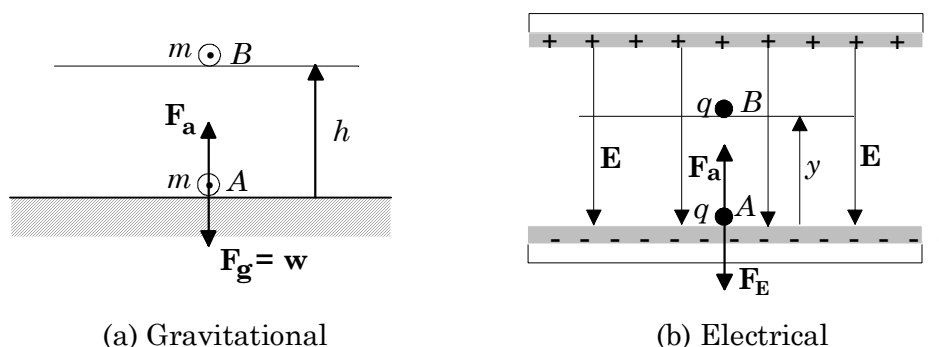


# Chapter 5 The Electric Potential

## 5.1 Potential Energy

Although the electric field is defined as a force on a unit test charge, it is extremely difficult to measure fields in this way. Let us return to the energy concepts studied in the mechanics section of college physics and apply them to electric charges in an electric field to develop another way to describe electric fields.

Recall from college physics that the *gravitational potential energy was defined as the energy that a body possessed by virtue of its position in a gravitational field, and that potential energy was equal to the work that had to be done to place the body in that position in the gravitational field. The work done was defined as the product of the applied force  $F_a$ , in the direction of the displacement, and the displacement  $y$  that the body moved through while the force was acting.* As an example, the gravitational potential energy of the body of mass  $m$  at point  $B$ , figure 5.1(a), is equal to the work  $W$  done against gravity to lift the body, with an applied force  $F_a$ ,



**Figure 5.1** Potential Energy.

from point  $A$  to point  $B$ , through the distance  $h$ . That is,

$$PE = W = F_a y \quad (5.1)$$

But the applied force  $F_a$  is the force necessary to overcome the force due to gravity  $F_g$ , that is,  $F_a = F_g$ , hence the potential energy is

$$PE = F_g y$$

However,  $F_g$  is equal to the weight  $w$  of the body, which in turn is equal to the product  $mg$ , while the distance  $y$  that the force acts is just the height  $h$  that the body is lifted, thus

$$\begin{aligned} PE &= wh \\ PE &= mgh \end{aligned} \quad (5.2)$$

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Hence the gravitational potential energy of an object placed in a uniform gravitational field is given by equation 5.2.

Just as a mass has a potential energy when placed in a gravitational field an electric charge  $q$  has a potential energy when placed in an electric field  $\mathbf{E}$ . If an electric charge is placed in a parallel, uniform electric field, such as the electric field between the plates of a parallel plate capacitor, *the electrical potential energy of that charge can be defined as the energy it possesses by virtue of its position in the electric field. The potential energy is equal to the work that must be done to place that charge into that position in the electric field.* Figure 5.1(b) shows a charge  $q$  in a uniform electric field  $\mathbf{E}$  that emanates from the positive plate at the top, points downward, and terminates on the negative plate at the bottom. The potential energy that a positive charge  $q$  has at position  $B$  is the work  $W$  that must be done by an external agent as it exerts an applied force  $F_a$  to move that charge from the bottom plate,  $A$ , to the position  $B$ , a distance  $y$ . That is, the electric potential energy is

$$\text{PE} = W = F_a y \quad (5.3)$$

But the magnitude of the applied external force  $F_a$  is just equal to the magnitude of the electric force on the charge, i.e.,

$$F_a = F_E = qE \quad (5.4)$$

Replacing equation 5.4 into 5.3 yields the electric potential energy of the charge at  $B$  as

$$\text{PE} = qEy \quad (5.5)$$

Just as the gravitational potential energy of a mass  $m$  was given by equation 5.2, the electrical potential energy of a charge  $q$  is given by equation 5.5.

### **Example 5.1**

*The potential energy of a point charge.* A point charge  $q = 8.00 \times 10^{-9} \text{ C}$  is placed at the position  $y = 5.00 \text{ mm}$  above the negative plate of a parallel plate capacitor that has an electric field intensity  $E = 4.00 \times 10^4 \text{ N/C}$ . Find the potential energy of the point charge at this location.

### **Solution**

The potential energy of the point charge is found from equation 5.5 as

$$\begin{aligned} \text{PE} &= qEy = (8.00 \times 10^{-9} \text{ C})(4.00 \times 10^4 \text{ N/C})(5.00 \times 10^{-3} \text{ m}) \\ \text{PE} &= 1.60 \times 10^{-6} \text{ J} \end{aligned}$$

**To go to this Interactive Example click on this sentence.**

## 5.2 The Electric Potential and the Potential Difference

Because the potential energy depends upon the charge  $q$ , and it is sometimes difficult to work directly with electric charges, it is desirable to define a new quantity that is independent of the charge. Hence, *the electric potential  $V$  is defined as the potential energy per unit charge, i.e.,*

$$V = \frac{PE}{q} = \frac{W}{q} \quad (5.6)$$

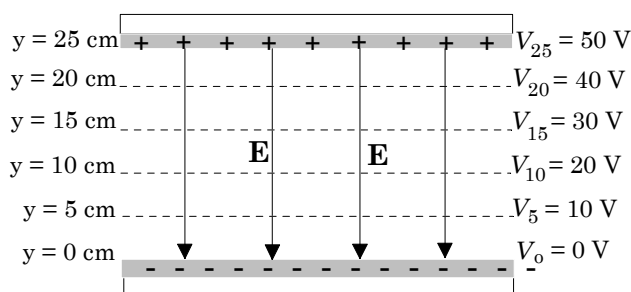
Note that the potential  $V$  is a potential energy per unit charge while the electric field  $E$  is a force per unit charge. The SI unit of potential is defined to be the volt, where

$$1 \text{ volt} = 1 \text{ joule/coulomb} = 1 \text{ J/C}$$

For the parallel plate configuration of figure 5.2, the potential becomes

$$V = \frac{PE}{q} = \frac{qEy}{q} \quad (5.7)$$

Just as the electric field exists in a region around an electric charge, we can also talk about a potential field existing in a region about an electric charge. Whereas the electric field is a vector field, the potential field, however, is a scalar field. As an example, if the electric field intensity between the plates of figure 5.2 is 200 N/C,



**Figure 5.2** The equipotential lines between the plates of a parallel plate capacitor.

let us find the potential field at intervals of 5.00 cm between the plates. Because the distance  $y$  is measured from the negative plate, the potential at the negative plate, equation 5.7, becomes

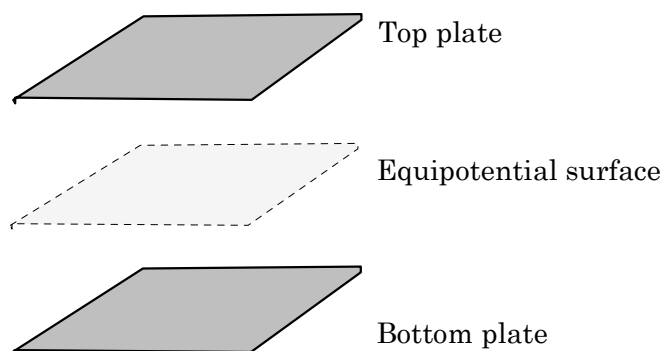
$$V_0 = Ey = (200 \text{ N/C})(0) = 0$$

The negative plate is therefore the zero of our potential. At a height of 5.00 cm above the negative plate the potential is

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$$\begin{aligned}V_5 &= E_y = (200 \text{ N/C})(0.0500 \text{ m}) \\V_5 &= 10.0 \text{ N m/C} = 10.0 \text{ J/C} \\V_5 &= 10.0 \text{ V}\end{aligned}$$

Therefore, on a line 5.00 cm above the bottom plate the potential is 10.0 V everywhere. *This dotted line in the figure is called an equipotential line, a line of equal or constant potential.* If this figure were drawn in three dimensions, the equipotential line would be an equipotential surface, as shown in figure 5.3. The



**Figure 5.3** An equipotential surface.

potentials at 10.0 cm, 15.0 cm, 20.0 cm, and 25.0 cm are found as

$$\begin{aligned}V_{10} &= E_y = (200 \text{ N/C})(0.100 \text{ m}) = 20.0 \text{ V} \\V_{15} &= E_y = (200 \text{ N/C})(0.150 \text{ m}) = 30.0 \text{ V} \\V_{20} &= E_y = (200 \text{ N/C})(0.200 \text{ m}) = 40.0 \text{ V} \\V_{25} &= E_y = (200 \text{ N/C})(0.250 \text{ m}) = 50.0 \text{ V}\end{aligned}$$

All these equipotentials are shown in figure 5.2 as dotted lines.

A very interesting result can be observed in figure 5.2. *The equipotential lines are everywhere perpendicular to the electric field lines.* Although this has been determined for the field between the parallel plates of a capacitor, it is true in general for any electric field and its equipotentials as will be proved later.

The value of the potential, like gravitational potential energy depends upon a particular reference position. In this example, the potential was zero at the negative plate because the value of  $y$  for equation 5.7 was zero there. In practical problems, such as the electrical wiring in your home or office, the zero of potential is usually taken to be that of the surface of the earth, literally “the ground”. When you plug an appliance into a wall outlet in your home, it sees a potential of 120 V. One wire, the “hot” wire is 120 V above the other wire, the ground or zero wire. (The ground wire, is truly a ground wire. If you look into the electric box servicing your home you will see the “ground” wire, usually an uninsulated wire. If this ground wire is followed, you will see that it is connected to the cold water pipe of your home. The cold water

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pipe is eventually buried in the ground. Thus, the zero of potential in your home is in fact the potential of the ground). When dealing with individual point charges the zero of potential is usually taken at infinity, a step that will be justified later.

The introduction of energy concepts into the description of electric fields now pays its dividend. Solving equation 5.7 for  $E$ , the electric field intensity between the plates, gives

$$E = \frac{V}{y} \quad (5.8)$$

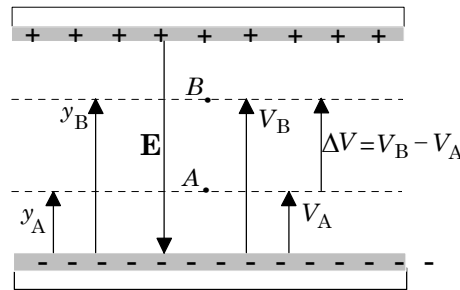
That is, knowing the potential between the plates, something that is easily measured with a voltmeter, and the distance separating the plates, a new indirect way of determining the electric field intensity is obtained.

When equation 5.8 is used to determine the electric field, an equivalent unit of electric field intensity, the volt per meter, can be used. To show that this is equivalent note that

$$\frac{\text{volt}}{\text{meter}} = \frac{\text{joule/coulomb}}{\text{meter}} = \frac{\text{newton meter}}{\text{coulomb meter}} = \frac{\text{newton}}{\text{coulomb}}$$

### The Potential Difference

Instead of knowing the actual potential at a particular point, it is sometimes more desirous to know the difference in potential between two points,  $A$  and  $B$ . If the point  $A$  is at the ground potential, then the potential and potential difference is the same. If  $A$  is not the ground potential then the potential difference between point  $A$  and point  $B$  is just the difference between the potential at  $B$  and the potential at  $A$  as shown in figure 5.4. From equation 5.7



**Figure 5.4** The potential difference.

$$V_B = Ey_B$$

while

$$V_A = Ey_A$$

Therefore, the potential difference between points A and B is

$$\Delta V = V_B - V_A = Ey_B - Ey_A$$

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$$\begin{aligned}\Delta V &= E(y_B - y_A) \\ \Delta V &= E\Delta y\end{aligned}\tag{5.9}$$

In general, then, the magnitude of the electric field intensity can be found from equation 5.9 as

$$E = \frac{\Delta V}{\Delta y}\tag{5.10}$$

Equation 5.10 gives the average value of the magnitude of the electric field intensity. For the parallel plate configuration the electric field intensity is a constant and hence the average value is the same as the constant value. When dealing with fields that are not constant equation 5.10 will give us an average value of the field over the interval  $\Delta y$ . To make the average value closer to the actual value of  $E$  at a point, the interval  $\Delta y$  would have to be made smaller and smaller, until in the limit the actual value of  $E$  at any point will be given by

$$E = \lim_{\Delta y \rightarrow 0} \frac{\Delta V}{\Delta y} = \frac{dV}{dy}\tag{5.11}$$

Recall that the limit of  $\Delta V/\Delta y$  as  $\Delta y$  approaches zero is the definition of a derivative in calculus. Thus, *the electric field intensity  $E$  is given by the derivative of the electric potential  $V$  with respect to the distance  $y$ .*

