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EE 5337

Computational Electromagnetics

Lecture #2

Maxwell's Equations

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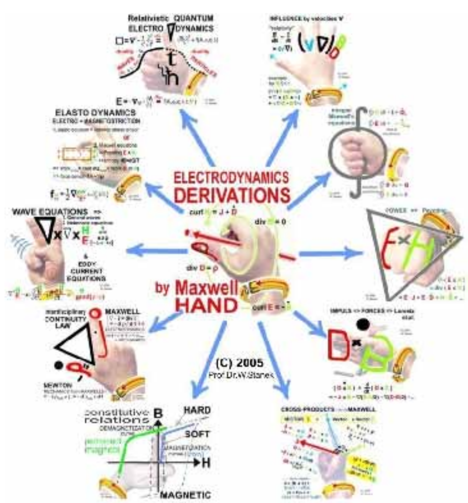
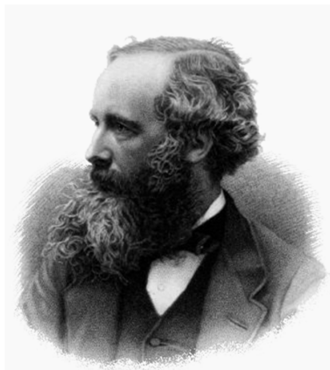
Outline

- Maxwell's equations
- Physical Boundary conditions
- Parameter relations
- Preparing Maxwell's equations for CEM
- The wave equation and its solutions
- Scaling properties of Maxwell's equations



Maxwell's Equations



James Clerk Maxwell

Born June 13,1831
Edinburgh, Scotland

Died November 5, 1879
Cambridge, England

Sign Conventions for Waves

To describe a wave propagating the positive z direction, we have two choices:

$$E(z,t) = A\cos(\omega t - kz)$$ Most common in engineering

$$E(z,t) = A\cos(-\omega t + kz)$$ Most common science and physics

Both are correct, but we must choose a convention and be consistent with it. For time-harmonic signals, this becomes

$$E(z) = A\exp(-jkz)$$ Negative sign convention

$$E(z) = A\exp(+jkz)$$ Positive sign convention

SIGN CONVENTIONS FOR EM WAVES

EE 3321 Electromagnetic Field Theory

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EQUATION(S)	ELECTRICAL ENGINEERING (Negative Sign Convention)	PHYSICS / SCIENCE (Positive Sign Convention)
	$j \leftrightarrow i$	
Wave Propagating in $+z$ Direction	$\cos(\omega t \mp kz)$ − forward wave $\exp(+jkz)$ + backward wave	$\cos(-\omega t \pm kz)$ − backward wave $\exp(+ikz)$ + forward wave
Maxwell's Equations	$\nabla \times \vec{E} = -j\omega\vec{B}$ $\nabla \bullet \vec{D} = \rho_v$ $\nabla \times \vec{H} = \vec{J} + j\omega\vec{D}$ $\nabla \bullet \vec{B} = 0$ $\vec{D} = \varepsilon\vec{E}$ $\vec{B} = \mu\vec{H}$	$\nabla \times \vec{E} = i\omega\vec{B}$ $\nabla \bullet \vec{D} = \rho_v$ $\nabla \times \vec{H} = -\vec{J} - i\omega\vec{D}$ $\nabla \bullet \vec{B} = 0$ $\vec{D} = \varepsilon\vec{E}$ $\vec{B} = \mu\vec{H}$
Wave Vector	$k = \beta - j\alpha$ $\alpha < 0$ gain (grow) $\alpha > 0$ loss (decay)	$\beta < 0$ backward $\beta > 0$ forward $k = \beta + i\alpha$
Refractive Index	$\tilde{n} = n - j\kappa$ $n < 0$ negative index $n > 0$ positive index	$\kappa < 0$ gain (growth) $\kappa > 0$ loss (decay) $\tilde{n} = n + i\kappa$
Lorentz Model	$\tilde{\varepsilon}_r(\omega) = 1 + \dfrac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\Gamma}$ $\Gamma < 0$ gain (grow) $\Gamma > 0$ loss (decay)	$\tilde{\varepsilon}_r(\omega) = 1 + \dfrac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}$

GOVERNING EQUATIONS FOR CLASSICAL ELECTROMAGNETICS

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	Integral Form	Differential Form	Name
Time-Domain	$Q_v(t) = \oiint_S \vec{D}(t) \bullet d\vec{s} = \iiint_V \rho_v(t)\,dv$	$\nabla \bullet \vec{D}(t) = \rho_v(t)$	Gauss' Law
	$\oiint_S \vec{B}(t) \bullet d\vec{s} = 0$	$\nabla \bullet \vec{B}(t) = 0$	No Magnetic Charge
	$V_{emf}(t) = \oint_L \vec{E}(t) \bullet d\vec{\ell} = -\iint_S \left[\dfrac{\partial \vec{B}(t)}{\partial t}\right] \bullet d\vec{s}$	$\nabla \times \vec{E}(t) = -\dfrac{\partial \vec{B}(t)}{\partial t}$	Faraday's Law
	$I(t) = \oint_L \vec{H}(t) \bullet d\vec{\ell} = \iint_S \left[\vec{J}(t) + \dfrac{\partial \vec{D}(t)}{\partial t}\right] \bullet d\vec{s}$	$\nabla \times \vec{H}(t) = \vec{J}(t) + \dfrac{\partial \vec{D}(t)}{\partial t}$	Ampere's Circuit Law
	$\oiint_S \vec{J} \bullet d\vec{s} = -\dfrac{\partial Q_v}{\partial t}$	$\nabla \bullet \vec{J} = -\dfrac{\partial \rho_v}{\partial t}$	Continuity of Current
	$\vec{D}(t) = [\varepsilon(t)] * \vec{E}(t)$ Electric Response $\vec{B}(t) = [\mu(t)] * \vec{H}(t)$ Magnetic Response		Constitutive Relations
Frequency-Domain	$Q_v = \oiint_S \vec{D} \bullet d\vec{s} = \iiint_V \rho_v\,dv$	$\nabla \bullet \vec{D} = \rho_v$	Gauss' Law
	$\oiint_S \vec{B} \bullet d\vec{s} = 0$	$\nabla \bullet \vec{B} = 0$	No Magnetic Charge
	$V_{emf} = \oint_L \vec{E} \bullet d\vec{\ell} = -\iint_S \left[j\omega\vec{B}\right] \bullet d\vec{s}$	$\nabla \times \vec{E} = -j\omega\vec{B}$	Faraday's Law
	$I = \oint_L \vec{H} \bullet d\vec{\ell} = \iint_S \left[\vec{J} + j\omega\vec{D}\right] \bullet d\vec{s}$	$\nabla \times \vec{H} = \vec{J} + j\omega\vec{D}$	Ampere's Circuit Law
	$\oiint_S \vec{J} \bullet d\vec{s} = -j\omega Q_v$	$\nabla \bullet \vec{J} = -j\omega\rho_v$	Continuity of Current
	$\vec{D} = [\varepsilon]\vec{E}$ Electric Response $\vec{B} = [\mu]\vec{H}$ Magnetic Response		Constitutive Relations

