Chapter 1: Vectors

1.1 Introduction

Electromagnetic Theory is described in terms of the language of vector analysis. Let us, therefore, start this course by reviewing some of the concepts of vectors that were discussed in your College Physics course. As you recall, of all the varied quantities that are observed in nature, some have characteristics of scalar quantities while others have the characteristics of vector quantities. *A **scalar** quantity is a quantity that can be completely described by a magnitude, i.e., by a number and a unit.* The characteristic of scalar quantities is that they add up like ordinary numbers. That is, if we have a mass $m_1 = 3$ kg and an another mass $m_2 = 4$ kg then the sum of the two masses is

$$m = m_1 + m_2 = 3 \text{ kg} + 4 \text{ kg} = 7 \text{ kg}$$

*A **vector** quantity, on the other hand, is a quantity that needs both a magnitude and a direction to completely describe it.* A vector is represented in a text book by boldface script, that is, **A**. Because you cannot write in boldface script on note paper or a blackboard, a vector is written there as the letter with an arrow over it. A vector can be represented on a diagram by an arrow. A picture of this vector can be obtained by drawing an arrow from the origin of a Cartesian coordinate system. The length of the arrow represents the magnitude of the vector while the direction of the arrow represents the direction of the vector. The direction is specified by the angle θ that the vector makes with an axis, usually the x-axis, and is shown in figure 1.1. The magnitude of the vector **A** is written as the

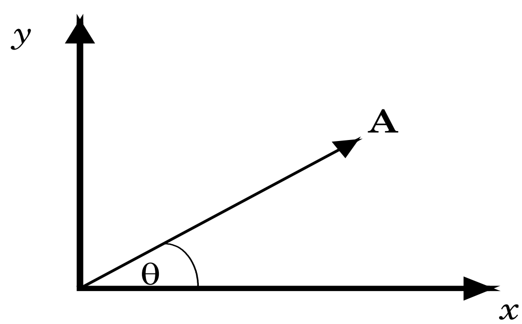


Figure 1.1 Representation of a vector.

absolute value of **A**, namely $|\mathbf{A}|$, or simply by the letter A without boldfacing. One of the defining characteristics of vector quantities is that they must be added in a way that takes their direction into account.

Probably the simplest vector that can be discussed is the displacement vector. Whenever a body moves from one position to another it undergoes a displacement.

*The **displacement** can be represented as a vector that describes how far the body has been displaced from its original position.* As an example, if you walk 3 miles due east, then this walk can be represented as a vector that is 3 units long and points due east. It is shown as $\mathbf{d}_1$ in figure 1.2. Suppose you now walk 4 miles due north. This distance of 4 miles in a northerly direction can be represented as another displacement vector, $\mathbf{d}_2$, and is also shown in figure 1.2. The result of these two

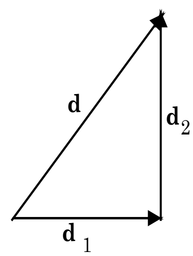


Figure 1.2 The displacement vector.

displacements is a final displacement vector that shows the total displacement from the original position. It is shown as the vector $\mathbf{d}$ in the figure.

We now ask how far did you walk? Well, you walked 3 miles east and 4 miles north and hence you have walked a total distance of 7 miles. But how far are you from where you started? It is certainly not 7 miles as anyone who recalls any high school geometry knows. In fact the final displacement, $\mathbf{d}$, is a vector of magnitude d and that distance, d, can be immediately determined by simple geometry. Applying the Pythagorean theorem to the right triangle of figure 1.2 we get

$$d = \sqrt{d_1^2 + d_2^2}$$
$$d = \sqrt{3^2 + 4^2} = \sqrt{25}$$

and thus $d = 5$ miles

Even though you may have walked a total distance of 7 miles, you are only 5 miles away from where you started. Hence, when the vector displacements are added

$$\boldsymbol{d} = \boldsymbol{d}_1 + \boldsymbol{d}_2$$

you do not get 7 miles for the magnitude of the final displacement, but instead you get 5 miles. It should now be obvious that vectors do not add like ordinary scalar numbers. When the two masses $m_1 = 3$ kg and $m_2 = 4$ kg are added, the sum is 7 kg. But when you add the two displacement vectors of magnitudes $d_1 = 3$ mile and $d_2 = 4$ mile the magnitude of the vector sum is 5 mile. To solve physical problems associated with vectors it is necessary to deal with vector algebra.

1.2 Vector Algebra - The Addition of Vectors

Let us now add any two arbitrary vectors **a** and **b**. The result of adding the two vectors **a** and **b** is to form a new **resultant** vector **R**, which is the sum of **a** and **b**. This can be shown graphically by laying off the first vector **a** in the horizontal direction and then placing the tail of the second vector **b** at the tip of vector **a** as shown in figure 1.3.

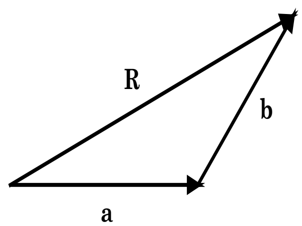


Figure 1.3 The addition of vectors.

The resultant vector **R** is drawn from the origin of the first vector to the tip of the last vector. The resultant vector is written mathematically as

$$\boldsymbol{R} = \boldsymbol{a} + \boldsymbol{b} \tag{1.1}$$

Remember that in this sum we do not mean scalar addition. The resultant vector is the vector sum of the individual vectors **a** and **b**.

Although a vector is a quantity that has both magnitude and direction, it does not have a position. Consequently a vector may be moved parallel to itself without changing the characteristic of the vector. Because the magnitude of the moved vector is still the same, and its direction is still the same, the vector is the same.

Hence, when adding the vectors **a** and **b**, the vector **a** can be moved parallel to itself until the tip of **a** touches the tip of **b**. Similarly the vector **b** can be moved parallel to itself until the tip of **b** touches the tail end of the top vector **a**. In moving the vectors parallel to themselves a parallelogram has been formed, as shown in figure 1.4.

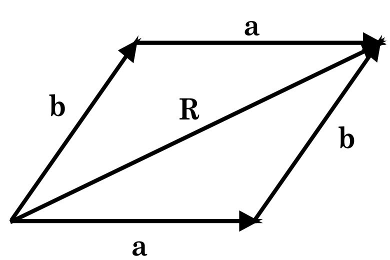


Figure 1.4 The addition of vectors by the parallelogram method.

From what was said before about the resultant of **a** and **b**, the resultant of the two vectors is seen to be the main diagonal of the parallelogram formed by the vectors **a** and **b**, hence we call this process **the parallelogram method of vector addition**. It is sometimes stated as part of the definition of a vector, that vectors obey the parallelogram law of addition. Note from the diagram that

$$\boldsymbol{R} = \boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a} \tag{1.2}$$

i.e., vectors can be added in any order. Mathematicians would say vector addition is commutative.

1.3 Vector Subtraction - The Negative of a Vector

If we are given a vector **a**, as shown in figure 1.5, then the vector minus **a**, (–**a**) is a vector of the same magnitude as **a** but in the opposite direction. That is, if

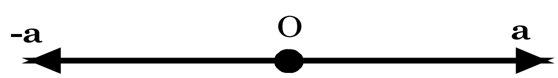
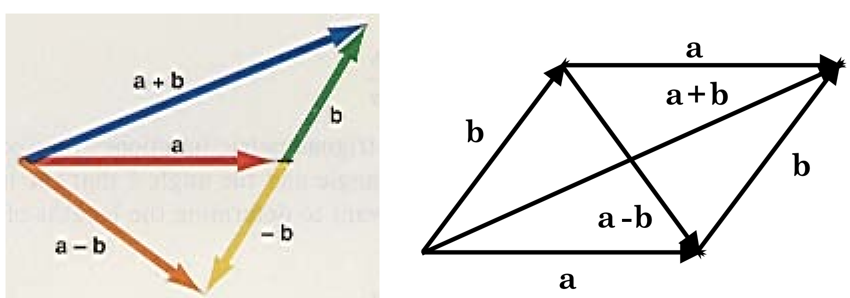


Figure 1.5 The negative of a vector.

vector **a** points to the right, then the vector –**a** points to the left. The vector –**a** is called the negative of the vector **a**. By defining the negative of a vector in this way, the process of vector subtraction can now be determined. The subtraction of the vector **b** from the vector **a** is defined as

$$\mathrm{a} - \mathrm{b} = \mathrm{a} + (-\mathrm{b}) \tag{1.3}$$

That is, the subtraction of **b** from **a** is equivalent to adding the vector **a** and the negative vector (–**b**). This is shown graphically in figure 1.6(a). If we complete the



(a) (b)

Figure 1.6 The subtraction of vectors.

parallelogram for the addition of **a** + **b**, we see that we can move the vector **a** – **b** parallel to itself until it becomes minor diagonal of the parallelogram, figure 1.6.

1.4 Addition of Vectors by the Polygon Method

To find the sum of any number of vectors graphically, the polygon method is used. In the polygon method each vector is added to the preceding vector by placing the tail of one vector to the head of the previous vector as shown in figure 1.7. The

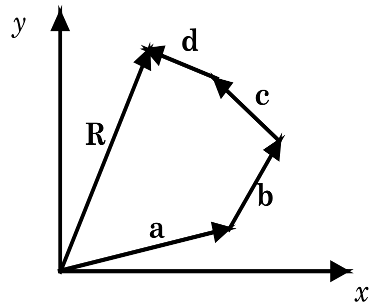


Figure 1.7 Addition of vectors by the polygon method.

resultant vector **R** is the sum of all these vectors. That is,

$$\mathbf{R} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} \tag{1.4}$$

R is found by drawing the vector from the origin of the coordinate system to the tip of the final vector.

Vectors are usually added analytically or mathematically. In order to do that, the components of a vector must be defined.

1.5 Resolution of a Vector into its Components

An arbitrary vector **a** is drawn onto an x, y-coordinate system in figure 1.8. The vector **a** makes an angle θ with the x-axis. To find the x-component a_x of the vector **a**, the vector **a** is projected down onto the x-axis, i.e., a perpendicular is dropped from the tip of **a** to the x-axis. One way of visualizing this concept of a **component**

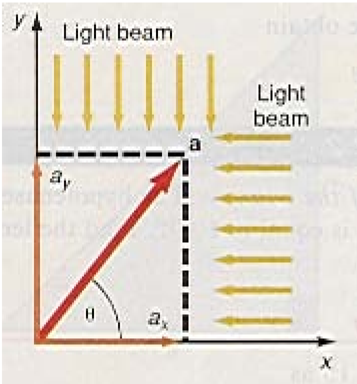


Figure 1.8 Defining the components of a vector.

of a vector is to place a parallel beam of light above the vector **a** and parallel to the y-axis. The light hitting the vector **a** will not make it to the x-axis, and will therefore leave a shadow on the x-axis. This shadow on the x-axis will be called the x-component of the vector **a** and will be denoted by a_x. The component is shown as the thick black line on the x-axis in figure 1.8. It should be emphasized here that a_x is a number, a length along the x-axis.

In the same way, the y-component of the vector **a**, a_y, can be determined by projecting **a** onto the y-axis, figure 1.8. That is, a perpendicular from the tip of **a** is dropped onto the y-axis. Again, this can be visualized by projecting light, which is parallel to the x-axis, onto the vector **a**. The shadow of the vector **a** on the y-axis is the y-component, a_y. It is shown in figure 1.8 as the thick black line on the y-axis. It is a number, the length of the shadow along the y-axis.