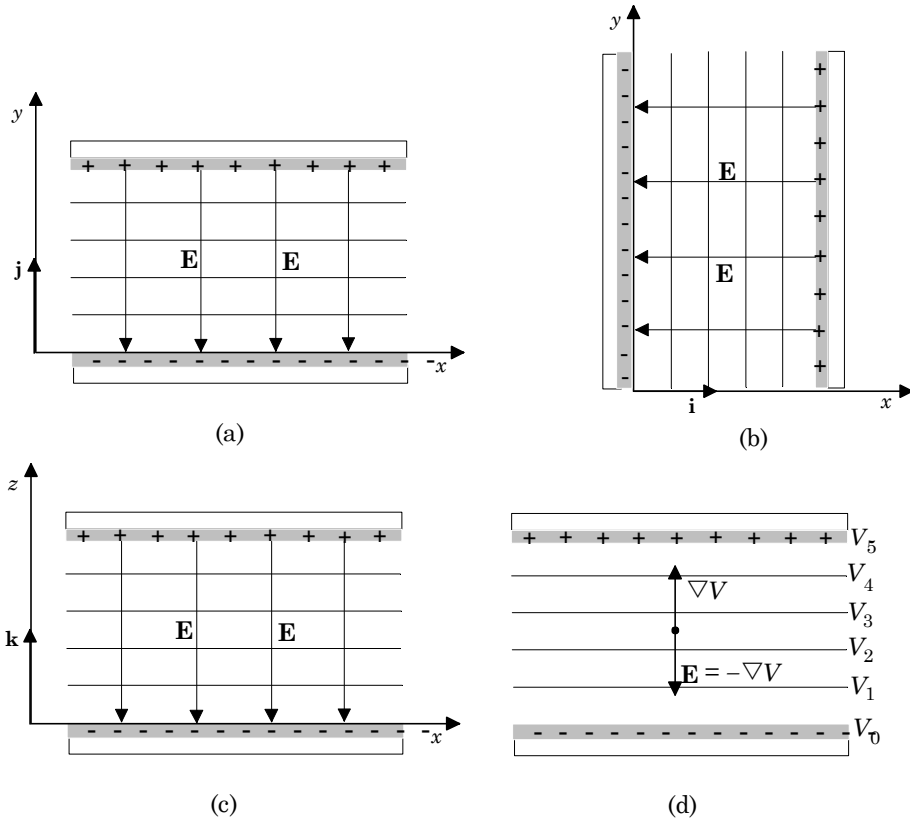
Note from figure 5.4 and equation 5.10 that  $\Delta V$  is positive and  $\Delta y$  is positive, therefore  $\Delta V/\Delta y$  is a positive quantity. Yet the vector  $\mathbf{E}$  is a negative quantity since it points downward in the figure. If we use the unit vector  $\mathbf{j}$  that points upward in the positive  $y$ -direction, then we can rewrite equation 5.11 in vector notation as

$$\mathbf{E} = -\frac{dV}{dy}\mathbf{j}\tag{5.12}$$

Equation 5.12 says that the magnitude of the electric field intensity is given by the derivative  $dV/dy$  and its direction by  $-\mathbf{j}$ . The unit vector  $\mathbf{j}$  and the electric field vector  $\mathbf{E}$  is shown in figure 5.5(a). If the parallel plates are rotated through  $90^\circ$  the equipotentials and electric field vectors  $\mathbf{E}$  are shown in figure 5.5(b). For this case the equipotential lines are each drawn for a fixed value of  $x$ . The derivative  $dV/dx$  is a positive quantity, yet the electric field vector  $\mathbf{E}$  points in the negative  $x$ -direction. Therefore the electric field vector  $\mathbf{E}$  is given by

$$\mathbf{E} = -\frac{dV}{dx}\mathbf{i}\tag{5.13}$$

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**Figure 5.5** The electric field, the unit vectors, and the gradient.

If we were to rearrange the parallel plates again, so that they appear as in figure 5.5(c), then we would find, in the same way, that the electric field vector  $\mathbf{E}$  would be given by

$$\mathbf{E} = -\frac{dV}{dz}\mathbf{k} \quad (5.14)$$

For an arbitrary electric field in 3-dimensional space the electric field vector  $\mathbf{E}$  would be given by the superposition of equations 5.12, 5.13, and 5.14 as

$$\mathbf{E} = -\frac{\partial V}{\partial x}\mathbf{i} - \frac{\partial V}{\partial y}\mathbf{j} - \frac{\partial V}{\partial z}\mathbf{k} \quad (5.15)$$

where ordinary derivatives have been replaced by partial derivatives because  $V$  would no longer be a function of a single variable but rather of the three variables  $x$ ,  $y$ , and  $z$ . Equation 5.15 can be simplified into the form

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right) \quad (5.16)$$

But recall from equation 1-66 that

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$$\left( \frac{\partial V}{\partial x} \mathbf{i} + \frac{\partial V}{\partial y} \mathbf{j} + \frac{\partial V}{\partial z} \mathbf{k} \right) = \nabla V \quad (1-66)$$

where  $\nabla V$  was called the gradient of  $V$ . Replacing equation 1-66 into equation 5.16 gives

$$\mathbf{E} = -\nabla V \quad (5.17)$$

Equation 5.17 says that *the electric field  $\mathbf{E}$  is equal to the negative of the gradient of the potential  $V$* . The quantity  $\nabla V$  is a directional derivative that points in the direction of the increasing potential, as seen in figure 5.5(d). The electric field  $\mathbf{E}$  points in the opposite direction, the direction in which  $V$  is decreasing. Since the potential  $V$  is the potential energy per unit charge, the electric field vector  $\mathbf{E}$  points in the direction of decreasing potential energy and a charge placed in the electric field would “fall” from a region of high potential energy to a region of lower potential energy.

### Example 5.2

*The potential difference and the electric field.* The potential difference  $\Delta V$  between two plates of a parallel plate capacitor is 400 V. If the plate separation is 1.00 mm, what is the magnitude of the electric field intensity  $E$  between the plates?

### Solution

The magnitude of the electric field intensity is determined from equation 5.10 as

$$E = \frac{\Delta V}{\Delta y} = \frac{400 \text{ V}}{1.00 \times 10^{-3} \text{ m}} \\ E = 4.00 \times 10^5 \text{ V/m}$$

**To go to this Interactive Example click on this sentence.**

### Example 5.3

*The force on a charge in an electric field.* What force would act on an electron placed in the field of example 5.2?

### Solution

The magnitude of the force is found from

$$F = qE = (1.60 \times 10^{-19} \text{ C})(4.00 \times 10^5 \text{ V/m}) \frac{(\text{J/C})(\text{N} \cdot \text{m})}{(\text{V})(\text{J})}$$



$$F = 6.40 \times 10^{-14} \text{ N}$$

The direction of the force on the electron would be from the negative plate to the positive plate.

**To go to this Interactive Example click on this sentence.**

### 5.3 Generalization of the Formulation for the Potential Difference

The determination of the electrical potential energy and hence the electric potential of a charge in a uniform electric field, section 5.2, was essentially easy because the electric field between the plates was a constant. That is, the potential  $V$  was defined in equation 5.6 as the potential energy PE per unit charge. But the potential energy was just equal to the work  $W$  that had to be done to place the charge  $q$  into that position in the electric field, that is

$$V = \frac{\text{PE}}{q} = \frac{W}{q} \quad (5.6)$$

The work that had to be done was calculated from

$$W = F_a y = F_E y = qEy$$

But because the electric field  $E$  was a constant, the work done was simply the product of the force  $F$  times the displacement  $y$ . Thus the force acting on the electric charge was a constant and the work done,  $W = Fy$ , was also a constant. If the force acting on the charge is not a constant then a new formulation for the work done and hence the electric potential is required.

As an example suppose we have two point charges and we wish to move the second charge toward the first charge. How much work will be done? The force acting on the second charge caused by the first charge is given by Coulomb's law, equation 2-1 as

$$F = k \frac{q_1 q_2}{r^2} \quad (2-1)$$

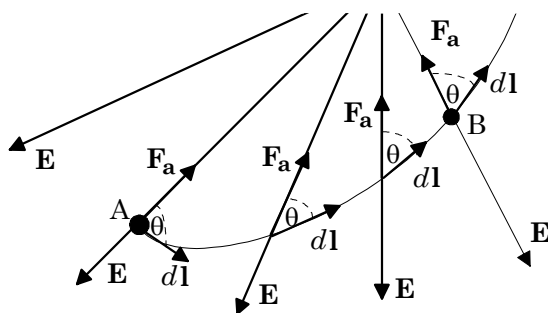
Notice that the force is not a constant but varies with the distance  $r$  between the two charges. Therefore, for each value of  $r$  a different amount of work would be done. The only way to solve this problem is to break the distance that we will move the charge, into a series of smaller intervals  $dl$  for which the force  $F$  is effectively constant. A small amount of work  $dW$  will be done in the small interval  $dl$  given by

$$dW = F dl \quad (5.18)$$

and the total amount of work that will be done will be equal to the sum or integral of all these  $dW$ 's. That is,

$$W = \int dW = \int F dl \quad (5.19)$$

Equation 5.19 is on the right track to determine the work done for a variable force but it is not complete. Recall from the concept of work, that work is the product of the force in the direction of the displacement, times the displacement. Equations 5.18 and 5.19 assume that the direction of the force is in the direction of the displacement. The most general case would be where the displacement and the force are not in the same direction. This case is shown in figure 5.6, which shows a portion of an electric field that varies in both magnitude and direction. A charge  $q$



**Figure 5.6** The force and displacement are not in the same direction.

is located at the point  $A$ . The charge is moved from position  $A$  to position  $B$  by an external force  $\mathbf{F}_a$ . It is desired to determine the amount of work done in moving the charge from  $A$  to  $B$ . The first thing that we note is that the force is varying in magnitude and direction as we proceed from  $A$  to  $B$ . The small amount of work  $dW$  done at position  $A$  is given by

$$dW = \mathbf{F}_a \cdot d\mathbf{l} \quad (5.20)$$

where  $d\mathbf{l}$  is the small distance of the path, over which it is assumed  $\mathbf{F}_a$  is constant. At each of the next intersections the amount of work is also  $dW = \mathbf{F}_a \cdot d\mathbf{l}$ . Of course the magnitude of the force changes at each intersection as well as the angle  $\theta$  between the force vector  $\mathbf{F}_a$  and the displacement  $d\mathbf{l}$ . The total work done in moving the charge  $q$  from the point  $A$  to the point  $B$  is the sum of all the  $dW$ 's from every point along the path  $AB$ . That is,

$$W_{AB} = \int dW = \int \mathbf{F}_a \cdot d\mathbf{l} \quad (5.21)$$

The change in the potential  $dV$  for each point along the path becomes

$$dV = \frac{dW}{q} \quad (5.22)$$

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Calling the potential at the point  $A$ ,  $V_A$  and at  $B$ ,  $V_B$  the total change in potential between  $A$  and  $B$  becomes

$$\int_{V_A}^{V_B} dV = \int \frac{dW}{q} = \int \frac{\mathbf{F}_a \cdot d\mathbf{l}}{q} \quad (5.23)$$

But the applied force  $\mathbf{F}_a$  is equal and opposite to the electric force  $\mathbf{F}_E$ , that is,

$$\mathbf{F}_a = -\mathbf{F}_E = -q\mathbf{E} \quad (5.24)$$

Equation 5.23 therefore becomes

$$\begin{aligned} V_B - V_A &= - \int \frac{q\mathbf{E} \cdot d\mathbf{l}}{q} \\ V_B - V_A &= - \int \mathbf{E} \cdot d\mathbf{l} \end{aligned} \quad (5.25)$$

Equation 5.25 gives the general equation for the difference in potential  $V_B - V_A$  between two points  $A$  and  $B$ , when both the magnitude and direction of the electric field varies along the path  $AB$ .

### Example 5.4

*Potential difference.* Using the generalized equation for the difference in potential, equation 5.25, find the difference in potential between the two points  $A$  and  $B$  lying between the two parallel plates in figure 5.2.

### Solution

The parallel plates are reproduced in the figure below. Notice from that diagram, that  $d\mathbf{l}$  point upward while  $\mathbf{E}$  points downward and hence the angle between  $d\mathbf{l}$  and  $\mathbf{E}$  is equal to  $180^\circ$ . Equation 5.25 becomes

$$V_B - V_A = - \int \mathbf{E} \cdot d\mathbf{l} = - \int Edl \cos 180^\circ = - \int Edl(-1) = \int Edl$$

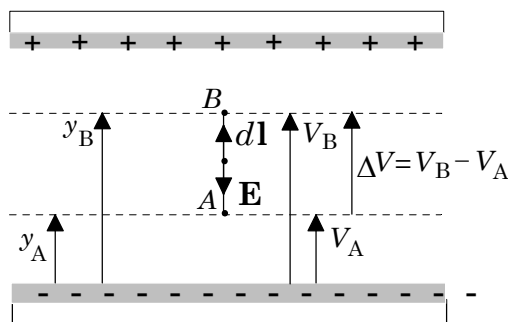


Diagram for example 5.4

As shown in chapter 3 the electric field  $E$  between the parallel plates is a constant and can come out of the integral sign. The difference in potential between the points  $A$  and  $B$  now becomes

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$$\begin{aligned} V_B - V_A &= \int E dl = E \int_{y_A}^{y_B} dl = El \Big|_{y_A}^{y_B} \\ V_B - V_A &= \Delta V = E(y_B - y_A) = E\Delta y \\ \Delta V &= E\Delta y \end{aligned}$$

This is the same potential difference that we found earlier in section 5.2.