Parameter Definitions

Electric Field Intensity, E (V/m)
Electric Flux Density, D (C/m²)
Magnetic Field Intensity, H (A/m)
Magnetic Flux Density, B (Wb/m²)
Electric Current Density, J (A/m²)
Volume Charge Density, ρ_v (C/m³)
Permittivity, ε (F/m)
Permeability, μ (H/m)
Electrical Conductivity, σ (1/Ω·m)

Constants

Permittivity: $[\varepsilon] = \varepsilon_0[\varepsilon_r]$
$\varepsilon_0 = 8.8541878176 \times 10^{-12}$ (F/m)

Permeability: $[\mu] = \mu_0[\mu_r]$
$\mu_0 \approx 4\pi \times 10^{-7}$ (H/m)
$\mu_0 = 1.2566370614 \times 10^{-6}$ (H/m)

Impedance: $\eta_0 \approx 120\pi$ (Ω)
$\eta_0 = 376.73031346177$ (Ω)

Speed of Light: $c_0 = 299{,}792{,}458$ (m/s)

Lorentz Force Law

$\vec{F} = q\vec{E} + q\left(\vec{v} \times \vec{B}\right)$

Sign Convention

e^{-jkz} For propagation in the +z direction.

Lorentz Force Law

One additional equation is needed to completely describe classical electromagnetism...

$$\vec{F} = \underbrace{q\vec{E}}_{\text{Electric Force}} + \underbrace{q\vec{v} \times \vec{B}}_{\text{Magnetic Force}}$$

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Alternate Forms of Maxwell's Equations

Maxwell's Equations with Gaussian Units

$$\nabla \bullet \vec{D} = 4\pi\rho_v \qquad \nabla \times \vec{E} = -\frac{1}{c_0}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \bullet \vec{B} = 4\pi\rho_v \qquad \nabla \times \vec{H} = \frac{1}{c_0}\left(4\pi\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

Relativistic Maxwell's Equations

$$\partial_\alpha \vec{F}^{\alpha\beta} = \mu_0 \vec{J}^{\beta}$$

$$\partial_\alpha \left(\tfrac{1}{2} e^{\alpha\beta\gamma\delta} \vec{F}_{\gamma\delta}\right) = 0 \qquad \vec{J}^{\nu}_{\text{free}} = D^{\mu\nu}$$

Maxwell's Equations in Moving Media

$$\nabla \bullet \vec{D} = 4\pi\rho_v \qquad \nabla \times \vec{E} = -\frac{1}{c_0}\left(\frac{\partial \vec{B}}{\partial t} + \nabla \times \alpha\vec{B} \times \vec{v}\right)$$

$$\nabla \bullet \vec{B} = 4\pi\rho_v \qquad \nabla \times \vec{H} = \frac{1}{c_0}\left(4\pi\vec{J} + \frac{\partial \vec{D}}{\partial t} + \nabla \times \alpha\vec{D} \times \vec{v}\right)$$

$$\alpha = \frac{\mu\varepsilon - 1}{\mu\varepsilon}$$

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Time-Harmonic Maxwell's Equations

Time-Domain

$$\nabla \bullet \vec{D} = \rho_v \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Frequency-Domain
(e^{+jkz} convention)

$$\nabla \bullet \vec{D} = \rho_v \qquad \nabla \times \vec{E} = j\omega \vec{B}$$

$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = \vec{J} - j\omega \vec{D}$$

Frequency-Domain
(e^{-jkz} convention)

$$\nabla \bullet \vec{D} = \rho_v \qquad \nabla \times \vec{E} = -j\omega \vec{B}$$

$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = \vec{J} + j\omega \vec{D}$$

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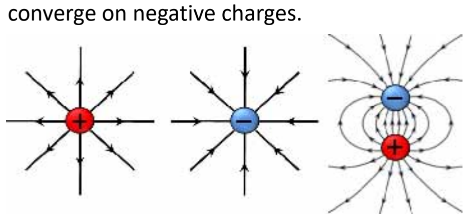
Gauss's Law

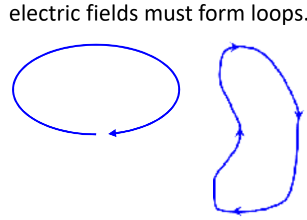
$$\boxed{\nabla \bullet \vec{D} = \rho_v}$$

$$\nabla \bullet \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Electric fields diverge from positive charges and converge on negative charges.



If there are no charges, electric fields must form loops.



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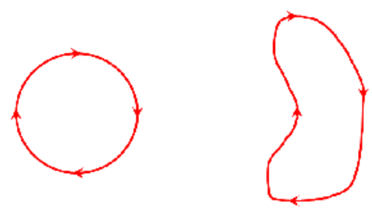
Gauss's Law for Magnetism

$$\boxed{\nabla \bullet \vec{B} = 0}$$

$$\nabla \bullet \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

Magnetic fields always form loops.



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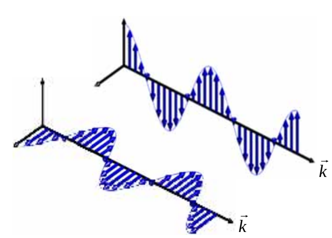
Consequence of Zero Divergence

The divergence theorems force the **D** and **B** fields to be perpendicular to the propagation direction of a plane wave.

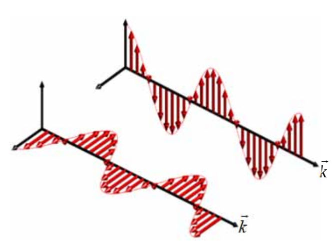
$$\nabla \bullet \vec{D} = 0$$
$$\nabla \bullet \left(\vec{d} e^{-j\vec{k} \bullet \vec{r}} \right) = 0$$
$$\underbrace{\cancel{\nabla \bullet \vec{d}}}_{\text{no charges}} - j\vec{k} \bullet \vec{d} = 0$$
$$\vec{k} \bullet \vec{d} = 0$$

$$\boxed{\vec{k} \perp \vec{D}}$$



$$\nabla \bullet \vec{B} = 0$$
$$\nabla \bullet \left(\vec{b} e^{-j\vec{k} \bullet \vec{r}} \right) = 0$$
$$\underbrace{\cancel{\nabla \bullet \vec{b}}}_{\text{no charges}} - j\vec{k} \bullet \vec{b} = 0$$
$$\vec{k} \bullet \vec{b} = 0$$

$$\boxed{\vec{k} \perp \vec{B}}$$



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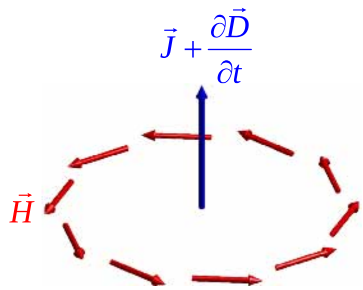
Ampere's Law with Maxwell's Correction

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

$$\nabla \times \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \hat{a}_z$$

Circulating magnetic fields induce currents and/or time varying electric fields.

Currents and/or time varying electric fields induce circulating magnetic fields.

