Chapter 3: Quantum Physics

"Newton himself was better aware of the weakness inherent in his intellectual edifice than the generations which followed him. This fact has always aroused my admiration." Albert Einstein

3.1 The Particle Nature of Waves

Up to now in our study of physics, we considered (1) the motion of particles and their interaction with other particles and their environment and (2) the nature, representation, and motion of waves. We considered particles as little hard balls of matter while a wave was a disturbance that was spread out through a medium. There was certainly a significant difference between the two concepts, and one of the most striking of these is illustrated in figure 3.1. In figure 3.1(a), two

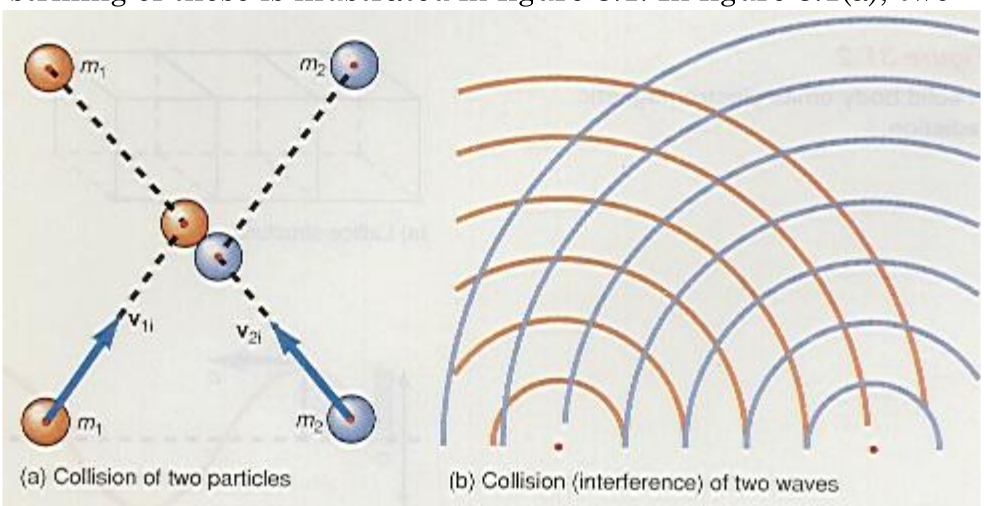

Figure 3.1 Characteristics of particles and waves.

particles collide, bounce off each other, and then continue in a new direction. In figure 3.1(b), two waves collide, but they do not bounce off each other. They add together by the principle of superposition, and then each continues in its original direction as if the waves never interacted with each other.

Another difference between a particle and a wave is that the total energy of the particle is concentrated in the localized mass of the particle. In a wave, on the other hand, the energy is spread out throughout the entire wave. Thus, there is a very significant difference between a particle and a wave.

We have seen that light is an electromagnetic wave. The processes of interference, diffraction, and polarization are characteristic of wave phenomena and have been studied and verified in the laboratory many times over. Yet there has appeared with time, some apparent contradictions to the wave nature of light. We will discuss the following three of these physical phenomena:

1. Blackbody radiation.
2. The photoelectric effect.

3. Compton scattering.

3.2 Blackbody Radiation

All bodies emit and absorb radiation. (Recall that radiation is heat transfer by electromagnetic waves.) The Stefan-Boltzmann law showed that the amount of energy radiated is proportional to the fourth power of the temperature, but did not say how the heat radiated was a function of the wavelength of the radiation. Because the radiation consists of electromagnetic waves, we would expect that the energy should be distributed evenly among all possible wavelengths. However, the energy distribution is not even but varies according to wavelength and frequency. All attempts to account for the energy distribution by classical means failed.

