Appendix C: Review of Wave Motion

C.1 Introduction

Everyone has observed that when a rock is thrown into a pond of water, waves are produced that move out from the point of the disturbance in a series of concentric circles. *The **wave** is a propagation of the disturbance through the medium without any net displacement of the medium.* In this case the rock hitting the water initiates the disturbance and the water is the medium through which the wave travels. Of the many possible kinds of waves, the simplest to understand, and the one that we will analyze, is the transverse wave.

*A **transverse wave** is a wave in which the particles of the medium execute simple harmonic motion in a direction perpendicular to its direction of propagation.* The vibrating string shown in figure C.1 is an example of a transverse wave. As the

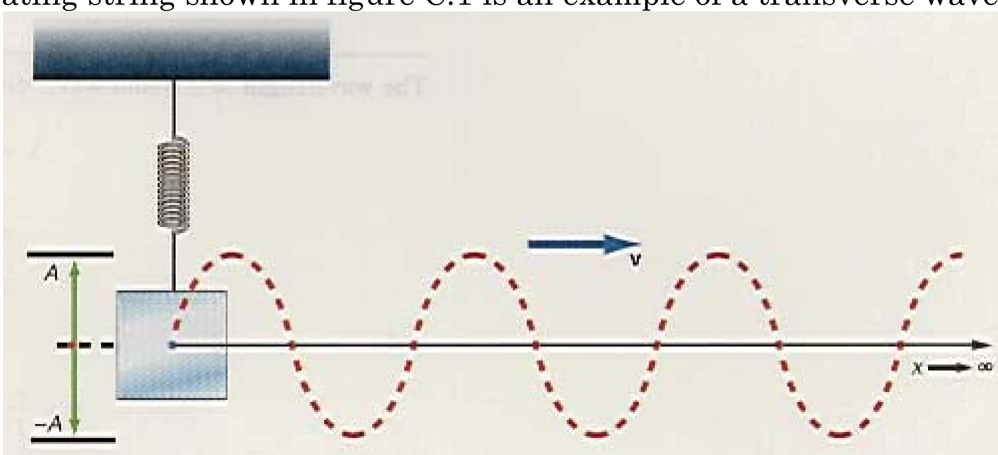

Figure C.1 A transverse wave.

end of the string moves up and down, it causes the particle next to it to follow suit. It, in turn, causes the next particle to move. Each particle transmits the motion to the next particle along the entire length of the string. The resulting wave propagates in the horizontal direction with a velocity v, while any one particle of the string executes simple harmonic motion in the vertical direction. The particle of the string is moving perpendicular to the direction of wave propagation, and is not moving in the direction of the wave. *The wave causes a transfer of energy from one point in the medium to another point in the medium without the actual transfer of matter between these points.*

Using figure C.2 let us now define the characteristics of a transverse wave moving in a horizontal direction:

*The **displacement** of any particle of the wave is the displacement of that particle from its equilibrium position* and is measured by the vertical distance y.

*The **amplitude** of the wave is the maximum value of the displacement* and is denoted by A in figure C.2.

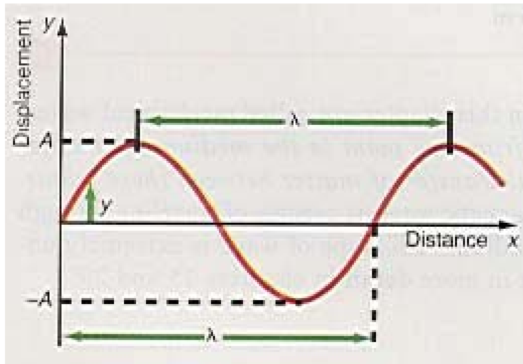

Figure C.2 Characteristics of a simple wave.

*The **wavelength** of a wave is the distance, in the direction of propagation, in which the wave repeats itself* and is denoted by λ.

*The **period T** of a wave is the time it takes for one complete wave to pass a particular point.*

*The **frequency f** of a wave is defined as the number of waves passing a particular point per second.*

It is obvious from the definitions that the frequency is the reciprocal of the period, that is,

$$f = \frac{1}{T} \tag{C.1}$$

The speed of propagation of the wave is the distance the wave travels in unit time. Because a wave of one wavelength, passes a point in a time of one period, its speed of propagation is

$$v = \frac{\text{distance traveled}}{\text{time}} = \frac{\lambda}{T} \tag{C.2}$$

Using equation C.1 this becomes

$$v = \lambda f \tag{C.3}$$

Equation C.3 is the fundamental equation of wave propagation. It relates the speed of the wave to its wavelength and frequency.