### 5.4 The Potential of a Positive Point Charge

The electric field of a positive point charge is shown in figure 5.7. We would like to determine the difference in potential between the two points  $A$  and  $B$  in the figure. The difference in potential is found from equation 5.25 as

$$V_B - V_A = - \int \mathbf{E} \cdot d\mathbf{l}$$

The electric field  $\mathbf{E}$ , emanating from the point charge points to the right along the path  $AB$  while  $d\mathbf{l}$ , the element of path, points in the direction that we are moving along the path as we move from  $A$  to  $B$ . Hence  $d\mathbf{l}$  points to the left and thus the angle  $\theta$  between  $\mathbf{E}$  and  $d\mathbf{l}$  is  $180^\circ$  as seen in figure 5.7(b). Therefore

$$\begin{aligned} V_B - V_A &= - \int_A^B \mathbf{E} \cdot d\mathbf{l} = - \int_A^B E dl \cos 180^\circ = - \int_A^B E dl (-1) \\ V_B - V_A &= \int_A^B E dl \end{aligned} \tag{5.26}$$

But the electric field of a point charge was found to be

$$E = \frac{kq}{r^2} \tag{3-2}$$

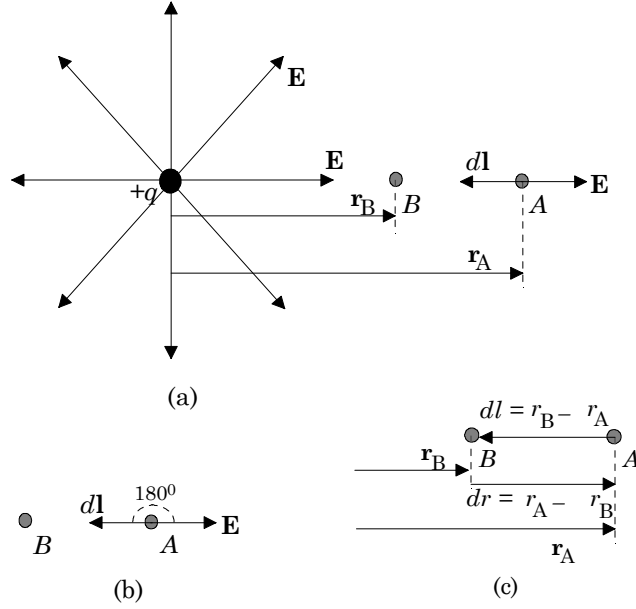
Replacing equation 3-2 into equation 5.26 yields

$$V_B - V_A = \int_A^B E dl = \int_A^B \frac{kq}{r^2} dl \tag{5.27}$$

We have a slight problem here because  $E$  is a function of  $r$  and yet the integration is over the path  $l$ . But  $dl$  and  $dr$  are related as can be seen in figure 5.7(c). The coordinate  $r$  is measured from the charge outward. Its value at  $B$  is  $r_B$  and at  $A$ ,  $r_A$  where  $r_A > r_B$ . The element of  $r$  is thus

$$dr = r_A - r_B \tag{5.28}$$

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**Figure 5.7** The electric field of a positive point charge.

The element of path  $dl$  extends from  $A$  to  $B$  and is seen to be given by

$$dl = r_B - r_A \quad (5.29)$$

Comparing equations 5.28 and 5.29 we see that

$$dl = -dr \quad (5.30)$$

Replacing equations 5.30 into equation 5.27 gives

$$V_B - V_A = - \int_{r_A}^{r_B} \frac{kq}{r^2} dr \quad (5.31)$$

Notice that the value of the limits at  $A$  and  $B$  are  $r_A$  and  $r_B$  respectively and are now the limits of integrations in equation 5.31. Evaluating the integral we obtain

$$\begin{aligned} V_B - V_A &= - \int_{r_A}^{r_B} \frac{kq}{r^2} dr = - \int_{r_A}^{r_B} kqr^{-2} dr = -kq \left( \frac{r^{-2+1}}{-2+1} \right) \Big|_{r_A}^{r_B} \\ V_B - V_A &= -kq \left( -\frac{1}{r} \right) \Big|_{r_A}^{r_B} = \frac{kq}{r} \Big|_{r_A}^{r_B} \\ V_B - V_A &= kq \left( \frac{1}{r_B} - \frac{1}{r_A} \right) \end{aligned} \quad (5.32)$$

Equation 5.32 gives the difference in potential between the point  $A$  and the point  $B$  in the electric field of a positive point charge.

**Example 5.5**

*The difference in potential.* Find the difference in potential between the points  $A$  and  $B$  for a point charge  $q = 2.50 \times 10^{-6}$  C. The distance  $r_B = 10.0$  cm and  $r_A = 25.0$  cm.

**Solution**

The difference in potential is found from equation 5.32 as

$$\begin{aligned} V_B - V_A &= kq \left( \frac{1}{r_B} - \frac{1}{r_A} \right) \\ \Delta V &= \left( 9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2} \right) (2.50 \times 10^{-6} \text{ C}) \left( \frac{1}{0.100 \text{ m}} - \frac{1}{0.250 \text{ m}} \right) \\ \Delta V &= 1.35 \times 10^5 \text{ V} \end{aligned}$$

**To go to this Interactive Example click on this sentence.**

We can use the result of equation 5.32 to determine the potential of an isolated point charge. Rewriting equation 5.32 as

$$V_B - V_A = \frac{kq}{r_B} - \frac{kq}{r_A} \quad (5.33)$$

We now let the field point  $A$  extend out to infinity, i.e.,  $r_A \rightarrow \infty$ . The term  $kq/r_A$  will approach zero as  $r_A$  approaches infinity, i.e.,

$$\lim_{r_A \rightarrow \infty} \frac{kq}{r_A} \rightarrow 0$$

We associate  $V_B$  with  $kq/r_B$  and  $V_A$  with  $kq/r_A$ . So when  $kq/r_A \rightarrow 0$  we set  $V_A = 0$ . That is, *the zero of potential for a point charge is taken at infinity*, and hence we can refer to the potential at an arbitrary point  $B$  as

$$V_B = \frac{kq}{r_B}$$

But what is so special about  $B$ ?  $B$  is just an arbitrary point. Therefore we drop the subscript  $B$  and say that this is *the electric potential  $V$  at any position  $r$  for a point charge  $q$* , that is,

$$V = \frac{kq}{r} \quad (5.34)$$

**Example 5.6**

*The potential of a point charge.* Find the potential  $V$  at the point  $r = 20.0$  cm from a positive point charge of  $3.85 \times 10^{-6}$  C.

**Solution**

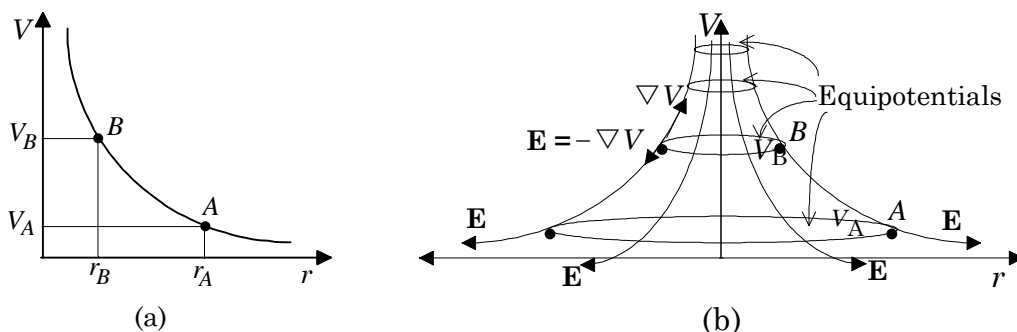
The potential is found from equation 5.34 as

$$V = \frac{kq}{r} = \frac{(9 \times 10^9 \text{ N m}^2/\text{C}^2)(3.85 \times 10^{-6} \text{ C})}{0.200 \text{ m}}$$

$$V = 1.73 \times 10^5 \text{ J}$$

**To go to this Interactive Example click on this sentence.**

Notice from equation 5.34 that the potential  $V$  of a point charge  $q$  varies as  $1/r$ . For a constant distance  $r$  from the point charge, the potential is a constant. We can visualize the electric potential of a point charge  $q$  by plotting the potential  $V$  as a function of the distance  $r$  to get figure 5.8(a). The location of the points  $A$  and  $B$ ,

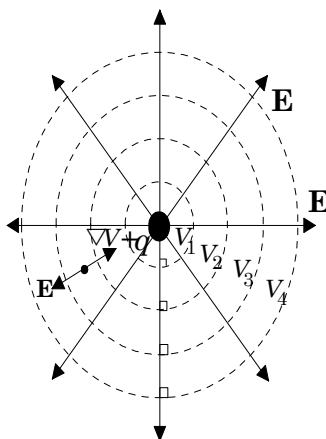


**Figure 5.8** Plot of the potential function for a positive point charge.

the radii  $r_A$  and  $r_B$ , and the potentials  $V_A$  and  $V_B$  are shown in the figure. We can see that as  $r_A$  approaches infinity, the potential  $V_A$  approaches zero. Figure 5.8(b) is a plot of this same potential function, except it is shown with respect to a two-dimensional  $x,y$  space. The points  $A$  and  $B$ , and the potentials  $V_A$  and  $V_B$  are also shown. This diagram is generated by rotating the diagram of figure 5.8(a) about the potential  $V$  as an axis. Notice that the potential looks like a hill or a mountain and work must be done to bring another positive charge up the potential hill. Also observe that the gradient  $\nabla V$  points up the potential hill to the regions of higher potential while  $-\nabla V$  points downward in the direction of the electric field, since  $\mathbf{E} = -\nabla V$ .

## Chapter 5: The Electric Potential

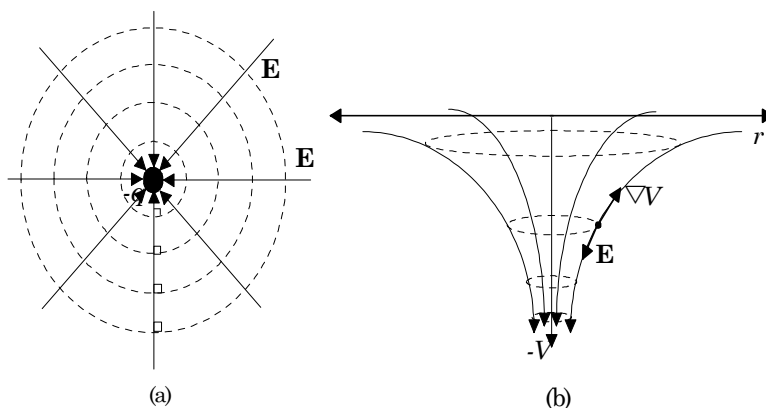
If figure 5.8(b) is projected onto the  $x$ - $y$  plane we obtain a two-dimensional picture of the potential field of a point charge, as shown in figure 5.9. Notice that it



**Figure 5.9** Two dimensional picture of the potential field of a positive point charge.

consists of a family of concentric circles, around the point charge. Each of these circles is an equipotential line. Note that the electric field is everywhere perpendicular to the equipotential lines. Also observe that the gradient  $\nabla V$  points toward the center of the circle, to the regions of higher potential while  $-\nabla V$  points outward in the direction of the electric field, since  $\mathbf{E} = -\nabla V$ . In three dimensions, there would be a family of equipotential spheres surrounding the point charge.

The equipotential surfaces for a negative charge are shown in figure 5.10. A positive charge would have to be held back to prevent it from falling down the potential well.



**Figure 5.10** The potential well of a negative point charge.



**Example 5.7**

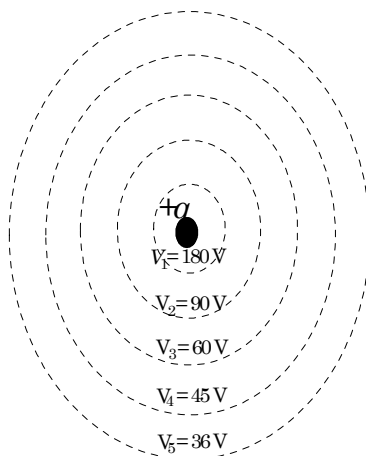
*The potential field of a point charge.* Find the potential field for a positive point charge of 2.00 nC at  $r = 10.0, 20.0, 30.0, 40.0$ , and  $50.0$  cm.

**Solution**

The potential for a positive point charge, found from equation 5.34, is

$$\begin{aligned}
 V_1 &= \frac{kq}{r_1} = (9.00 \times 10^9 \text{ N m}^2/\text{C}^2) \frac{(2.00 \times 10^{-9} \text{ C})}{0.100 \text{ m}} = 180 \text{ V} \\
 V_2 &= \frac{kq}{r_2} = (9.00 \times 10^9 \text{ N m}^2/\text{C}^2) \frac{(2.00 \times 10^{-9} \text{ C})}{0.200 \text{ m}} = 90.0 \text{ V} \\
 V_3 &= \frac{kq}{r_3} = (9.00 \times 10^9 \text{ N m}^2/\text{C}^2) \frac{(2.00 \times 10^{-9} \text{ C})}{0.300 \text{ m}} = 60.0 \text{ V} \\
 V_4 &= \frac{kq}{r_4} = (9.00 \times 10^9 \text{ N m}^2/\text{C}^2) \frac{(2.00 \times 10^{-9} \text{ C})}{0.400 \text{ m}} = 45.0 \text{ V} \\
 V_5 &= \frac{kq}{r_5} = (9.00 \times 10^9 \text{ N m}^2/\text{C}^2) \frac{(2.00 \times 10^{-9} \text{ C})}{0.500 \text{ m}} = 36.0 \text{ V}
 \end{aligned}$$

These equipotential lines are drawn in figure 5.11.



**Figure 5.11** Finding the potential field of a point charge.

## 5.5 The Point Charge and the Gradient of the Potential

We just saw in example 5.7 how to find the potential field for a positive point charge. We will now show some interesting characteristics of the potential field and

## Chapter 5: The Electric Potential

the gradient of the potential field. We saw in chapter 1 that when  $\nabla$  operates on a scalar function  $V$ , the gradient of  $V$  is obtained, equation 1-66.

$$\nabla V = \mathbf{i} \frac{\partial V}{\partial x} + \mathbf{j} \frac{\partial V}{\partial y} + \mathbf{k} \frac{\partial V}{\partial z} \quad (1-66)$$

Because  $\nabla$  is a vector, and  $V$  is a scalar, the gradient  $\nabla V$  is a vector, as can be seen in equation 1-66. Let us now look into the physical characteristics of the gradient.

The first equipotential line in figure 5.11 is drawn at a potential  $V = 180$  V. Everywhere on this line the potential is exactly 180 V. This is the top of the potential hill. The next equipotential line is drawn at  $V = 90$  V. Everywhere on this line the potential is exactly 90 V. Between the 180 V equipotential and the 90 V equipotential the potential varies between 180 V and 90 V. The equipotential lines for 60 V, 45 V, and 36 V are also drawn in the figure. The potentials indicated by the equipotential lines are an example of a scalar field, in that they depend only upon the potential, the magnitude at any point and not a direction. Equation 1-103 can now represent the gradient of the potential  $V$ .