Let us consider for a moment how a body can radiate energy. We know that an oscillating electric charge generates an electromagnetic wave. A body can be considered to be composed of a large number of atoms in a lattice structure as shown in figure 3.2(a). For a metallic material the positively ionized atom is located

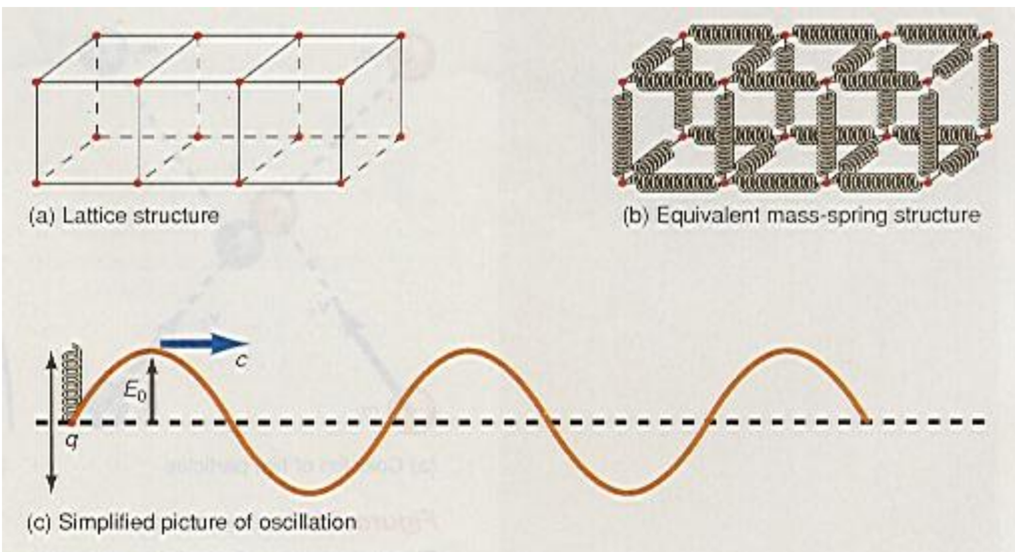

Figure 3.2 A solid body emits electromagnetic radiation.

at the lattice site and the outermost electron of the atom moves throughout the lattice as part of the electron gas. Each atom of the lattice is in a state of equilibrium under the action of all the forces from all its neighboring atoms. The atom is free to vibrate about this equilibrium position. A mechanical analogue to the lattice structure is shown in figure 3.2(b) as a series of masses connected by springs. Each mass can oscillate about its equilibrium position. To simplify the picture further, let us consider a single ionized atom with a charge q and let it oscillate in simple harmonic motion, as shown in figure 3.2(c). The oscillating charge generates an electromagnetic wave that is emitted by the body. Each ionized atom is an oscillator and each has its own fixed frequency and emits radiation of this frequency. Because the body is made up of millions of these oscillating charges, the body always emits radiation of all these different frequencies, and hence the emission spectrum should be continuous. The intensity of the radiation depends on the amplitude of the oscillation. As you recall from general physics, a typical radiated wave is given by

$$E = E_0 \sin(kx - \omega t)$$

where

$$k = \frac{2\pi}{\lambda} \tag{12.9}$$

and

$$\omega = 2\pi\nu \tag{12.12}$$

Thus, the frequency of the oscillating charge is the frequency of the electromagnetic wave. The amplitude of the wave E_0 depends on the amplitude of the simple harmonic motion of the oscillating charge. When the body is heated, the heat energy causes the ionized atoms to vibrate with greater amplitude about their equilibrium position. The energy density of the emitted waves is given by

$$u = \varepsilon_0 E^2$$

or

$$u = \varepsilon_0 E_0^2 \sin^2(kx - \omega t) \tag{3.1}$$

Thus, when the amplitude of the oscillation E_0 increases, more energy is emitted. When the hot body is left to itself it loses energy to the environment by this radiation process and the amplitude of the oscillation decreases. The amplitude of the oscillation determines the energy of the electromagnetic wave. Because of the extremely large number of ionized atoms in the lattice structure that can participate in the oscillations, all modes of vibration of the lattice structure are possible and hence all possible frequencies are present. Thus, the classical picture of blackbody radiation permits all frequencies and energies for the electromagnetic waves. However, this classical picture does not agree with experiment.

