays. The field (EM) can be viewed as the combination of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{H}$, that is, these are three-dimensional vector fields that have a value defined at every point of space and time: $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H} = \mathbf{H}(\mathbf{r}, t)$, where $\mathbf{r}$ represents a point in 3-d space $\mathbf{r} = (x, y, z)$. The electric field is produced by stationary charges, and the magnetic field by moving charges (currents); these two are often described as the sources of the field. The way in which $\mathbf{E}$ and $\mathbf{H}$ interact is described by Maxwell's equations (see [Som52, p. 22]):

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \text{Gauss's law} \tag{1.1}$$

$$\nabla \cdot \mathbf{H} = 0, \quad \text{Gauss's law for magnetism} \tag{1.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}, \quad \text{Faraday's law} \tag{1.3}$$

$$\nabla \times \mathbf{H} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \quad \text{Ampère-Maxwell's law.} \tag{1.4}$$

Here

$\nabla = (\partial_x, \partial_y, \partial_z)$	the gradient
$\rho = \rho(\mathbf{r}, t)$	charge density
ϵ_0	permittivity of free space
$\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$	current density vector
μ_0	permeability of free space

We have $c = 1/\sqrt{\epsilon_0 \mu_0}$, the speed of light in vacuum.

1.1. **General case.** In several situations is necessary to consider a medium where the magnetic permeability $\mu = \mu(x, y, z)$* and the electric permittivity $\epsilon = \epsilon(x, y, z)$† are not constants. This is the case when the physical properties of the medium change from point to point, in particular, this happens in geometric optics when materials of different refractive indices are considered. In such case the Maxwell equations have the form:

$$\nabla \times \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t}, \tag{1.5}$$

$$\nabla \times \mathbf{H} = \frac{2\pi}{c}\sigma \mathbf{E} + \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{1.6}$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 4\pi\rho \tag{1.7}$$

$$\nabla \cdot (\mu \mathbf{H}) = 0, \tag{1.8}$$

c being the speed of light in vacuum. Recall that substances for which $\sigma \neq 0$ are conductors and if σ is negligibly small, the substances are called insulators or dielectrics, see [BW59][Section 1.1.2]. Under certain assumptions on the field and the physical set up we have that $\mathbf{J} = \sigma \mathbf{E}$, see [BW59][Section 1.1.2, formula (9)].

It is important to notice that these equations are written in Gaussian units, and the Maxwell equations in the first section written in SI units.

*For values of μ for different substances see `http://en.wikipedia.org/wiki/Permeability_(electromagnetism)#Values_for_some_common_materials`.

†For relative permittivity of some substances see `http://en.wikipedia.org/wiki/Relative_permittivity`.

1.2. **Maxwell equations in integral form.** Points in $\mathbb{R}^4$ are denoted by (x, y, z, t), and suppose $D \subset \mathbb{R}^4$ is a domain for which the divergence theorem holds, for example, the boundary is piecewise smooth, that is, a finite union of C^1 surfaces. For a point $P = (x, y, z, t)$ on the boundary ∂D, the unit outer normal at P is denoted by $\nu = (\nu_x, \nu_y, \nu_z, \nu_t)$. From equation (1.6)

$$\nabla \times \mathbf{H} - \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{2\pi}{c}\sigma \mathbf{E}. \tag{1.9}$$

Recall we assume $\epsilon = \epsilon(x, y, z)$, and we want to derive an integral form of the last equation that does not require differentiability of the fields. In order to do that, we initially assume the fields are smooth and applying the divergence theorem we will obtain formulas independent of the derivatives of the fields. Set $\mathbf{H} = (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3)$. We have

$$\begin{aligned}
&\int_D \nabla \times \mathbf{H}\, dxdydzdt \\
&= \mathbf{i} \int_D (\partial_y \mathbf{H}_3 - \partial_z \mathbf{H}_2) - \mathbf{j} \int_D (\partial_x \mathbf{H}_3 - \partial_z \mathbf{H}_1) + \mathbf{k} \int_D (\partial_x \mathbf{H}_2 - \partial_y \mathbf{H}_1) \\
&= \mathbf{i} \int_D \text{div } (0, \mathbf{H}_3, -\mathbf{H}_2, 0) - \mathbf{j} \int_D \text{div } (\mathbf{H}_3, 0, -\mathbf{H}_1, 0) + \mathbf{k} \int_D \text{div } (\mathbf{H}_2, -\mathbf{H}_1, 0, 0) \\
&= \mathbf{i} \int_{\partial D} (0, \mathbf{H}_3, -\mathbf{H}_2, 0) \cdot (\nu_x, \nu_y, \nu_z, \nu_t)\, d\sigma \\
&\qquad - \mathbf{j} \int_{\partial D} (\mathbf{H}_3, 0, -\mathbf{H}_1, 0) \cdot (\nu_x, \nu_y, \nu_z, \nu_t)\, d\sigma + \mathbf{k} \int_{\partial D} (\mathbf{H}_2, -\mathbf{H}_1, 0, 0) \cdot (\nu_x, \nu_y, \nu_z, \nu_t)\, d\sigma \\
&= \int_{\partial D} (\nu_x, \nu_y, \nu_z) \times (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3)\, d\sigma.
\end{aligned}$$

So integrating (1.9) over D yields

$$\begin{aligned}
&\int_{\partial D} (\nu_x, \nu_y, \nu_z) \times (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3)\, d\sigma - \int_D \frac{\epsilon}{c} \mathbf{E}_t\, dxdydzdt \\
&= \int_{\partial D} (\nu_x, \nu_y, \nu_z) \times (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3)\, d\sigma - \int_D \left(\frac{\epsilon}{c}\mathbf{E}\right)_t\, dxdydzdt \\
&= \int_{\partial D} (\nu_x, \nu_y, \nu_z) \times (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3)\, d\sigma - \int_{\partial D} \frac{\epsilon}{c} \mathbf{E}\, \nu_t\, d\sigma \\
&= \int_{\partial D} \left((\nu_x, \nu_y, \nu_z) \times (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3) - \frac{\epsilon}{c}\mathbf{E}\, \nu_t\right) d\sigma = \int_D \frac{2\pi}{c} \sigma \mathbf{E}\, dxdydzdt.
\end{aligned}$$

Therefore the surface integral

$$\int_{\partial D} \left((\nu_x, \nu_y, \nu_z) \times (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3) - \frac{\epsilon}{c}\mathbf{E}\, \nu_t\right) d\sigma = \int_D \frac{2\pi}{c} \sigma \mathbf{E}\, dxdydzdt, \tag{1.10}$$

for each closed hyper-surface ∂D in $\mathbb{R}^4$. Proceeding in the same way with equation (1.5) we obtain that the surface integral

$$\int_{\partial D} \left((\nu_x, \nu_y, \nu_z) \times (\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3) + \frac{\mu}{c} \mathbf{H}\, \nu_t \right) d\sigma = 0, \tag{1.11}$$

for each closed hyper-surface ∂D in $\mathbb{R}^4$.