It is interesting to note that if each potential  $V$  in figure 5.11, were multiplied by  $q$ , the equipotential lines would be lines of constant potential energy. Hence, figure 5.11 can also be thought of as a plot of potential energies. Everywhere on a line of constant  $V$ , the potential energy is also a constant. As you advance up the hill, the potential energy increases as you would expect.

To get a clearer picture of the gradient let us assume that we are located at the point  $P$  and make a small displacement  $d\mathbf{r}$  away from  $P$ , figure 5.12. What will our new potential be? Let us dot multiply the gradient of the potential field,  $\nabla V$ , by the displacement  $d\mathbf{r}$  and obtain

$$\nabla V \cdot d\mathbf{r} = \left( \mathbf{i} \frac{\partial V}{\partial x} + \mathbf{j} \frac{\partial V}{\partial y} + \mathbf{k} \frac{\partial V}{\partial z} \right) \cdot (i dx + j dy + k dz) \quad (5.35)$$

Using the result of the scalar product, equation 1-61 this becomes

$$\nabla V \cdot d\mathbf{r} = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \quad (5.36)$$

Notice the right-hand side of equation 5.36. It is the total differential of the function  $V$  and is given in the calculus by

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \quad (5.37)$$

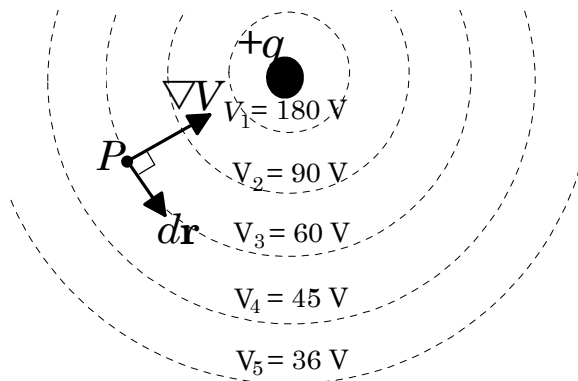
Let us examine equation 5.37 in more detail to see what it means physically. Let us assume that  $V$  is a function, like the electric potential function, that varies as you move in the  $x$ ,  $y$ , and  $z$  directions. The first partial derivative term,  $\partial V / \partial x$ , represents the rate at which the function  $V$  changes as you move in the  $x$ -direction,

and  $dx$  represents the total distance you move in the  $x$ -direction. When you multiply  $(\partial V / \partial x)$  by  $dx$ , you get the total change in the function  $V$  as you move in the  $x$ -direction. The second partial derivative term,  $\partial V / \partial y$  represents the rate at which the function  $V$  changes as you move in the  $y$ -direction, and  $dy$  represents the total distance you move in the  $y$ -direction. When you multiply  $(\partial V / \partial y)$  by  $dy$ , you get the total change in the function,  $V$ , as you move in the  $y$ -direction. Finally, the third partial derivative term,  $(\partial V / \partial z)$ , represents the rate at which the function changes as you move in the  $z$ -direction, and  $dz$  represents the total distance you move in the  $z$ -direction. When you multiply  $(\partial V / \partial z)$  by  $dz$ , you get the total change in the function  $V$  as you move in the  $z$ -direction. The sum of the three terms represents the total change in the function  $V$  as you move through space.

But the right hand side of this equation is the definition of the total differential of  $V$  as seen in equation 5.37. Hence, equation 5.36 becomes

$$\nabla V \cdot d\mathbf{r} = dV \quad (5.38)$$

If the displacement,  $d\mathbf{r}$ , is along an equipotential line, as seen in the enlarged diagram of figure 5.12, then there is no change in the potential  $dV$ , because the equipotential line is a line of constant potential. Hence, for a displacement along an equipotential line equation 5.38 becomes



**Figure 5.12** Displacement along an equipotential line.

$$\nabla V \cdot d\mathbf{r} = 0 \quad (5.39)$$

hence

$$\nabla V \perp d\mathbf{r}$$

That is, the dot product of two vectors is only zero when the two vectors are perpendicular to each other. This means that the gradient  $\nabla V$  is perpendicular to the equipotential lines. This is shown in figure 5.12. Using the definition of the dot product equation 5.38 can also be written as

$$dV = \nabla V \cdot d\mathbf{r} = |\nabla V| dr \cos\theta \quad (5.40)$$

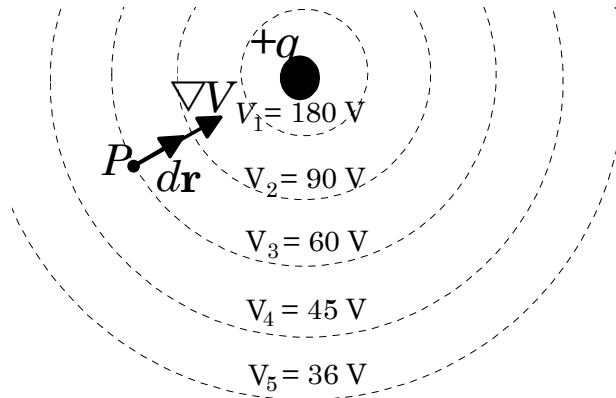
## Chapter 5: The Electric Potential

As can be observed in equation 5.40, the maximum change in potential  $dV$  is obtained when the  $\cos\theta$  is equal to one. But the  $\cos\theta = 1$  for  $\theta = 0^\circ$ , which is the case when the displacement  $d\mathbf{r}$  is parallel to the gradient  $\nabla V$ . *Hence the maximum change of potential occurs when the displacement  $d\mathbf{r}$  is in the direction of the gradient  $\nabla V$ , figure 5.34. The gradient is the slope of the potential hill at any point  $P$  and is directed upward towards the highest value of the potential.* For this special case equation 5.40 becomes

$$dV = \nabla V \cdot d\mathbf{r} = |\nabla V| dr \cos 0^\circ = |\nabla V| dr_{||} \quad (5.41)$$

where the subscript  $||$  has been placed on  $d\mathbf{r}$  to remind us that this is the special case of  $d\mathbf{r}$  being parallel to  $\nabla V$ . The maximum change of potential at the point  $P$  can be found from equation 5.41 as

$$\frac{dV}{dr_{||}} = |\nabla V| \quad (5.42)$$

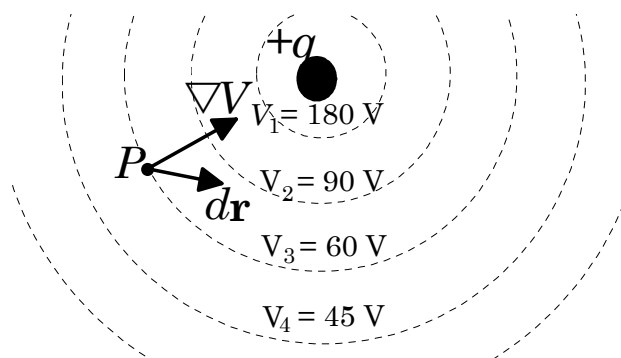


**Figure 5.13** The maximum change of potential occurs when the displacement  $d\mathbf{r}$  is in the direction of the gradient, which means the displacement is perpendicular to the equipotential line.

While the magnitude of the gradient can be found in the same way that you obtain the magnitude of any vector, namely

$$|\nabla V| = \sqrt{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2} \quad (5.43)$$

For the general case of a displacement  $d\mathbf{r}$  in any direction from the point  $P$ , figure 5.14, the change in potential  $dV$  is obtained from equation 5.40.



**Figure 5.14** An arbitrary displacement  $d\mathbf{r}$ .

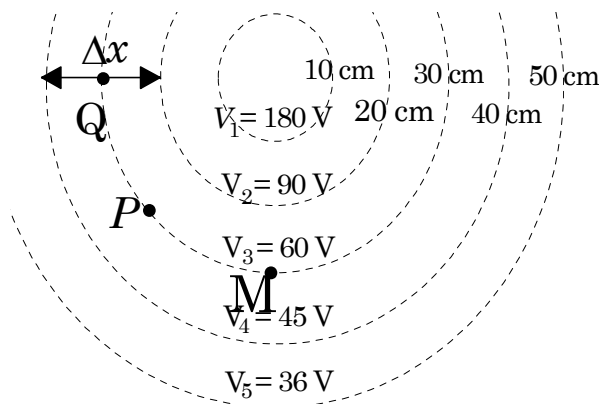
Let us summarize the characteristics of the gradient of the potential field:

- 1) The gradient  $\nabla V$  is perpendicular to the lines of constant potential  $V$ .
- 2) The gradient  $\nabla V$  points in the direction in which the potential function  $V$  increases most rapidly.
- 3) The magnitude of  $\nabla V$  is equal to the rate of change of  $V$  with respect to distance in the direction of  $\nabla V$ .
- 4) Once the gradient is known, the electric field is determined from the equation

$$\mathbf{E} = -\nabla V$$

### Example 5.8

A  $75.0 \mu\text{C}$  charge is located at the point  $Q$ , which is on the  $60 \text{ V}$  equipotential line in the symmetrical potential distribution of figure 5.15. (a) Find the value of  $\Delta x$



**Figure 5.15**

between the two equipotentials shown. (b) Find the gradient between the equipotential lines. (c) Find the value of the electric field  $\mathbf{E}$  at the point  $Q$ . (d) If the charge is moved  $1.50 \text{ cm}$  in the direction of the gradient, by how much will the potential change? (e) If the charge is moved a distance of  $1.50 \text{ cm}$  at an angle of  $35.0^\circ$  to the

direction of the gradient, by how much will the potential change? (f) If the charge is moved along the 60 V equipotential line, by how much will the potential change? (g) What is the change in potential energy of the 75.0  $\mu\text{C}$  charge if it is moved from the 60 V equipotential to the 90 V equipotential in the direction of the gradient?

### ***Solution***

(a) the value of  $\Delta x$  between the two equipotentials shown is obtained from the figure as

$$\begin{aligned}\Delta x &= 40.0 \text{ cm} - 20.0 \text{ cm} \\ \Delta x &= 20.0 \text{ cm}\end{aligned}$$

(b) The gradient is obtained from a modification of equation 1-66 as

$$\begin{aligned}\nabla V &= \mathbf{i} \frac{\partial V}{\partial x} = \mathbf{i} \frac{\Delta V}{\Delta x} = \mathbf{i} \frac{(90 \text{ V} - 45 \text{ V})}{20.0 \text{ cm}} \\ \nabla V &= (2.25 \text{ V/cm})\mathbf{i}\end{aligned}$$

That is, if a charge is at the point  $Q$  in figure 5.15, and is moved in the  $x$ -direction, its potential will increase by 2.25 V for every cm the charge is moved in the  $x$ -direction. Notice that the gradient is in the  $x$ -direction, pointing toward the highest region of potentials. If the charge were located at the point  $M$  in figure 5.15, the gradient would be  $\nabla V = \mathbf{j} 2.25 \text{ V/cm}$ . This means that the gradient would now point in the  $y$ -direction. At some other point between  $Q$  and  $M$ , such as  $P$ , the gradient would have an  $x$  and  $y$  component. However, wherever you are located, the gradient always points toward the center of the circle in figure 5.15.

(c) The electric field  $\mathbf{E}$  is found as the negative of the gradient, i.e.,

$$\begin{aligned}\mathbf{E} &= -\nabla V \\ \mathbf{E} &= -(2.25 \text{ V/cm})\mathbf{i} = -(225 \text{ V/m})\mathbf{i}\end{aligned}$$

Notice that in this problem we did not specify the charge that caused the potential distribution nor its electric field. The important thing that concerns us here is the potential distribution. Once *the potential field is known, without any knowledge about either the charge distribution that created the field, or the field itself, the electric field can still be computed from*  $\mathbf{E} = -\nabla V$ .

(d) To determine how much the potential changes when the charge is moved 1.50 cm in the direction of the gradient, we use equation 5.38 as

$$\begin{aligned}dV &= \nabla V \cdot d\mathbf{r} = (\nabla V)(\Delta r)\cos 0^\circ \\ dV &= (2.25 \text{ V/cm})(1.5 \text{ cm}) \\ dV &= 3.38 \text{ V}\end{aligned}$$

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(e) To determine the change in the potential when the charge is moved a distance of 1.50 cm at an angle of  $35.0^\circ$  to the direction of the gradient, we use equation 5.40 as

$$\begin{aligned}dV &= \nabla V \cdot dr = |\nabla V|(\Delta r)\cos\theta \\dV &= (2.25 \text{ V/cm})(1.5 \text{ cm})\cos 35.0^\circ \\dV &= 2.76 \text{ V}\end{aligned}$$

(f) If the charge is moved along the 60 V equipotential line, the potential of that charge does not change at all, ( $dV = 0$ ) by virtue of the definition of the equipotential line as a line along which the potential is a constant.

(g) The change in potential energy of a  $75.0 \mu\text{C}$  charge as it is moved from the 60 V equipotential to the 90 V equipotential is found as

$$\begin{aligned}\Delta(\text{PE}) &= \text{PE}_{90} - \text{PE}_{60} = qV_{90} - qV_{60} \\ \Delta(\text{PE}) &= q(V_{90} - V_{60}) \\ \Delta(\text{PE}) &= (75.0 \times 10^{-6} \text{ C})(90 \text{ V} - 60 \text{ V}) \\ \Delta(\text{PE}) &= 2.25 \times 10^{-3} \text{ J}\end{aligned}$$

### Example 5.9

*Determining the electric field from a potential distribution.* If a potential field  $V$  is given by

$$V = \frac{C}{\sqrt{x^2 + y^2 + z^2}}$$

where  $C$  is a constant and  $x, y, z$  are coordinates, find the electric field  $\mathbf{E}$  associated with this potential distribution.