The components of the vector are found mathematically by noting that the vector and its components constitute a triangle as seen in figure 1.9. From

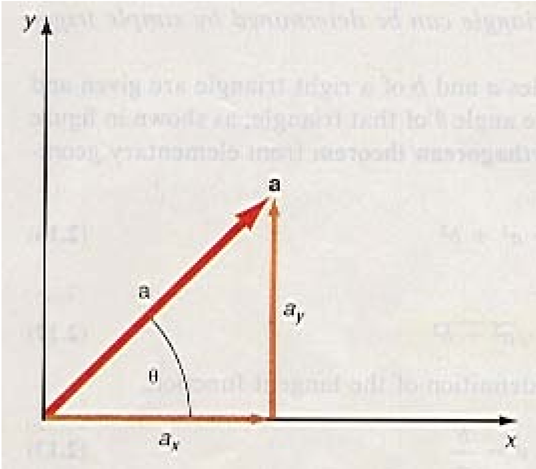


Figure 1.9 Finding the components of a vector mathematically.

trigonometry, the x-component of the vector **a** is found from

$$\cos\theta = \frac{a_x}{a}$$

Upon solving for a_x, the x-component of the vector **a** is obtained as

$$a_x = a\cos\theta \tag{1.5}$$

The y-component of vector **a** is found from

$$\sin\theta = \frac{a_y}{a}$$

Hence, the y-component of the vector **a** is

$$a_y = a \sin \theta \tag{1.6}$$

Example 1.1

Finding the components of a vector. The magnitude of vector **a** is 10.0 units and the vector makes an angle of 30.0^0 with the x-axis. Find the components of **a**.

Solution

The x-component of vector **a** is found from equation 1.5 as

$$a_x = a \cos \theta = 10.0 \text{ units} \cos 30.0^0 = 8.66 \text{ units}$$

The y-component of **a** is found from equation 1.6 as

$$a_y = a \sin \theta = 10.0 \text{ units} \sin 30.0^0 = 5.00 \text{ units}$$

To go to this interactive example click on this sentence.

1.6 Determination of a Vector from its Components

If the components a_x and a_y of a vector are given and it is desired to find the vector **a** itself, that is, its magnitude |**a**| and its direction θ, then the process will be the inverse of the technique used in the last section. The components a_x and a_y of the vector **a** are seen in figure 1.10. By forming the triangle with sides a_x and a_y, the

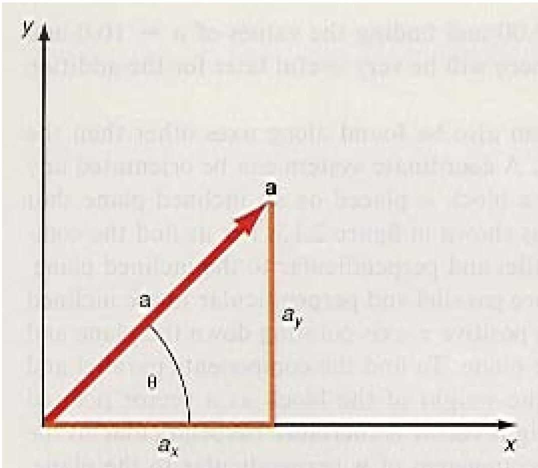


Figure 1.10 Determining a vector from its components.

hypotenuse of that triangle is the magnitude of the vector, and is determined by the Pythagorean Theorem as

$$a^2 = a_x^2 + a_y^2 \tag{1.7}$$

Hence, the magnitude of vector **a** is

$$a = \sqrt{a_x^2 + a_y^2} \tag{1.8}$$

It is thus very simple to find the magnitude of a vector once its components are known.

To find the angle θ that the vector **a** makes with the *x*-axis the definition of the tangent is used, namely,

$$\tan\theta = \frac{a_y}{a_x}$$

The angle θ is found by the inverse tangent as

$$\theta = \tan^{-1}\frac{a_y}{a_x} \tag{1.9}$$

Example 1.2

Finding a vector from its components. The components of a certain vector are given as $a_x = 7.55$ and $a_y = 3.25$. Find the magnitude of the vector and the angle θ that it makes with the *x*-axis.

Solution

The magnitude of the vector **a** is found from equation 1.8 as

$$\begin{aligned} a = \sqrt{a_x^2 + {a_y}^2} &= \sqrt{(7.55)^2 + (3.25)^2} \\ &= \sqrt{57.0 + 10.6} \\ &= 8.22 \end{aligned} \tag{1.8}$$

The angle θ is found from equation 1.9 as

$$\begin{aligned} \theta = \tan^{-1}\frac{a_y}{a_x} &= \tan^{-1}\frac{3.25}{7.55} = \tan^{-1} 0.430 \\ &= 23.3^0 \end{aligned}$$

Therefore, the magnitude of vector **a** is 8.22 and the angle θ is 23.3^0. The techniques developed here will be very useful later for the addition of any number of vectors. The components of a vector can also be found along axes other than the *x*, *y*-axes.

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1.7 Unit Vectors

It is convenient in our analysis to introduce the concept of unit vectors. *A unit vector is a vector that has a magnitude of one, hence the name, unit. It is used to specify a particular direction.* The most common of the unit vectors are the **i j k** system of vectors shown in figure 1.11. The unit vector **i** is a unit vector in the

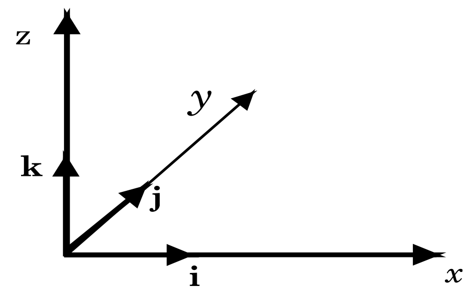


Figure 1.11 The **i j k** system of unit vectors.

x-direction, the unit vector **j** is a unit vector in the y-direction, and the unit vector **k** is a unit vector in the z-direction. Also notice from figure 1.11, the coordinate system is a right handed coordinated system. This means that if you place your right hand along the x-axis with your palm facing the y-axis, and then rotate your hand toward the y-axis, your thumb will face in the z-direction. Let us now consider an arbitrary vector **a** in two dimensions, as shown in figure 1.12. The components a_x

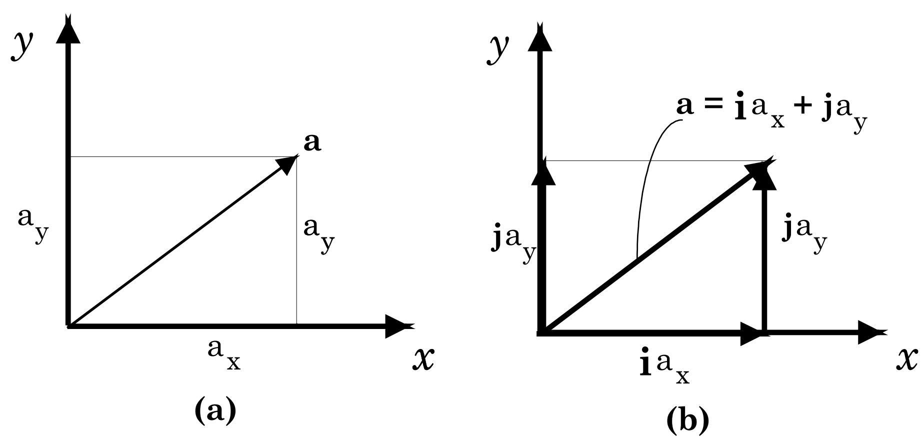


Figure 1.12 A vector in terms of unit vectors.

and a_y of the vector **a** are shown in figure 1.12a. If the x-component, a_x, is multiplied by the unit vector **i**, the vector $\mathbf{i}a_x$ is obtained. The vector $\mathbf{i}a_x$ is a vector in the x-direction with magnitude a_x as shown in figure 1.12b. Multiplying the y-component of the vector **a**, a_y, by the unit vector **j** yields the vector **j**a_y. The vector **j**a_y is a vector in the y-direction with magnitude a_y and is also shown in figure 1.12b. As can be seen from figure 1.12b, adding the vector **i**a_x to the vector **j**a_y yields the original vector **a**. This can be written mathematically as

$$\mathbf{a} = \mathbf{i}a_x + \mathbf{j}a_y \tag{1.10}$$

Equation 1.10 is the form of the equation used for writing a vector in terms of its components and the unit vectors. We will use this form of the equation for writing many of the vectors in this course. Besides the **i j k** unit vectors there are other unit vectors specifying other directions. These will be defined later as the need arises.

Example 1.3

Finding the magnitude and direction of a vector when expressed in terms of unit vectors. Given the vector

$$\mathbf{a} = 3\mathbf{i} + 6\mathbf{j}$$

Find the magnitude of the vector **a** and the direction it makes with the x-axis.

Solution

The magnitude of the vector is found from equation 1.8 as

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{3^2 + 6^2} = \sqrt{45} = 6.71$$

The direction is found from equation 1.9 as

$$\theta = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\frac{6}{3} = \tan^{-1}2$$
$$\theta = 63.4^0$$

To go to this interactive example click on this sentence.

1.8 The Addition of Vectors by the Component Method

A very important technique for the addition of vectors is the **addition of vectors by the component method.** Let us assume that we are given two vectors **a** and **b**, and it is desired to find their vector sum. The sum of the vectors is the resultant vector **R** given by

$$\mathbf{R} = \mathbf{a} + \mathbf{b} \tag{1.11}$$

and is shown in figure 1.13. Let us first look at the addition from a geometrical point of view. The components a_x and a_y of the vector **a** are found by taking the

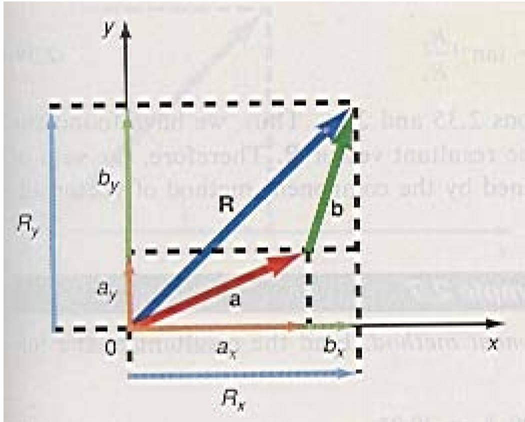


Figure 1.13 The addition of vectors by the component method.

projections onto the x and y-axis respectively. To take the components of the vector **b**, a projection is again made onto the x and y-axis but note that the tail of the vector **b** is not at the origin of coordinates, but rather at the tip of **a**. So both the tip and the tail of **b** are projected onto the x-axis as shown to get b_x, the x-component of **b**. In the same way, **b** is projected onto the y-axis to get b_y, the y-component of **b**. All these components are shown in figure 1.13.

The resultant vector **R** is given by equation 1.11, and because **R** is a vector it has components R_x and R_y, which are the projections of **R** on the x and y-axis respectively and they are shown in figure 1.14. Now go back to the original diagram (figure 1.13) and project **R** onto the x-axis. R_x is shown a little distance below the x-axis, so as not to confuse R_x with the other components that are already there. Similarly, **R** is projected onto the y-axis to get R_y. R_y is slightly displaced from the y-axis, so as not to confuse R_y with the other components already there.

Look very carefully at figure 1.13. Note that the length of R_x is equal to the length of a_x plus the length of b_x. Because components are numbers and hence add like ordinary numbers, this addition can be written simply as

$$R_x = a_x + b_x \tag{1.12}$$

That is, *the x-component of the resultant vector is equal to the sum of the x-components of the individual vectors.*

In the same manner, look at the geometry on the y-axis of figure 1.13. The length R_y is equal to the sum of the lengths of a_y and b_y and therefore

$$R_y = a_y + b_y \tag{1.13}$$

Thus, *the y-component of the resultant vector is equal to the sum of the components of the individual vectors*.