Example C.1

Wavelengths of sound. The human ear can hear sounds from a low of 20.0 Hz up to a maximum frequency of about 20,000 Hz. If the speed of sound in air at a temperature of 0^0 C is 331 m/s, find the wavelengths associated with these frequencies.

Solution

Solution. The wave length of a sound wave, determined from equation C.3, is

$$\lambda = \frac{v}{f}$$

$$\lambda = \frac{331 \text{ m/s}}{20.0 \text{ cycles/s}}$$

$$\lambda = 16.6 \text{ m}$$

and

$$\lambda = \frac{v}{f} = \frac{331 \text{ m/s}}{20{,}000 \text{ cycles/s}}$$

$$\lambda = 0.0166 \text{ m}$$

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C.2 Mathematical Representation of a Wave

The simple wave shown in figure C.2 is a picture of a transverse wave in a string at a particular time, let us say at $t = 0$. The wave can be described as a sine wave and can be expressed mathematically as

$$y = A \sin x \tag{C.4}$$

The value of y represents the displacement of the string at every position x along the string and A is the maximum displacement, and is called the amplitude of the wave. Equation C.4 is plotted in figure C.3. We see that the wave repeats itself for $x = 360^0 = 2\pi$ rad. Also plotted in figure C.3 is $y = A \sin 2x$ and $y = A \sin 3x$. Notice

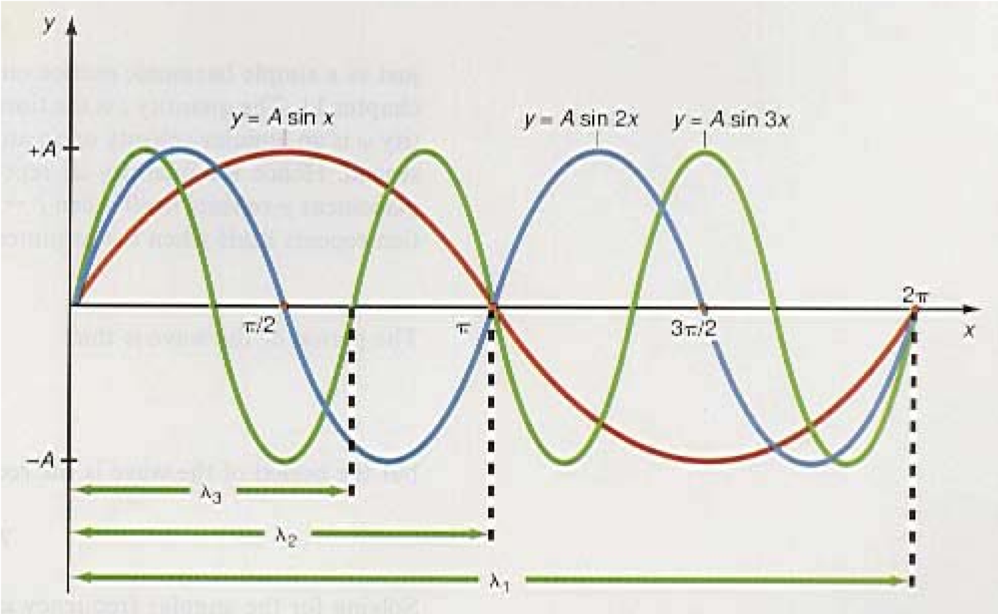

Figure C.3 Plot of $A \sin x$, $A \sin 2x$, and $A \sin 3x$.

from the figure that $y = A \sin 2x$ repeats itself twice in the same interval of 2π that $y = A \sin x$ repeats itself only once. Also note that $y = A \sin 3x$ repeats itself three times in that same interval of 2π. The wave $y = A \sin kx$ would repeat itself k times in the interval of 2π.