### Solution

The electric field  $\mathbf{E}$  is equal to the negative of the gradient of  $V$ . That is

$$\mathbf{E} = -\nabla V = -\left(\mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial z}\right)$$

where the partial derivatives are determined as

$$\begin{aligned}\frac{\partial V}{\partial x} &= \frac{\partial}{\partial x} \left( \frac{C}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \right) = \frac{\partial}{\partial x} C(x^2 + y^2 + z^2)^{-\frac{1}{2}} \\ \frac{\partial V}{\partial x} &= C\left(-\frac{1}{2}\right)(x^2 + y^2 + z^2)^{-\frac{3}{2}}(2x)\end{aligned}$$

## Chapter 5: The Electric Potential

$$\frac{\partial V}{\partial x} = \frac{-Cx}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

The partial derivatives of  $V$  with respect to  $y$  and  $z$  are found similarly and are given by

$$\begin{aligned}\frac{\partial V}{\partial y} &= \frac{-Cy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \\ \frac{\partial V}{\partial z} &= \frac{-Cz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\end{aligned}$$

The gradient of  $V$  is now found as

$$\begin{aligned}\nabla V &= \mathbf{i} \frac{-Cx}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \mathbf{j} \frac{-Cy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \mathbf{k} \frac{-Cz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \\ \nabla V &= \frac{-C(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\end{aligned}$$

But

$$(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) = \mathbf{r}$$

is the displacement vector and

$$(x^2 + y^2 + z^2)^{1/2} = r$$

is the magnitude of the displacement vector. Therefore the gradient of  $V$  becomes

$$\nabla V = \frac{-C\mathbf{r}}{r^3}$$

The value of the electric field is now found as

$$\begin{aligned}\mathbf{E} &= -\nabla V \\ \mathbf{E} &= -\nabla V = \frac{C\mathbf{r}}{r^3}\end{aligned}$$

Does this electric field look familiar to you? What do you think it might be?

## 5.6 Superposition of Potentials for Multiple Discrete Charges

The principle of the **superposition of potentials** is stated: If there are a number of point charges present, the total potential at any arbitrary point is the sum of the potentials for each point charge. That is,

$$V = V_1 + V_2 + V_3 + \dots \quad (5.44)$$



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This is, of course, the same superposition principle encountered in section 3.3 for the superposition of electric fields. However, *because the potentials are scalar quantities they add according to the rules of ordinary arithmetic*. Recall that the superposition of the electric field for a number of point charges consisted in the process of vector addition. Thus, the computation of the total potential of a number of point charges is much simpler than the computation of the vector resultant of the electric field of a number of point charges by the superposition principle.

### Example 5.10

*Superposition of potentials.* Find the potential at point A in the diagram if  $q_1 = 2.00 \mu\text{C}$ ,  $q_2 = -6.00 \mu\text{C}$ , and  $q_3 = 8.00 \mu\text{C}$ .

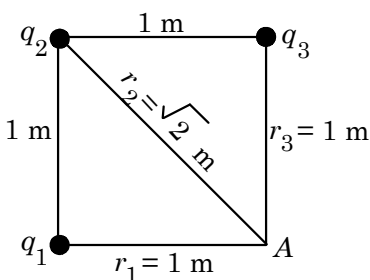


Diagram for example 5.10

### Solution

By the superposition principle the potential at point A is

$$V = V_1 + V_2 + V_3$$

$$V = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} + \frac{kq_3}{r_3}$$

$$V = \frac{(9.00 \times 10^9 \text{ N m}^2/\text{C}^2)(2.00 \times 10^{-6} \text{ C})}{1.00 \text{ m}} - \frac{(9.00 \times 10^9 \text{ N m}^2/\text{C}^2)(6.00 \times 10^{-6} \text{ C})}{\sqrt{2.00} \text{ m}} + \frac{(9.00 \times 10^9 \text{ N m}^2/\text{C}^2)(8.00 \times 10^{-6} \text{ C})}{1.00 \text{ m}}$$

$$V = 1.80 \times 10^4 \frac{\text{N m}}{\text{C}} - 3.82 \times 10^4 \frac{\text{J}}{\text{C}} + 7.20 \times 10^4 \text{ V}$$

$$V = 5.18 \times 10^4 \text{ V}$$

Notice that the second term in the computation was negative because  $q_2$  was negative.

[To go to this Interactive Example click on this sentence.](#)

### Example 5.11

*Work done in moving a charge from infinity.* How much work is required to bring a charge of  $q = 3.50 \mu\text{C}$  from infinity to point  $A$  in example 5.10?

### Solution

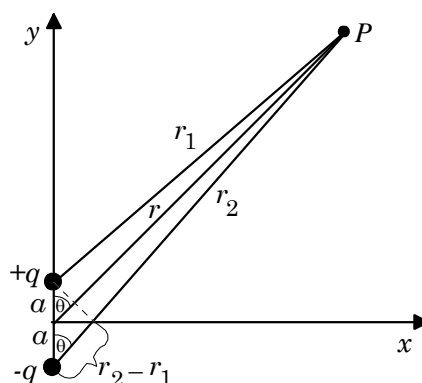
Recall from the definition of the potential that the potential is equal to the potential energy per unit charge or the work done per unit charge. Thus, the work done is equal to the charge multiplied by the potential. The zero of potential for a point charge was taken at infinity, so if a point charge is moved from infinity to a point such as  $A$ , the work done is simply the charge  $q$  multiplied by the potential at  $A$ , that is,

$$W = qV = (3.50 \times 10^{-6} \text{ C})(5.18 \times 10^4 \text{ V}) = 0.181 \text{ J}$$

[To go to this Interactive Example click on this sentence.](#)

## 5.7 The Potential of an Electric Dipole

As an example of the superposition of potentials for multiple discrete charges let us find the potential at an arbitrary point  $P$  in the field of an electric dipole as shown in figure 5.16. Since an electric dipole is a combination of a  $+q$  charge and a  $-q$



**Figure 5.16** Finding the potential of an electric dipole.

charge, the potential at  $P$  will be the sum of the potentials of each point charge. That is,

## Chapter 5: The Electric Potential

$$\begin{aligned}
 V &= V_1 + V_2 \\
 V &= \frac{kq}{r_1} + \frac{k(-q)}{r_2} \\
 V &= k\left(\frac{q}{r_1} - \frac{q}{r_2}\right) = kq\left(\frac{1}{r_1} - \frac{1}{r_2}\right)
 \end{aligned} \tag{5.45}$$

$$V = kq\left(\frac{r_2 - r_1}{r_1 r_2}\right) \tag{5.46}$$

But as can be seen in figure 5.16

$$(r_2 - r_1) \approx 2a \cos \theta \tag{5.47}$$

and

$$r_1 r_2 \approx r^2 \tag{5.48}$$

Replacing equations 5.47 and 5.48 into equation 5.46 yields

$$V = kq\left(\frac{2a \cos \theta}{r^2}\right)$$

or

$$V = \frac{k2aq \cos \theta}{r^2} \tag{5.49}$$

But the quantity  $2aq$  was defined in equation 3-15 to be the electric dipole moment  $p$ . That is,

$$p = 2aq \tag{3-15}$$

Replacing equation 3-15 into equation 5.49 gives

$$V = \frac{kp \cos \theta}{r^2} \tag{5.50}$$

*Equation 5.50 gives the electric potential for an electric dipole at any arbitrary point P. Notice that in equation 5.50 the potential is given in terms of the polar coordinates  $r$  and  $\theta$ .*

The tremendous advantage of determining the potential for a particular charge distribution is that the potential is a scalar quantity and is relatively easy to determine. The electric field for the electric dipole can now be found from equation 5.17 as

$$\mathbf{E} = -\nabla V \tag{5.17}$$

Up to now, we have been using the del operator in the rectangular coordinates  $x$ ,  $y$ , and  $z$ . In this problem we used the polar coordinates  $r$  and  $\theta$ , so it is necessary to express the del operator in terms of polar coordinates. The del operator in polar coordinates is given by

$$\nabla = \mathbf{r}_o \frac{\partial}{\partial r} + \boldsymbol{\theta}_o \frac{1}{r} \frac{\partial}{\partial \theta} \tag{5.51}$$

## Chapter 5: The Electric Potential

where  $\mathbf{r}_o$  is a unit vector in the  $r$ -direction, and  $\boldsymbol{\theta}_o$  is a unit vector in the direction of increasing  $\theta$ . Applying the del operator from equation 5.51 to the potential function  $V$  yields

$$\nabla V = \mathbf{r}_o \frac{\partial V}{\partial r} + \boldsymbol{\theta}_o \frac{1}{r} \frac{\partial V}{\partial \theta} \quad (5.52)$$

and the electric field now becomes

$$\mathbf{E} = -\nabla V = -\mathbf{r}_o \frac{\partial V}{\partial r} - \boldsymbol{\theta}_o \frac{1}{r} \frac{\partial V}{\partial \theta} \quad (5.53)$$

The value of the electric field at the arbitrary point  $P$  is now found from equation 5.53. Let us start by first evaluating the partial derivatives of  $V$  from equation 5.50. Thus,

$$\begin{aligned} \frac{\partial V}{\partial r} &= kp \cos \theta \frac{\partial(r^{-2})}{\partial r} = kp \cos \theta (-2r^{-3}) \\ \frac{\partial V}{\partial r} &= \frac{-2kp \cos \theta}{r^3} \end{aligned} \quad (5.54)$$

and

$$\begin{aligned} \frac{1}{r} \frac{\partial V}{\partial \theta} &= \frac{1}{r} \frac{kp}{r^2} \frac{\partial(\cos \theta)}{\partial \theta} = \frac{kp}{r^3} (-\sin \theta) \\ \frac{1}{r} \frac{\partial V}{\partial \theta} &= \frac{-kp \sin \theta}{r^3} \end{aligned} \quad (5.55)$$

Replacing equations 5.54 and 5.55 into equation 5.53 gives

$$\begin{aligned} \mathbf{E} = -\nabla V &= -\mathbf{r}_o \frac{\partial V}{\partial r} - \boldsymbol{\theta}_o \frac{1}{r} \frac{\partial V}{\partial \theta} = -\mathbf{r}_o \frac{-2kp \cos \theta}{r^3} - \boldsymbol{\theta}_o \frac{-kp \sin \theta}{r^3} \\ \mathbf{E} &= \frac{2kp \cos \theta}{r^3} \mathbf{r}_o + \frac{kp \sin \theta}{r^3} \boldsymbol{\theta}_o \end{aligned} \quad (5.56)$$

Equation 5.56 gives the electric field of an electric dipole at any arbitrary point  $P$ .

### Example 5.12

Using the general result for the electric field of an electric dipole, equation 5.56, show that it reduces to the special case of the electric field along the perpendicular bisector of an electric dipole studied in section 3.4, and in particular gives the same result as in equation 3-17.

### Solution

Finding the field along the perpendicular bisector is equivalent to letting  $\theta = 90^\circ$  in equation 5.56 (remember that  $\theta$  was measured from the  $z$ -axis). Also, with  $\theta = 90^\circ$ ,

this is equivalent to having  $r$  lie on the  $x$ -axis. Thus,  $y = 0$ , and  $r = x$ ,  $\cos 90^\circ = 0$ , and  $\sin 90^\circ = 1$ . Replacing these values into equation 5.56 yields

$$\begin{aligned} \mathbf{E} &= \frac{2kp \cos \theta}{r^3} \mathbf{r}_o + \frac{kp \sin \theta}{r^3} \boldsymbol{\theta}_o \\ \mathbf{E} &= 0 + \frac{kp}{r^3} \boldsymbol{\theta}_o = \frac{kp}{x^3} \boldsymbol{\theta}_o \end{aligned}$$

But when the point  $P$  is on the  $x$ -axis,  $\boldsymbol{\theta}_o$  point upward in the positive  $\mathbf{j}$  direction. Therefore the electric field becomes

$$\mathbf{E} = \frac{kp}{x^3} \mathbf{j}$$

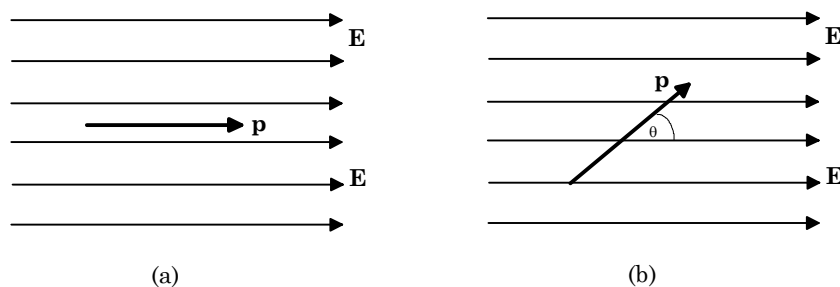
which is the same result obtained in equation 3-17. Notice how equation 5.56 is so much more general, in that it gives you the electric field at any point and yet it is much easier to deal with than the vector addition necessary in chapter 3.

## 5.8 The Potential Energy of an Electric Dipole in an External Electric Field

In section 3.5 we saw that when an electric dipole  $p$  is placed in an external electric field  $\mathbf{E}$ , it experiences a torque given by

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} \quad (3-24)$$

This torque acts to rotate the dipole until it is aligned with the external electric field. Because the natural position of  $\mathbf{p}$  is parallel to the field, as shown in figure 5.17(a), work must be done to rotate  $\mathbf{p}$  in the external electric field. When work



**Figure 5.17** An electric dipole in an external electric field  $\mathbf{E}$ .

was done in lifting a rock in a gravitational field, the rock then possessed potential energy. In the same way, since work must be done by an external agent to change the orientation of the dipole, the work done in rotating the dipole in the electric field shows up as potential energy of the dipole, figure 5.17(b). That is, the dipole now

## Chapter 5: The Electric Potential

possesses an additional potential energy associated with the work done in rotating  $\mathbf{p}$ .