The same results can be obtained mathematically by writing the vectors in terms of the unit vectors. That is,

$$\mathbf{a} = \mathbf{i}a_x + \mathbf{j}a_y \tag{1.14}$$

and

$$\mathbf{b} = \mathbf{i}b_x + \mathbf{j}b_y \tag{1.15}$$

The addition of the two vectors is simply

$$\mathbf{a} + \mathbf{b} = \mathbf{i}a_x + \mathbf{j}a_y + \mathbf{i}b_x + \mathbf{j}b_y$$

gathering the terms with the same unit vectors together we obtain

$$\mathbf{a} + \mathbf{b} = \mathbf{i}a_x + \mathbf{i}b_x + \mathbf{j}a_y + \mathbf{j}b_y$$

and simplifying

$$\mathbf{a} + \mathbf{b} = \mathbf{i}(a_x + b_x) + \mathbf{j}(a_y + b_y) \tag{1.16}$$

But the resultant vector **R** can also be written in terms of the unit vectors as

$$\mathbf{R} = \mathbf{i}R_x + \mathbf{j}R_y \tag{1.17}$$

However, since $\mathbf{R} = \mathbf{a} + \mathbf{b}$, it follows that

$$\mathbf{i}R_x = \mathbf{i}(a_x + b_x)$$

and therefore,

$$R_x = a_x + b_x \tag{1.18}$$

Notice that equation 1.18, obtained mathematically, agrees with equation 1.12, which was obtained geometrically.

The y-component of the resultant vector is obtained similarly, i.e.,

$$\mathbf{j}R_y = \mathbf{j}(a_y + b_y)$$
$$R_y = a_y + b_y \tag{1.19}$$

Again, notice that equation 1.19, obtained mathematically, agrees with equation 1.13, which was obtained geometrically.

This demonstration of the addition of vectors was done for two vectors because it is easier to see the results in figure 1.13 for two vectors than it is for many vectors. However, the technique would be the same for the addition of any number of vectors. For the general case where there are many vectors, equations 1.18 and 1.19 for R_x and R_y can be generalized to

$$R_x = a_x + b_x + c_x + d_x + \dots \tag{1.20}$$

and

$$R_y = a_y + b_y + c_y + d_y + \ldots \tag{1.21}$$

The plus sign and the dots that appear in equations 1.20 and 1.21 indicate additional components can be added for any additional vectors.

We now have R_x and R_y, the components of the resulting vector **R,** as shown in figure 1.14. But if the components of **R** are known the resultant vector can be

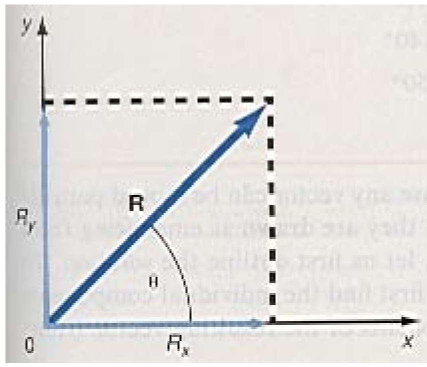


Figure 1.14 The resultant vector.

written as

$$\mathbf{R} = \mathbf{i}R_x + \mathbf{j}R_y \tag{1.22}$$

If it is desired to determine the magnitude of the resultant vector, **R**, this can be done by using the Pythagorean Theorem, i.e.,

$$R = \sqrt{R_x^2 + R_y^2} \tag{1.23}$$

The angle θ in figure 1.14 is found from the geometry to be

$$\tan\theta = \frac{R_y}{R_x}$$

Thus,

$$\theta = \tan^{-1}\frac{R_y}{R_x} \tag{1.24}$$

where R_x and R_y are given by equations 1.20 and 1.21. The magnitude R and the direction θ, of the resultant vector **R**, have thus been found. Therefore, the sum of any number of vectors can be determined by the component method of vector addition.

Example 1.4

The addition of vectors by the component method. Find the resultant of the following four vectors:

$$A = 100\ ;\ \ \theta_1 = 30.0^0$$
$$B = 200\ ;\ \ \theta_2 = 60.0^0$$
$$C = 75.0\ ;\ \ \theta_3 = 140^0$$
$$D = 150\ ;\ \ \theta_4 = 250^0$$

Solution

The four vectors are drawn in figure 1.15. Because any vector can be moved parallel to itself, all the vectors have been so moved and are thus drawn as emanating from

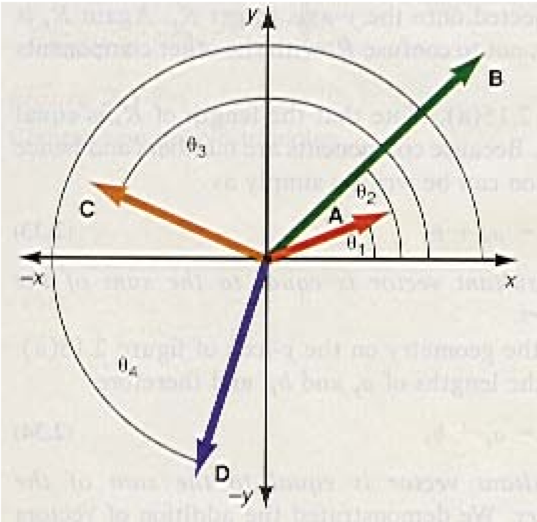


Figure 1.15 Addition of four vectors.

the origin. Before actually solving the problem, let us first outline the solution. To find the resultant of these four vectors the individual components of each vector must first be found, then the x and y-components of the resulting vector are found from

$$R_x = A_x + B_x + C_x + D_x \tag{1.20}$$

$$R_y = A_y + B_y + C_y + D_y \tag{1.21}$$

We then find the resulting vector from

$$R = \sqrt{R_x^2 + R_y{}^2} \tag{1.23}$$

and

$$\theta = \tan^{-1}\frac{R_y}{R_x} \tag{1.24}$$

The actual solution of the problem is found as follows: the individual x-components are found as

$$A_x = A\cos\theta_1 = 100\cos 30.0^0 = 100(0.866) = 86.6$$
$$B_x = B\cos\theta_2 = 200\cos 60.0^0 = 200(0.500) = 100.0$$
$$C_x = C\cos\theta_3 = 75\cos 140^0 = 75(-0.766) = -57.5$$
$$D_x = D\cos\theta_4 = 150\cos 250^0 = 150(-0.342) = -51.3$$
$$R_x = A_x + B_x + C_x + D_x = 77.8$$

While the y-components are:

$$A_y = A\sin\theta_1 = 100\sin 30.0^0 = 100(0.500) = 50.0$$
$$B_y = B\sin\theta_2 = 200\sin 60.0^0 = 200(0.866) = 173.0$$
$$C_y = C\sin\theta_3 = 75\sin 140^0 = 75(0.643) = 48.2$$
$$D_y = D\sin\theta_4 = 150\sin 250^0 = 150(-0.940) = -141.0$$
$$R_y = A_y + B_y + C_y + D_y = 130.2$$

The x and y-components of the vector **R** are shown in figure 1.16. Because R_x and R_y are both positive, the vector **R** is found in the first quadrant. If R_x were negative, **R** would have been in the second quadrant. It is a good idea for the student to plot the components R_x and R_y for any addition so that the direction of **R** is immediately apparent.

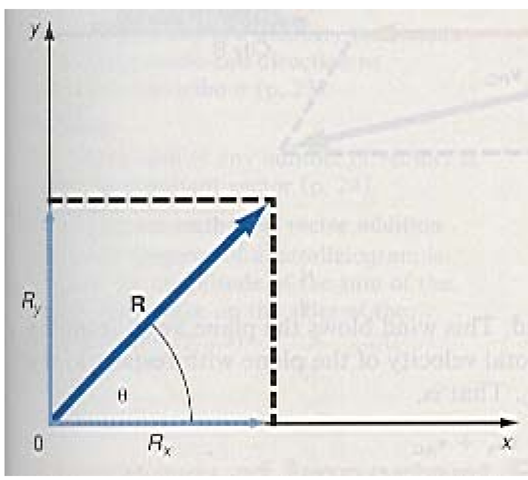


Figure 1.16 The resultant vector.

The resultant vector can be written in terms of the unit vectors as

$$\mathbf{R} = \mathbf{i}R_x + \mathbf{j}R_y = 77.8\mathbf{i} + 130\mathbf{j}$$

The magnitude of the resultant vector is found from equation 1.23 as

$$\begin{aligned} R &= \sqrt{R_x^2 + R_y{}^2} = \sqrt{(77.8)^2 + (130.2)^2} \\ &= \sqrt{23,004.8} \\ &= 152 \end{aligned}$$

The angle θ, that the vector **R** makes with the *x*-axis is found as

$$\theta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{130.2}{77.8} = \tan^{-1} 1.674$$
$$= 59.1^0$$

and is seen in figure 1.16.

It is important to note here that the components C_x, D_x, and D_y are negative numbers. This is because C_x and D_x lie along the negative *x*-axis and D_y lies along the negative *y*-axis.

To go to this interactive example click on this sentence.

Example 1.5

The addition and subtraction of vectors. Given the two vectors

$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$$
$$\mathbf{b} = 6\mathbf{i} + 3\mathbf{j}$$

Find the following vector, its magnitude and its direction for (a) **a** + **b**, (b) **a** – **b**, and (c) **b** – **a**.

Solution

(a) The sum of the two vectors is found as

$$\mathbf{a} + \mathbf{b} = (3 + 6)\mathbf{i} + (2 + 3)\mathbf{j}$$
$$\mathbf{a} + \mathbf{b} = 9\mathbf{i} + 5j$$

Its magnitude is

$$|\mathbf{a} + \mathbf{b}| = \sqrt{9^2 + 5^2} = \sqrt{106} = 10.3$$

Its direction is found as

$$\theta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{5}{9} = \tan^{-1} 0.556$$
$$\theta = 29.1^0$$

(b) The difference between the two vectors is

$$\mathbf{a} - \mathbf{b} = (3 - 6)\mathbf{i} + (2 - 3)$$
$$\mathbf{a} - \mathbf{b} = -3\mathbf{i} - \mathbf{j}$$

Its magnitude is

$$|\mathbf{a} - \mathbf{b}| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10.0} = 3.16$$

Its direction is found as

$$\theta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{-1}{-3} = \tan^{-1} 0.333$$
$$\theta = 18.4^0$$

Notice that both R_x and R_y are negative indicating that the angle θ is an angle in the third quadrant, between the negative x-axis and the vector (**a** – **b**). The angle that the vector (**a** – **b**) makes with the positive x-axis is found by adding 180^0 to the angle θ, giving an angle of 198.4^0 with respect to the positive x-axis.

(c) The difference between the two vectors is

$$\mathbf{b} - \mathbf{a} = (6 - 3)\mathbf{i} + (3 - 2)\mathbf{j}$$
$$\mathbf{b} - \mathbf{a} = 3\mathbf{i} + \mathbf{j} = -(\mathbf{a} - \mathbf{b})$$

Its magnitude is the same as the magnitude of **a** – **b**,

$$|\mathbf{b} - \mathbf{a}| = \sqrt{(3)^2 + (1)^2} = \sqrt{10.0} = 3.16$$

Its direction is found as

$$\theta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{1}{3} = \tan^{-1} 0.333$$
$$\theta = 18.4^0$$

Notice that in this case both R_x and R_y are positive indicating that the angle θ is in the first quadrant, and hence the angle should be considered as positive.