We call the space interval in which $y = A \sin x$ repeats itself its wavelength, denoted by λ_1. Thus, when $x = \lambda_1 = 2\pi$, the wave starts to repeat itself. The wave represented by $y = A \sin 2x$ repeats itself for $2x = 2\pi$, and hence its wavelength is

$$\lambda_2 = x = \frac{2\pi}{2} = \pi$$

The wave $y = A \sin 3x$ repeats itself when $3x = 2\pi$, hence its wavelength is

$$\lambda_3 = x = \frac{2\pi}{3}$$

Using this notation any wave can be represented as

$$y = A \sin kx \tag{C.5}$$

where k is a number, called the *wave number*. The wave repeats itself whenever

$$kx = 2\pi \tag{C.6}$$

Because the value of x for a wave to repeat itself is its wavelength λ, equation C.6 can be written as

$$k\lambda = 2\pi \tag{C.7}$$

We can obtain the wavelength λ from equation C.7 as

$$\lambda = \frac{2\pi}{k} \tag{C.8}$$

Note that equation C.8 gives the wavelengths in figure C.3 by letting k have the values 1, 2, 3, and so forth, that is,

$$\lambda_1 = \frac{2\pi}{1}$$
$$\lambda_2 = \frac{2\pi}{2}$$
$$\lambda_3 = \frac{2\pi}{3}$$
$$\lambda_4 = \frac{2\pi}{4}$$

We observe that the wave number k is the number of waves contained in the interval of 2π. We can express the wave number k in terms of the wavelength λ by rearranging equation C.8 into the form

$$k = \frac{2\pi}{\lambda} \tag{C.9}$$

Note that in order for the units to be consistent, the wave number must have units of m^{-1}. The quantity x in equation C.5 represents the location of any point on the string and is measured in meters. The quantity kx in equation C.5, has the units ($m^{-1}m = 1$) and is a thus a dimensionless quantity and represents an angle measured in radians.

Equation C.5 represents a snapshot of the wave at $t = 0$. That is, it gives the displacement of every particle of the string at time $t = 0$. As time passes, this wave, and every point on it, moves. Since each particle of the string executes simple harmonic motion in the vertical, we can look at the particle located at the point $x = 0$ and see how that particle moves up and down with time. Because the particle executes simple harmonic motion in the vertical (along the y-axis), it is reasonable to represent the displacement of the particle of the string at any time t as

$$y = A \sin \omega t \tag{C.10}$$

(Recall from college physics that just as a simple harmonic motion on the x-axis was represented by the equation $x = A \cos \omega t$, a simple harmonic motion on the y-axis is represented by $y = A \sin \omega t$.) The quantity t is the time and is measured in seconds, whereas the quantity ω is an angular velocity or an angular frequency and is measured in radians per second. Hence the quantity ωt represents an angle measured in radians. The displacement y repeats itself when $t = T$, the period of the wave. Since the sine function repeats itself when the argument is equal to 2π, we have

$$\omega T = 2\pi \tag{C.11}$$

The period of the wave is thus

$$T = \frac{2\pi}{\omega}$$

but the period of the wave is the reciprocal of the frequency. Therefore,

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

Solving for the angular frequency ω in terms of the frequency f, we get

$$\omega = 2\pi f \tag{C.12}$$

Notice that the wave is periodic in both space and time. The space period is represented by the wavelength λ, and the time by the period T.

Equation C.5 represents every point on the string at $t = 0$, while equation C.10 represents the point $x = 0$ for every time t. Obviously the general equation for a wave must represent every point x of the wave at every time t. We can arrange this by combining equations C.5 and C.10 into the one equation for a wave given by

$$y = A\sin(kx - \omega t) \tag{C.13}$$

The reason for the minus sign for ωt is explained below. We can find the relation between the wave number k and the angular frequency ω by combining equations C.7 and C.11 as

$$k\lambda = 2\pi \tag{C.7}$$

$$\omega T = 2\pi \tag{C.11}$$

Thus,

$$\omega T = k\lambda$$

and

$$\omega = \frac{k\lambda}{T}$$

However, the wavelength λ, divided by the period T is equal to the velocity of propagation of the wave v, equation C.2. Therefore, the angular frequency becomes

$$\omega = kv \tag{C.14}$$

Now we can write equation C.13 as

$$y = A\sin(kx - kvt) \tag{C.15}$$

or

$$y = A\sin k(x - vt) \tag{C.16}$$

The minus sign before the velocity v determines the direction of propagation of the wave. As an example, consider the wave