Max Planck (1858-1947), a German physicist, tried to “fit” the experimental results to the theory. However, he found that he had to break with tradition and propose a new and revolutionary concept. *Planck assumed that the atomic oscillators cannot take on all possible energies, but could only oscillate with certain discrete amounts of energy given by*

$$E = nh\nu \tag{3.2}$$

where h is a constant, now called *Planck’s constant,* and has the value

$$h = 6.625 \times 10^{-34} \text{ J s}$$

In equation 3.2 ν is the frequency of the oscillator and n is an integer, a number, now called a *quantum number.* The energies of the vibrating atom are now said to be quantized, or limited to only those values given by equation 3.2. Hence, the atom can have energies $h\nu$, $2h\nu$, $3h\nu$, and so on, but never an energy such as 2.5 $h\nu$. This concept of quantization is at complete variance with classical electromagnetic theory. In the classical theory, as the oscillating charge radiates energy it loses energy and the amplitude of the oscillation decreases continuously. If the energy of the oscillator is quantized, the amplitude cannot decrease continuously and hence the oscillating charge cannot radiate while it is in this quantum state. If the oscillator now drops down in energy one quantum state, the difference in energy between the two states is now available to be radiated away. Hence, *the assumption of discrete energy states entails that the radiation process can only occur when the oscillator jumps from one quantized energy state to another quantized energy state.* As an example, if the oscillating charge is in the quantum state 4 it has an energy

$$E_4 = 4h\nu$$

When the oscillator drops to the quantum state 3 it has the energy

$$E_3 = 3h\nu$$

When the oscillator drops from the 4 state to the 3 state it can emit the energy

$$\Delta E = E_4 - E_3 = 4h\nu - 3h\nu = h\nu$$

Thus, the amount of energy radiated is always in small bundles of energy of amount $h\nu$. *This little bundle of radiated electromagnetic energy was called a quantum of energy. Much later, this bundle of electromagnetic energy came to be called a* ***photon.***

Although this quantum hypothesis led to the correct formulation of blackbody radiation, it had some serious unanswered questions. Why should the energy of the oscillator be quantized? If the energy from the blackbody is emitted as a little bundle of energy how does it get to be spread out into Maxwell's electromagnetic wave? How does the energy, which is spread out in the wave, get compressed back into the little quantum of energy so it can be absorbed by an atomic oscillator? These and other questions were very unsettling to Planck and the physics community in general. Although Planck started what would be eventually called *quantum mechanics,* and won the Nobel Prize for his work, he spent many years trying to disprove his own theory.

Example 3.1

Applying the quantum condition to a vibrating spring. A weightless spring has a spring constant k of 29.4 N/m. A mass of 300 g is attached to the spring and is then displaced 5.00 cm. When the mass is released, find (a) the total energy of the mass, (b) the frequency of the vibration, (c) the quantum number n associated with this energy, and (d) the energy change when the oscillator changes its quantum state by one value, that is, for $n = 1$.

Solution

a. The total energy of the vibrating spring comes from its potential energy, which it obtained when work was done to stretch the spring to give an amplitude A of 5.00 cm. The energy, with $x = A$, is

$$E_{\text{total}} = \text{PE} = \frac{1}{2}kA^2$$

$$E = \frac{1}{2}(29.4 \text{ N/m})(0.0500 \text{ m})^2$$

$$= 3.68 \times 10^{-2} \text{ J}$$

b. The frequency ν of the vibration, is

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{29.4 \text{ N/m}}{0.300 \text{ kg}}}$$

$$= 1.58 \text{ Hz}$$

c. The quantum number n associated with this energy, found from equation 3.2, is

$$E = nh\nu$$

$$n = \frac{E}{h\nu}$$

$$= \frac{3.68 \times 10^{-2} \text{ J}}{6.625 \times 10^{-34} \text{ J s} \times 1.58 \text{ s}^{-1}}$$

$$= 3.52 \times 10^{31}$$

This is an enormously large number. Therefore, the effect of a quantum of energy is very small unless the vibrating system itself is very small, as in the case of the vibration of an atom.

d. The energy change associated with the oscillator changing one energy state, found from equation 3.2, is

$$E = nh\nu = h\nu$$

$$= (6.625 \times 10^{-34} \text{ J s})(1.58 \text{ s}^{-1})$$

$$= 1.05 \times 10^{-33} \text{ J}$$

This change in energy is so small that for all intents and purposes, the energy of a vibrating spring-mass system is continuous.

To go to this Interactive Example click on this sentence.

Example 3.2

The energy of a photon of light. An atomic oscillator emits radiation of 700.0-nm wavelength. How much energy is associated with a photon of light of this wavelength?