Concerning equations (1.7) and (1.8), proceeding in the same way as before we obtain that

$$\int_{\partial D} \epsilon \mathbf{E} \cdot \nu \, d\sigma = 4\pi \int_D \rho \, dxdydzdt \tag{1.12}$$

$$\int_{\partial D} \mu \mathbf{H} \cdot \nu \, d\sigma = 0, \tag{1.13}$$

for each domain $D \subset \mathbb{R}^4$ for which the divergence theorem holds. These formulas make sense as long as the fields $\mathbf{E}, \mathbf{H}$ and the coefficients μ and ϵ are piecewise continuous over ∂D and bounded. Equations (1.10), (1.11), (1.12), and (1.13) are Maxwell's equations in integral form.

1.3. **Boundary conditions at a surface of discontinuity.** Let us consider a point $P_0 = (x_0, y_0, z_0, t_0)$, a hyper-surface Γ_0 passing through P_0 and suppose that the fields $\mathbf{H}$ and $\mathbf{E}$, solutions to the Maxwell equations in integral form, as well as the functions ϵ and μ, are discontinuous on Γ_0. Suppose that all these quantities are defined locally around P_0 say in the 4-dimensional ball $B_R(P_0)$. This situation is typical when we have two media with different indices of refraction and the surface Γ_0 is the one separating the two media. The surface Γ_0 divides the open ball $B_R(P_0)$ into two open pieces: B_R^+ and B_R^-. In order to make sense of the integrals we assume the surface Γ_0 is C^1, the fields $\mathbf{E}$ and $\mathbf{H}$, and ϵ and μ are bounded in $B_R(P_0)$, and all continuous on $B_R(P_0) \setminus \Gamma_0$. We assume also that for each $Q \in \Gamma_0 \cap B_R(P_0)$ the following limits exist and are finite:

$$\lim_{P \to Q, P \in B_R^+} \mathbf{E}(P) = \mathbf{E}^+(Q), \qquad \lim_{P \to Q, P \in B_R^+} \mathbf{H}(P) = \mathbf{H}^+(Q)$$
$$\lim_{P \to Q, P \in B_R^-} \mathbf{E}(P) = \mathbf{E}^-(Q), \qquad \lim_{P \to Q, P \in B_R^-} \mathbf{H}(P) = \mathbf{H}^-(Q),$$

and similar quantities for ϵ and μ. Let us call $\Gamma_+(R)$ the boundary of B_R^+, and $\Gamma_-(R)$ the boundary of B_R^-. If we let $\mathbf{E}^+(Q) = \mathbf{E}(Q)$ for $Q \in B_R^+$ and $\mathbf{E}^+(Q) = \lim_{P \to Q, P \in B_R^+} \mathbf{E}(P)$ for $Q \in \Gamma_0 \cap B_R$, and similarly $\mathbf{H}^+$, μ^+, and ϵ^+, then all these functions are continuous in B_R^+. In a similar way, we define $\mathbf{E}^-, \mathbf{H}^-, \mu_-$ and ϵ_- in B_R^- that are continuous in B_R^-. Hence applying (1.10) with $D = B_R^+$ yields

$$\int_{\Gamma_+(R) \cup (\Gamma_0 \cap B_R(P_0))} \left((\nu_x, \nu_y, \nu_z) \times (\mathbf{H}_1^+, \mathbf{H}_2^+, \mathbf{H}_3^+) - \frac{\epsilon^+}{c} \mathbf{E}^+ \, \nu_t \right) d\sigma = \int_{B_R^+} \frac{2\pi}{c} \sigma \mathbf{E}^+ \, dxdydzdt, \tag{1.14}$$

and
(1.15)
$$\int_{\Gamma_-(R)\cup(\Gamma_0\cap B_R(P_0))}\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1^-,\mathbf{H}_2^-,\mathbf{H}_3^-)-\frac{\epsilon^-}{c}\mathbf{E}^-\,\nu_t\right)d\sigma=\int_{B_R^-}\frac{2\pi}{c}\sigma\mathbf{E}^-\,dxdydzdt.$$

Now

$$\begin{aligned}\int_{\Gamma_+(R)\cup(\Gamma_0\cap B_R(P_0))}&\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1^+,\mathbf{H}_2^+,\mathbf{H}_3^+)-\frac{\epsilon^+}{c}\mathbf{E}^+\,\nu_t\right)d\sigma\\ &=\int_{\Gamma_+(R)}\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3)-\frac{\epsilon}{c}\mathbf{E}\,\nu_t\right)d\sigma\\ &\quad+\int_{\Gamma_0\cap B_R(P_0)}\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1^+,\mathbf{H}_2^+,\mathbf{H}_3^+)-\frac{\epsilon^+}{c}\mathbf{E}^+\,\nu_t\right)d\sigma,\end{aligned}\tag{1.16}$$

where in the integral over $\Gamma_0\cap B_R(P_0)$, $\nu:=(\nu_x,\nu_y,\nu_z,\nu_t)$ is the downward unit normal to Γ_0; and

$$\begin{aligned}\int_{\Gamma_-(R)\cup(\Gamma_0\cap B_R(P_0))}&\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1^-,\mathbf{H}_2^-,\mathbf{H}_3^-)-\frac{\epsilon^-}{c}\mathbf{E}^-\,\nu_t\right)d\sigma\\ &=\int_{\Gamma_-(R)}\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3)-\frac{\epsilon}{c}\mathbf{E}\,\nu_t\right)d\sigma\\ &\quad+\int_{\Gamma_0\cap B_R(P_0)}\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1^-,\mathbf{H}_2^-,\mathbf{H}_3^-)-\frac{\epsilon^-}{c}\mathbf{E}^-\,\nu_t\right)d\sigma,\end{aligned}\tag{1.17}$$

where in the integral over $\Gamma_0\cap B_R(P_0)$, ν is the upward unit normal to Γ_0. Adding (1.16) and (1.17), and using (1.14) and (1.15) yields

$$\begin{aligned}&\int_{B_R^+}\frac{2\pi}{c}\sigma\mathbf{E}^+\,dxdydzdt+\int_{B_R^-}\frac{2\pi}{c}\sigma\mathbf{E}^-\,dxdydzdt\\ &=\int_{\Gamma_+(R)}\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3)-\frac{\epsilon}{c}\mathbf{E}\,\nu_t\right)d\sigma+\int_{\Gamma_-(R)}\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3)-\frac{\epsilon}{c}\mathbf{E}\,\nu_t\right)d\sigma\\ &+\quad\int_{\Gamma_0\cap B_R(P_0)}\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}^+-\mathbf{H}^-)-\frac{1}{c}\left(\epsilon^+\,\mathbf{E}^+-\epsilon^-\,\mathbf{E}^-\right)\,\nu_t\right)d\sigma.\end{aligned}$$

On the other hand, applying (1.10) with $D=B_R$ yields

$$\int_{\Gamma_+(R)\cup\Gamma_-(R)}\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_3)-\frac{\epsilon}{c}\mathbf{E}\,\nu_t\right)d\sigma=\int_{B_R}\frac{2\pi}{c}\sigma\mathbf{E}\,dxdydzdt.$$

Since the field $\mathbf{E}$ is discontinuous only on Γ_0, which is a set of measure zero, we therefore obtain

$$\int_{\Gamma_0\cap B_R(P_0)}\left((\nu_x,\nu_y,\nu_z)\times(\mathbf{H}^+-\mathbf{H}^-)-\frac{1}{c}\left(\epsilon^+\,\mathbf{E}^+-\epsilon^-\,\mathbf{E}^-\right)\,\nu_t\right)d\sigma=0$$

for all R sufficiently small. Now letting $R \to 0$ we obtain the following equation valid at P_0

$$(\nu_x, \nu_y, \nu_z) \times (\mathbf{H}^+ - \mathbf{H}^-) - \frac{1}{c}(\epsilon^+ \mathbf{E}^+ - \epsilon^- \mathbf{E}^-)\, \nu_t = 0, \tag{1.18}$$

where ν is the normal to the interface Γ_0 at the point P_0.