$$y_1 = A\sin k(x - vt) \tag{C.17}$$

We will now see that this is the equation of a wave traveling to the right with a speed v at any time t. A little later in time, Δt, the wave will have moved a distance Δx to the right such that the same point of the wave now has the coordinates $x + \Delta x$ and $t + \Delta t$, figure C.4(a). Then we represent the wave as

$$y_2 = A\sin k[(x + \Delta x) - v(t + \Delta t)]$$

or

$$y_2 = A\sin k[(x - v\mathrm{t} + \Delta x - v\Delta t] \tag{C.18}$$

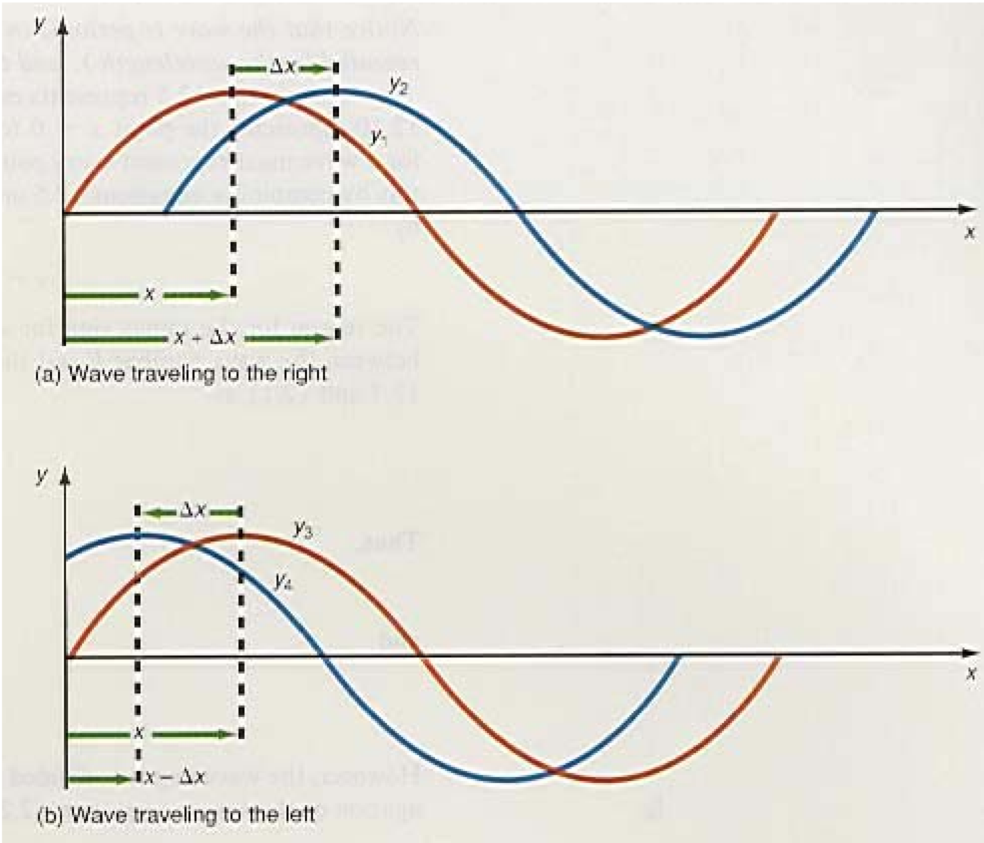

Figure C.4 A traveling wave.

If this equation for y_2 is to represent the same wave as y_1, then y_2 must be equal to y_1. It is clear from equations C.18 and C.17 that if

$$v = \frac{\Delta x}{\Delta t} \tag{C.19}$$

the velocity of the wave to the right, then

$$\Delta x - v\Delta t = \Delta x - \frac{\Delta x}{\Delta t}\Delta t = 0$$

and y_2 is equal to y_1. Because the term $\Delta x - v\Delta t$ is indeed equal to zero, y_2 is the same wave as y_1 only displaced a distance Δx to the right in the time Δt. Thus, equation C.17 represents a wave traveling to the right with a velocity of propagation v.