Solution

The energy of the photon, given by equation 3.2, is

$$E = h\nu$$

but since the frequency ν can be written as c/λ, the energy of the photon can also be written as

$$E = h\nu = \frac{hc}{\lambda}$$

$$= \frac{\left(6.625\times10^{-34}\text{ J s}\right)\left(3.00\times10^{8}\text{ m/s}\right)}{700.0\text{ nm}}\left(\frac{1\text{ nm}}{10^{-9}\text{ m}}\right)$$

$$= 2.84 \times 10^{-19}\text{ J}$$

Thus, the photon of light is indeed a small bundle of energy.

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3.3 The Photoelectric Effect

When Heinrich Hertz performed his experiments in 1887 to prove the existence of electromagnetic waves, he accidentally found that when light fell on a metallic surface, the surface emitted electrical charges. *This effect, whereby light falling on a metallic surface produces electrical charges, is called the **photoelectric effect.*** The photoelectric effect was the first proof that light consists of small particles called photons. Thus, the initial work that showed light to be a wave would also show that light must also be a particle.

Further experiments by Philipp Lenard in 1900 confirmed that these electrical charges were electrons. These electrons were called *photoelectrons*. The photoelectric effect can best be described by an experiment, the schematic diagram of which is shown in figure 3.3. The switch S is thrown to make the anode of the phototube positive and the cathode negative. Monochromatic light (light of a single frequency ν) of intensity I_1, is allowed to shine on the cathode of the phototube, causing electrons to be emitted.

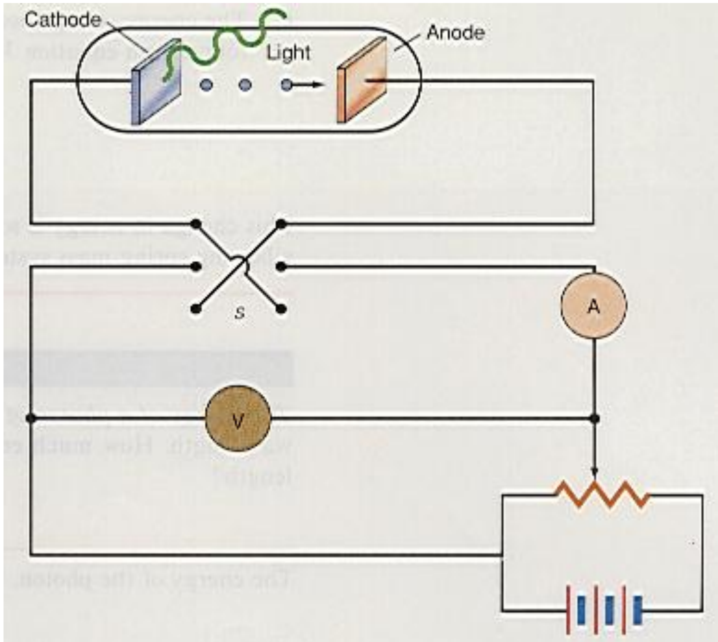

Figure 3.3 Schematic diagram for the photoelectric effect.

The positive anode attracts these electrons, and they flow to the anode and then through the connecting circuit. The ammeter in the circuit measures this current. Starting with a positive potential V, the current is observed for decreasing values of V. When the potential V is reduced to zero, the switch S is reversed to make the anode negative and the cathode positive. The negative anode now repels the photoelectrons as they approach the anode. If this potential is made more and more negative, however, a point is eventually reached when the kinetic energy of the electrons is not great enough to overcome the negative stopping potential, and no more electrons reach the anode. The current i, therefore, becomes zero. A plot of the current i in the circuit, as a function of the potential between the plates, is shown in figure 3.4. If we increase the intensity of the light to I_2 and repeat the experiment, we obtain the second curve shown in the figure.

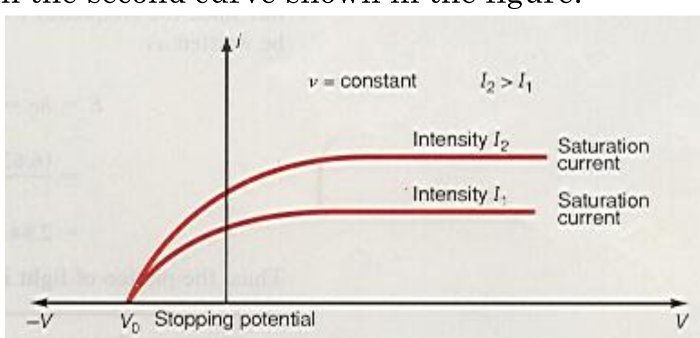

Figure 3.4 Current i as a function of voltage V for the photoelectric effect.