The potential energy of the dipole in an external electric field  $\mathbf{E}$  is found by computing the work that must be done to rotate the dipole in the external electric field. That is,

$$\text{PE} = W = \int dW \quad (5.57)$$

Just as the element of work  $dW = \mathbf{F} \cdot d\mathbf{s}$  for translational motion, the element of work for rotational motion is given by

$$dW = \boldsymbol{\tau} \cdot d\boldsymbol{\theta} \quad (5.58)$$

where  $\boldsymbol{\tau}$  is the torque acting on the dipole to cause it to rotate and  $d\boldsymbol{\theta}$  is the element of angle turned through. Both the torque vector  $\boldsymbol{\tau}$  and the element of angle vector are perpendicular to the plane of the paper, and hence the angle between the two vectors are zero and their dot product is simply  $\tau d\theta$ . The increased potential energy becomes

$$\text{PE} = \int dW = \int \tau d\theta \quad (5.59)$$

The magnitude of the torque is found from equation 3-24 as

$$\tau = pE \sin\theta \quad (5.60)$$

Replacing equation 5.60 into equation 5.59 gives

$$\text{PE} = \int_{\theta_0}^{\theta} \tau d\theta = \int_{90^\circ}^{\theta} pE \sin\theta d\theta \quad (5.61)$$

We have made the lower limit of integration  $\theta_0$  to be  $90^\circ$ , and  $\theta$  the upper limit. Equation 5.61 becomes

$$\begin{aligned} \text{PE} &= -pE \cos\theta \Big|_{90^\circ}^{\theta} \\ \text{PE} &= -pE \cos\theta \end{aligned} \quad (5.62)$$

Noticing the form of this equation, we can write it more generally as

$$\text{PE} = -\mathbf{p} \cdot \mathbf{E} \quad (5.63)$$

*Equation 5.63 gives the potential energy of an electric dipole in an external electric field.*

**Example 5.13**

*The potential energy of an electric dipole.* Find the potential energy of an electric dipole in an external field when (a) it is antiparallel to  $\mathbf{E}$  (i.e.,  $\theta = 180^\circ$ ), (b) it is perpendicular to  $\mathbf{E}$  (i.e.,  $\theta = 90^\circ$ ), and (c) it is aligned with  $\mathbf{E}$  (i.e.,  $\theta = 0^\circ$ ).

**Solution**

The potential energy of the dipole, found from equation 5.62, is

(a)

$$PE = -pE \cos 180^\circ$$

$$PE = +pE$$

(b)

$$PE = -pE \cos 90^\circ$$

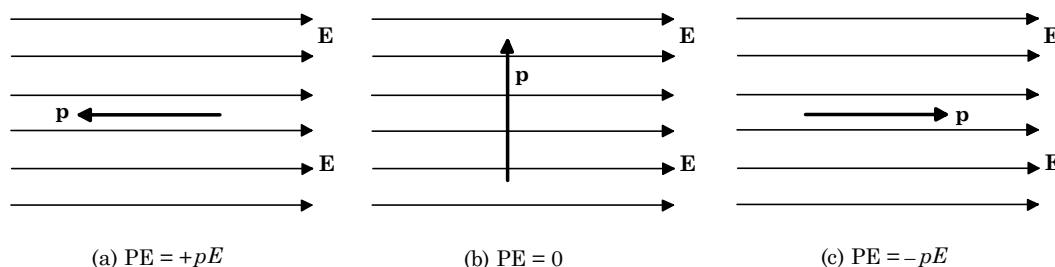
$$PE = 0$$

(c)

$$PE = -pE \cos 0^\circ$$

$$PE = -pE$$

Thus, the dipole has its highest potential energy when it is antiparallel ( $180^\circ$ ), decreases to zero when it is perpendicular ( $90^\circ$ ), and decreases to its lowest potential energy, a negative value, when it is aligned with the electric field,  $\theta = 0^\circ$ . This is shown in figure 5.18. So, just as the rock falls from a position of high potential energy to the ground where it has its lowest potential energy, the dipole, if given a slight push to get it started, rotates from its highest potential energy (antiparallel) to its lowest potential energy (parallel).



**Figure 5.18** Potential energy of an electric dipole in an external electric field.

## 5.9 The Potential for a Continuous Distribution of Charge

As we saw in section 5.6, when there are multiple discrete charges in a region, the electric potential produced by those charges at any point is found by the algebraic

## Chapter 5: The Electric Potential

sum of the potential fields associated with each of the charges. That is, the resultant potential for a group of point charges was given by equation 5.44 as

$$V = V_1 + V_2 + V_3 + \dots \quad (5.44)$$

Equation 5.44 can be written in the shorthand notation as

$$V = \sum_{i=1}^N V_i \quad (5.64)$$

where, again,  $\Sigma$  means “the sum of” and the sum goes from  $i = 1$  to  $i = N$ , the total number of charges present.

If the charge distribution is a continuous one, the field it sets up at any point  $P$  can be computed by dividing the continuous distribution of charge into a large number of infinitesimal elements of charge  $dq$ . Each element of charge  $dq$  acts like a point charge and since the potential of a point charge is  $V = kq/r$ , the element of charge  $dq$  will produce an element of the electric potential  $dV$  at the point  $P$ , given by

$$dV = k \frac{dq}{r} \quad (5.65)$$

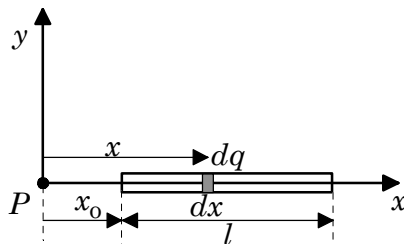
where  $r$  is the distance from the element of charge  $dq$  to the field point  $P$ . The total electric potential  $V$  at the point  $P$  caused by the potential fields from the entire distribution of all the  $dq$ 's is again a sum, but since the elements of charge  $dq$  are infinitesimal, the sum becomes the integral of all the elements of potential  $dV$ . That is, *the total potential of a continuous distribution of charge is found as*

$$V = \int dV = \int k \frac{dq}{r} \quad (5.66)$$

We will now look at some specific examples of the electric potential caused by continuous charge distributions.

### 5.10 The Potential on Axis for a Charged Rod

Let us find the potential at the point  $P$ , the origin of our coordinate system in figure 5.19, for a rod of charge that lies along the  $x$ -axis. The charge  $q$  is distributed



**Figure 5.19** Potential on axis for a rod of charge.



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uniformly over the rod. We divide the rod up into small elements of charge  $dq$  as shown. Each of these elements of charge will produce an element of electric potential  $dV$ . The element of charge  $dq$  located at the position  $x$  will produce the element of potential  $dV$  given by

$$dV = k \frac{dq}{x} \quad (5.67)$$

The total potential at the point  $P$  is the sum or integral of each of these  $dV$ 's and is given by equation 5.66 as

$$V = \int dV = \int k \frac{dq}{r} \quad (5.66)$$

$$V = \int dV = \int k \frac{dq}{x} \quad (5.68)$$

The *linear charge density*  $\lambda$  is defined as the charge per unit length and can be written for the rod as

$$\lambda = q/x$$

The total charge on the rod can now be written as

$$q = \lambda x$$

and its differential by

$$dq = \lambda dx \quad (5.69)$$

Substituting equation 5.69 back into equation 5.68 gives

$$V = \int k \frac{dq}{x} = \int k \frac{\lambda dx}{x}$$

The integration is over  $x$  and as can be seen from the figure, the limits of integration will go from  $x_o$  to  $x_o + l$ , where  $l$  is the length of the rod. That is,

$$V = \int_{x_o}^{x_o+l} k\lambda \frac{dx}{x}$$

But  $k$  is a constant and  $\lambda$ , the charge per unit length, is also a constant because the charge is distributed uniformly over the rod and can be taken outside of the integral. Therefore,

$$\begin{aligned} V &= k\lambda \int_{x_o}^{x_o+l} \frac{dx}{x} \\ V &= k\lambda [\ln x]_{x_o}^{x_o+l} \\ V &= k\lambda [\ln(x_o + l) - \ln x_o] \end{aligned}$$

Hence, the potential at the point  $P$  is found to be

$$V = k\lambda \ln \left[ \frac{x_o + l}{x_o} \right] \quad (5.70)$$

If we prefer we can write the potential in terms of the total charge  $q$  on the rod instead of the linear charge density  $\lambda$ , since  $\lambda = q/l$ . That is, the total potential at the point  $P$  caused by the rod of charge is also given by

$$V = k \frac{q}{l} \ln \left[ \frac{x_o + l}{x_o} \right] \quad (5.71)$$

### Example 5.14

*The potential of a rod of charge.* A rod of uniform charge density of  $200 \mu\text{C/m}$  is located on the  $x$  axis at  $x_o = 10.0 \text{ cm}$ . The rod has a length of  $12.7 \text{ cm}$ . Find the potential at the origin

### Solution

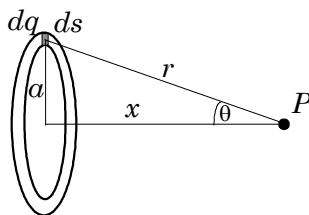
The electric potential at the point  $P$  is found from equation 5.70 as

$$\begin{aligned} V &= k\lambda \ln \left[ \frac{x_o + l}{x_o} \right] \\ V &= (9.00 \times 10^9 \text{ N m}^2/\text{C}^2)(200 \times 10^{-6} \text{ C/m}) \ln \left[ \frac{(0.10 \text{ m} + 0.127 \text{ m})}{(0.10 \text{ m})} \right] \\ V &= 1.48 \times 10^6 \text{ V} \end{aligned}$$

**To go to this Interactive Example click on this sentence.**

## 5.11 The Potential on Axis for a Ring of Charge

Let us determine the potential at the point  $P$ , a distance  $x$  from the center of a ring of charge of radius “ $a$ ” as shown in figure 5.20. We will assume that the charge is



**Figure 5.20** The electric potential for a ring of charge.

distributed uniformly along the ring, and the ring contains a total charge  $q$ . The charge per unit length of the ring,  $\lambda$ , is defined as

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$$\lambda = q/s \quad (5.72)$$

where  $s$  is the entire length or arc of the ring (circumference). Let us now consider a small element  $ds$  at the top of the ring that contains a small element of charge  $dq$ . The total charge contained in this element  $dq$  is found from equation 5.72 as

$$q = \lambda s \quad (5.73)$$

Hence,

$$dq = \lambda ds \quad (5.74)$$

This element of charge  $dq$  can be considered as a point charge and it sets up a differential potential  $dV$  given by

$$dV = k \frac{dq}{r} \quad (5.75)$$

and is shown in figure 5.20. The total potential at the point  $P$  is obtained by adding up, integrating, all the small element  $dV$ 's caused by all the  $dq$ 's. That is,

$$V = \int dV \quad (5.76)$$

Replacing equation 5.75 into equation 5.76 we get

$$V = \int dV = \int k \frac{dq}{r} \quad (5.77)$$

Replacing equation 5.74 into equation 5.77 yields

$$V = \int k \frac{dq}{r} = \int k \frac{\lambda ds}{r}$$

But  $k$ ,  $\lambda$ , and  $r$  are constants and can be taken outside of the integral. Thus,

$$V = \int k \frac{\lambda ds}{r} = \frac{k\lambda}{r} \int ds \quad (5.78)$$

The integration is over  $ds$  which is an element of arc of the ring. But the sum of all the  $ds$ 's of the ring is just the circumference of the ring itself, that is,

$$\int ds = 2\pi a \quad (5.79)$$

Replacing equation 5.79 into equation 5.78 gives

$$V = \frac{k\lambda}{r} (2\pi a) \quad (5.80)$$

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But the charge per unit length,  $\lambda = q/s = q/(2\pi a)$  from equation 5.72, and as can be seen in figure 5.20,  $r = \sqrt{a^2 + x^2}$ , hence substituting these back into equation 5.80 we get

$$V = \frac{k\lambda}{r}(2\pi a) = \frac{k}{\sqrt{a^2 + x^2}} \left( \frac{q}{2\pi a} \right) (2\pi a)$$

Therefore the potential at the point  $x$  due to a ring of charge of radius “ $a$ ” carrying a total charge  $q$  is given by

$$V = \frac{kq}{\sqrt{a^2 + x^2}} \quad (5.81)$$

### Example 5.15

*Electric field of a ring of charge from the potential of the ring of charge.* Starting with equation 5.81, find the electric field on axis for a ring of charge.

### Solution

The electric field is found from equation 5.17 as

$$\mathbf{E} = -\nabla V = -\left(\mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial z}\right)$$

where  $V$  is

$$V = \frac{kq}{\sqrt{a^2 + x^2}}$$

Notice that  $V$  is only a function of  $x$  and hence the electric field is found as

$$\begin{aligned} \mathbf{E} &= -\nabla V = -\mathbf{i}\frac{\partial V}{\partial x} = -\frac{dV}{dx}\mathbf{i} \\ \mathbf{E} &= -\frac{d}{dx}\left(\frac{kq}{\sqrt{a^2 + x^2}}\right)\mathbf{i} = -kq\frac{d}{dx}\left((a^2 + x^2)^{-1/2}\right)\mathbf{i} = -kq(-1/2)((a^2 + x^2)^{-3/2})(2x)\mathbf{i} \end{aligned}$$

and the electric field on axis becomes

$$\mathbf{E} = \frac{kqx}{(a^2 + x^2)^{3/2}}\mathbf{i}$$

Notice that this is the same value of the electric field that we found in chapter 3, equation 3-48.

**Example 5.16**

*The electric potential at the center of a ring of charge.* Find the electric potential at the center of a ring of charge.