A wave traveling to the left is depicted in figure C.4(b) and we will begin by representing it as

$$y_3 = A \sin k(x - vt) \tag{C.20}$$

In a time Δt, the wave y_3 moves a distance $-\Delta x$ to the left. The coordinates (x,t) of a point on y_3 now has the coordinates $(x - \Delta x)$ and $(t + \Delta t)$ for the same point on y_4. We can now write the new wave as

$$y_4 = A \sin k[(x - \Delta x) - v(t + \Delta t)]$$

or

$$y_4 = A \sin k[(x - vt) + (-\Delta x - v\Delta t)] \tag{C.21}$$

The wave y_4 represents the same wave as y_3, providing $-\Delta x - v\Delta t = 0$ in equation C.21. If $v = -\Delta x/\Delta t$, the velocity of the wave to the left, then

$$-\Delta x - v\Delta t = -\Delta x - \left(\frac{\Delta x}{\Delta t}\right)\Delta t = 0$$

Thus, $-\Delta x - v\Delta t$ is indeed equal to zero and wave y_4 represents the same wave as y_3 only it is displaced a distance $-\Delta x$ to the left in the time Δt. Instead of writing the equation C.20 as a wave to the left with v a negative number, it is easier to write the equation for the wave to the left as

$$y = A \sin k(x + vt) \tag{C.22}$$

where v is now a positive number. Therefore, equation C.22 represents a wave traveling to the left, with a speed v. In summary, a wave traveling to the right can be represented either as

$$y = A \sin k(x - vt) \tag{C.23}$$

or

$$y = A \sin(kx - \omega t) \tag{C.24}$$

and a wave traveling to the left can be represented as either

$$y = A \sin k(x + vt) \tag{C.25}$$

or

$$y = A \sin(kx + \omega t) \tag{C.26}$$

Example C.2

Characteristics of a wave. A particular wave is given by

$$y = (0.200 \text{ m}) \sin[(0.500 \text{ m}^{-1})x - (8.20 \text{ rad/s})t]$$

Find (a) the amplitude of the wave, (b) the wave number, (c) the wavelength, (d) the angular frequency, (e) the frequency, (f) the period, (g) the velocity of the wave (i.e., its speed and direction), and (h) the displacement of the wave at $x = 10.0$ m and $t = 0.500$ s.

Solution

The characteristics of the wave are determined by writing the wave in the standard form

$$y = A \sin(kx - \omega t)$$

(a) The amplitude A is determined by inspection of both equations as $A = 0.200$ m.
(b) The wave number k is found from inspection to be $k = 0.500\ \text{m}^{-1}$ or a half a wave in an interval of 2π.
(c) The wavelength λ found from equation C.8, is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.500\ \text{m}^{-1}}$$
$$\lambda = 12.6\ \text{m}$$

(d) The angular frequency ω, found by inspection, is

$$\omega = 8.20\ \text{rad/s}$$

(e) The frequency f of the wave, found from equation C.12, is

$$f = \frac{\omega}{2\pi} = \frac{8.20\ \text{rad/s}}{2\pi\ \text{rad}} = 1.31\ \text{cycles/s} = 1.31\ \text{Hz}$$

(f) The period of the wave is the reciprocal of the frequency, thus

$$T = \frac{1}{f} = \frac{1}{1.31\ \text{Hz}} = 0.766\ \text{s}$$

(g) The speed of the wave, found from equation C.14, is

$$v = \frac{\omega}{k} = \frac{8.20\ \text{rad/s}}{0.500\ \text{m}^{-1}} = 16.4\ \text{m/s}$$

We could also have determined this by

$$v = f\lambda = (1.31\ \frac{1}{\text{s}})(\text{C.6 m}) = 16.4\ \text{m/s}$$

The direction of the wave is to the right because the sign in front of ω is negative.
(h) The displacement of the wave at $x = 10.0$ m and $t = 0.500$ s is

$$y = (0.200\ \text{m})\sin[(0.500\ \text{m}^{-1})(10.0\text{m}) - (8.20\ \text{rad/s})(0.500\ \text{s})]$$
$$y = (0.200\ \text{m})\sin[0.900\ \text{rad}] = (0.200\ \text{m})(0.783)$$

$$y = 0.157 \text{ m}$$

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