An analysis of figure 3.4 shows that when the value of V is high and positive, the current i is a constant. This occurs because all the photoelectrons formed at the cathode are reaching the anode. By increasing the intensity I, we obtain a higher constant value of current, because more photoelectrons are being emitted per unit time. This shows that the number of electrons emitted (the current) is proportional to the intensity of the incident light, that is,

$$i \propto I$$

Notice that when the potential is reduced to zero, there is still a current in the tube. Even though there is no electric field to draw them to the anode, many of the photoelectrons still reach the anode because of the initial kinetic energy they possess when they leave the cathode. As the switch S in figure 3.3 is reversed, the potential V between the plates becomes negative and tends to repeal the photoelectrons. As the retarding potential V is made more negative, the current i (in figure 3.4) decreases, indicating that fewer and fewer photoelectrons are reaching the anode. When V is reduced to V_0, there is no current at all in the circuit; V_0 is called the *stopping potential.* Note that it is the same value regardless of the intensity. (Both curves intersect at V_0.) Hence the stopping potential is independent of the intensity of light, or stated another way, the stopping potential is not a function of the intensity of light. Stated mathematically this becomes,

$$V_0 \neq V_0(I) \tag{3.3}$$

The retarding potential is related to the kinetic energy of the photoelectrons. For the electron to reach the anode, its kinetic energy must be equal to the potential energy between the plates. (A mechanical analogy might be helpful at this point. If we wish to throw a ball up to a height h, where it will have the potential energy PE = mgh, we must throw the ball with an initial velocity v_0 such that the initial kinetic energy of the ball $\text{KE} = \frac{1}{2}mv_0^2$, is equal to the final potential energy of the ball.) Hence, the kinetic energy of the electron must be

$$\text{KE of electron} = \text{PE between the plates}$$

or

$$\text{KE} = eV \tag{3.4}$$

where e is the charge on the electron and V is the potential between the plates.

The retarding potential acts on electrons that have less kinetic energy than that given by equation 3.4. When $V = V_0$, the stopping potential, even the most energetic electrons (those with maximum kinetic energy) do not reach the anode. Therefore,

$$\text{KE}_{\textbf{max}} = eV_0 \tag{3.5}$$

As equations 3.3 and 3.5 show, the maximum kinetic energy of the photoelectrons is not a function of the intensity of the incident light, that is,

$$\text{KE}_{\textbf{max}} \neq \text{KE}_{\textbf{max}}(I)$$

It is also found experimentally that there is essentially no time lag between the time the light shines on the cathode and the time the photoelectrons are emitted.

If we keep the intensity constant and perform the experiment with different frequencies of light, we obtain the curves shown in figure 3.5. As the graph in figure 3.5 shows, the saturation current (the maximum current) is the same for

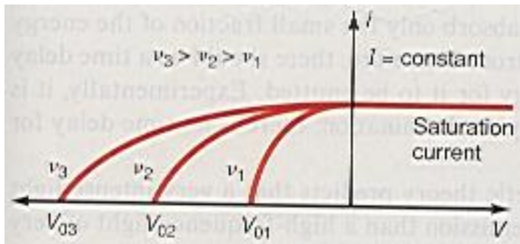

Figure 3.5 Current i as a function of voltage V for different light frequencies.

any frequency of light, as long as the intensity is constant. But the stopping potential is different for each frequency of the incident light. Since the stopping potential is proportional to the maximum kinetic energy of the photoelectrons by equation 3.5, the maximum kinetic energy of the photoelectrons should be proportional to the frequency of the incident light. The maximum kinetic energy of the photoelectrons is plotted as a function of frequency in the graph of figure 3.6.