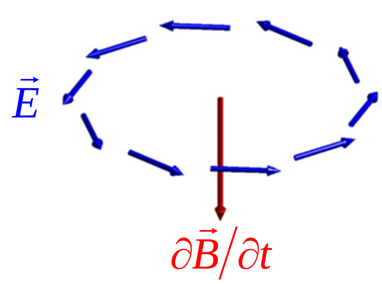


Faraday's Law of Induction

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

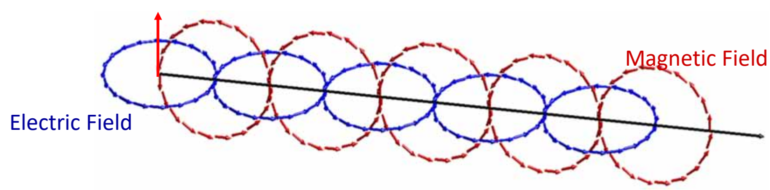
$$\nabla \times \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{a}_z$$

Circulating electric fields induce time varying magnetic fields.
Time varying magnetic fields induce circulating electric fields.



Consequence of Curl Equations

The curl equations predict electromagnetic waves!!



The Constitutive Relations

Electric Response

$$\vec{D} = \varepsilon \vec{E}$$

- Electric field intensity (V/m)
- Initial electric "push."
- Induced electric field.
- Electric energy in vacuum.

- Permittivity (F/m)
- Measure of how well a material stores electric energy.

- Electric flux density (C/m^2)
- Pretends as if all electric energy is displaced charge.
- Includes electric energy in vacuum and matter.

Magnetic Response

$$\vec{B} = \mu \vec{H}$$

- Magnetic field intensity (A/m)
- Initial magnetic "push."
- Induced magnetic field.
- Magnetic energy in vacuum.

- Permeability (H/m)
- Measure of how well a material stores magnetic energy.

- Magnetic flux density (Wb/m^2)
- Pretends as if all magnetic energy is tilted magnetic dipoles.
- Includes magnetic energy in vacuum and matter.

Material Classifications

Linear, isotropic and non-dispersive materials:

$$\vec{D}(t) = \varepsilon \vec{E}(t)$$

We will use this almost exclusively

Dispersive materials:

$$\vec{D}(t) = \varepsilon(t) * \vec{E}(t)$$

A key point is that you can wrap all of the complexities associated with modeling strange materials into this single equation. This will make your code more modular and easier to modify. It may not be as efficient as it could be though.

Anisotropic materials:

$$\vec{D}(t) = [\varepsilon]\vec{E}(t)$$

Nonlinear materials:

$$D(t) = \varepsilon_0 \chi_e^{(1)} E(t) + \varepsilon_0 \chi_e^{(2)} E^2(t) + \varepsilon_0 \chi_e^{(3)} E^3(t) + \cdots$$

Types of Anisotropy

Isotropic

$$\vec{D}(t) = \varepsilon \vec{E}(t)$$
$$\vec{B}(t) = \mu \vec{H}(t)$$

Anisotropic

$$\vec{D}(t) = [\varepsilon]\vec{E}(t)$$
$$\vec{B}(t) = [\mu]\vec{H}(t)$$

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{aa} & \varepsilon_{ab} & \varepsilon_{ac} \\ \varepsilon_{ba} & \varepsilon_{bb} & \varepsilon_{bc} \\ \varepsilon_{ca} & \varepsilon_{cb} & \varepsilon_{cc} \end{bmatrix}$$

electrically anisotropic

$$[\mu] = \begin{bmatrix} \mu_{aa} & \mu_{ab} & \mu_{ac} \\ \mu_{ba} & \mu_{bb} & \mu_{bc} \\ \mu_{ca} & \mu_{cb} & \mu_{cc} \end{bmatrix}$$

magnetically anisotropic

Gryrotropic

$$\vec{D}(t) = [\varepsilon]\vec{E}(t)$$
$$\vec{B}(t) = [\mu]\vec{H}(t)$$

$$[\varepsilon] = \begin{bmatrix} \varepsilon_1 & -j\varepsilon_2 & 0 \\ j\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

gyroelectric

$$[\mu] = \begin{bmatrix} \mu_1 & -j\mu_2 & 0 \\ j\mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}$$

gyromagnetic

Bi-Isotropic

$$\vec{D}(t) = \varepsilon \vec{E}(t) + \xi \vec{H}$$
$$\vec{B}(t) = \mu \vec{H}(t) + \zeta \vec{E}$$

Bi-Anisotropic

$$\vec{D}(t) = [\varepsilon]\vec{E}(t) + [\xi]\vec{H}$$
$$\vec{B}(t) = [\mu]\vec{H}(t) + [\zeta]\vec{E}$$

isotropic

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{\text{iso}} & 0 & 0 \\ 0 & \varepsilon_{\text{iso}} & 0 \\ 0 & 0 & \varepsilon_{\text{iso}} \end{bmatrix} = \varepsilon_{\text{iso}}$$

uniaxial

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{\text{o}} & 0 & 0 \\ 0 & \varepsilon_{\text{o}} & 0 \\ 0 & 0 & \varepsilon_{\text{e}} \end{bmatrix}$$

biaxial

$$[\varepsilon] = \begin{bmatrix} \varepsilon_a & 0 & 0 \\ 0 & \varepsilon_b & 0 \\ 0 & 0 & \varepsilon_c \end{bmatrix}$$

All Together Now…

Divergence Equations

$$\nabla \bullet \vec{B} = 0$$

$$\nabla \bullet \vec{D} = \rho_v$$

Curl Equations

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

What produces fields

Constitutive Relations

$$\vec{D}(t) = [\varepsilon(t)] * \vec{E}(t)$$

$$\vec{B}(t) = [\mu(t)] * \vec{H}(t)$$

$*$ means convolution

[ ] means tensor

How fields interact with materials

Maxwell's Equations in Cartesian Coordinates (1 of 4)

Vector Terms

$$\vec{E} = E_x\hat{a}_x + E_y\hat{a}_y + E_z\hat{a}_z \qquad \vec{H} = H_x\hat{a}_x + H_y\hat{a}_y + H_z\hat{a}_z \qquad \vec{J} = J_x\hat{a}_x + J_y\hat{a}_y + J_z\hat{a}_z$$

$$\vec{D} = D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z \qquad \vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$$

Divergence Equations

$$\nabla \bullet \vec{D} = 0 \qquad \nabla \bullet \vec{B} = 0$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0 \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

Maxwell's Equations in Cartesian Coordinates (2 of 4)

CEM

Constitutive Relations

$$\vec{D} = [\varepsilon]\vec{E}$$

$$D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z = \left(\varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z\right)\hat{a}_x + \left(\varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z\right)\hat{a}_y + \left(\varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \varepsilon_{zz}E_z\right)\hat{a}_z$$

$$D_x = \varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z$$
$$D_y = \varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z$$
$$D_z = \varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \varepsilon_{zz}E_z$$

$$\vec{B} = [\mu]\vec{H} \longrightarrow \begin{aligned} B_x &= \varepsilon_{xx}H_x + \varepsilon_{xy}H_y + \varepsilon_{xz}H_z \\ B_y &= \varepsilon_{yx}H_x + \varepsilon_{yy}H_y + \varepsilon_{yz}H_z \\ B_z &= \varepsilon_{zx}H_x + \varepsilon_{zy}H_y + \varepsilon_{zz}H_z \end{aligned}$$