Suppose the interface is independent of time and is given by a function $\phi(x, y, z) = 0$, then the normal at a point is $\nu = (\phi_x, \phi_y, \phi_z, 0)$, therefore equation (1.18) becomes

$$\nabla\phi \times (\mathbf{H}^+ - \mathbf{H}^-) = 0.$$

We can write $\mathbf{H}^\pm = \mathbf{H}^\pm_{\tan} + \mathbf{H}^\pm_{\mathrm{perp}}$, where $\mathbf{H}^\pm_{\mathrm{perp}}$ is the component in the direction of the normal $\nabla\phi$ and $\mathbf{H}^\pm_{\tan}$ is the component perpendicular to the normal. We have $\nabla\phi \times \mathbf{H}^\pm = \nabla\phi \times \mathbf{H}^\pm_{\tan} + \nabla\phi \times \mathbf{H}^\pm_{\mathrm{perp}} = \nabla\phi \times \mathbf{H}^\pm_{\tan}$. So

$$0 = \nabla\phi \times (\mathbf{H}^+ - \mathbf{H}^-) = \nabla\phi \times (\mathbf{H}^+_{\tan} - \mathbf{H}^-_{\tan}) = |\nabla\phi|\, |\mathbf{H}^+_{\tan} - \mathbf{H}^-_{\tan}|,$$

since the vectors are perpendicular. So if $\nabla\phi \neq 0$, the obtain the important relation that

$$\mathbf{H}^+_{\tan} - \mathbf{H}^-_{\tan} = 0,$$

that is, *the tangential components of the magnetic field are continuous across the boundary*.

Proceeding in the same manner this time with (1.11) yields the equation

$$(\nu_x, \nu_y, \nu_z) \times (\mathbf{E}^+ - \mathbf{E}^-) + \frac{1}{c}(\mu^+ \mathbf{H}^+ - \mu^- \mathbf{H}^-)\, \nu_t = 0, \tag{1.19}$$

where ν is the normal to the interface Γ_0 at the point P_0. If the interface is independent of t, proceeding exactly as before, we obtain

$$\nabla\phi \times (\mathbf{E}^+ - \mathbf{E}^-) = 0,$$

and

$$\mathbf{E}^+_{\tan} - \mathbf{E}^-_{\tan} = 0,$$

that is, also *the tangential components of the electric field are continuous across the boundary*.

In regard to equations (1.12) and (1.13), we obtain similarly that

$$(\epsilon^+ \mathbf{E}^+ - \epsilon^- \mathbf{E}^-) \cdot \nu = 0, \text{ and } (\mu^+ \mathbf{H}^+ - \mu^- \mathbf{H}^-) \cdot \nu = 0.$$

Since $\mathbf{H}^\pm \cdot \nu = \mathbf{H}^\pm_{\mathrm{perp}} \cdot \nu$, and similarly for $\mathbf{E}$, assuming $\phi = \phi(x, y, z)$ yields

$$0 = (\epsilon^+ \mathbf{E}^+_{\mathrm{perp}} - \epsilon^- \mathbf{E}^-_{\mathrm{perp}}) \cdot \nabla\phi = |\epsilon^+ \mathbf{E}^+_{\mathrm{perp}} - \epsilon^- \mathbf{E}^-_{\mathrm{perp}}|\, |\nabla\phi|,$$

and

$$0 = (\mu^+ \mathbf{H}^+_{\mathrm{perp}} - \mu^- \mathbf{H}^-_{\mathrm{perp}}) \cdot \nabla\phi = |\mu^+ \mathbf{H}^+_{\mathrm{perp}} - \mu^- \mathbf{H}^-_{\mathrm{perp}}|\, |\nabla\phi|,$$

and therefore

$$|\epsilon^+ \mathbf{E}^+_{\mathrm{perp}} - \epsilon^- \mathbf{E}^-_{\mathrm{perp}}| = |\mu^+ \mathbf{H}^+_{\mathrm{perp}} - \mu^- \mathbf{H}^-_{\mathrm{perp}}| = 0.$$

Therefore *the perpendicular components of the fields $\epsilon\mathbf{E}$ and $\mu\mathbf{H}$ are continuous across the interface*.

1.4. **Maxwell's equations in the absence of charges.** This is the case when $\rho = 0$ and $\mathbf{J} = 0$. So the equations become

$$\nabla \cdot \mathbf{E} = 0, \tag{1.20}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{1.21}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}, \tag{1.22}$$

$$\nabla \times \mathbf{H} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{1.23}$$

1.5. **The wave equation.** Recall the following formula from vector analysis for a vector $\mathbf{A} = \mathbf{A}(x, y, z) = (\mathbf{A}_x(x, y, z), \mathbf{A}_y(x, y, z), \mathbf{A}_z(x, y, z))$:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A}. \tag{1.24}$$

Denote $\nabla \cdot \nabla = \nabla^2$, the Laplacian, and so

$$\nabla^2 \mathbf{A} =$$
$$\left(\frac{\partial^2 \mathbf{A}_x}{\partial x^2} + \frac{\partial^2 \mathbf{A}_x}{\partial y^2} + \frac{\partial^2 \mathbf{A}_x}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 \mathbf{A}_y}{\partial x^2} + \frac{\partial^2 \mathbf{A}_y}{\partial y^2} + \frac{\partial^2 \mathbf{A}_y}{\partial z^2}\right)\mathbf{j} + \left(\frac{\partial^2 \mathbf{A}_z}{\partial x^2} + \frac{\partial^2 \mathbf{A}_z}{\partial y^2} + \frac{\partial^2 \mathbf{A}_z}{\partial z^2}\right)\mathbf{k}.$$

From Faraday's law and Ampère's law

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial(\nabla \times \mathbf{H})}{\partial t} = -\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

and so from formula (1.24) we obtain that $\mathbf{E}$ satisfies the wave equation

$$\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E}.$$

Proceeding in the same manner for $\mathbf{H}$ we obtain

$$\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} = \nabla^2 \mathbf{H}.$$

That is, both the electric and magnetic fields satisfy the wave equation. We have from physics that

$$v = c = \frac{1}{\sqrt{\epsilon_0 \mu_0}},$$

c being the speed of propagation of light in free space, which in this case is the velocity v of propagation. If free space is changed by a material with other values of μ_0 and ϵ_0, the velocity v represent the speed of propagation of waves in this material.‡

‡The relative permittivity is ϵ/ϵ_0 and the relative permeability is μ/μ_0; the index of refraction is defined by $n = \sqrt{\epsilon_r \mu_r}$. The velocity of propagation $v = \frac{1}{\sqrt{\epsilon \mu}}$. Since $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$, we get that $n = c/v$.