To go to this interactive example click on this sentence.

Although most problems treated in this text will be two dimensional, we do live in a three dimensional world and hence most vectors are 3-dimensional. For those problems requiring three dimensional vectors the extension to three dimensions is quite simple. A three dimensional vector **a** is shown in figure 1.17.

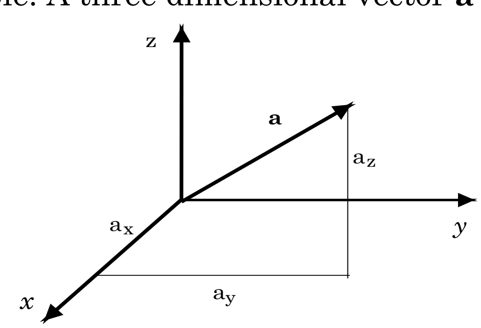


Figure 1.17 A vector in three dimensions.

Notice that it has the component a_x in the x-direction, a_y in the y-direction, and a_z in the z-direction.

The vector **a** is written mathematically as

$$\mathbf{a} = \mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z \tag{1.25}$$

while another arbitrary vector, **b**, is written as

$$\mathbf{b} = \mathbf{i}b_x + \mathbf{j}b_y + \mathbf{k}b_z \tag{1.26}$$

The sum of the two vectors is determined as before

$$\mathbf{R} = \mathbf{a} + \mathbf{b} = \mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z + \mathbf{i}b_x + \mathbf{j}b_y + \mathbf{k}b_z$$

$$\mathbf{R} = \mathbf{a} + \mathbf{b} = \mathbf{i}(a_x + b_x) + \mathbf{j}(a_y + b_y) + \mathbf{k}(a_z + b_z) \tag{1.27}$$

And the resultant vector **R** can also be written in terms of the unit vectors as

$$\mathbf{R} = \mathbf{i}R_x + \mathbf{j}R_y + \mathbf{k}R_z \tag{1.28}$$

And as determined for the two dimensional case,

$$R_x = a_x + b_x \tag{1.29}$$
$$R_y = a_y + b_y \tag{1.30}$$
$$R_z = a_z + b_z \tag{1.31}$$

Finally, the magnitude of the resultant vector is found by the extension of the Pythagorean theorem to three dimensions as

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \tag{1.32}$$

Example 1.6

The magnitude of a three dimensional vector. Given the vector

$$\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$$

Find the magnitude of the vector **a**.

Solution

The magnitude of the vector is found from equation 1.32 as

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{4^2 + 3^2 + 6^2}$$

$$a = \sqrt{61} = 7.81$$

To go to this interactive example click on this sentence.

Example 1.7

The addition of three dimensional vectors. Given the two vectors

$$\mathbf{a} = 7\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$$
$$\mathbf{b} = -\,4i + 5\mathbf{j} - 3\mathbf{k}$$

Find the resultant vector and its magnitude.

Solution

The sum of the two vectors is found as

$$\mathbf{R} = \mathbf{a} + \mathbf{b} = (7 + (-4))\mathbf{i} + (2 + 5)\mathbf{j} + (8 - 3)\mathbf{k}$$
$$\mathbf{R} = \mathbf{a} + \mathbf{b} = 3\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$$

Its magnitude is found from equation 1.32 as

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$
$$R = \sqrt{3^2 + 7^2 + 5^2} = \sqrt{83.0} = 9.11$$

To go to this interactive example click on this sentence.

1.9 The Multiplication of Vectors

Not only can vectors be added and subtracted they can also be multiplied. The multiplication of vectors is important because there are many physical laws that are expressed as the product of vectors.

There are three types of multiplication involving vectors. They are:

1) The multiplication of a vector by a scalar.

Consider the vector **a** in figure 1.18a. This vector has both magnitude and direction. If **a** is multiplied by a scalar k then the product k**a** is a new vector, say **b**, where

$$\mathbf{b} = k\mathbf{a} \tag{1.33}$$

If k is a positive number greater than one, then the product represents a vector in the same direction as **a** but elongated by a factor k, as shown in figure 1.18b. If k is less than one, but greater than zero, the new vector is shorter than **a**, figure 1.18c, and if k is negative, the new vector is in the opposite direction of **a**, figure 1.18d.

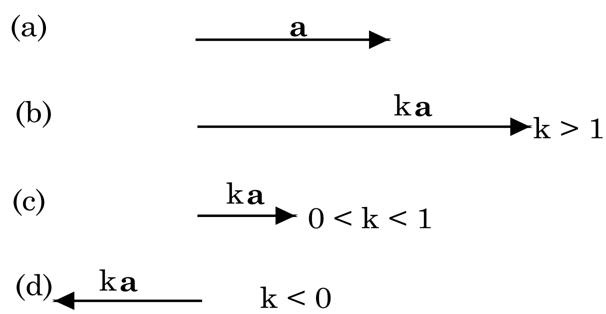


Figure 1.18 Multiplication of a vector by a scalar.

2) The scalar product.

The scalar product is the multiplication of two vectors, the result of which is a scalar. The scalar product will be discussed in detail in the next section.

3) The vector product.

The vector product is the multiplication of two vectors, the result of which is a vector. The vector product will be discussed in detail in section 1.11.

1.10 The Scalar Product or Dot Product

The scalar product of two vectors **a** and **b**, figure 1.19, is defined to be

$$\mathbf{a} \bullet \mathbf{b} = ab\cos\theta \tag{1.34}$$

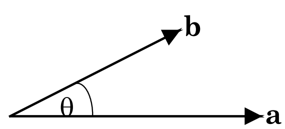


Figure 1.19 The scalar product.

where θ is the angle between the two vectors when they are drawn tail-to-tail, and *a* and *b* are the magnitudes of the two vectors. The symbol for this type of multiplication is the dot, • , between the vectors **a** and **b**. Hence, the scalar product is also called the dot product. The definition, equation 1.34 might seem quite arbitrary with the inclusion of the cosθ. You might ask, “why not use the tanθ or some other trigonometric function?” The answer is quite simple; as will be shown shortly, equation 1.46 can represent different physical concepts and phenomena. As can be observed in figure 1.19, *b* cosθ is the component of the vector **b** in the **a** direction. Thus, the scalar product is the component of the vector **b** in the direction of the vector **a**, multiplied by the magnitude of the vector **a** itself. It can also be stated that the quantity, *a* cosθ, is the component of the vector **a** in the **b** direction and hence the scalar product is also the component of the vector **a** in the **b** direction, multiplied by the magnitude of the vector **b** itself. Either description is correct.

The inclusion of the cosθ in the definition of the scalar product gives rise to some interesting special cases. If the vectors **a** and **b** are parallel to each other, then the angle θ between **a** and **b** is equal to zero. In that case, the scalar product becomes

$$\mathbf{a} \bullet \mathbf{b} = ab \cos 0^0$$

But the $\cos 0^0 = 1$, therefore

$$\mathbf{a} \bullet \mathbf{b} = ab \qquad (\text{for } \mathbf{a} \mid\mid \mathbf{b}) \tag{1.35}$$

(Note that as a special case, $\mathbf{a} \bullet \mathbf{a} = a^2$, and hence $a = \sqrt{\mathbf{a} \bullet \mathbf{a}}$ defines the magnitude of any vector.) On the other hand, if **a** is perpendicular to **b**, then the angle between the two vectors is 90^0, and the scalar product becomes

$$\mathbf{a} \bullet \mathbf{b} = ab \cos 90^0$$

But the $\cos 90^0 = 0$, and hence

$$\mathbf{a} \bullet \mathbf{b} = 0 \qquad (\text{for } \mathbf{a} \perp \mathbf{b}) \tag{1.36}$$

As an example, if the vector **a** has a magnitude of 2 units and vector **b** a magnitude of 3 units, then their dot product can take on any value between –6 and +6 depending on the angle θ between the two vectors. For scalars, on the other hand, 2 times 3 is always equal to 6 and nothing else.

Example 1.8

The scalar product. The vector **a** has a magnitude of 5.00 units and the vector **b** a magnitude of 7.00 units. If the angle between the vectors is 53.0^0, find their scalar product.

Solution

The scalar product of the two vectors is found, by equation 1.34, to be

$$\mathbf{a} \bullet \mathbf{b} = ab \cos\theta$$
$$\mathbf{a} \bullet \mathbf{b} = (5.00)(7.00)\cos 53.0^0$$
$$\mathbf{a} \bullet \mathbf{b} = 21.1$$

To go to this interactive example click on this sentence.

One of the most important examples of the dot product is found in the concept of work. As you recall from your course in college physics, the work done by a force **F**, in moving a block through a displacement **s**, figure 1.20, is determined by multiplying the component of the force in the direction of the displacement, by the displacement itself. The component of the force in the direction of the displacement

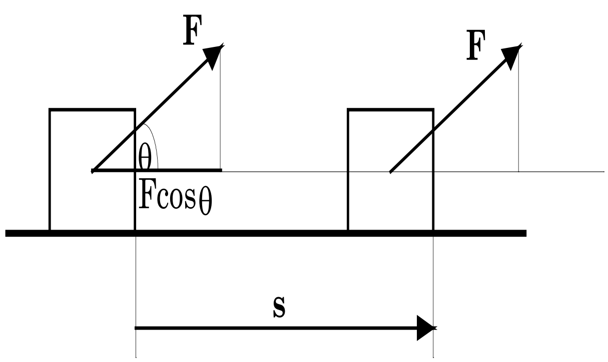


Figure 1.20 The concept of work.

is seen from figure 1.20, to be

$$F_x = F\cos\theta$$

From the definition, the work done becomes

$$W = (F\cos\theta)s = Fs\cos\theta$$

But $Fs\cos\theta$ is the dot product of $\mathbf{F} \bullet \mathbf{s}$. Therefore, *work should be defined as the scalar or dot product of the force* ***F*** *and the displacement* ***s****, i.e.,*

$$W = \mathbf{F} \bullet \mathbf{s} = Fs\cos\theta \tag{1.37}$$

It is now obvious why the scalar product was defined the way it was in equation 1.34, because it does indeed represent a physical concept.

Example 1.9

The work done. A force of 50.5 N acts on a block and causes a displacement of 3.5 m. The angle between the force and the displacement is 35^0. Find the work done by the force during the displacement.

Solution

The work done by the force is given by equation 1.37 as

$$W = \mathbf{F} \bullet \mathbf{s} = Fs\cos\theta$$
$$W = (50.5\text{ N})(3.5\text{ m})\cos 35^0$$

$$W = 145 \text{ N m} = 145 \text{ J}$$

To go to this interactive example click on this sentence.

Another very important application of the scalar product is in the definition of the concept of flux. *Flux is a quantitative measure of the number of lines of a vector field that passes perpendicularly through a surface.* Figure 1.21, shows an electric field **E** passing through a portion of a surface of area **A**.

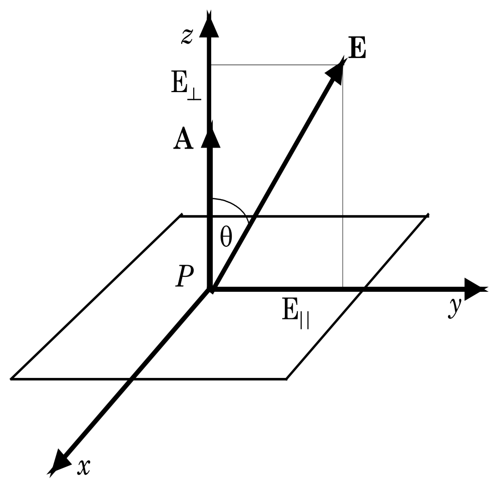


Figure 1.21 Electric flux for an angle θ.