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Maxwell's Equations in Cartesian Coordinates (3 of 4)

CEM

Curl Equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{a}_z = -\frac{\partial}{\partial t}\left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{a}_z = -\frac{\partial B_x}{\partial t}\hat{a}_x - \frac{\partial B_y}{\partial t}\hat{a}_y - \frac{\partial B_z}{\partial t}\hat{a}_z$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$$
$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t}$$

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Maxwell's Equations in Cartesian Coordinates (4 of 4)

CEM

Curl Equations

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{a}_z = \left(J_x\hat{a}_x + J_y\hat{a}_y + J_z\hat{a}_z\right) + \frac{\partial}{\partial t}\left(D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z\right)$$

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{a}_z = \left(J_x + \frac{\partial D_x}{\partial t}\right)\hat{a}_x + \left(J_y + \frac{\partial D_y}{\partial t}\right)\hat{a}_y + \left(J_z + \frac{\partial D_z}{\partial t}\right)\hat{a}_z$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x + \frac{\partial D_x}{\partial t}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + \frac{\partial D_y}{\partial t}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + \frac{\partial D_z}{\partial t}$$

Alternative Form of Maxwell's Equations in Cartesian Coordinates (1 of 2)

CEM

Alternate Curl Equations

$$\nabla \times \vec{H} = [\varepsilon]\frac{\partial \vec{E}}{\partial t}$$

$$\begin{aligned}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{a}_z = &\left(\varepsilon_{xx}\frac{\partial E_x}{\partial t} + \varepsilon_{xy}\frac{\partial E_y}{\partial t} + \varepsilon_{xz}\frac{\partial E_z}{\partial t}\right)\hat{a}_x \\ &+ \left(\varepsilon_{yx}\frac{\partial E_x}{\partial t} + \varepsilon_{yy}\frac{\partial E_y}{\partial t} + \varepsilon_{yz}\frac{\partial E_z}{\partial t}\right)\hat{a}_y \\ &+ \left(\varepsilon_{zx}\frac{\partial E_x}{\partial t} + \varepsilon_{zy}\frac{\partial E_y}{\partial t} + \varepsilon_{zz}\frac{\partial E_z}{\partial t}\right)\hat{a}_z\end{aligned}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \varepsilon_{xx}\frac{\partial E_x}{\partial t} + \varepsilon_{xy}\frac{\partial E_y}{\partial t} + \varepsilon_{xz}\frac{\partial E_z}{\partial t}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \varepsilon_{yx}\frac{\partial E_x}{\partial t} + \varepsilon_{yy}\frac{\partial E_y}{\partial t} + \varepsilon_{yz}\frac{\partial E_z}{\partial t}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \varepsilon_{zx}\frac{\partial E_x}{\partial t} + \varepsilon_{zy}\frac{\partial E_y}{\partial t} + \varepsilon_{zz}\frac{\partial E_z}{\partial t}$$

Alternative Form of Maxwell's Equations in Cartesian Coordinates (2 of 2)

Alternate Curl Equations

$$\nabla \times \vec{E} = -[\mu]\frac{\partial \vec{H}}{\partial t}$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{a}_z = -\left(\mu_{xx}\frac{\partial H_x}{\partial t} + \mu_{xy}\frac{\partial H_y}{\partial t} + \mu_{xz}\frac{\partial H_z}{\partial t}\right)\hat{a}_x$$
$$-\left(\mu_{yx}\frac{\partial H_x}{\partial t} + \mu_{yy}\frac{\partial H_y}{\partial t} + \mu_{yz}\frac{\partial H_z}{\partial t}\right)\hat{a}_y$$
$$-\left(\mu_{zx}\frac{\partial H_x}{\partial t} + \mu_{zy}\frac{\partial H_y}{\partial t} + \mu_{zz}\frac{\partial H_z}{\partial t}\right)\hat{a}_z$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu_{xx}\frac{\partial H_x}{\partial t} - \mu_{xy}\frac{\partial H_y}{\partial t} - \mu_{xz}\frac{\partial H_z}{\partial t}$$
$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu_{yx}\frac{\partial H_x}{\partial t} - \mu_{yy}\frac{\partial H_y}{\partial t} - \mu_{yz}\frac{\partial H_z}{\partial t}$$
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu_{zx}\frac{\partial H_x}{\partial t} - \mu_{zy}\frac{\partial H_y}{\partial t} - \mu_{zz}\frac{\partial H_z}{\partial t}$$

Physical Boundary Conditions

Physical Boundary Conditions

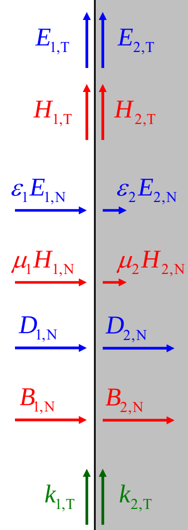
μ_1 and ε_1 | μ_2 and ε_2

Tangential components of E and H are continuous across an interface.

E and H fields normal to the interface are discontinuous across an interface.

Note: Normal components of D and B are continuous across the interface.

Tangential components of the wave vector are continuous across an interface.



These are more complicated boundary conditions, physically and analytically.

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Parameter Relations

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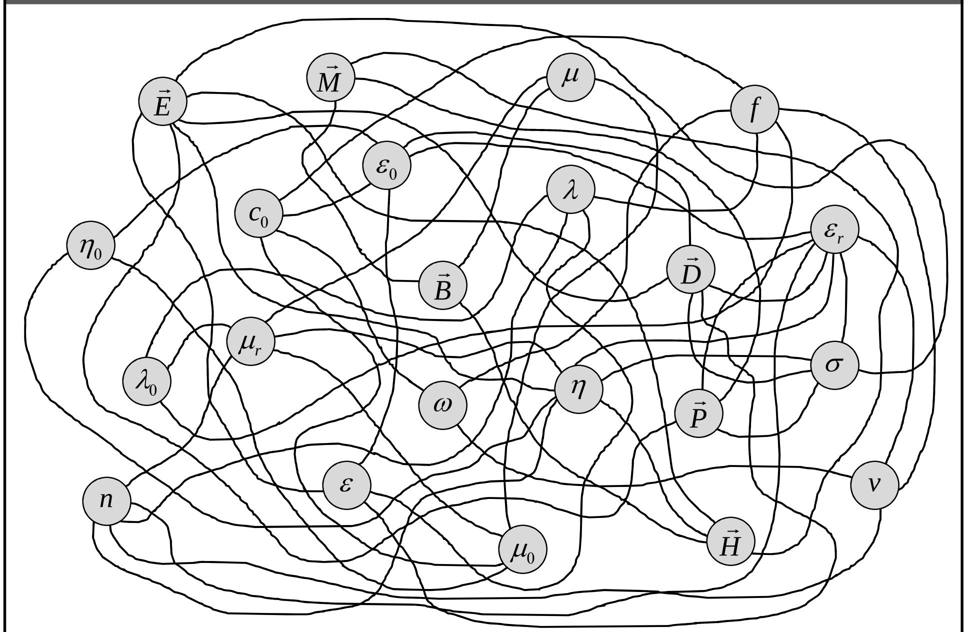
Map of Parameter Relations



The Relative Permittivity

The permittivity is a measure of how well a material stores electric energy. A circulating magnetic field induces an electric field at the center of the circulation in proportion to the permittivity.