The first thing to observe is that the maximum kinetic energy of the photoelectrons is proportional to the frequency of the incident light. That is,

$$KE_{\mathbf{max}} \propto \nu$$

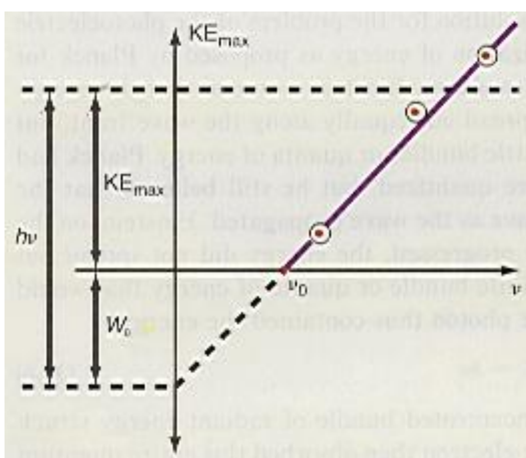

Figure 3.6 Maximum kinetic energy (KE_{max}) as a function of frequency ν for the photoelectric effect.

The second thing to observe is that there is a cutoff frequency ν_0 below which there is no photoelectronic emission. That is, no photoelectric effect occurs unless the incident light has a frequency higher than the threshold frequency ν_0. For most metals, ν_0 lies in the ultraviolet region of the spectrum, but for the alkali metals it lies in the visible region.

Failure of the Classical Theory of Electromagnetism to Explain the Photoelectric Effect

The classical theory of electromagnetism was initially used to try to explain the results of the photoelectric effect. The results of the experiment are compared with the predictions of classical electromagnetic theory in table 3.1. The only agreement

Table 3.1
The Photoelectric Effect

Experimental Results	Theoretical Predictions of Classical Electromagnetism	Agreement
$i \propto I$	$i \propto I$	Yes
Cutoff frequency ν_0	There should not be a cutoff frequency	No
No time lag for emission of electrons	There should be a time lag	No
$KE_{\mathbf{max}} \propto \nu$	$KE_{\mathbf{max}}$ not $\propto \nu$	No
$KE_{\mathbf{max}} \neq KE_{\mathbf{max}}(I)$	$KE_{\mathbf{max}} \propto I$	No

between theory and experiment is the fact that the photocurrent is proportional to the intensity of the incident light. According to classical theory, there should be no minimum threshold frequency ν_0 for emission of photoelectrons. This prediction does not agree with the experimental results.

According to classical electromagnetic theory, energy is distributed equally throughout the entire electric wave front. When the wave hits the electron on the cathode, the electron should be able to absorb only the small fraction of the energy of the total wave that is hitting the electron. Therefore, there should be a time delay to let the electron absorb enough energy for it to be emitted. Experimentally, it is found that emission occurs immediately on illumination; there is no time delay for emission.

Finally, classical electromagnetic theory predicts that a very intense light of very low frequency will cause more emission than a high-frequency light of very low intensity. Again the theory fails to agree with the experimental result. Therefore, classical electromagnetic theory cannot explain the photoelectric effect.

Einstein's Theory of the Photoelectric Effect

In the same year that Einstein published his special theory of relativity, 1905, he also proposed a new and revolutionary solution for the problem of the photoelectric effect. Using the concept of the quantization of energy as proposed by Planck for the solution to the blackbody radiation problem, Einstein assumed that the energy of the electromagnetic wave was not spread out equally along the wave front, but that it was concentrated into Planck's little bundles or quanta of energy. Planck had assumed that the atomic radiators were quantized, but he still believed that the energy became spread out across the wave as the wave propagated. Einstein, on the other hand, assumed that as the wave progressed, the energy did not spread out with the wave front, but stayed in the little bundle or quanta of energy that would later become known as the photon. The photon thus contained the energy

$$E = h\nu \tag{3.6}$$

Einstein assumed that this concentrated bundle of radiant energy struck an electron on the metallic surface. The electron then absorbed this entire quantum of energy ($E = h\nu$). A portion of this energy is used by the electron to break away from the solid, and the rest shows up as the kinetic energy of the electron. That is,

$$\begin{aligned}&\text{(incident absorbed energy)} - \text{(energy to break away from solid)}\\ &\qquad = \text{(maximum KE of electron)}\end{aligned} \tag{3.7}$$

We call the energy for the electron to break away from the solid the *work function of the solid* and denote it by W_0. We can state equation 3.7 mathematically as

$$E - W_0 = \mathrm{KE}_{\mathbf{max}} \tag{3.8}$$

or

$$h\nu - W_0 = \mathrm{KE}_{\mathbf{max}} \tag{3.9}$$

We find the final maximum kinetic energy of the photoelectrons from equation 3.9 as

$$\mathrm{KE}_{\mathbf{max}} = h\nu - W_0 \tag{3.10}$$

Equation 3.10 is known as Einstein's photoelectric equation.