The area of the surface is represented by a vector **A**, whose magnitude is the area A of the surface, and whose direction is perpendicular to the surface. That an area can be represented by a vector will be justified in a moment. The electric field will be defined in the next chapter, however the student should already have some knowledge of the electric field from his/her course in college physics.

The electric flux is defined to be

$$\Phi_E = \mathbf{E} \bullet \mathbf{A} = EA\cos\theta \tag{1.38}$$

and is a quantitative measure of the number of lines of the electric field **E** that pass normally through the surface area **A**. The number of lines represents the strength of the field. The vector **E**, at the point P of figure 1.21, can be resolved into the components $E_\perp$, the component perpendicular to the area, and $E_{||}$. the parallel component. The perpendicular component is given by

$$E_\perp = E\cos\theta$$

while the parallel component is given by

$$E_{||} = E \sin\theta$$

The parallel component, $E_{||}$, lies in the surface itself and therefore does not pass through the surface, while the perpendicular component, $E_{\perp}$, completely passes through the surface at the point P. The product of the perpendicular component and the area

$$E_{\perp}A = (E \cos\theta)A = EA \cos\theta = \mathbf{E} \bullet \mathbf{A} = \Phi_E \tag{1.39}$$

is therefore a quantitative measure of the number of lines of **E** passing normally through the entire area **A**. If the angle θ in equation 1.38 is zero, then **E** is parallel to the vector **A** and all the lines of **E** pass normally through the area **A**, as seen in figure 1.22.

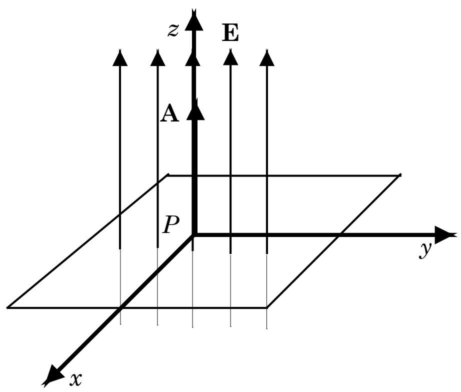


Figure 1.22 Electric flux for $\theta = 0^0$.

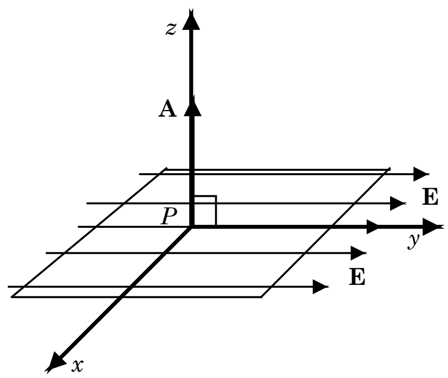


Figure 1.23 Electric flux for $\theta = 90^0$.

If the angle θ in equation 1.38 is 90^0 then **E** is perpendicular to the area vector **A**, and none of the lines of **E** pass through the surface, **A**, as seen in figure 1.23. The concept of flux is a very important one, and one that will be used frequently later. For now, consider it as another example of a scalar product.

Example 1.10

The electric flux. An electric field of 500 V/m makes an angle of 30.0^0 with the surface vector, which has a magnitude of 0.500 m^2. Find the electric flux that passes through the surface.

Solution

The electric flux passing through the surface is given by equation 1.38 as

$$\Phi_E = \mathbf{E} \bullet \mathbf{A} = EA \cos\theta$$
$$\Phi_E = (500 \text{ V/m})(0.500 \text{ m}^2)\cos 30.0^0$$
$$\Phi_E = 217 \text{ V m}$$

Notice that the unit of electric flux is a volt times a meter.

To go to this Interactive Example click on this sentence.

If the vectors are expressed in terms of the unit vectors, the dot product becomes

$$\mathbf{a} \bullet \mathbf{b} = (\mathbf{i}a_x + \mathrm{j}a_y + \mathbf{k}a_z)\bullet (\mathbf{i}b_x + \mathbf{j}b_y + \mathbf{k}b_z)$$

$$\begin{aligned}\mathbf{a} \bullet \mathbf{b} = {} & \mathbf{i} \bullet \mathbf{i}a_xb_x + \mathbf{i} \bullet \mathbf{j}a_xb_y + \mathbf{i} \bullet \mathbf{k}a_xb_z \\ & + \mathbf{j} \bullet \mathbf{i}a_yb_x + \mathbf{j} \bullet \mathbf{j}a_yb_y + \mathbf{j} \bullet \mathbf{k}a_yb_z \\ & + \mathbf{k} \bullet \mathbf{i}a_zb_x + \mathbf{k} \bullet \mathbf{j}a_zb_y + \mathbf{k} \bullet \mathbf{k}a_zb_z\end{aligned}$$

The dot product of the unit vectors are

$$\begin{gathered}\mathbf{i} \bullet \mathbf{i} = (1)(1)\cos 0^0 = 1 \\ \mathbf{j} \bullet \mathbf{j} = (1)(1)\cos 0^0 = 1 \\ \mathbf{k} \bullet \mathbf{k} = (1)(1)\cos 0^0 = 1 \\ \mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{i} = (1)(1)\cos 90^0 = 0 \\ \mathbf{i} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = (1)(1)\cos 90^0 = 0 \\ \mathbf{k}\bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = (1)(1)\cos 90^0 = 0\end{gathered} \tag{1.40}$$

Notice that the dot product of a unit vector by itself is equal to one, because the magnitude of the unit vectors is equal to one and the angle between the unit vector and itself is equal to zero, and of course, the cosine of zero degrees is equal to one. Also notice that the *dot product of two different unit vectors is equal to zero*, because the angle between two different unit vectors is 90^0 and the cosine of 90 degrees is equal to zero. Using the results of equations 1.40, the dot product of two vectors becomes

$$\mathbf{a}\bullet\mathbf{b} = a_xb_x + a_yb_y + a_zb_z \tag{1.41}$$

A special case of equation 1.41 is the dot product of a vector by itself. That is,

$$\mathbf{a} \bullet \mathbf{a} = a_xa_x + a_ya_y + a_za_z$$

$$\mathbf{a}\bullet\mathbf{a} = a_x{}^2 + a_y{}^2 + a_z{}^2 = a^2 \tag{1.42}$$

and hence the magnitude of any vector is given by,

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{1.43}$$

Example 1.11

Dot product of three dimensional vectors. Find the dot product of the following vectors

$$\mathbf{a} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{b} = -4\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$$

Solution

The dot product is found from equation 1.41 as

$$\mathbf{a} \bullet \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$
$$\mathbf{a} \bullet \mathbf{b} = (3)(-4) + (5)(7) + (2)(5)$$
$$\mathbf{a} \bullet \mathbf{b} = 33$$

Notice that the result of the dot product of two vectors is a scalar, that is, a number.

To go to this Interactive Example click on this sentence.

Example 1.12

The work done in three dimensions. Find the work done by the force

$$\mathbf{F} = (2.5\text{ N})\mathbf{i} + (3.7\text{ N})\mathbf{j} + (2.0\text{ N})\mathbf{k}$$

that acts through the displacement

$$\mathbf{s} = (7.3\text{ m})\mathbf{i} - (3.4\text{ m})\mathbf{j} + (4.5\text{ m})\mathbf{k}$$

Solution

The dot product is found from equation 1.41 as

$$\mathbf{F} \bullet \mathbf{s} = F_x s_x + F_y s_y + F_z s_z$$
$$\mathbf{F} \bullet \mathbf{s} = (2.5\text{ N})(7.3\text{ m}) + (3.7\text{ N})(-3.4\text{ m}) + (2.0\text{ N})(4.5\text{ m})$$
$$\mathbf{F} \bullet \mathbf{s} = 14.7\text{ N m} = 14.7\text{ J}$$

To go to this Interactive Example click on this sentence.

1.11 The Vector Product or Cross Product

The magnitude of the vector product of two vectors **a** and **b** is defined to be

$$\mathbf{a} \times \mathbf{b} = ab \sin\theta \tag{1.44}$$

where θ is the angle between the two vectors when they are drawn tail-to-tail, and is shown in figure 1.24. The symbol for the multiplication is designated by the cross sign, "×", between the two vectors, and hence the name, cross product. The result of

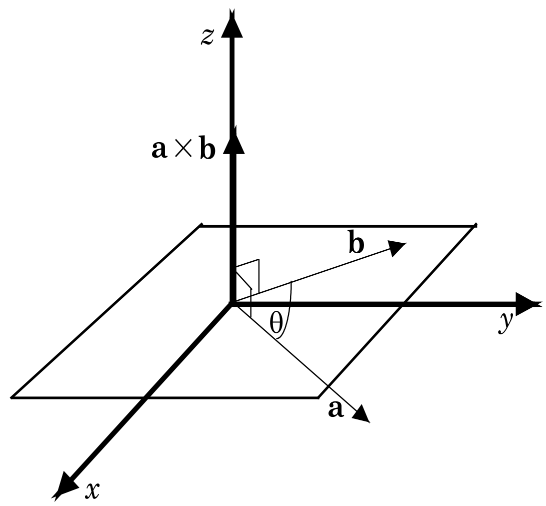


Figure 1.24 The cross product.

the cross product of two vectors, **a** and **b**, is another vector, **a** × **b**, which is perpendicular to the plane made by the vectors **a** and **b**.

The direction of **a** × **b** is found by taking your right hand with the fingers in the same direction as the vector **a**, your palm facing toward the vector **b**, and then rotating your right hand through the angle between **a** and **b**. Your thumb will then point in the direction of **a** × **b** as seen in figure 1.25. The new vector is both

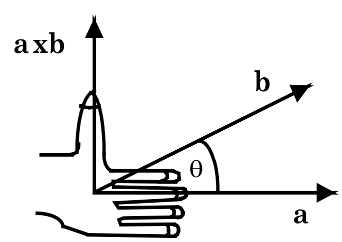


Figure 1.25 The right hand rule.

perpendicular to the vector **a** and the vector **b**. This rule is called the right hand rule and it is imperative that the right hand be used.

One of the characteristics of this vector product, is that the order of the multiplication is very important, for as can be seen from figure 1.26

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} \tag{1.45}$$

The magnitude of **a** cross **b** is the same as the magnitude of **b** cross **a** from the defining equation 1.44, but their directions are opposite to each other as seen in figure 1.26. This result is often stated as: *vectors are non-commutative under vector multiplication*. If the vectors **a** and **b** are parallel to each other then $\theta = 0^0$ and

$$|\mathbf{a} \times \mathbf{b}| = ab \sin 0^0$$

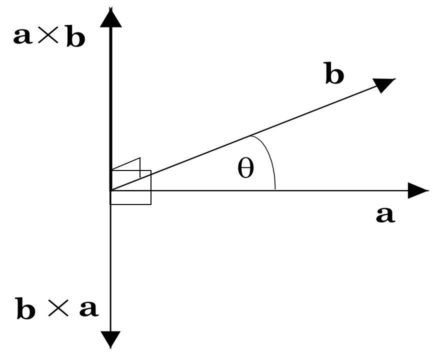


Figure 1.26 The directions of the cross products.