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad \tilde{\varepsilon} = \varepsilon' - j\varepsilon''$$

The dielectric constant of a material is its permittivity relative to the permittivity of free space.

$$\varepsilon = \varepsilon_0 \varepsilon_r \qquad \varepsilon_0 = 8.854187817 \times 10^{-12} \quad \text{F/m}$$

$$1 \leq \varepsilon_r \leq \infty$$

ε_r is the relative permittivity or dielectric constant

The Relative Permeability

The permeability is a measure of how well a material stores magnetic energy. A circulating electric field induces a magnetic field at the center of the circulation in proportion to the permeability.

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad \tilde{\mu} = \mu' - j\mu''$$

The relative permeability of a material is its permeability relative to the permeability of free space.

$$\mu = \mu_0 \mu_r \qquad \mu_0 = 1.256637061 \times 10^{-6} \quad \text{H/m}$$

$$1 \le \mu_r \le \infty$$

μ_r is the relative permeability

Conductivity σ

Conductivity is the measure of a material's ability to support electric current. This term is responsible for ohmic loss in materials.

It appears in Ampere's Circuit Law.

$$\nabla \times \vec{H} = \vec{J} + j\omega\vec{D}$$

The current density $\vec{J}$ is related to conductivity and the electric field intensity through Ohm's Law.

$$\vec{J} = \sigma\vec{E}$$

ε'-$j\varepsilon''$ Vs. ε and σ

It is redundant to have a complex dielectric constant along with a conductivity term, although it happens. We should use either a complex dielectric constant or a real dielectric constant and a conductivity.

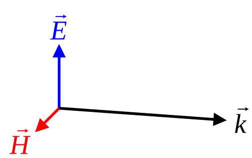
$$\nabla \times \vec{H} = j\omega\left(\varepsilon' - j\varepsilon''\right)\vec{E}$$
$$\nabla \times \vec{H} = \sigma\vec{E} + j\omega\varepsilon\vec{E}$$

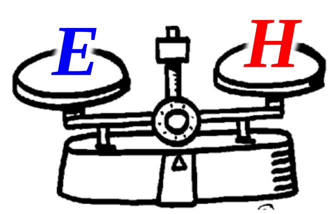
$$\Rightarrow \quad j\omega\left(\varepsilon' - j\varepsilon''\right) = \sigma + j\omega\varepsilon$$
$$\Downarrow$$
$$\varepsilon' - j\varepsilon'' = \frac{\sigma}{j\omega} + \varepsilon$$
$$\Downarrow$$
$$\varepsilon' = \varepsilon \qquad \varepsilon'' = \frac{\sigma}{\omega}$$

Material Impedance

The material impedance is the parameter which describes the balance between the electric and magnetic field amplitudes.

$$\eta = \frac{\left|\vec{E}\right|}{\left|\vec{H}\right|}$$





It is calculated from the permeability and permittivity of the material.

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}$$

$\eta_0 \equiv$ free space impedance
$= 376.73031346177\ \Omega$

Impedance tells us that E and H are three orders of magnitude different.

$$\eta = |\eta|\measuredangle\theta$$

Phase between E and H (θ)

Amplitude between E and H ($|\eta|$)

$$\eta = \eta' + j\eta''$$

Reactive component (η'')

Resistive component. (η')

The Complex Refractive Index

The permittivity and permeability appear in Maxwell's equations so they are the most fundamental material properties. However, it is difficult to determine physical meaning from them in terms of how waves propagate (i.e. speed, loss, etc.). In this case, the refractive index is a more meaningful quantity.

$$\tilde{n} = \sqrt{\mu_r \varepsilon_r}$$

In the frequency-domain, the refractive index is a complex quantity.

$$\tilde{n} = n_\mathrm{o} - j\kappa$$

n_o is the *ordinary refractive index*. It quantifies how quickly a wave propagates.

$$E(z) = E_0 e^{-jk_0\tilde{n}z}$$

κ is the *extinction coefficient*. It quantifies loss and how quickly a wave decays.

* Note: when only the refractive index *n* is specified for a material, assume $\mu_r = 1.0$.

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The Complex Propagation Constant, γ

The propagation constant is very close to the complex refractive index. It describes the speed and decay of a wave.

$$E(z) = E_0 e^{-\gamma z}$$

The propagation constant has a real and imaginary part.

$$\gamma = \alpha + j\beta$$

α is the *attenuation coefficient*. It quantifies how quickly the amplitude of a wave decays.

$$E(z) = E_0 e^{-\alpha z} e^{-j\beta z}$$

β is the *propagation constant*. It quantifies how quickly a wave accumulates phase.

It is related to the complex refractive index through

$$\gamma = jk_0\tilde{n}$$

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The Absorption Coefficient, α

The absorption coefficient describes how quickly the power in a wave decays.

$$P(z) = P_0 e^{-\alpha z}$$

WARNING: Notice the unfortunate reuse of the symbol α for two different things. This is easily confused!!

The attenuation coefficient and absorption coefficient are related through

$$\alpha_{\text{abs}} = 2\alpha_{\text{att}}$$

The absorption coefficient and extinction coefficient are related through

$$\alpha_{\text{abs}} = 2k_0\kappa$$

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Loss Tangent

Sometimes material loss is given in terms of a "loss tangent."

$$\tan\delta = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon}$$

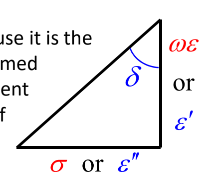
$$P(z) = P_0 e^{-\delta k_0 n z}$$

Recall that interpreting wave properties (velocity and loss) is not intuitive using just the complex dielectric function. To do this, we preferred the complex refractive index.

It turns out that the loss tangent and the extinction coefficient are essentially the same.

$$\boxed{\delta = \frac{2\kappa}{n} = \frac{\alpha_{\text{abs}}}{k_0 n}}$$

It is called a loss tangent because it is the angle in the complex plane formed between the resistive component and the reactive component of the electromagnetic field.