Notice from figure 3.6, when the $\mathrm{KE}_{\mathbf{max}}$ of the photoelectrons is equal to zero, the frequency ν is equal to the cutoff frequency ν_0. Hence, equation 3.10 becomes

$$0 = h\nu_0 - W_0$$

Thus, we can also write the work function of the metal as

$$W_0 = h\nu_0 \tag{3.11}$$

Hence, we can also write Einstein's photoelectric equation as

$$\mathrm{KE}_{\mathbf{max}} = h\nu - h\nu_0 \tag{3.12}$$

For light frequencies equal to or less than ν_0, there is not enough energy in the incident wave to remove the electron from the solid, and hence there is no photoelectric effect. This explains why there is a threshold frequency below which there is no photoelectric effect.

When Einstein proposed his theory of the photoelectric effect, there were not enough quantitative data available to prove the theory. In 1914, R. A. Millikan performed experiments (essentially the experiment described here) that confirmed Einstein's theory of the photoelectric effect.

Einstein's theory accounts for the absence of a time lag for photoelectronic emission. As soon as the electron on the metal surface is hit by a photon, the electron absorbs enough energy to be emitted immediately. Einstein's equation also correctly predicts the fact that the maximum kinetic energy of the photoelectron is dependent on the frequency of the incident light. Thus, Einstein's equation completely predicts the experimental results.

Einstein's theory of the photoelectric effect is outstanding because it was the first application of quantum concepts. *Light should be considered as having not only a wave character, but also a particle character. (The photon is the light particle.)*

For his explanation of the photoelectric effect, Einstein won the Nobel Prize in physics in 1921. As mentioned earlier, Einstein's paper on the photoelectric effect was also published in 1905 around the same time as his paper on special relativity. Thus, he was obviously thinking about both concepts at the same time. It is no wonder then that he was not too upset with dismissing the concept of the ether for the propagation of electromagnetic waves. Because he could now picture light as a particle, a photon, he no longer needed a medium for these waves to propagate in.

Example 3.3

The photoelectric effect. Yellow light of 577.0-nm wavelength is incident on a cesium surface. It is found that no photoelectrons flow in the circuit when the cathode-anode voltage drops below 0.250 V. Find (a) the frequency of the incident photon, (b) the initial energy of the photon, (c) the maximum kinetic energy of the photoelectron, (d) the work function of cesium, (e) the threshold frequency, and (f) the corresponding threshold wavelength.

Solution

a. The frequency of the photon is found from

$$\nu = \frac{c}{\lambda} = \left(\frac{3.00\times10^{8}\ \text{m/s}}{577.0\ \text{nm}}\right)\left(\frac{1\ \text{nm}}{10^{-9}\ \text{m}}\right)$$
$$= 5.20 \times 10^{14}\ \text{Hz}$$

b. The energy of the incident photon, found from equation 3.6, is

$$E = h\nu = (6.625 \times 10^{-34}\ \text{J s})(5.20 \times 10^{14}\ \text{s}^{-1})$$
$$= 3.45 \times 10^{-19}\ \text{J}$$

c. The maximum kinetic energy of the photoelectron, found from equation 3.5, is

$$\begin{aligned} KE_{max} &= eV_0 \\ &= (1.60 \times 10^{-19}\ \text{C})(0.250\ \text{V}) \\ &= 4.00 \times 10^{-20}\ \text{J} \end{aligned}$$

d. The work function of cesium is found by rearranging Einstein's photoelectric equation, 3.8, as

$$\begin{aligned} W_0 &= E - KE_{max} \\ &= 3.45 \times 10^{-19}\ \text{J} - 4.00 \times 10^{-20}\ \text{J} \\ &= 3.05 \times 10^{-19}\ \text{J} \\ &= 1.91\ \text{eV} \end{aligned}$$