**Solution**

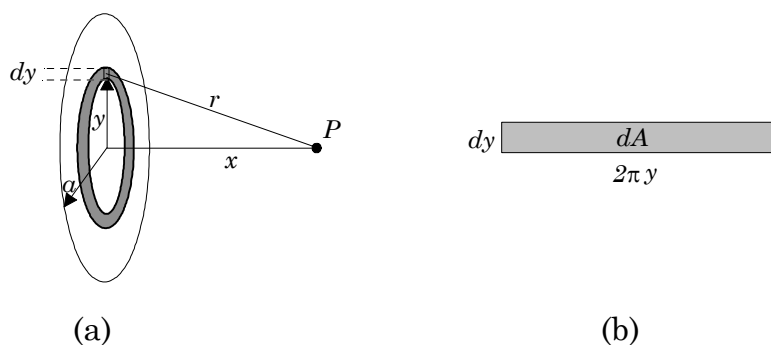
At the center of the ring  $x = 0$ . Therefore the potential at the center of a ring of charge is found from equation 5.81 with  $x$  set equal to zero. That is,

$$V = \frac{kq}{\sqrt{a^2 + x^2}} = \frac{kq}{\sqrt{a^2}} = \frac{kq}{a}$$

Notice that the potential at the center of the ring is a constant nonzero value. We found in example 3.8 that the electric field at the center of a ring of charge was zero.

**5.12 The Potential on Axis for a Disk of Charge**

Let us find the potential on axis at the point  $P$  in figure 5.21(a) for a uniform disk of charge. Since a disk can be generated by adding up many rings of different radii, *the potential of a disk of charge can be generated by adding up (integrating) the*



**Figure 5.21** The potential of a disk of charge.

*potentials of many rings of charge.* Thus the potential of a disk of charge will be given by

$$V_{\text{disk}} = \int dV_{\text{ring}} \quad (5.82)$$

We found in the last section that the potential on axis at the distance  $x$  from the center of a ring of charge of radius “ $a$ ” was given by

$$V = \frac{kq}{\sqrt{a^2 + x^2}} \quad (5.81)$$

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In the present problem the ring will be of radius  $y$  and we will add up all the rings from a radius of  $y = 0$  to the radius  $y = a$ , the radius of the disk. Hence, equation 5.81 will be written as

$$V_{ring} = \frac{kq}{\sqrt{y^2 + x^2}} \quad (5.83)$$

We now consider the charge on this ring to be a small element  $dq$  of the total charge that will be found on the disk. This element of charge  $dq$  will then produce an element of potential  $dV$  on the axis of the disk at the point  $P$ . That is,

$$dV_{ring} = \frac{k dq}{\sqrt{y^2 + x^2}} \quad (5.84)$$

Replacing equation 5.84 back into equation 5.82 for the potential of the disk we get

$$V_{disk} = \int dV_{ring} = \int \frac{k dq}{\sqrt{y^2 + x^2}} \quad (5.85)$$

When dealing with a rod of charge or a ring of charge which has the charge distributed along a line, we introduced the concept of the *linear charge density*  $\lambda$  as the *charge per unit length*. When dealing with a disk, the electric charge is distributed across a surface. Therefore, *the surface charge density*  $\sigma$  is defined as the *charge per unit area* and it is given by

$$\sigma = \frac{q}{A}$$

In terms of the surface charge density, the charge on the disk is given by

$$q = \sigma A$$

Its differential  $dq$ , is the amount of charge on the ring, i.e.,

$$dq = \sigma dA \quad (5.86)$$

where  $dA$  is the area of the ring. To determine the area of the ring, let us take the ring in figure 5.21a and unfold it as shown in figure 5.21b. The length of the ring is the circumference of the inner circle of the ring,  $2\pi y$ , and its width is the differential thickness of the ring,  $dy$ . The area of the ring  $dA$  is then given by the product of its length and width as

$$dA = (2\pi y) dy \quad (5.87)$$

Replacing equation 5.87 back into 5.86 gives for the element of charge of the ring

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$$dq = \sigma(2\pi y) dy \quad (5.88)$$

Replacing equation 5.88 back into equation 5.85 we get for the potential of the disk

$$V_{disk} = \int \frac{k dq}{\sqrt{y^2 + x^2}} = \int \frac{k \sigma 2\pi y dy}{\sqrt{y^2 + x^2}}$$

Taking the constants outside of the integral we get

$$\begin{aligned} V_{disk} &= k 2\pi \sigma \int \frac{y dy}{\sqrt{y^2 + x^2}} \\ V_{disk} &= \left( \frac{1}{4\pi \epsilon_o} \right) 2\pi \sigma \int \frac{y dy}{\sqrt{y^2 + x^2}} \\ V_{disk} &= \left( \frac{\sigma}{2\epsilon_o} \right) \int_0^a \frac{y dy}{\sqrt{y^2 + x^2}} \end{aligned} \quad (5.89)$$

Note that we have introduced the limits of integration 0 to  $a$ , that is, we add up all the rings from a radius of 0 to the radius “ $a$ ”, the radius of the disk. To determine the potential it is necessary to solve the integral

$$I = \int \frac{y dy}{\sqrt{y^2 + x^2}}$$

We could solve this directly by making the appropriate substitution, however let us simply the process by using the table of integrals in the appendix and find that

$$I = \int \frac{y dy}{\sqrt{y^2 + x^2}} = \sqrt{y^2 + x^2}$$

Hence

$$\begin{aligned} I &= \int_0^a \frac{y dy}{\sqrt{y^2 + x^2}} = \sqrt{y^2 + x^2} \Big|_0^a \\ I &= \int_0^a \frac{y dy}{\sqrt{y^2 + x^2}} = \sqrt{a^2 + x^2} - \sqrt{0 + x^2} \\ I &= \int_0^a \frac{y dy}{\sqrt{y^2 + x^2}} = \sqrt{a^2 + x^2} - x \end{aligned} \quad (5.90)$$

Replacing equation 5.90 back into equation 5.89 we obtain

$$\begin{aligned} V_{disk} &= \left( \frac{\sigma}{2\epsilon_o} \right) \int_0^a \frac{y dy}{\sqrt{y^2 + x^2}} = \left( \frac{\sigma}{2\epsilon_o} \right) [\sqrt{a^2 + x^2} - x] \\ V_{disk} &= \left( \frac{\sigma}{2\epsilon_o} \right) [\sqrt{a^2 + x^2} - x] \end{aligned} \quad (5.91)$$

Equation 5.91 gives the potential at the position  $x$  on the axis of a disk of charge of radius “ $a$ ”, carrying a uniform surface charge density  $\sigma$ .

### Example 5.17

*The potential of a disk of charge.* (a) Find the potential at the point  $x = 15.0$  cm in front of a disk of 10.0 cm radius carrying a uniform surface charge density of  $200 \mu\text{C}/\text{m}^2$ .

### Solution

(a) The potential of the disk of charge is found from equation 5.91 as

$$V_{\text{disk}} = \left( \frac{\sigma}{2\epsilon_0} \right) \left[ \sqrt{a^2 + x^2} - x \right] \quad (5.91)$$

$$V_{\text{disk}} = \left( \frac{200 \times 10^{-6} \text{ C}/\text{m}^2}{2(8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2)} \right) \left[ \sqrt{(0.100 \text{ m})^2 + (0.150 \text{ m})^2} - 0.150 \text{ m} \right]$$

$$V_{\text{disk}} = (1.13 \times 10^7 \text{ N/C}) [0.180 - 0.150]$$

$$V_{\text{disk}} = 3.42 \times 10^5 \text{ V}$$

**To go to this Interactive Example click on this sentence.**

## Summary of Important Concepts

**Electric potential** - The electric potential is defined as the potential energy per unit charge. It is measured in volts. The electric field intensity  $E$  is given by the derivative of the electric potential  $V$  with respect to the distance  $y$ .

**Equipotential line** - A line along which the electric potential is the same everywhere. The equipotential lines are everywhere perpendicular to the electric field lines.

**Equipotential surface** - A surface along which the electric potential is the same everywhere.

**Potential difference** - The difference in electric potential between two points.

**Superposition of potentials** - When there are several point charges present, the total potential at any arbitrary point is the algebraic sum of the potentials for each of the various point charges. Because the potentials are scalar quantities they add according to the rules of ordinary arithmetic.



**The gradient of the potential** - If  $V$  is the potential function, then  $\nabla V$  is called the gradient of  $V$ . The gradient is essentially a three dimensional slope of a surface and is directed upward towards the highest point of the surface. The quantity  $-\nabla V$  is directed downward towards the lowest point of the surface. The maximum change of  $V$  occurs when the displacement  $d\mathbf{r}$  is in the direction of the gradient. To summarize the characteristics of the gradient:

- 1) The gradient  $\nabla V$  is perpendicular to the lines of constant  $V$ .
- 2) The gradient  $\nabla V$  points in the direction in which the function  $V$  increases most rapidly.
- 3) The magnitude of  $\nabla V$  is equal to the rate of change of  $V$  with respect to distance in the direction of  $\nabla V$ .
- 4) the electric field  $\mathbf{E}$  is equal to the negative of the gradient of the potential  $V$ .

## Summary of Important Equations

The electric field  $\mathbf{E}$  is equal to the negative of the gradient of the potential  $V$ .

$$\mathbf{E} = -\nabla V \quad (5.17)$$

Rectangular coordinates  $\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right) \quad (5.16)$

Polar coordinates  $\mathbf{E} = -\nabla V = -\mathbf{r}_o \frac{\partial V}{\partial r} - \boldsymbol{\theta}_o \frac{1}{r} \frac{\partial V}{\partial \theta} \quad (5.53)$

The general equation for the difference in potential  $V_B - V_A$  between two points  $A$  and  $B$ , when both the magnitude and direction of the electric field varies along the path  $AB$ .

$$V_B - V_A = -\int \mathbf{E} \cdot d\mathbf{l} \quad (5.25)$$

The difference in potential between the point  $A$  and the point  $B$  in the electric field of a positive point charge.

$$V_B - V_A = kq\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \quad (5.32)$$

The electric potential  $V$  at any position  $r$  for a point charge  $q$

$$V = \frac{kq}{r} \quad (5.34)$$

Superposition of potentials

$$V = V_1 + V_2 + V_3 + \dots \quad (5.44)$$

The electric potential for an electric dipole at any arbitrary point

$$V = \frac{kp \cos \theta}{r^2} \quad (5.50)$$

## Chapter 5: The Electric Potential

The electric field of an electric dipole at any arbitrary point  $P$ .

$$\mathbf{E} = \frac{2kp \cos \theta}{r^3} \mathbf{r}_o + \frac{kp \sin \theta}{r^3} \boldsymbol{\theta}_o \quad (5.56)$$

Potential energy of an electric dipole in an external electric field.

$$\text{PE} = -\mathbf{p} \cdot \mathbf{E} \quad (5.63)$$

$$\text{PE} = -pE \cos \theta \quad (5.62)$$

The total potential of a continuous distribution of charge

$$V = \int dV = \int k \frac{dq}{r} \quad (5.66)$$

The potential at the point  $x$  due to a ring of charge of radius “ $a$ ”

$$V = \frac{kq}{\sqrt{a^2 + x^2}} \quad (5.81)$$

The potential of a disk of charge

$$V_{\text{disk}} = \int dV_{\text{ring}} \quad (5.82)$$

$$V_{\text{disk}} = \left( \frac{\sigma}{2\epsilon_0} \right) \left[ \sqrt{a^2 + x^2} - x \right] \quad (5.91)$$

## Questions for Chapter 5

1. If the electric potential is equal to zero at a point, must the electric field also be zero there?
2. Can two different equipotential lines ever cross?
3. If the electric potential is a constant, what does this say about the electric field?

## Problems for Chapter 5

1. Two charged parallel plates are separated by a distance of 2.00 cm. If the potential difference between the plates is 300 V, what is the value of the electric field between the plates?

2. A charge of 3.00 pC is placed at point  $A$  in the diagram. The electric field is 200 N/C downward. Find the work done in moving the charge along the path  $ABC$  and the work done in going from  $A$  to  $C$  directly.

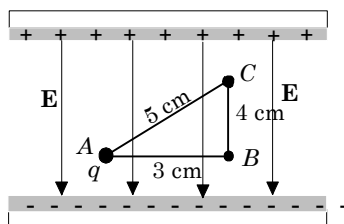


Diagram for problem 2.

## Chapter 5: The Electric Potential

3. A point charge of  $+2.00\ \mu\text{C}$  is  $30.0\ \text{cm}$  from a charge of  $+3.00\ \mu\text{C}$ . Where is the electric field between the charges equal to zero? What is the value of the potential there?

4. A charge of  $1.53 \times 10^{-8}\ \text{C}$  is placed at the origin of a coordinate system. (a) Find the potential at point  $A$  located on the  $x$ -axis at  $x = -5.00\ \text{cm}$ , and at point  $B$  located at  $x = 20.0\ \text{cm}$ . (b) Find the difference in potential between points  $A$  and  $B$ .

5. Repeat problem 4 but with point  $A$  located at the coordinates  $x = 0$  and  $y = -5.00\ \text{cm}$ .

6. How much work is done in moving a charge of  $3.00\ \mu\text{C}$  from a point where the potential is  $50.0\ \text{V}$  to another point where the potential is (a)  $150\ \text{V}$  and (b)  $-150\ \text{V}$ ?

7. Find the electric potential at point  $A$  in the diagram if (a)  $q_1 = 2.00\ \mu\text{C}$ , and  $q_2 = 3.00\ \mu\text{C}$ . and (b)  $q_1 = 2.00\ \mu\text{C}$  and  $q_2 = -3.00\ \mu\text{C}$ .

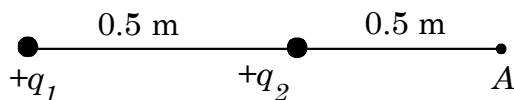


Diagram for problem 7.

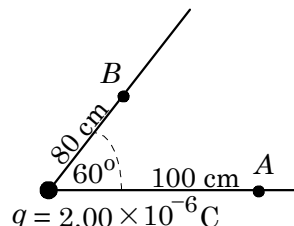


Diagram for problem 8.

8. (a) Find the potential at the points  $A$  and  $B$  shown in the diagram. (b) Find the potential difference between the points  $A$  and  $B$ . (c) Find the work required to move a charge of  $1.32\ \mu\text{C}$  from point  $A$  to point  $B$ . (d) Find the work required to move the same charge from point  $B$  to point  $A$ .