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ω versus f

ω is the angular frequency measured in radians per second. It relates more directly to phase and k. Think $\cos(\omega t)$.

f is the ordinary frequency measured in cycles per second. It relates more directly to time. Think $\cos(2\pi f t)$ and $\tau = 1/f$.

$$\boxed{\omega = 2\pi f}$$

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Wavelength and Frequency

The frequency f and free space wavelength λ_0 are related through

$$c_0 = f\lambda_0 \qquad c_0 = 299792458\ \tfrac{\text{m}}{\text{s}} \equiv \text{speed of light in vacuum}$$

Inside a material, the wave slows down according to the refractive index as follows.

$$v = \frac{c_0}{n}$$

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Summary of Parameter Relations

Permittivity

$\varepsilon = \varepsilon_0 \varepsilon_r$

$\varepsilon_0 = 8.854187817 \times 10^{-12}$ F/m

Permeability

$\mu = \mu_0 \mu_r$

$\mu_0 = 1.256637061 \times 10^{-6}$ H/m

Refractive Index

$n = \sqrt{\mu_r \varepsilon_r}$

Impedance

$\eta = \eta_0 \sqrt{\mu_r / \varepsilon_r}$

$\eta_0 = \sqrt{\mu_0 / \varepsilon_0} = 376.73031346177\ \Omega$

Wave Velocity

$v = \dfrac{c_0}{n}$

$c_0 = 299792458$ m/s

Exact

Frequency and Wavelength

$\omega = 2\pi f$

$c_0 = f \lambda_0$

Wave Number

$k_0 = \dfrac{2\pi}{\lambda_0}$

Table of Dielectric Constants and Loss Tangents

TABLE 2-5
Dielectric constants and loss tangents of typical dielectric materials

Material	ε_r'	$\tan\delta$
Air	1.0006	
Alcohol (ethyl)	25	0.1
Aluminum oxide	8.8	6×10^{-4}
Bakelite	4.74	22×10^{-3}
Carbon dioxide	1.001	
Germanium	16	
Glass	4–7	1×10^{-3}
Ice	4.2	0.1
Mica	5.4	6×10^{-4}
Nylon	3.5	2×10^{-2}
Paper	3	8×10^{-3}
Plexiglas	3.45	4×10^{-2}
Polystyrene	2.56	5×10^{-5}
Porcelain	6	14×10^{-3}
Pyrex glass	4	6×10^{-4}
Quartz (fused)	3.8	7.5×10^{-4}
Rubber	2.5–3	2×10^{-3}
Silica (fused)	3.8	7.5×10^{-4}
Silicon	11.8	
Snow	3.3	0.5
Sodium chloride	5.9	1×10^{-4}
Soil (dry)	2.8	7×10^{-2}
Styrofoam	1.03	1×10^{-4}
Teflon	2.1	3×10^{-4}
Titanium dioxide	100	15×10^{-4}
Water (distilled)	80	4×10^{-2}
Water (sea)	81	4.64
Water (dehydrated)	1	0
Wood (dry)	1.5–4	1×10^{-2}

Constantine A. Balanis, Advanced Engineering Electromagnetics, Wiley, 1989.

Table of Permeabilities

TABLE 2-2
Approximate static relative permeabilities of magnetic materials

Material	Class	Relative permeability (μ_{sr})
Bismuth	Diamagnetic	0.999834
Silver	Diamagnetic	0.99998
Lead	Diamagnetic	0.999983
Copper	Diamagnetic	0.999991
Water	Diamagnetic	0.999991
Vacuum	Nonmagnetic	1.0
Air	Paramagnetic	1.0000004
Aluminum	Paramagnetic	1.00002
Nickel chloride	Paramagnetic	1.00004
Palladium	Paramagnetic	1.0008
Cobalt	Ferromagnetic	250
Nickel	Ferromagnetic	600
Mild steel	Ferromagnetic	2,000
Iron	Ferromagnetic	5,000
Silicon iron	Ferromagnetic	7,000
Mumetal	Ferromagnetic	100,000
Purified iron	Ferromagnetic	200,000
Supermalloy	Ferromagnetic	1000,000

Constantine A. Balanis, Advanced Engineering Electromagnetics, Wiley, 1989.

Duality Between E-D and H-B

Electric Field	Magnetic Field
E	H
D	B
P	M
ε	μ

Preparing Maxwell's Equations for CEM

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Simplifying Maxwell's Equations

1. Assume no charges or current sources: $\rho_v = 0,\ \vec{J} = 0$

$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = \partial \vec{D}/\partial t \qquad \vec{D}(t) = [\varepsilon(t)] * \vec{E}(t)$$
$$\nabla \bullet \vec{D} = 0 \qquad \nabla \times \vec{E} = -\partial \vec{B}/\partial t \qquad \vec{B}(t) = [\mu(t)] * \vec{H}(t)$$

2. Transform Maxwell's equations to frequency-domain:

$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = j\omega \vec{D} \qquad \vec{D} = [\varepsilon]\vec{E}$$
$$\nabla \bullet \vec{D} = 0 \qquad \nabla \times \vec{E} = -j\omega \vec{B} \qquad \vec{B} = [\mu]\vec{H}$$

Convolution becomes simple multiplication

Note: We have chose to proceed with the negative sign convention.

3. Substitute constitutive relations into Maxwell's equations:

$$\nabla \bullet \left([\mu]\vec{H}\right) = 0 \qquad \nabla \times \vec{H} = j\omega[\varepsilon]\vec{E}$$
$$\nabla \bullet \left([\varepsilon]\vec{E}\right) = 0 \qquad \nabla \times \vec{E} = -j\omega[\mu]\vec{H}$$

Note: It is useful to retain μ and ε and not replace them with refractive index *n*.

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Isotropic Materials

For anisotropic materials, the permittivity and permeability terms are tensor quantities.

$$[\varepsilon]=\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}\end{bmatrix} \qquad [\mu]=\begin{bmatrix}\mu_{xx} & \mu_{xy} & \mu_{xz}\\ \mu_{yx} & \mu_{yy} & \mu_{yz}\\ \mu_{zx} & \mu_{zy} & \mu_{zz}\end{bmatrix}$$

For isotropic materials, the tensors reduce to a single scalar quantity.

$$[\varepsilon]=\begin{bmatrix}\varepsilon & 0 & 0\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon\end{bmatrix}=\varepsilon \qquad [\mu]=\begin{bmatrix}\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu\end{bmatrix}=\mu$$

Maxwell's equations can then be written as

$$\nabla\bullet\left(\mu_r\vec{H}\right)=0 \qquad \nabla\times\vec{H}=j\omega\varepsilon_0\varepsilon_r\vec{E}$$

$$\nabla\bullet\left(\varepsilon_r\vec{E}\right)=0 \qquad \nabla\times\vec{E}=-j\omega\mu_0\mu_r\vec{H}$$

ε_0 and μ_0 dropped from these equations because they are constants and do not vary spatially.