1.6. **Dispersion equation.** Suppose $\mathbf{E}$ and $\mathbf{H}$ solve the Maxwell equations (1.5), (1.6), (1.7), (1.8) with $\rho = 0$ and $\sigma = 0$. Then

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\mu}{c}\nabla \times \mathbf{H}_t = -\frac{\mu}{c}(\nabla \times \mathbf{H})_t = -\frac{\epsilon\mu}{c^2}\mathbf{E}_{tt}.$$

On the other hand, from (1.24), $\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E}$ and so

$$\nabla^2 \mathbf{E} - \frac{\epsilon\mu}{c^2}\mathbf{E}_{tt} = 0,$$

and similarly

$$\nabla^2 \mathbf{H} - \frac{\epsilon\mu}{c^2}\mathbf{H}_{tt} = 0.$$

If $\mathbf{E} = A\cos(\mathbf{r}\cdot\mathbf{k} + \omega t)$, then we obtain the dispersion equation

$$|\mathbf{k}|^2 = \epsilon\mu\left(\frac{\omega}{c}\right)^2. \tag{1.25}$$

1.7. **Plane waves.** Let $\mathbf{s}$ be a unit vector. Any solution to the wave equation

$$\frac{1}{v^2}\partial_t^2 V = \nabla^2 V,$$

of the form $V(\mathbf{r}, t) = F(\mathbf{r}\cdot\mathbf{s}, t)$ is called a *plane wave*, since at each time t, V is constant on each plane of the form $\mathbf{r}\cdot\mathbf{s}$=constant. That is, for each t the vector $V(\mathbf{r}, t)$ is the same on each plane $\mathbf{r}\cdot\mathbf{s}$=constant. The plane wave propagates in the direction $\mathbf{s}$. It can be proved that any solution to the wave equation of this form can be written as

$$V(\mathbf{r}, t) = V_1(\mathbf{r}\cdot\mathbf{s} - vt) + V_2(\mathbf{r}\cdot\mathbf{s} + vt)$$

where V_1, V_2 are arbitrary functions, see [BW59][Section 1.3.1]. Since the fields $\mathbf{E}$ and $\mathbf{H}$ both satisfy the wave equation, it is then natural to consider the case when

$$\mathbf{E} = \mathbf{E}(\mathbf{r}\cdot\mathbf{s} - vt), \quad \mathbf{H} = \mathbf{H}(\mathbf{r}\cdot\mathbf{s} - vt),$$

that is, $\mathbf{E}$ and $\mathbf{H}$ are functions of the scalar variable $\mathbf{r}\cdot\mathbf{s} - vt$. We have

$$\frac{\partial \mathbf{E}}{\partial t} = -v\mathbf{E}', \text{ and } \nabla \times \mathbf{E} = s \times \mathbf{E}';$$

and similarly for $\mathbf{H}$ under the assumption that $\mathbf{J} = 0$. Thus, from the Faraday and Ampère laws, and since $v^2 = \dfrac{1}{\epsilon_0\mu_0}$, we obtain the equations

$$s \times \mathbf{E}' = v\mathbf{H}'$$

$$s \times \mathbf{H}' = -\frac{1}{v}\mathbf{E}'.$$

Since s is a constant vector $s \times \mathbf{E}' = (s \times \mathbf{E})'$, and so the equations are

$$(s \times \mathbf{E})' = v\mathbf{H}'$$

$$(s \times \mathbf{H})' = -\frac{1}{v}\mathbf{E}'.$$

Integrating these equations and taking constants of integration zero (which amounts to neglect constant fields), we obtain the very important equations relating the electric and magnetic fields

$$\mathbf{E} = -v(s \times \mathbf{H}) \tag{1.26}$$

$$\mathbf{H} = \frac{1}{v}(s \times \mathbf{E}). \tag{1.27}$$

This shows that $\mathbf{s} \cdot \mathbf{E} = \mathbf{s} \cdot \mathbf{H} = \mathbf{0}$, that means, the electric and magnetic field are always *perpendicular* to the direction of propagation $\mathbf{s}$. In addition, $\mathbf{E} \cdot \mathbf{H} = v(s \times \mathbf{H}) \cdot \mathbf{H} = 0$, that is, $\mathbf{E}$ and $\mathbf{H}$ are always perpendicular. We also obtain taking absolute values that

$$|\mathbf{E}| = v|\mathbf{H}|.$$

2. Fresnel formulas

We consider plane waves whose components have the form

$$a \cos\left(\omega\left(t - \frac{\mathbf{r} \cdot \mathbf{s}}{v}\right) + \delta\right) = a \cos\left(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta\right),$$

that is, $\mathbf{k} = \dfrac{\omega}{v}\mathbf{s}$ and a, δ are real numbers. The quantity $\omega t - \mathbf{k} \cdot \mathbf{r} + \delta$ is called the *phase*, and a is called the *amplitude*.

Let $\mathbf{s}^i$ be the direction (unit) of an incident plane wave traveling for a while in media I with velocity of propagation v_1 that hits, at a point P, a boundary Γ between I and another media II where the velocity of propagation is v_2 (I and II are also called dielectrics as they are materials with zero conductivity, that is $\sigma = 0$ and so the current density vector $\mathbf{J} = 0$, see Subsection 1.1). Then the wave splits into two waves: a *transmitted wave* propagating in media II and a *reflected wave* propagated back into media I. We shall assume that these two waves are also plane. The plane determined by ν and $\mathbf{s}^i$ is called the *incidence plane*.

It is important to remark that for our analysis, we will choose a spacial local system of coordinates around the point P. Indeed, we are going to write all fields as functions of $\mathbf{r}$ (position) and t, with $\mathbf{r}$ close to zero, such that the coordinates of P in this system are $\mathbf{r} = 0$. In particular, the fields will be calculated near the point P.

We choose a rectangular right-hand system of coordinates x, y, z such that the normal ν is on the z-axis, and the x and y axes are on the plane perpendicular to ν and in such a way that the vector $\mathbf{s}^i$ lies on the xz-plane. This means that the tangent plane to Γ at P is the xy-plane; in particular, $P = (0, 0, 0)$. So we assume that

$$\mathbf{s}^i = \sin\theta_i \mathbf{i} + \cos\theta_i \mathbf{k}$$

that is, $\mathbf{s}^i$ lives on the xz-plane and so the direction of propagation is perpendicular to the y-axis and θ_i is the angle between the normal vector ν to the boundary at P (the z-axis) and the incident direction $\mathbf{s}^i$ (as usual $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit coordinate vectors).