But $\sin 0^0 = 0$, therefore

$$|\mathbf{a} \times \mathbf{b}| = 0 \qquad (\text{for } \mathbf{a} \parallel \mathbf{b}) \tag{1.46}$$

If the vector **a** is perpendicular to the vector **b**, then $\theta = 90^0$ and

$$|\mathbf{a} \times \mathbf{b}| = ab \sin 90^0$$

But the $\sin 90^0 = 1$, therefore

$$|\mathbf{a} \times \mathbf{b}| = ab \qquad (\text{for } \mathbf{a} \perp \mathbf{b}) \tag{1.47}$$

Note that these two cases are the opposite of what was found for the dot product. This is because the dot product contains the $\cos\theta$ term, while the cross product contains the $\sin\theta$. The $\cos\theta$ and $\sin\theta$ are complementary functions and they are out of phase with each other by 90^0.

Example 1.13

The cross product of two vectors. The vector **a** has a magnitude of 5.00 units and the vector **b** of 7.00 units. If the angle between the vectors is 53.0^0, find the magnitude of their cross product.

Solution

The magnitude of their cross product is found from equation 1.44 to be

$$|\mathbf{a} \times \mathbf{b}| = ab \sin\theta$$
$$|\mathbf{a} \times \mathbf{b}| = (5.00)(7.00)\sin 53.0^0$$
$$|\mathbf{a} \times \mathbf{b}| = 28.0$$

The direction of $\mathbf{a} \times \mathbf{b}$ is as shown in figure 1.24.

To go to this interactive example click on this sentence.

The area of a surface can be represented by the cross product of the two vectors that generate the area. As an example, consider the two vectors, **a** and **b**, in figure 1.27. If vectors **a** and **b** are moved parallel to themselves they are still the

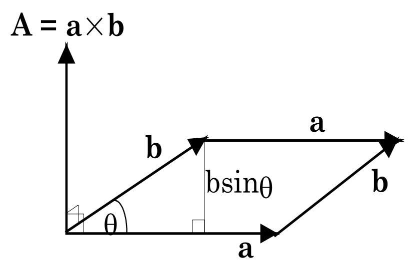


Figure 1.27 The area of a surface can be represented by a vector.

same vectors, since they still have the same magnitude and direction. But in the process of moving them parallel to themselves they have generated a parallelogram. As you recall from elementary geometry, the area of a parallelogram is equal to the product of its base times its altitude, i.e.,

$$A = (\text{base})(\text{altitude})$$

The base of the parallelogram is given by the magnitude of the vector **a**, and the altitude is given by

$$\text{altitude} = b \sin\theta$$

Therefore, the area of the parallelogram is given by

$$A = ab \sin\theta$$

But

$$ab \sin\theta = |\mathbf{a} \times \mathbf{b}|$$

Therefore, the area of a parallelogram generated by sides **a** and **b** is

$$A = |\mathbf{a} \times \mathbf{b}| = ab \sin\theta \tag{1.48}$$

Because $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the surface generated by the vectors **a** and **b**, the area of the surface can be represented as the vector **A**, where

$$\mathbf{A} = |\mathbf{a} \times \mathbf{b}| \tag{1.49}$$

The area vector **A** is thus perpendicular to the surface generated by the vectors **a** and **b**. (This is why an area was previously represented as a vector in the definition of the electric flux.)

Another very important application of the cross product is in the definition of the concept of torque. As you recall from your course in college physics, *a torque* $\boldsymbol{\tau}$ *was defined as the product of a force* $\boldsymbol{F}$ *acting at the point A, times the lever arm* $\boldsymbol{r}_\perp$*, as shown in figure 1.28. It was also defined as the perpendicular component of the*

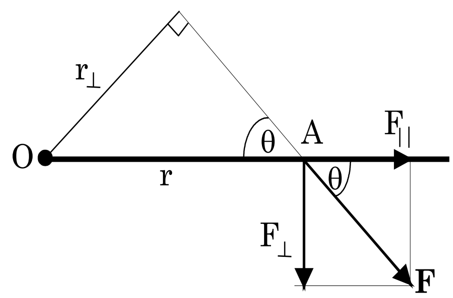


Figure 1.28 Torque.

force $F_\perp$ *times the distance r from the axis of rotation, O.* In either case it is represented mathematically as

$$\tau = r_\perp F = rF_\perp = rF\sin\theta \tag{1.50}$$

From our knowledge of the vector product, we see from equation 1.50, that *the torque acting about an axis should be defined as the cross product of* $\boldsymbol{r}$ *and* $\boldsymbol{F}$*,* i.e.,

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \tag{1.51}$$

with magnitude

$$\tau = |\tau| = |\mathbf{r} \times \mathbf{F}| = rF\sin\theta \tag{1.52}$$

The torque vector lies along the axis of rotation. It is perpendicular to the page and points into the page in figure 1.28.

As an example of the vector concept of torque, consider the screw with right handed threads shown in figure 1.29. A screw driver head is attached to a ratchet wrench and is inserted into the slot at the top of the screw. A force **F** is exerted at the end of the ratchet as shown. If **r** is the vector distance from the axis of the screw to the point of application of the force, then a torque is created given by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ The torque vector $\boldsymbol{\tau}$ lies along the axis of rotation and points upward, in the figure, in the direction that the screw will move while the torque is applied. A force in the direction shown will cause the screw to rotate in a counterclockwise direction, which causes the screw to move out from the surface toward you. If the direction of the force is reversed by 180^0, the screw will rotate in a clockwise direction, the torque vector will point downward along the axis of the screw, and the screw will move in that direction into the surface. (It should be pointed out that there is not

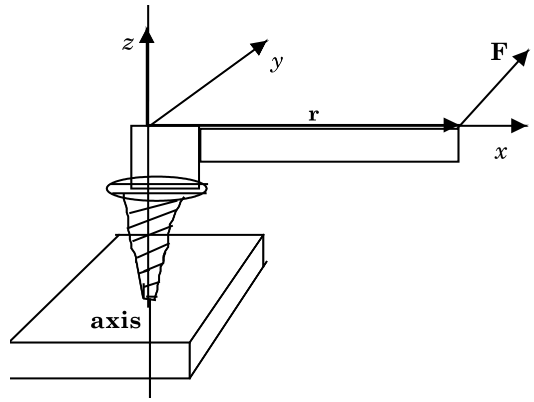


Figure 1.29 Torque and the cross product.

always a linear motion in the direction of the torque vector. The screw moves in that direction because of the threads of the screw. If the same torque were applied in the same way to a cylinder, the torque would lie along the axis of the cylinder but the cylinder would not translate in the direction of the torque vector.)

If the vectors are expressed in terms of the unit vectors, the cross product becomes

$$\mathbf{a}\times\mathbf{b} = (\mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z)\times(\mathbf{i}b_x + \mathbf{j}b_y + \mathbf{k}b_z) \tag{1.53}$$

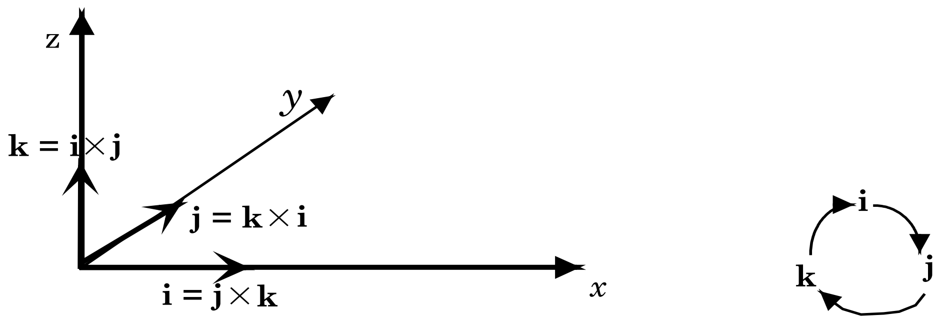
$$\begin{aligned}\mathbf{a}\times\mathbf{b} = {} & \mathbf{i}\times\mathbf{i}a_xb_x + \mathbf{i}\times\mathbf{j}a_xb_y + \mathbf{i}\times\mathbf{k}a_xb_z + \\ & \mathbf{j}\times\mathbf{i}a_yb_x + \mathbf{j}\times\mathbf{j}a_yb_y + \mathbf{j}\times\mathbf{k}a_yb_z + \\ & \mathbf{k}\times\mathbf{i}a_zb_x + \mathbf{k}\times\mathbf{j}a_zb_y + \mathbf{k}\times\mathbf{k}a_zb_z\end{aligned} \tag{1.54}$$

The magnitudes of the cross product of the unit vectors are

$$\begin{aligned}|\mathbf{i}\times\mathbf{i}| &= (1)(1)\sin 0^0 = 0\\ |\mathbf{j}\times\mathbf{j}| &= (1)(1)\sin 0^0 = 0\\ |\mathbf{k}\times\mathbf{k}| &= (1)(1)\sin 0^0 = 0\\ |\mathbf{i}\times\mathbf{j}| &= |\mathbf{j}\times\mathbf{i}| = (1)(1)\sin 90^0 = 1\\ |\mathbf{i}\times\mathbf{k}| &= |\mathbf{k}\times\mathbf{i}| = (1)(1)\sin 90^0 = 1\\ |\mathbf{k}\times\mathbf{j}| &= |\mathbf{j}\times\mathbf{k}| = (1)(1)\sin 90^0 = 1\end{aligned} \tag{1.55}$$

Notice that the cross product of a unit vector by itself is equal to zero, because the magnitude of the unit vector is equal to one and the angle between the unit vector and itself is equal to zero, and of course, the sine of zero degrees is equal to zero. Also notice that *the magnitude of the cross product of two different unit vectors is equal to one,* because the angle between two different unit vectors is 90^0 and the sine of 90 degrees is equal to one.

Because the cross product of two vectors yields another vector, *the cross product of two different unit vectors must be another unit vector*. It is now necessary to determine the direction of this unit vector. The direction of the new unit vector obtained by the cross product of two unit vectors is obtained with the help of figure 1.30(a). The direction of the cross product of $\mathbf{i} \times \mathbf{j}$ is obtained by placing the right



(a) **(b)**

Figure 1.30 The cross product of the unit vectors.

hand in the direction of the unit vector **i**, with the palm facing toward the unit vector **j**, and then rotating the hand toward the vector **j**. The thumb will point in the direction of the new vector. But as can be seen in figure 1.30(a), the thumb would point in the direction of the unit vector **k**. Since the cross product of the unit vectors **i** and **j** must be a unit vector, and the direction of this vector has been shown to be in the **k** direction, the new unit vector must be the unit vector **k**. Hence,

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \tag{1.56}$$

That is, *the cross product of the unit vectors **i** and **j** is equal to the unit vector **k**.* Using the same technique and figure 1.30(a), it can be seen that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \tag{1.57}$$

and

$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \tag{1.58}$$

Because the cross product is non-commutative, as shown in equation 1.45, the cross product of $\mathbf{j} \times \mathbf{i}$ will point in the negative **k** direction, as can be seen in figure 1.30(a). Therefore, we also have

$$\mathbf{j} \times \mathbf{i} = -\,\mathbf{k} \tag{1.59}$$

and

$$\mathbf{i} \times \mathbf{k} = -\,\mathbf{j} \tag{1.60}$$

and

$$\mathbf{k} \times \mathbf{j} = -\,\mathbf{i} \tag{1.61}$$

Using the results of equations 1.55 through 1.61 in equation 1.54 gives the cross product of two vectors as

$$\mathbf{a} \times \mathbf{b} = (0)a_x b_x + (\mathbf{k})a_x b_y + (-\,\mathbf{j})a_x b_z + (-\,\mathbf{k})a_y b_x + (0)a_y b_y + (\mathbf{i})a_y b_z + (\mathbf{j})a_z b_x + (-\,\mathbf{i})a_z b_y + (0)a_z b_z \qquad (1.62)$$

Upon gathering like terms together, this becomes

$$\mathbf{a} \times \mathbf{b} = \mathbf{i}(a_y b_z - a_z b_y) + \mathbf{j}(a_z b_x - a_x b_z) + \mathbf{k}(a_x b_y - a_y b_x) \qquad (1.63)$$

Every student should make this calculation at least once to see how the cross product of the unit vectors are determined. However, once this has been accomplished it is easy to remember the sequence of the result by using the little circle shown in figure 1.30(b). If you go around the circle in a clockwise manor from **i** to **j** to **k** to **i**, the cross product of the first two vectors will yield the next unit vector in the cycle. Thus, **i** × **j** = **k**; **j** × **k** = **i**; and **k** × **i** = **j**. If the circle is traversed in a counterclockwise direction, the resultant unit vector will be negative. Hence, **i** × **k** = – **j**; **k** × **j** = – **i**; and **j** × **i** = – **k**.