Expand Maxwell's Equations

Divergence Equations

$$\nabla\bullet\left(\mu_r\vec{H}\right)=0$$
$$\downarrow$$
$$\frac{\partial(\mu_r H_x)}{\partial x}+\frac{\partial(\mu_r H_y)}{\partial y}+\frac{\partial(\mu_r H_z)}{\partial z}=0$$

$$\nabla\bullet\left(\varepsilon_r\vec{E}\right)=0$$
$$\downarrow$$
$$\frac{\partial(\varepsilon_r E_x)}{\partial x}+\frac{\partial(\varepsilon_r E_y)}{\partial y}+\frac{\partial(\varepsilon_r E_z)}{\partial z}=0$$

Curl Equations

$$\nabla\times\vec{H}=j\omega\varepsilon_0\varepsilon_r\vec{E}$$
$$\downarrow$$
$$\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=j\omega\varepsilon_0\varepsilon_r E_x$$
$$\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=j\omega\varepsilon_0\varepsilon_r E_y$$
$$\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=j\omega\varepsilon_0\varepsilon_r E_z$$

$$\nabla\times\vec{E}=-j\omega\mu_0\mu_r\vec{H}$$
$$\downarrow$$
$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-j\omega\mu_0\mu_r H_x$$
$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-j\omega\mu_0\mu_r H_y$$
$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-j\omega\mu_0\mu_r H_z$$

Normalize the Magnetic Field

Standard form of "Maxwell's Curl Equations"

$$\nabla \times \vec{E} = -j\omega\mu_0\mu_r\vec{H} \qquad \nabla \times \vec{H} = j\omega\varepsilon_0\varepsilon_r\vec{E}$$

Normalized Magnetic Field

$$\frac{|\vec{E}|}{|\vec{H}|} \cong \frac{377}{n} \qquad \vec{\tilde{H}} = \underbrace{-j}_{\text{Eliminates } j\omega \text{; No sign inconsistency; Just have } k_0}\underbrace{\sqrt{\frac{\mu_0}{\varepsilon_0}}}_{\text{Equalizes } E \text{ and } H \text{ amplitudes}}\vec{H}$$

Note:
$k_0 = \omega\sqrt{\mu_0\varepsilon_0}$

- Eliminates $j\omega$
- No sign inconsistency
- Just have k_0

Equalizes E and H amplitudes

Normalized Maxwell's Equations

$$\nabla \times \vec{E} = k_0\mu_r\vec{\tilde{H}} \qquad \nabla \times \vec{\tilde{H}} = k_0\varepsilon_r\vec{E}$$

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Starting Point for Most CEM

We arrive at the following set of equations that are the same regardless of the sign convention used.

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0\mu_{xx}\tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = k_0\varepsilon_{xx}E_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = k_0\mu_{yy}\tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = k_0\varepsilon_{yy}E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = k_0\mu_{zz}\tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = k_0\varepsilon_{zz}E_z$$

The manner in which the magnetic field is normalized does depend on the sign convention chosen.

$$\vec{\tilde{H}} = \begin{cases} -j\eta_0\vec{H} & \text{negative sign convnetion} \\ +j\eta_0\vec{H} & \text{positive sign convnetion} \end{cases}$$

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The Wave Equation and Its Solutions

Derivation of the Wave Equation

We start with Maxwell's curl equations.

$$\nabla \times \vec{E} = -j\omega\mu_0\mu_r\vec{H} \quad \text{Eq. (1)} \qquad \nabla \times \vec{H} = j\omega\varepsilon_0\varepsilon_r\vec{E} \quad \text{Eq. (2)}$$

Equation (1) is solved for the magnetic field.

$$\vec{H} = \frac{j}{\omega\mu_0\mu_r}\left(\nabla \times \vec{E}\right) \quad \text{Eq. (3)}$$

Equation (3) is substituted into Eq. (2).

$$\nabla \times \left[\frac{j}{\omega\mu_0\mu_r}\left(\nabla \times \vec{E}\right)\right] = j\omega\varepsilon_0\varepsilon_r\vec{E}$$

$$\nabla \times \left[\frac{1}{\mu_r}\left(\nabla \times \vec{E}\right)\right] = k_0^2\varepsilon_r\vec{E}$$

Two Different Wave Equations

We can derive a wave equation for both E and H.

$$\nabla \times \mu_r^{-1} \nabla \times \vec{E} = k_0^2 \varepsilon_r \vec{E} \qquad \nabla \times \varepsilon_r^{-1} \nabla \times \vec{H} = k_0^2 \mu_r \vec{H}$$

It is not actually possible to simplify these equations further without making an approximation. Assuming a linear homogeneous isotropic (LHI) material, the wave equations reduce to

$$\nabla \times \nabla \times \vec{E} = k_0^2 \mu_r \varepsilon_r \vec{E} \qquad \nabla \times \nabla \times \vec{H} = k_0^2 \mu_r \varepsilon_r \vec{H}$$

$$\nabla \left(\cancel{\nabla \bullet \vec{E}} \right) - \nabla^2 \vec{E} = k_0^2 \mu_r \varepsilon_r \vec{E} \qquad \nabla \left(\cancel{\nabla \bullet \vec{H}} \right) - \nabla^2 \vec{H} = k_0^2 \mu_r \varepsilon_r \vec{H}$$

$$\nabla^2 \vec{E} + k_0^2 \mu_r \varepsilon_r \vec{E} = 0 \qquad \nabla^2 \vec{H} + k_0^2 \mu_r \varepsilon_r \vec{H} = 0$$

We see that these equations will have the same solution since it is the same differential equation! So, we only have to solve one of them.

Plane Wave Solution in Homogeneous Media

Given the wave equation in an LHI material,

$$\nabla^2 \vec{E} + k_0^2 \mu_r \varepsilon_r \vec{E} = 0$$

The solution is a plane wave.

$$\vec{E}(\vec{r}) = \vec{E}_0 \exp\left(\pm j\vec{k} \bullet \vec{r} \right)$$

$$\vec{H}(\vec{r}) = \vec{H}_0 \exp\left(\pm j\vec{k} \bullet \vec{r} \right)$$

Amplitude Relation

Given plane wave functions of the form

$$\vec{E}(\vec{r}) = \vec{E}_0 \exp\left(-j\vec{k} \bullet \vec{r}\right)$$

$$\vec{H}(\vec{r}) = \vec{H}_0 \exp\left(-j\vec{k} \bullet \vec{r}\right)$$

The amplitudes are related through Maxwell's equations.

$$\nabla \times \vec{E} = -j\omega\mu_0\mu_r\vec{H}$$

$$\nabla \times \left(\vec{E}_0 e^{-j\vec{k}\bullet\vec{r}}\right) = -j\omega\mu_0\mu_r\left(\vec{H}_0 e^{-j\vec{k}\bullet\vec{r}}\right)$$

$$-j\left(\vec{k} \times \vec{E}_0\right)e^{-j\vec{k}\bullet\vec{r}} = -j\omega\mu_0\mu_r\vec{H}_0 e^{-j\vec{k}\bullet\vec{r}}$$

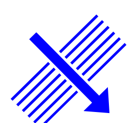
$$\vec{k} \times \vec{E}_0 = \omega\mu_0\mu_r\vec{H}_0$$

$$\vec{H}_0 = \frac{\vec{k} \times \vec{E}_0}{\omega\mu_0\mu_r} = \frac{\vec{k} \times \vec{E}_0}{k_0\eta_0\mu_r}$$

IMPORTANT: Plane Waves are of Infinite Extent

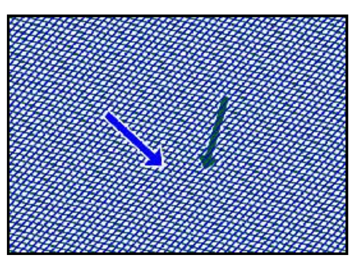
Many times we just draw rays or sometime rays with perpendicular lines to represent the wave fronts.

ray



ray + perpendicular lines

Unfortunately, this suggests the wave is confined spatially. In reality, plane waves are of infinite extent. Think more this way…



Solving the Wave Equation as a Scattering Problem

Scattering problems cast the wave equation into the following matrix form.