The electric field corresponding to this incident field is

$$\mathbf{E}^i(\mathbf{r},t) = (-I_\| \cos\theta_i, I_\perp, I_\| \sin\theta_i)\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right) = \mathbf{E}_0^i\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right). \tag{2.1}$$

Notice that $\mathbf{E}$ has this form because, as is was proved in Subsection 1.7, $\mathbf{E}$ is always perpendicular to the direction of propagation $\mathbf{s}^i$. Notice also that the field $\mathbf{E}^i$ has a component that is perpendicular to the plane of incidence and a component that is parallel to this plane, indeed, we write

$$\mathbf{E}_\perp^i = I_\perp \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right)\mathbf{j},$$

and

$$\mathbf{E}_\|^i = (-I_\| \cos\theta_i\, \mathbf{i} + I_\| \sin\theta_i\, \mathbf{k}) \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right).$$

Also notice that

$$|\mathbf{E}^i|^2 = \left(I_\|^2 + I_\perp^2\right)\cos^2\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right)$$

From (1.27), the magnetic field is then

$$\mathbf{H}^i(\mathbf{r},t) = \frac{1}{v_1}(-I_\perp \cos\theta_i, -I_\|, I_\perp \sin\theta_i)\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right) = \mathbf{H}_0^i\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^i}{v_1}\right)\right).$$

Let us now introduce $\mathbf{s}^t$, the direction of propagation of the transmitted wave, and θ_t the angle between the normal ν and $\mathbf{s}^t$, and similarly, $\mathbf{s}^r$ is the direction of propagation of the reflected wave and θ_r is the angle between the normal ν and $\mathbf{s}^r$. We have from the Snell law that $\mathbf{s}^r = \sin\theta_r\mathbf{i} + \cos\theta_r\mathbf{k} = \sin\theta_i\mathbf{i} - \cos\theta_i\mathbf{k}$. Then the electric and magnetic fields corresponding to transmission are

(2.2)

$$\mathbf{E}^t(\mathbf{r},t) = (-T_\| \cos\theta_t, T_\perp, T_\| \sin\theta_t)\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^t}{v_2}\right)\right) = \mathbf{E}_0^t\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^t}{v_2}\right)\right)$$

$$\mathbf{H}^t(\mathbf{r},t) = \frac{1}{v_2}(-T_\perp \cos\theta_t, -T_\|, T_\perp \sin\theta_t)\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^t}{v_2}\right)\right) = \mathbf{H}_0^t\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^t}{v_2}\right)\right);$$

and similarly the fields corresponding to reflection are

(2.3)

$$\mathbf{E}^r(\mathbf{r},t) = (-R_\| \cos\theta_r, R_\perp, R_\| \sin\theta_r)\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^r}{v_1}\right)\right) = \mathbf{E}_0^r\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^r}{v_1}\right)\right)$$

$$\mathbf{H}^r(\mathbf{r},t) = \frac{1}{v_1}(-R_\perp \cos\theta_r, -R_\|, R_\perp \sin\theta_r)\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^r}{v_1}\right)\right) = \mathbf{H}_0^r\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s}^r}{v_1}\right)\right).$$

Recall that from Snell's law all vectors $\mathbf{s}^i, \mathbf{s}^t, \mathbf{s}^r$ and ν all live on the same plane, that is, the xz-plane. Each of the electric and magnetic fields can be decomposed uniquely as a sum of a component in the direction of the normal (normal component) or on the z-axis, plus another component perpendicular to the normal (tangential component) or on the xy-plane. From the integral form of Maxwell's equations, as it shown in Subsection 1.3, the tangential components of $\mathbf{E}$ (and also of $\mathbf{H}$ if $\mathbf{J} = 0$) at the interface are continuous (see also [BW59][Section 1.1.3, formula (23)]). Since the electric field on medium 1 near Γ equals $\mathbf{E}^i + \mathbf{E}^r$, we get $\mathbf{E}^i_{\tan} + \mathbf{E}^r_{\tan} = \mathbf{E}^t_{\tan}$ on Γ, since $\mathbf{E}^t_{\tan}$ is the transmitted electric field in media 2 near Γ. From the configuration we have, we can write $\mathbf{E}^i = \mathbf{E}^i_{\text{normal}}\,\mathbf{k} + \mathbf{E}^i_{\tan}$, and so $\mathbf{k} \times \mathbf{E}^i = \mathbf{k} \times \mathbf{E}^i_{\tan}$. Similarly, $\mathbf{k} \times \mathbf{E}^r = \mathbf{k} \times \mathbf{E}^r_{\tan}$ and $\mathbf{k} \times \mathbf{E}^t = \mathbf{k} \times \mathbf{E}^t_{\tan}$. So $\mathbf{k} \times \mathbf{E}^i + \mathbf{k} \times \mathbf{E}^r = \mathbf{k} \times \mathbf{E}^t$. Then

$$\mathbf{k} \times \mathbf{E}^i_0 \cos\left(\omega\left(t - \frac{\mathbf{r} \cdot \mathbf{s}^i}{v_1}\right)\right) + \mathbf{k} \times \mathbf{E}^r_0 \cos\left(\omega\left(t - \frac{\mathbf{r} \cdot \mathbf{s}^r}{v_1}\right)\right) = \mathbf{k} \times \mathbf{E}^t_0 \cos\left(\omega\left(t - \frac{\mathbf{r} \cdot \mathbf{s}^t}{v_2}\right)\right),$$

for all $\mathbf{r}$ close to zero and all t. The interface point P is $\mathbf{r} = (0, 0, 0)$, so in particular, we obtain

$$\mathbf{k} \times \mathbf{E}^i_0 \cos(\omega t) + \mathbf{k} \times \mathbf{E}^r_0 \cos(\omega t) = \mathbf{k} \times \mathbf{E}^t_0 \cos(\omega t)$$

for all t. Eliminating the cosines we get

$$\mathbf{k} \times \mathbf{E}^i_0 + \mathbf{k} \times \mathbf{E}^r_0 = \mathbf{k} \times \mathbf{E}^t_0. \tag{2.4}$$

Since we are assuming the current density vector $\mathbf{J} = 0$, as it was mentioned earlier, the tangential component of the magnetic field is also continuous across the interface. So as before with the electric field, we have $\mathbf{H}^i_{\tan} + \mathbf{H}^r_{\tan} = \mathbf{H}^t_{\tan}$ on Γ, and so

$$\mathbf{k} \times \mathbf{H}^i_0 + \mathbf{k} \times \mathbf{H}^r_0 = \mathbf{k} \times \mathbf{H}^t_0. \tag{2.5}$$

From (2.4) we obtain the equations

$$I_\perp + R_\perp = T_\perp, \qquad \cos\theta_i (I_\parallel - R_\parallel) = \cos\theta_t T_\parallel;$$

and from (2.5) we obtain

$$\frac{I_\parallel}{v_1} + \frac{R_\parallel}{v_1} = \frac{T_\parallel}{v_2}, \qquad \cos\theta_i \left(\frac{I_\perp}{v_1} - \frac{R_\perp}{v_1}\right) = \cos\theta_t \frac{T_\perp}{v_2}.$$

We have $n_1 = c/v_1$ and $n_2 = c/v_2$ so solving the last two sets of equations yields

$$\begin{aligned}
T_\parallel &= \frac{2n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t}\, I_\parallel \\
T_\perp &= \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}\, I_\perp \\
R_\parallel &= \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t}\, I_\parallel \\
R_\perp &= \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}\, I_\perp.
\end{aligned}$$

These are the *Fresnel equations* expressing the amplitudes of the reflected and transmitted waves in terms of the amplitude of the incident wave.