The same cyclic relation can be used to remember the form of equation 1.63 because **i** is in the *x*-direction, **j** is in the *y* direction, and **k** is in the *z* direction. Hence, the cycle **i j k** is the same as the cycle *x y z*. The first term in equation 1.63 starts the cycle with **i**, which is associated with *x*, the next term in the parenthesis has the subscripts *y* and *z*. The minus term within the parenthesis then reverses the subscripts to *z* and *y*. The second term in equation 1.63 starts with **j** which is associated with *y* and the next term in the parenthesis has the subscripts *z* and *x*. The minus term within the parenthesis then reverses the subscripts to *x* and *z*. Finally the last term in equation 1.63 starts with **k** which is associated with *z*, and the next term in the parenthesis has the subscripts *x* and *y*. The minus term within the parenthesis then reverses the subscripts to *y* and *x*. Hence by remembering the cycle **i j k** and *x y z* it is easy to remember the form of equation 1.63.

Example 1.14

The vector product of three dimensional vectors. Given the two vectors

$$\mathbf{a} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{b} = -4\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$$

Find their vector product.

Solution

The cross product is found from equation 1.63 as

$$\mathbf{a} \times \mathbf{b} = \mathbf{i}(a_y b_z - a_z b_y) + \mathbf{j}(a_z b_x - a_x b_z) + \mathbf{k}(a_x b_y - a_y b_x)$$
$$\mathbf{a} \times \mathbf{b} = \mathbf{i}\{(5)(5) - (2)(7)\} + \mathbf{j}\{(2)(-4) - (3)(5)\} + \mathbf{k}\{(3)(7) - (5)(-4)\}$$
$$\mathbf{a} \times \mathbf{b} = 11\mathbf{i} - 23\mathbf{j} + 41\mathbf{k}$$

Notice that the result of the cross product is indeed a vector.

To go to this interactive example click on this sentence.

Example 1.15

The cross product and a determinant. Show that the cross product of two vectors can be represented by a determinant as

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{1.64}$$

Solution

Using the rules for expansion of a determinant, equation 1.64 can be expressed as

$$\mathbf{a}\times\mathbf{b} = \mathbf{i}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

$$\mathbf{a}\times\mathbf{b} = \mathbf{i}(a_yb_z - a_zb_y) - \mathbf{j}(a_xb_z - a_zb_x) + \mathbf{k}(a_xb_y - a_yb_x)$$

Factoring a minus one from inside the second term this becomes

$$\mathbf{a}\times\mathbf{b} = \mathbf{i}(a_yb_z - a_zb_y) + \mathbf{j}(a_zb_x - a_xb_z) + \mathbf{k}(a_xb_y - a_yb_x)$$

which is identical to equation 1.63. Thus, the cross product can also be represented by a determinant.

1.12 The Del Operator

We are now in a position where we can define a new quantity called the del operator. The del operator is defined as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{1.65}$$

The del operator is an operator in that it operates on whatever comes after it, just as the derivative d/dx operates on whatever comes after it[1] When ∇ operates on a quantity, it takes the partial derivative of that quantity with respect to the variables x, y, and z. However, because the del operator contains the unit vectors **i j k**, *the del operator is a vector derivative. If the function that ∇ operates on is a scalar the result is called the gradient. Hence, if V is a scalar point function, then ∇V is called the gradient of V and is given by*

$$\nabla V = \mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial z} \tag{1.66}$$

Since ∇ is a vector we can think of ∇V as the product of del times a scalar quantity V. The gradient will be discussed in more detail in chapter 5.

If the function V is only a function of one variable, say x, then the gradient will reduce to an ordinary derivative. That is, if $V = V(x)$, then $\partial V/\partial y = 0$, and $\partial V/\partial z = 0$ and

$$\nabla V = \mathbf{i}\frac{\partial V}{\partial x} = \mathbf{i}\frac{dV}{dx} \tag{1.67}$$

Because ∇ is a vector it can be multiplied by other vectors. However, because ∇ is an operator it must operate on the vector being multiplied. Hence, ∇ must always be the first term in the multiplication process. Since there are two different types of multiplication, there are two different quantities that can be obtained depending upon the type of multiplication involved. If the scalar product of ∇ and a vector, say **E**, is taken, the result is called the divergence of **E** and is given by

$$\nabla \bullet \mathbf{E} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet \left(\mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z\right)$$

Recalling the result of the dot product of two vectors, Equation 1.41, this becomes

$$\nabla \bullet \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{1.68}$$

Equation 1.*68 is called the divergence of E* and will be discussed in more detail in chapter 4.

If ∇ is crossed multiplied by a vector **E**, the result is

$$\nabla \times \mathbf{E} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times \left(\mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z\right)$$

[1]For a review of ordinary and partial derivatives see appendix B.

Recalling the result of the cross product of two vectors, equation 1.63, this becomes

$$\nabla \times \mathbf{E} = \mathbf{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_x}{\partial y}\right) \tag{1.69}$$

*Equation 1.69 is called the curl of the vector **E*** and will be described in detail in chapter 10.

Summary of Important Concepts

Scalar Product - The multiplication of two vectors, the result of which is a scalar.

Vector Product - The multiplication of two vectors, the result of which is a vector.

Work - The scalar product of the force and the displacement.

Electric Flux - Flux is a quantitative measure of the number of lines of an electric field that pass perpendicularly through a surface.

The del operator ∇ is a vector derivative. If the function that ∇ operates on is a scalar the result is called the gradient. If the function that ∇ operates on is a vector the result is either the divergence or the curl of that vector.

Summary of Important Equations

Addition of vectors	$\mathbf{R} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$	(1.4)
Vector addition is commutative	$\mathbf{R} = \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	(1.2)
Subtraction of vectors	$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$	(1.3)
x-component of a vector	$a_x = a\cos\theta$	(1.5)
y-component of a vector	$a_y = a\sin\theta$	(1.6)
Magnitude of a vector	$a = \sqrt{a_x^2 + a_y^2}$	(1.8)
Direction of a vector	$\theta = \tan^{-1}\frac{a_y}{a_x}$	(1.9)
Vector in terms of unit vectors	$\mathbf{a} = \mathbf{i}a_x + \mathbf{j}a_y$	(1.10)

x-component of resultant vector	$R_x = a_x + b_x + c_x + d_x + \ldots$	(1.20)
y-component of resultant vector	$R_y = a_y + b_y + c_y + d_y + \ldots$	(1.21)
Resultant vector	$\mathbf{R} = \mathbf{i}R_x + \mathbf{j}R_y$	(1.22)
Magnitude of resultant vector	$R = \sqrt{R_x^2 + R_y^2}$	(1.23)
Direction of resultant vector	$\theta = \tan^{-1}\dfrac{R_y}{R_x}$	(1.24)
Three dimensional vector	$\mathbf{a} = \mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z$	(1.25)
Resultant vector	$\mathbf{R} = \mathbf{i}R_x + \mathbf{j}R_y + \mathbf{k}R_z$	(1.28)
Scalar product	$\mathbf{a} \bullet \mathbf{b} = ab\cos\theta$	(1.34)
Work	$W = \mathbf{F} \bullet \mathbf{s} = Fs\cos\theta$	(1.37)
Electric Flux	$\Phi_E = \mathbf{E} \bullet \mathbf{A} = EA\cos\theta$	(1.38)
Scalar product	$\mathbf{a} \bullet \mathbf{b} = a_xb_x + a_yb_y + a_zb_z$	(1.41)
Magnitude of a vector	$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$	(1.43)
Magnitude of cross product	$\lvert\mathbf{a}\times\mathbf{b}\rvert = ab\sin\theta$	(1.44)
Vector product is non-commutative	$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$	(1.45)
Area of a parallelogram	$A = \lvert\mathbf{a}\times\mathbf{b}\rvert = ab\sin\theta$	(1.48)
Torque	$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$	(1.51)
with magnitude	$\tau = \lvert\boldsymbol{\tau}\rvert = \lvert\mathbf{r}\times\mathbf{F}\rvert = rF\sin\theta$	(1.52)
Vector product	$\mathbf{a}\times\mathbf{b} = \mathbf{i}(a_yb_z - a_zb_y) + \mathbf{j}(a_zb_x - a_xb_z) + \mathbf{k}(a_xb_y - a_yb_x)$	(1.63)
Vector product	$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$	(1.64)

Del operator
$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{1.65}$$

Gradient
$$\nabla V = \mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial z} \tag{1.66}$$

Divergence of **E**
$$\nabla \bullet \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{1.68}$$

Curl of **E**
$$\nabla \times \mathbf{E} = \mathbf{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_x}{\partial y}\right) \tag{1.69}$$

Problems for Chapter 1

1. A force of 100 N acts at an angle of 35.0^0 above the horizontal. (a) Find the vertical and horizontal components of this force, and (b) write the vector in terms of unit vectors.

2. A 50 N force is directed at an angle of 50.0^0 above the horizontal. (a) Resolve this force into vertical and horizontal components, and (b) write the vector in terms of unit vectors..

3. A displacement vector makes an angle of 35.0^0 with respect to the x-axis. If the y-component of the vector is equal to 150 cm, what is the magnitude of the displacement vector?

4. If $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, what is the angle between **a** and **b**?

5. The vector **A** has a magnitude of 50.0 m and points in a direction of 50.0^0 north of east. What are the magnitudes and directions of the vectors, (a) 2**A**, (b) 0.5**A**, (c) – **A**, (d) –5**A**, (e) **A** + 4**A**, (f) **A** – 4**A**.

6. Express the following three displacements in vector notation in terms of the unit vectors **i** and **j**: (a) 3 km due east, (b) 6 km east- northeast, and (c) 7 km northwest.

7. Find the resultant of the following forces; 5 N at an angle of 33.0^0 above the horizontal and 20 N at an angle of 97.0^0 counterclockwise from the horizontal. Express the answer in terms of (a) the unit vectors, and (b) in terms of the magnitude and direction of the resultant vector.

8. A girl drives 3 km north, then 12 km to the northwest, and finally 5 km south-southwest. How far has she traveled? What is her displacement? Express the answer in terms of (a) the unit vectors, and (b) in terms of the magnitude and direction of the resultant vector.

9. Find the resultant of the following forces: (a) 30 N at an angle of 40.0^0 with respect to the x-axis, (b) 120 N at an angle of 135^0, and (c) 60 N at an angle of 260^0.

Express the answer in terms of (a) the unit vectors, and (b) in terms of the magnitude and direction of the resultant vector.