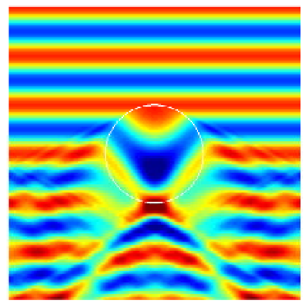
$$\nabla \times \mu_r^{-1} \nabla \times \vec{E} - k_0^2 \varepsilon_r \vec{E} = g$$

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{A} = \left(\nabla \times \mu_r^{-1} \nabla \times - k_0^2 \varepsilon_r \right)$$

$$\mathbf{x} = \vec{E} \qquad \mathbf{b} = g$$

- A source **b** is needed
- Only one solution exists



Lecture 2 Slide 57

Solving the Wave Equation as an Eigen-Value Problem

The wave equation can also be solved as an eigen-value problem. This approach is used when "modes" are being calculated.

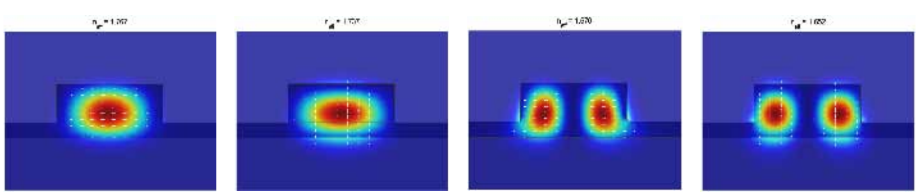
$$\nabla \times \mu_r^{-1} \nabla \times \vec{E} = k_0^2 \varepsilon_r \vec{E}$$

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{B}\mathbf{x}$$

$$\mathbf{A} = \nabla \times \mu_r^{-1} \nabla \times \qquad \mathbf{B} = \varepsilon_r$$

$$\mathbf{x} = \vec{E} \qquad \lambda = k_0^2$$

- No source is needed
- Multiple solutions exist



Lecture 2 Slide 58

Wave Equation Vs. Maxwell's Equations

Wave Equation

The most generalized wave equations are

$$\nabla \times \mu_r^{-1} \nabla \times \vec{E} = k_0^2 \varepsilon_r \vec{E}$$

$$\nabla \times \varepsilon_r^{-1} \nabla \times \vec{H} = k_0^2 \mu_r \vec{H}$$

In LHI materials, these reduce to

$$\nabla^2 \vec{E} + k_0^2 \mu_r \varepsilon_r \vec{E} = 0$$

$$\nabla^2 \vec{H} + k_0^2 \mu_r \varepsilon_r \vec{H} = 0$$

Today, it is rare to see the wave equations solved in this form because it leads to **spurious solutions**.

The "fixes" to the spurious solutions problem are incorporated into Maxwell's equations before a wave equation is derived.

Maxwell's Equations

Maxwell's equations expanded into Cartesian coordinates are

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0 \mu_{xx} \tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = k_0 \varepsilon_{xx} E_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = k_0 \mu_{yy} \tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = k_0 \varepsilon_{yy} E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = k_0 \mu_{zz} \tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = k_0 \varepsilon_{zz} E_z$$

These are often written in matrix form as

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = k_0 \begin{bmatrix} \mu_{xx} & 0 & 0 \\ 0 & \mu_{yy} & 0 \\ 0 & 0 & \mu_{zz} \end{bmatrix} \begin{bmatrix} \tilde{H}_x \\ \tilde{H}_y \\ \tilde{H}_z \end{bmatrix} \qquad \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} \tilde{H}_x \\ \tilde{H}_y \\ \tilde{H}_z \end{bmatrix} = k_0 \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Typically, "fixes" are incorporated here and then a wave equation is derived.

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} \mu_{xx} & 0 & 0 \\ 0 & \mu_{yy} & 0 \\ 0 & 0 & \mu_{zz} \end{bmatrix}^{-1} \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = k_0^2 \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Scaling Properties in Maxwell's Equations

Scaling Properties of Maxwell's Equations

There is no fundamental length scale in Maxwell's equations.

Devices may be scaled to operate at different frequencies just by scaling the mechanical dimensions or material properties in proportion to the change in frequency.

This assumes it is physically possible to scale systems in this manner. In practice, building larger or smaller features may not be practical. Further, the properties of the materials may be different at the new operating frequency.

Scaling Dimensions

We start with the wave equation and write the parameters dependence on position explicitly.

$$\nabla\times\frac{1}{\mu_r(\vec{r})}\nabla\times\vec{E}(\vec{r})=\omega^2\mu_0\varepsilon_0\cdot\varepsilon_r(\vec{r})\cdot\vec{E}(\vec{r})$$

Next, we scale the dimensions by a factor a.

$$(a\nabla)\times\frac{1}{\mu_r(\vec{r}/a)}(a\nabla)\times\vec{E}(\vec{r}/a)=\omega^2\mu_0\varepsilon_0\cdot\varepsilon_r(\vec{r}/a)\cdot\vec{E}(\vec{r}/a)$$

$a>1$ stretch dimensions
$a<1$ compress dimensions

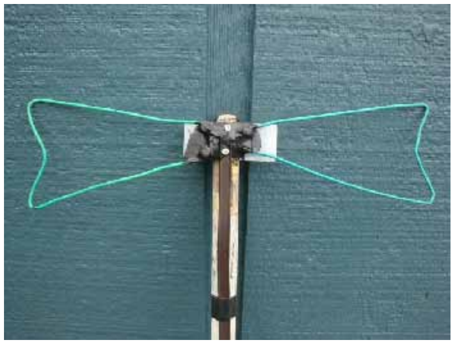
The scale factors multiplying the ∇ operators are moved to multiply the frequency term.

$$\nabla\times\frac{1}{\mu_r(\vec{r}')}\nabla\times\vec{E}(\vec{r}')=\left(\frac{\omega}{a}\right)^2\mu_0\varepsilon_0\cdot\varepsilon_r(\vec{r}')\cdot\vec{E}(\vec{r}') \qquad \vec{r}'=\frac{\vec{r}}{a}$$

The effect of scaling the dimensions is just a shift in frequency.

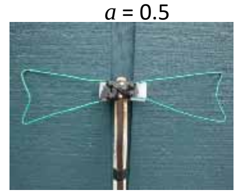
Visualization of Size Scaling

a = 1.0



f_c = 500 MHz

a = 0.5



f_c = 1000 MHz

Scaling μ and ε

We apply separate scaling factors to μ and ε.

$$\nabla \times \frac{1}{\left(a_\mu \mu_r\right)} \nabla \times \vec{E} = \omega^2 \mu_0 \varepsilon_0 \cdot \left(a_\varepsilon \varepsilon_r\right) \cdot \vec{E}$$

The scale factors are moved to multiply the frequency term.

$$\nabla \times \frac{1}{\mu_r} \nabla \times \vec{E} = \left(\omega \sqrt{a_\mu a_\varepsilon}\right)^2 \mu_0 \varepsilon_0 \cdot \varepsilon_r \cdot \vec{E}$$

The effect of scaling the material properties is just a shift in frequency.