2.1. **Rewriting the Fresnel Equations.** We will replace $\mathbf{s^i}$ by x and $\mathbf{s^t}$ by m, and we also set $\kappa = n_2/n_1$. Recall ν is the normal to the interface. We have $\cos\theta_i = x\cdot\nu$ and $\cos\theta_t = m\cdot\nu$. In addition, from the Snell law $x-\kappa m = \lambda\nu$, so the Fresnel equations have the form

$$\begin{aligned}
T_{\parallel} &= \frac{2\,x\cdot\nu}{\kappa\,x\cdot\nu + m\cdot\nu}\,I_{\parallel} = \frac{2\,x\cdot\nu}{(\kappa\,x+m)\cdot\nu}\,I_{\parallel} = \frac{2\,x\cdot(x-\kappa\,m)}{(\kappa\,x+m)\cdot(x-\kappa\,m)}\,I_{\parallel}\\
T_{\perp} &= \frac{2\,x\cdot\nu}{x\cdot\nu + \kappa\,m\cdot\nu}\,I_{\perp} = \frac{2\,x\cdot\nu}{(x+\kappa\,m)\cdot\nu}\,I_{\perp} = \frac{2\,x\cdot(x-\kappa\,m)}{(x+\kappa\,m)\cdot(x-\kappa\,m)}\,I_{\perp}\\
R_{\parallel} &= \frac{\kappa\,x\cdot\nu - m\cdot\nu}{\kappa\,x\cdot\nu + m\cdot\nu}\,I_{\parallel} = \frac{(\kappa\,x-m)\cdot\nu}{(\kappa\,x+m)\cdot\nu}\,I_{\parallel} = \frac{(\kappa\,x-m)\cdot(x-\kappa\,m)}{(\kappa\,x+m)\cdot(x-\kappa\,m)}\,I_{\parallel}\\
R_{\perp} &= \frac{x\cdot\nu - \kappa\,m\cdot\nu}{x\cdot\nu + \kappa\,m\cdot\nu}\,I_{\perp} = \frac{(x-\kappa\,m)\cdot\nu}{(x+\kappa\,m)\cdot\nu}\,I_{\perp} = \frac{(x-\kappa\,m)\cdot(x-\kappa\,m)}{(x+\kappa\,m)\cdot(x-\kappa\,m)}\,I_{\perp}.
\end{aligned}$$

Notice that the denominators of the perpendicular components are the same and likewise for the parallel components.

2.2. **The Poynting vector.** It is defined by

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H},$$

where c is the speed of light in free space. The vector $\mathbf{S}$ represents the flux of energy through a surface. Suppose dA is the area of a surface element at a point P and let ν be the normal at P. Then the flux of energy through dA at the point P is given by

$$dF = \mathbf{S}\cdot\nu\,dA.$$

From (1.27) we get that

$$\mathbf{S} = \frac{c}{4\pi v}\mathbf{E}\times(\mathbf{s}\times\mathbf{E}) = \frac{n}{4\pi}|\mathbf{E}|^2\mathbf{s}.$$

Using the form of the incident wave from the previous section, the amount of energy J^i flowing through a unit area of the boundary per second at P, of the incident wave $\mathbf{E}^i$ given in (2.1), is then

$$J^i = |\mathbf{S}^i|\cos\theta_i = \frac{n_1}{4\pi}|\mathbf{E}_0^i|^2\cos\theta_i.^{\S}$$

Similarly, the amount of energy in the reflected and transmitted waves (also given in the previous section) leaving a unit area of the boundary per second at P are

[§] Notice that from (2.1), the value of the field $\mathbf{E}^i$ at P is $\mathbf{E}^i(0,t) = \mathbf{E}_0^i\cos(\omega\,t)$. Hence we actually get $J^i = \frac{n_1}{4\pi}|\mathbf{E}_0^i|^2\cos\theta_i\,\cos^2(\omega\,t)$. Similarly, from (2.2), the value of the field $\mathbf{E}^t$ at P is $\mathbf{E}^t(0,t) = \mathbf{E}_0^t\cos(\omega\,t)$, and from (2.3), the value of the field $\mathbf{E}^r$ at P is $\mathbf{E}^r(0,t) = \mathbf{E}_0^r\cos(\omega\,t)$. Therefore we have $J^r = \frac{n_1}{4\pi}|\mathbf{E}_0^r|^2\cos\theta_i\,\cos^2(\omega\,t)$, and $J^t = \frac{n_2}{4\pi}|\mathbf{E}_0^t|^2\cos\theta_t\,\cos^2(\omega\,t)$. So we get formulas (2.6) because the factor $\cos^2(\omega t)$ cancels out.

given by

$$J^r = |\mathbf{S}^r| \cos\theta_i = \frac{n_1}{4\pi} |\mathbf{E}_0^r|^2 \cos\theta_i$$
$$J^t = |\mathbf{S}^t| \cos\theta_t = \frac{n_2}{4\pi} |\mathbf{E}_0^t|^2 \cos\theta_t.$$

The reflection and transmission coefficients are defined by

$$\mathcal{R} = \frac{J^r}{J^i} = \left(\frac{|\mathbf{E}_0^r|}{|\mathbf{E}_0^i|}\right)^2, \text{ and } \mathcal{T} = \frac{J^t}{J^i} = \frac{n_2 \cos\theta_t}{n_1 \cos\theta_i} \left(\frac{|\mathbf{E}_0^t|}{|\mathbf{E}_0^i|}\right)^2. \tag{2.6}$$

By conservation of energy or by direct verification $\mathcal{R} + \mathcal{T} = 1$.

2.3. **Polarization.** Polarization is a property of the field that describes the orientation of their oscillations. Since the electric vector is assumed a plane wave and as we showed it is perpendicular to the direction of propagation $\mathbf{s}$, then for each $\mathbf{r}$ in the plane $\mathbf{r} \cdot \mathbf{s} = c$ and t fixed, the vector $\mathbf{E}(\mathbf{r}, t)$ is constant. We visualize $\mathbf{E}(\mathbf{r}, t)$ as a vector with origin at the intersection of the direction $\mathbf{s}$ with the plane $\mathbf{r} \cdot \mathbf{s} = c$. That is, as a vector with origin at the point $(\mathbf{r} \cdot \mathbf{s})\mathbf{s}$ and terminal point $(\mathbf{r} \cdot \mathbf{s})\mathbf{s} + \mathbf{E}(\mathbf{r}, t)$. Then t is fixed and $\mathbf{r}$ runs over all space, the end point of this vector describes a curve in 3-d. If we now move t, this curve is shifted (and keeps the same shape) by changing the phase because of the presence of ωt in the cos function. So when $\mathbf{r} \cdot \mathbf{s} = c$ and t moves the vector $\mathbf{E}(\mathbf{r}, t)$ describes a curve in the plane $\mathbf{r} \cdot \mathbf{s} = c$. If this curve is an ellipse we say that the wave is *elliptically polarized* and when the ellipse is a circle we say the wave is *circularly polarized*, and if the the ellipse degenerates to a segment we say the wave is *linearly polarized*. If the wave describing the incident field has components that have different phases, then this changes the sense of circulation and inclination of the ellipse (for elliptically polarized light), see [BW59][Section 1.4.2]. See `http://en.wikipedia.org/wiki/Polarization` for pictures.