10. Find the resultant of the following set of forces. (a) $\mathbf{F}_1$ of 200 N at an angle of 53.0^0 with respect to the x-axis. (b) $\mathbf{F}_2$ of 300 N at an angle of 150^0 with respect to the x-axis. (c) $\mathbf{F}_3$ of 200 N at an angle of 270^0 with respect to the x-axis. (d) $\mathbf{F}_4$ of 350 N at an angle of 310^0 with respect to the x-axis. Express the answer in terms of (a) the unit vectors, and (b) in terms of the magnitude and direction of the resultant vector.

11. If $\mathbf{a} = 5\mathbf{i} + 6\mathbf{j}$, find the magnitude of $\mathbf{a}$ and the angle θ that the vector makes with the x-axis.

12. If $\mathbf{a} = 6\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$, find the magnitude of the vector.

13. Given the vectors $\mathbf{a}$ and $\mathbf{b}$, where $a = 50$, $\theta_1 = 33^0$ and $b = 80$, $\theta_2 = 128^0$, find (a) $\mathbf{a} + \mathbf{b}$, (b) $\mathbf{a} - \mathbf{b}$, (c) $\mathbf{a} - 2\mathbf{b}$, (d) $3\mathbf{a} + \mathbf{b}$, (e) $2\mathbf{a} - \mathbf{b}$, (f) $2\mathbf{b} - \mathbf{a}$.

14. Find the resultant of the following vectors.

$$\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$$
$$\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{c} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

What is the magnitude of the resultant vector.

15. Find the magnitude of the vector

$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

16. Find the resultant of the following vectors.

$$\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$$
$$\mathbf{b} = 2\mathbf{i} + 4\mathbf{j}$$
$$\mathbf{c} = \mathbf{i} + 2\mathbf{j}$$

What is the magnitude and direction of the resultant vector.

17. Vector $\mathbf{a}$ has a magnitude of 6 units and vector $\mathbf{b}$ has a magnitude of 8 units and makes an angle of 63.5^0 with vector $\mathbf{a}$. Find (a) the dot product, $\mathbf{a}\bullet\mathbf{b}$, and (b) the cross product $\mathbf{a}\times\mathbf{b}$.

18. A block rests upon a level surface when a force of 5.00 N acts on the block at an angle of 48.0^0 above the horizontal. If the block moves through a distance of 7.00 m, how much work is done on the block?

19. A force of 10.0 N acts on a body at an angle of 30.0^0 with the position vector $\mathbf{r}$. If $r = 25$ cm, find the torque acting on the body.

20. Find the area of a parallelogram generated by the vectors

$$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{b} = -2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$$

21. If an electric field of 100 V/m passes through a coil 20 cm by 30 cm, at an angle of 50^0, find the flux passing through the coil.

22. If

$$\mathbf{a} = 25\mathbf{i} + 15\mathbf{j}$$
$$\mathbf{b} = 3\mathbf{i} + 18\mathbf{j}$$

Find (a) $\mathbf{a} + \mathbf{b}$, (b) $\mathbf{a} - \mathbf{b}$, (c) $\mathbf{b} - \mathbf{a}$, (d) $\mathbf{a}\bullet\mathbf{b}$, (e) $\mathbf{a}\times\mathbf{b}$, and (f) the angle θ between the two vectors.

23. If

$$\mathbf{a} = 6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$
$$\mathbf{b} = 2\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$$

Find (a) $\mathbf{a} + \mathbf{b}$, (b) $|\mathbf{a} + \mathbf{b}|$, (c) $\mathbf{a} - \mathbf{b}$, (d) $|\mathbf{a} - \mathbf{b}|$, (e) $\mathbf{b} - \mathbf{a}$, (f) $\mathbf{a}\bullet\mathbf{b}$, (g) $\mathbf{a}\times\mathbf{b}$, and (h) the angle θ between the two vectors.

24. Show that the unit vectors can also be written as

$$\mathbf{i} = \nabla x$$
$$\mathbf{j} = \nabla y$$
$$\mathbf{k} = \nabla z$$

25. Given the vectors

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$
$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + c_z\mathbf{k}$$
$$\mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$$

Find the triple scalar product

$$\mathbf{a} \times \mathbf{b} \bullet \mathbf{c}.$$

Why isn't it necessary to place parentheses around any of the vectors?

26. Show that the triple scalar product

$$\mathbf{a}\times\mathbf{b}\bullet\mathbf{c}$$

is equal to the volume of a parallelepiped of sides **a**, **b**, **c**.

27. Show that the scalar triple product can be written as the determinant

$$\mathbf{a}\times\mathbf{b}\bullet\mathbf{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

28. Show that the triple vector product can be written as

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{b}(\mathbf{a} \bullet \mathbf{c}) - \mathbf{a}(\mathbf{b} \bullet \mathbf{c})$$

29. If **a**, **b**, **c** are coplanar vectors, show that the vector $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ lies in the same plane.

30. A point in space is located by the position vector

$$\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$$

Find a new unit vector $\mathbf{r}_o$ that points in the direction of **r**.

31. A vector is given by

$$\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

Find a new unit vector that points in the **a** direction.

32. If

$$V = 2xy + y$$

find the gradient of V.

33. In some problems in electromagnetic theory it is much simpler to express the problem in polar coordinates because of the radial and azimuthal symmetry inherent in the problem. In polar coordinates the del operator is given by

$$\nabla = \mathbf{r}_o\frac{\partial}{\partial r} + \boldsymbol{\theta}_o\frac{1}{r}\frac{\partial}{\partial \theta}$$

where $\mathbf{r}_o$ is a unit vector in the **r** direction and $\boldsymbol{\theta}_o$ is a unit vector in the direction of increasing θ. $\boldsymbol{\theta}_o$ is perpendicular to $\mathbf{r}_o$. If a function is given by

$$V = \frac{kq}{r}$$

where k and q are constants, find ∇V.

34. Using the value of ∇ from problem 33, find ∇V, if

$$V = \frac{kp}{r^2}\cos\theta$$

where k and p are constants.

35. If

$$V = \frac{kq}{\sqrt{x^2 + a^2}}$$

where k, q and a are constants, find ∇V.

36. Using the value of ∇ from problem 33, find $\nabla \bullet \mathbf{E}$ and $\nabla\times \mathbf{E}$ if

$$\mathbf{E} = \frac{kq\mathbf{r}_o}{r^2}$$

where k, and q are constants.

37. Using the value of ∇ from problem 33, find $\nabla \bullet \mathbf{E}$ and $\nabla \times \mathbf{E}$ if

$$\mathbf{E} = \frac{2kp\cos\theta}{r^3}\mathbf{r}_o + \frac{kp\sin\theta}{r^3}\boldsymbol{\theta}_o$$

where k, and p are constants.

38. If

$$\mathbf{E} = \frac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{j}$$

find $\nabla \bullet \mathbf{E}$ and $\nabla \times \mathbf{E}$.

39. If

$$\mathbf{E} = 2y\,\mathbf{i} + x\mathbf{j}$$

Find $\nabla \bullet \mathbf{E}$ and $\nabla \times \mathbf{E}$.

40. If

$$\mathbf{V} = 2yv\,\mathbf{i} + xv\mathbf{j}$$

find $\nabla \bullet \mathbf{V}$ and $\nabla \times \mathbf{V}$.

41. From the diagram and the dot product of $\mathbf{c}$ with itself, derive the law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

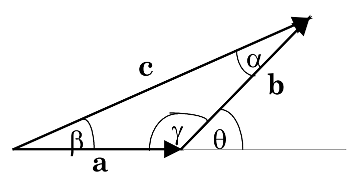


Diagram for problem 41.

42. From the diagram of problem 41, find the law of sines. Hint use $\mathbf{c} = \mathbf{a} + \mathbf{b}$, and some cross products to get

$$\frac{c}{\sin\theta} = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

43. Given the two vectors

$$\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$
$$\mathbf{w} = \mathbf{i} - 2\mathbf{j} + c\mathbf{k}$$

Find the value of $\mathbf{c}$ such that the vector $\mathbf{v}$ is perpendicular to the vector $\mathbf{w}$.

44. Given the two vectors

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$$
$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j}$$

Find the relation between the components such that the vector $\mathbf{a}$ will be perpendicular to the vector $\mathbf{b}$.

45. Show that the three unit vectors in the following diagram can be written as

$$\mathbf{u}_1 = \cos\alpha\ \mathbf{i} + \sin\alpha\ \mathbf{j}$$
$$\mathbf{u}_2 = \cos\beta\ \mathbf{i} + \sin\beta\ \mathbf{j}$$
$$\mathbf{u}_3 = \cos\beta\ \mathbf{i} - \sin\beta\ \mathbf{j}$$

Form the dot product $\mathbf{u}_1 \bullet \mathbf{u}_2$ and show that the trigonometric function for the cosine of the difference of two angles is given by

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

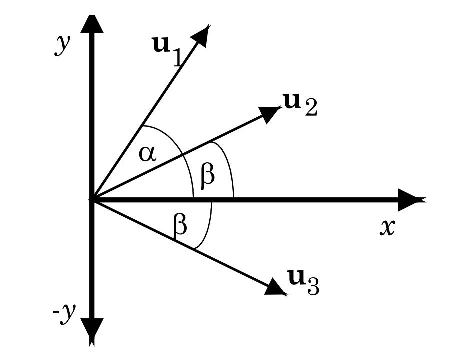


Diagram for problem 45.

46. Using the equations and diagram of problem 45, form the cross product $\mathbf{u}_1 \times \mathbf{u}_2$ and show that the trigonometric function for the sine of the difference of two angles is given by

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

47. Using the equations and diagram of problem 45, form the cross product $\mathbf{u}_1 \times \mathbf{u}_3$ and show that the trigonometric function for the sine of the sum of two angles is given by

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

48. Using the equations and diagram of problem 45, form the dot product $\mathbf{u}_1 \bullet \mathbf{u}_3$ and show that the trigonometric function for the cosine of the sum of two angles is given by

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

49. Using the equations and diagram of problem 45, form the dot product $\mathbf{u}_2 \bullet \mathbf{u}_3$ and show that the trigonometric function for the cosine of twice an angle is given by

$$\cos(2\beta) = \cos^2\beta - \sin^2\beta$$

50. Using the equations and diagram of problem 45, form the cross product $\mathbf{u}_2 \times \mathbf{u}_3$ and show that the trigonometric function for the sine of twice an angle is given by

$$\sin(2\beta) = 2\cos\beta\sin\beta$$

51. Show that if $\mathbf{E} = -\nabla V$ where $V = C/r$, and C is a constant and r is a distance in space then

$$\nabla \times \mathbf{E} = 0$$

52. Prove that

$$\nabla \bullet (\mathrm{a}\mathbf{E}) = \mathrm{a}\nabla \bullet \mathrm{E} + \mathbf{E} \bullet \nabla \mathrm{a}$$

53. Show that the divergence of the gradient is

$$\Delta \bullet (\nabla V) = \nabla^2 V = \frac{\partial^2 V}{\partial^2 x} + \frac{\partial^2 V}{\partial^2 y} + \frac{\partial^2 V}{\partial^2 z}$$

54. Show that the curl of the curl of a vector can be written as

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \bullet \mathrm{E}) - \nabla \bullet \nabla \mathrm{E})$$

55. Show that the divergence of the curl of a vector is equal to zero. That is, show that

$$\nabla \bullet (\nabla \times \mathbf{E}) = 0$$

56. If **r** is a displacement vector, find the divergence of **r**.

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