9. A point charge of  $2.00\ \mu\text{C}$  is  $30.0\ \text{cm}$  from a charge of  $-3.00\ \mu\text{C}$ . Where is the potential between the two charges equal to zero? How much work would be required to bring a charge of  $4.00\ \mu\text{C}$  to this point from infinity?

10. Find the potential at the apex of the equilateral triangle shown in the diagram if (a)  $q_1 = 2.00\ \mu\text{C}$  and  $q_2 = 3.00\ \mu\text{C}$  and (b) if  $q_1 = 2.00\ \mu\text{C}$  and  $q_2 = -3.00\ \mu\text{C}$ . (c) How much work is necessary to bring a charge of  $2.54\ \mu\text{C}$  from infinity to the point  $A$  in each case?

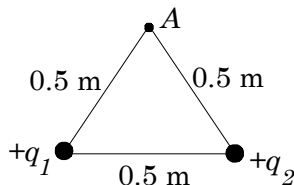


Diagram for Problem 10.

11. Electrons are located at the points  $(10.0\ \text{cm}, 0)$ ,  $(0, 10.0\ \text{cm})$ , and  $(10.0\ \text{cm}, 10.0\ \text{cm})$ . Find the value of the electric potential at the origin.

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12. Charges of  $2.00\ \mu\text{C}$ ,  $4.00\ \mu\text{C}$ ,  $-6.00\ \mu\text{C}$  and  $8.00\ \mu\text{C}$  are placed at the corner of a square of  $50.0\ \text{cm}$  length. Find the potential at the center of the square.

13. If a charge of  $2.00\ \mu\text{C}$  is separated by  $4.00\ \text{cm}$  from a charge of  $-2.00\ \mu\text{C}$  find the potential at a distance of  $5.00\ \text{m}$ , perpendicular to the axis of the dipole.

14. (a) Find the potential at point  $A$  in the diagram if charges  $q_1 = 2.63\ \mu\text{C}$  and  $q_2 = -2.63\ \mu\text{C}$ ,  $d = 10.0\ \text{cm}$ ,  $r_1 = 50.0\ \text{cm}$ ,  $r_2 = 42.2\ \text{cm}$ ,  $\theta_1 = 35.0^\circ$ , and  $\theta_2 = 42.8^\circ$ . (b) How much work is necessary to bring a charge of  $1.75\ \mu\text{C}$  from infinity to the point  $A$ ?

15. Find the potential at point  $A$  in the diagram if charge  $q_1 = 2.63\ \mu\text{C}$  and  $q_2 = -2.63\ \mu\text{C}$ ,  $d = 10.0\ \text{cm}$ ,  $r_1 = 50.0\ \text{cm}$ ,  $\theta_1 = 25.0^\circ$ . Hint: first find  $r_2$  by the law of cosines, then with  $r_2$  known, use the law of cosines again to find the angle  $\theta_2$ .

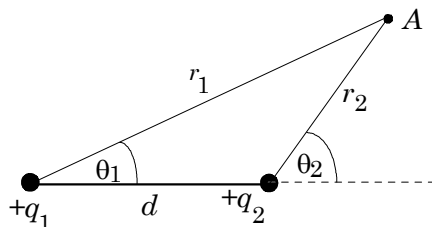


Diagram for problem 14, 15, and 16.

16. (a) Find the potential for an electric dipole at point  $A$  in the diagram for problem 14 if charges  $q_1 = 2.00\ \mu\text{C}$  and  $q_2 = -2.00\ \mu\text{C}$ ,  $d = 10.0\ \text{cm}$ ,  $r_1 = 50.0\ \text{cm}$ ,  $r_2 = 42.2\ \text{cm}$ ,  $\theta_1 = 35.0^\circ$ , and  $\theta_2 = 42.8^\circ$ . (b) Repeat part a with  $q_1$  now equal to  $6.00\ \mu\text{C}$ . Do you get the same result if you superimpose the potential field of the dipole of part a with the potential field of a point charge of  $4.00\ \mu\text{C}$  located at the same place as the original charge  $q_1$ ?

17. A point charge of  $2.00\ \mu\text{C}$  is  $30.0\ \text{cm}$  from a charge of  $3.00\ \mu\text{C}$ . Find the potential half way between the charges. How much work would be done in bringing a  $4.00\text{-}\mu\text{C}$  charge to this point from infinity?

18. In the Bohr theory of the hydrogen atom the electron circles the proton in a circular orbit of  $5.29 \times 10^{-11}\ \text{m}$  radius. Find the electric potential, produced by the proton, at this orbital radius. From the definition of the potential, determine the potential energy of the electron in this orbit.

19. Four  $2.50\ \mu\text{C}$  charges are located on the corners of a square  $1.00\ \text{m}$  on each side. How much work is required to move one of the charges to the center of the square?

20. How much work must be done to assemble four protons at the corners of a square of edge  $10.0\ \text{cm}$ ? (Assume that the protons start out very far apart.)

21. How much work is necessary to assemble three charges from infinity to each apex of the equilateral triangle of  $0.500\ \text{m}$  on a side shown in the diagram if  $q_1 = 3.00\ \mu\text{C}$ ,  $q_2 = 4.50\ \mu\text{C}$ , and  $q_3 = 6.53\ \mu\text{C}$ . Can you now talk about the potential energy of this charge configuration?

## Chapter 5: The Electric Potential

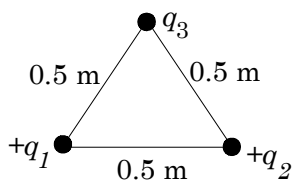


Diagram for problem 21.

22. Find the potential of a point charge of 3.00 pC at 10.0, 20.0, 25.0, 28.0, 29.0, 30.0, 31.0, 32.0, 35.0, 40.0, and 50.0 cm. Calculate the electric field at 30.0 cm by taking intervals of  $\Delta r$  from 40.0 cm down to 2.00 cm in the formula  $E = \Delta V / \Delta r$ . What is the value of  $E$  at 30.0 cm as computed by  $E = kq/r^2$ ?

23. Find the equation for the potential at the point  $P$  in figure 5-19 for a rod of charge that has a nonuniform linear charge density  $\lambda$  given by  $\lambda = Ax^2$ .

24. Find the equation for the potential at the point  $P$  for a ring of charge that has a *nonuniform* linear charge density  $\lambda$  given by  $\lambda = A \sin\phi$ , where  $\phi$  is the angle between the  $y$ -axis and the location of the element of charge  $dq$ , and  $A$  is a constant.

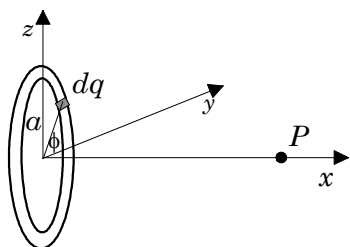


Diagram for problem 24.

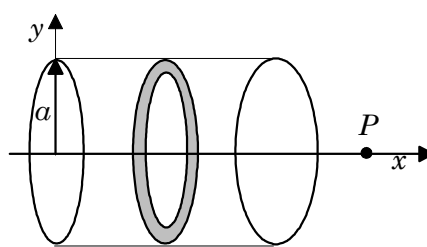


Diagram for problem 25.

25. Find the potential  $V$  on the  $x$ -axis at the point  $P$  of the cylindrical shell of radius  $a$  shown in the diagram. Hint: the cylindrical shell is made up of a sum of rings of charges.

26. A thin rod carrying a charge  $q$  spread uniformly along its length is bent into a semicircle of radius  $r$ . Find the potential at the center of the semicircle.

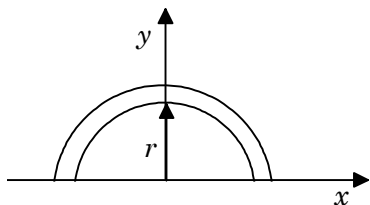


Diagram for problem 26.

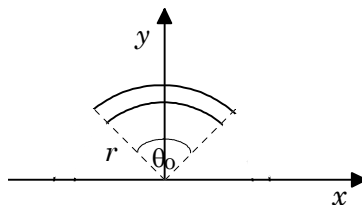


Diagram for problem 27.

27. A thin rod carrying a charge  $q$  spread uniformly along its length is bent into an arc of a circle of radius  $r$ . The arc subtends an angle  $\theta_0$ , as shown in the diagram. Find the potential at the center of the circle.

28. Find the point on the  $x$ -axis where the potential of a disk of charge takes on its maximum value.

## Chapter 5: The Electric Potential

29. Show that very far away from a disk of charge, the potential looks like the potential of a point charge. Hint: use the binomial theorem.

30. Starting with equation 5-91 for the potential on the  $x$ -axis for a uniform disk of charge, find the electric field on the  $x$ -axis.

31. Find the equation for the potential between two disks of charge. The first one carries the charge density  $+\sigma$  while the second carries the charge density  $-\sigma$ .

32. Starting with the equation for the potential of a uniform ring of charge, find the potential on the  $x$ -axis for a *nonuniform* disk of charge. The surface charge density on the rings vary linearly with the radius of the ring, i.e.,  $\sigma = Cy$  where  $y$  is the radius of each ring and  $C$  is a constant.

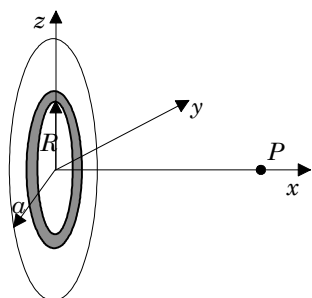


Diagram for problem 32.

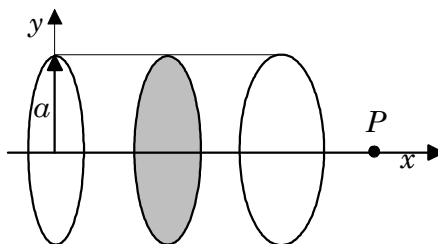


Diagram for problem 33.

33. Find the potential  $V$  on the  $x$ -axis at the point  $P$  of the solid cylinder of radius  $a$  shown in the diagram. Hint: the solid cylinder is made up of a sum of disks of charges.

34. Find the potential at the point  $P$  midway between a disk of charge carrying a surface charge density  $\sigma = 250 \mu\text{C}/\text{m}^2$  and a ring of charge carrying a total charge  $q = 5.60 \mu\text{C}$ . The radius of the disk is  $a_{\text{disk}} = 0.150 \text{ m}$ , and the radius of the ring is  $a_{\text{ring}} = 0.150 \text{ m}$ .

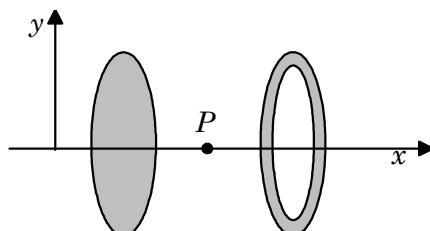


Diagram for problem 34.

35. Find the potential  $V$  at a point  $P$ , a distance  $x$  along the  $x$ -axis, for a uniform spherical distribution of volume charge density  $\rho$  of radius  $a$ . Hint: the element of volume charge density  $dv = 4\pi r^2 dr$ .

36. Find the potential  $V$  at a point  $P$ , a distance  $x$  along the  $x$ -axis, for a non-uniform spherical distribution of volume charge density  $\rho = A/r$  of radius  $R$ . Hint: the element of volume charge density  $dV = 4\pi r^2 dr$ .

37. Starting with equation 5-25 for the general relation for a potential difference, find the potential difference between two points  $A$  and  $B$  at distances  $r_A$  and  $r_B$

## Chapter 5: The Electric Potential

respectively from a line of charge of linear charge density  $\lambda$ . Hint: use Gauss's law first to determine  $\mathbf{E}$ .

38. Show that if

$$\mathbf{E} = -\nabla V$$

then

$$\nabla \times \mathbf{E} = 0$$

39. If  $\mathbf{E} = -\nabla V$ , find an equation for the divergence of  $\mathbf{E}$ ,  $\nabla \cdot \mathbf{E}$ .

40. Starting with the gradient in rectangular coordinates

$$\nabla V = \mathbf{i} \frac{\partial V}{\partial x} + \mathbf{j} \frac{\partial V}{\partial y}$$

show that this can be written in polar coordinates as

$$\nabla V = \mathbf{r}_o \frac{\partial V}{\partial r} + \boldsymbol{\theta}_o \frac{1}{r} \frac{\partial V}{\partial \theta}$$

Hint: from the relations between the rectangular coordinates and the polar coordinates

$$\begin{aligned} x &= r \cos \theta & r &= \sqrt{x^2 + y^2} \\ y &= r \sin \theta & \theta &= \tan^{-1} \frac{y}{x} \end{aligned}$$

Show that the rectangular partial derivative become

$$\begin{aligned} \frac{\partial V}{\partial x} &= \frac{\partial V}{\partial r} \cos \theta - \frac{\partial V}{\partial \theta} \frac{\sin \theta}{r} \\ \frac{\partial V}{\partial y} &= \frac{\partial V}{\partial r} \sin \theta + \frac{\partial V}{\partial \theta} \frac{\cos \theta}{r} \end{aligned}$$

and continue from there.

41. If  $V = kq/r$ , find  $\nabla \cdot \nabla V = \nabla^2 V$  which is called the Laplacian of  $V$ .

42. If the potential  $V$  in a certain region is given by

$$V = \frac{kq}{r} + \frac{kp \cos \theta}{r^2}$$

find the electric field  $\mathbf{E}$  there.

43. If the potential  $V$  in a certain region is given by

$$V = 2r^3 \cos \theta$$

find: (a) the electric field  $\mathbf{E}$  there, (b) the charge distribution  $\rho$  leading to this potential. Hint:  $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$

**To go to another chapter in the text, return to the table of contents by clicking on this sentence.**