Suppose for example that the wave is linearly polarized perpendicularly to the plane of incidence. That is, $I_\parallel = 0$. Then from Fresnel equations $T_\parallel = R_\parallel = 0$ and

$$\mathcal{R} = \left(\frac{|\mathbf{E}_0^r|}{|\mathbf{E}_0^i|}\right)^2 = \left(\frac{R_\perp}{I_\perp}\right)^2 = \left(\frac{|x - \kappa\, m|^2}{1 - \kappa^2}\right)^2,$$

and

$$\begin{aligned}\mathcal{T} &= \kappa \frac{m \cdot \nu}{x \cdot \nu} \left(\frac{T_\perp}{I_\perp}\right)^2 = \kappa \frac{m \cdot (x - \kappa\, m)}{x \cdot (x - \kappa\, m)} \left(\frac{2\, x \cdot (x - \kappa\, m)}{(x + \kappa\, m) \cdot (x - \kappa\, m)}\right)^2 \\ &= \frac{4\kappa}{(1 - \kappa^2)^2} \left(m \cdot (x - \kappa\, m)\right) \left(x \cdot (x - \kappa\, m)\right).\end{aligned}$$

For the case when no polarization is assumed, that is, radiation has no particular preference for the direction in which it vibrates, we have from Fresnel's equations that

$$|\mathbf{E}_0^r|^2 = R_\parallel^2 + R_\perp^2 = \left[\frac{(\kappa\,x - m)\cdot(x-\kappa\,m)}{(\kappa\,x + m)\cdot(x-\kappa\,m)}\right]^2 I_\parallel^2 + \left[\frac{(x-\kappa\,m)\cdot(x-\kappa\,m)}{(x+\kappa\,m)\cdot(x-\kappa\,m)}\right]^2 I_\perp^2,$$

and so

$$\begin{aligned}\mathcal{R} &= \left(\frac{|\mathbf{E}_0^r|}{|\mathbf{E}_0^i|}\right)^2 = \frac{R_\parallel^2 + R_\perp^2}{I_\parallel^2 + I_\perp^2}\\ &= \left[\frac{(\kappa\,x - m)\cdot(x-\kappa\,m)}{(\kappa\,x + m)\cdot(x-\kappa\,m)}\right]^2 \frac{I_\parallel^2}{I_\parallel^2 + I_\perp^2} + \left[\frac{(x-\kappa\,m)\cdot(x-\kappa\,m)}{(x+\kappa\,m)\cdot(x-\kappa\,m)}\right]^2 \frac{I_\perp^2}{I_\parallel^2 + I_\perp^2}\\ &= \frac{1}{(1-\kappa^2)^2}\left(\left[\frac{2\kappa}{x\cdot m} - (1+\kappa^2)\right]^2 \frac{I_\parallel^2}{I_\parallel^2 + I_\perp^2} + \left[1 - 2\kappa\,x\cdot m + \kappa^2\right]^2 \frac{I_\perp^2}{I_\parallel^2 + I_\perp^2}\right)\end{aligned}$$

which is a function only of $x \cdot m$. In principle the coefficients $I_\parallel$ and $I_\perp$ might depend on the direction x, in other words, for each direction x we would have a wave that changes its amplitude with the direction of propagation. The energy of the incident wave would be $f(x) = |\mathbf{E}_0^i|^2 = I_\parallel(x)^2 + I_\perp(x)^2$. Notice that if the incidence is normal, that is, $x = m$, then $\mathcal{R} = \left(\frac{1-\kappa}{1+\kappa}\right)^2$ which shows that even for radiation normal to the interface we lose energy by reflection. For example, if we go from air to glass, $n_1 = 1$ and $n_2 = 1.5$, we have $\kappa = 1.5$ so $\mathcal{R} = .04$, which means that 4% of the energy is lost in internal reflection.

2.4. **Estimation of the Fresnel coefficient.** Let

$$\phi(t) = \frac{1}{(1-\kappa^2)^2}\left(\left[\frac{2\kappa}{t} - (1+\kappa^2)\right]^2 \alpha + \left[1 - 2\kappa\,t + \kappa^2\right]^2 \beta\right),$$

where $\alpha = \dfrac{I_\parallel^2}{I_\parallel^2 + I_\perp^2}$, and $\beta = \dfrac{I_\perp^2}{I_\parallel^2 + I_\perp^2}$. We have $\phi(x\cdot m) = \mathcal{R}$. Assume $\kappa > 1$ and $\epsilon > 0$, we want to show that there exists a constant $0 < C_\epsilon < 1$ such that

$$\mathcal{R} \le C_\epsilon, \qquad \text{for all } \frac{1}{\kappa} + \epsilon \le x \cdot m \le 1. \tag{2.7}$$

This is equivalent to prove that

$$\phi(t) \le C_\epsilon, \qquad \text{for all } \frac{1}{\kappa} + \epsilon \le t \le 1.$$

We write

$$\begin{aligned}\phi(t) &= \frac{1}{(1-\kappa^2)^2}\left(\left((1+\kappa^2)^2 - 4\kappa(1+\kappa^2)\frac{1}{t} + \frac{4\kappa^2}{t^2}\right)\alpha + \left((1+\kappa^2)^2 - 4\kappa(1+\kappa^2)t + 4\kappa^2 t^2\right)\beta\right)\\ &= \frac{1}{(1-\kappa^2)^2}\left((1+\kappa^2)^2(\alpha+\beta) - 4\kappa(1+\kappa^2)\left(\frac{\alpha}{t} + \beta t\right) + 4\kappa^2\left(\frac{\alpha}{t^2} + \beta t^2\right)\right).\end{aligned}$$

If $I_{\|} = I_{\perp}$, then $\alpha = \beta = 1/2$ and

$$\begin{aligned}\phi(t) &= \frac{1}{2(1-\kappa^2)^2}\left(2(1+\kappa^2)^2 - 4\kappa(1+\kappa^2)\left(\frac{1}{t}+t\right) + 4\kappa^2\left(\frac{1}{t^2}+t^2\right)\right)\\ &= \frac{1}{(1-\kappa^2)^2}\left((1+\kappa^2)^2 - 2\kappa(1+\kappa^2)\left(\frac{1}{t}+t\right) + 2\kappa^2\left(\left(\frac{1}{t}+t\right)^2-2\right)\right).\end{aligned}$$

If we set $z(t) = \dfrac{1}{t} + t$ and

$$\psi(z) = \frac{1}{(1-\kappa^2)^2}\left((1+\kappa^2)^2 - 2\kappa(1+\kappa^2)z + +2\kappa^2\left(z^2-2\right)\right),$$

then $\phi(t) = \psi(z(t))$. We have that $z(t)$ is strictly decreasing in $(0,1)$ so

$$\max_{[(1/\kappa)+\epsilon,1]} z(t) = (1/\kappa) + \epsilon + \frac{1}{(1/\kappa)+\epsilon} = z_\epsilon,$$

and

$$\min_{[(1/\kappa)+\epsilon,1]} z(t) = 2.$$

On the other hand, $\psi'(z) = \dfrac{4\kappa^2}{(1-\kappa^2)^2}\left(z - \dfrac{1+\kappa^2}{2\kappa}\right)$, and so ψ is strictly increasing on $\left(\dfrac{1+\kappa^2}{2\kappa}, +\infty\right)$ and strictly decreasing on $\left(-\infty, \dfrac{1+\kappa^2}{2\kappa}\right)$. Suppose first that $\dfrac{1+\kappa^2}{2\kappa} \leq 2$. Then $[2, z_\epsilon] \subset \left(\dfrac{1+\kappa^2}{2\kappa}, +\infty\right)$, so $\phi(t) \leq \psi(z_\epsilon) < \psi\left(z(1/\kappa)\right) = 1$, for $(1/\kappa)+\epsilon \leq t \leq 1$. Suppose on the other hand that $\dfrac{1+\kappa^2}{2\kappa} > 2$. Since $\dfrac{1}{2}\left(\dfrac{1}{\kappa}+\kappa\right) < z_\epsilon$ for $\epsilon > 0$ small, we have that $\max_{[(1/\kappa)+\epsilon,1]} \phi(t) \leq \max\{\psi(2), \psi(z_\epsilon)\} < 1$ because $\psi(2) = \left(\dfrac{1-\kappa}{1+\kappa}\right)^2 < 1$. Therefore we have proved (2.7) when $\alpha = \beta = 1/2$.

2.5. **Estimation of the Fresnel coefficient BIS.** Let

$$\phi(t) = \frac{1}{(1-\kappa^2)^2}\left(\left[\frac{2\kappa}{t} - (1+\kappa^2)\right]^2 \alpha + \left[1 - 2\kappa\, t + \kappa^2\right]^2 \beta\right),$$

where $0 \leq \alpha, \beta \leq 1$ and $\alpha + \beta = 1$. Set

$$g(t) = \left[\frac{2\kappa}{t} - (1+\kappa^2)\right]^2, \qquad h(t) = \left[1 - 2\kappa\, t + \kappa^2\right]^2.$$

Suppose $\kappa < 1$ and $\kappa + \epsilon \leq t \leq 1$. We have $g'(t) = -4\kappa\left[\dfrac{2\kappa}{t} - (1+\kappa^2)\right]\dfrac{1}{t^2}$, so $g'(t) > 0$ for $t > \dfrac{2\kappa}{1+\kappa^2}$, and $g'(t) < 0$ for $t < \dfrac{2\kappa}{1+\kappa^2}$. Since $\kappa < 1$, we have $\kappa + \epsilon < \dfrac{2\kappa}{1+\kappa^2} < 1$

for $\epsilon > 0$ small. Therefore, g decreases in the interval $[\kappa+\epsilon, \dfrac{2\kappa}{1+\kappa^2}]$, and g increases in the interval $[\dfrac{2\kappa}{1+\kappa^2}, 1]$. Hence

$$\max_{[\kappa+\epsilon,1]} g(t) = \max\{g(\kappa+\epsilon), g(1)\}.$$

We have that $g(1) = (1-\kappa)^4$, and $g(\kappa+\epsilon) > g(1)$ for ϵ small, so

$$\max_{[\kappa+\epsilon,1]} g(t) = g(\kappa+\epsilon).$$

On the other hand, $h'(t) = -4\kappa\,[1 - 2\kappa\, t + \kappa^2]$, and so $h'(t) > 0$ for $t > \dfrac{1+\kappa^2}{2\kappa}$ and $h'(t) < 0$ for $t < \dfrac{1+\kappa^2}{2\kappa}$. Since $\dfrac{1+\kappa^2}{2\kappa} > 1$, the function h is decreasing in the interval $[\kappa+\epsilon, 1]$ and so

$$\max_{[\kappa+\epsilon,1]} h(t) = h(\kappa+\epsilon).$$

Therefore we obtain that

$$\max_{[\kappa+\epsilon,1]} \phi(t) \leq \frac{1}{(1-\kappa^2)^2} \left(\alpha\, g(\kappa+\epsilon) + \beta\, h(\kappa+\epsilon)\right).$$

It is easy to see that

$$g(\kappa+\epsilon) < (1-\kappa^2)^2, \quad \text{and} \quad h(\kappa+\epsilon) < (1-\kappa^2)^2$$

and so we obtain the bound

$$\max_{[\kappa+\epsilon,1]} \phi(t) \leq C_\epsilon < 1,$$

with

$$C_\epsilon = \max\left\{\frac{g(\kappa+\epsilon)}{(1-\kappa^2)^2}, \frac{h(\kappa+\epsilon)}{(1-\kappa^2)^2}\right\}$$

independent of α and β.

2.6. **Case** $\kappa > 1$**.** For ϵ small we have $\dfrac{1}{\kappa} + \epsilon < \dfrac{2\kappa}{1+\kappa^2} < 1$, so as before, g decreases in the interval $[\dfrac{1}{\kappa}+\epsilon, \dfrac{2\kappa}{1+\kappa^2}]$, and g increases in the interval $[\dfrac{2\kappa}{1+\kappa^2}, 1]$. Hence

$$\max_{[(1/\kappa)+\epsilon,1]} g(t) = \max\left\{g\left(\frac{1}{\kappa}+\epsilon\right), g(1)\right\}.$$

Since now $\kappa > 1$ we have that $g(1) > g\left(\dfrac{1}{\kappa}+\epsilon\right)$, for ϵ small, and so

$$\max_{[(1/\kappa)+\epsilon,1]} g(t) = g(1).$$

Since we always have $\dfrac{1+\kappa^2}{2\kappa} > 1$, the function h is decreasing in the interval $[(1/\kappa)+\epsilon, 1]$ and so

$$\max_{[(1/\kappa)+\epsilon,1]} h(t) = h((1/\kappa)+\epsilon).$$

Therefore we obtain that

$$\max_{[(1/\kappa)+\epsilon,1]} \phi(t) \leq \frac{1}{(1-\kappa^2)^2}\left(\alpha\, g(1) + \beta\, h((1/\kappa)+\epsilon)\right).$$

It is clear that $g(1) = (1-\kappa)^2 < (1-\kappa^2)^2$, and also $h((1/\kappa)+\epsilon) < (1-\kappa^2)^2$ when $0 < \epsilon < 1-(1/\kappa)$. So we obtain the bound

$$\max_{[(1/\kappa)+\epsilon,1]} \phi(t) \leq C_\epsilon < 1,$$

with

$$C_\epsilon = \max\left\{\left(\frac{1-\kappa}{1+\kappa}\right)^2, \frac{h((1/\kappa)+\epsilon)}{(1-\kappa^2)^2}\right\}$$

independent of α and β.

References

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