e. The threshold frequency is found by solving equation 3.11 for ν_0. Thus,

$$\begin{aligned} \nu_0 = \frac{W_0}{h} &= \frac{3.05 \times 10^{-19}\ \text{J}}{6.625 \times 10^{-34}\ \text{J s}} \\ &= 4.60 \times 10^{14}\ \text{Hz} \end{aligned}$$

f. The wavelength of light associated with the threshold frequency is found from

$$\lambda_0 = \frac{c}{\nu_0} = \left(\frac{3.00\times10^{8}\ \text{m/s}}{4.60\times10^{14}\ \text{s}^{-1}}\right)\left(\frac{1\ \text{nm}}{10^{-9}\ \text{m}}\right)$$
$$= 653\ \text{nm}$$

This wavelength lies in the red portion of the visible spectrum.

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3.4 The Properties of the Photon

According to classical physics light must be a wave. But the results of the photoelectric effect require light to be a particle, a photon. What then is light? Is it a wave or is it a particle?

If light is a particle then it must have some of the characteristics of particles, that is, it should possess mass, energy, and momentum. Let us first consider the mass of the photon. The relativistic mass of a particle was given by equation 1.86 as

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

But the photon is a particle of light and must therefore move at the speed of light c. Hence, its mass becomes

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = \frac{m_0}{0} \tag{3.13}$$

But division by zero is undefined. The only way out of this problem is to *define the rest mass of a photon as being zero,* that is,

Photon $\qquad m_0 = 0 \qquad$ (3.14)

At first this may seem a contradiction, but since the photon always moves at the speed c, it is never at rest, and therefore does not need a rest mass. With $m_0 = 0$, equation 3.13 becomes 0/0, which is an indeterminate form. Although the mass of the photon still cannot be defined by equation 1.86 it can be defined from equation 1.100, namely

$$E = mc^2$$

Hence,

$$m = \frac{E}{c^2} \tag{3.15}$$

The energy of the photon was given by

Energy of Photon $\qquad E = h\nu \qquad$ (3.6)

Therefore, the mass of the photon can be found by substituting equation 3.6 into equation 3.15, that is,

Mass of Photon $\qquad m = \dfrac{E}{c^2} = \dfrac{h\nu}{c^2} \qquad$ (3.16)

Example 3.4

The mass of a photon. Find the mass of a photon of light that has a wavelength of (a) 380.0 nm and (b) 720.0 nm.

Solution

a. For λ = 380.0 nm, the frequency of the photon is found from

$$\nu = \frac{c}{\lambda} = \left(\frac{3.00 \times 10^8 \text{ m/s}}{380.0 \text{ nm}}\right)\left(\frac{1 \text{ nm}}{10^{-9} \text{ m}}\right)$$
$$= 7.89 \times 10^{14} \text{ Hz}$$

Now we can find the mass from equation 3.16 as

$$m = \frac{E}{c^2} = \frac{h\nu}{c^2} = \frac{(6.625\times10^{-34}\ \text{J s})(7.89\times10^{14}\ \text{s}^{-1})}{(3.00\times10^{8}\ \text{m/s})^2}\left[\frac{(\text{kg m/s}^2)\ \text{m}}{\text{J}}\right]$$
$$= 5.81 \times 10^{-36}\ \text{kg}$$

b. For λ = 720.0 nm, the frequency is

$$\nu = \frac{c}{\lambda} = \left(\frac{3.00\times10^{8}\ \text{m/s}}{720.0\ \text{nm}}\right)\left(\frac{1\ \text{nm}}{10^{-9}\ \text{m}}\right)$$
$$= 4.17 \times 10^{14}\ 1/\text{s}$$

and the mass is

$$m = \frac{h\nu}{c^2} = \frac{(6.625\times10^{-34}\ \text{J s})(4.17\times10^{14}\ \text{s}^{-1})}{(3.00\times10^{8}\ \text{m/s})^2}$$
$$= 3.07 \times 10^{-36}\ \text{kg}$$

As we can see from these examples, the mass of the photon for visible light is very small.

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The momentum of the photon can be found as follows. Starting with the relativistic mass

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \tag{1.86}$$

we square both sides of the equation and obtain

$$m^2\left(1 - \frac{v^2}{c^2}\right) = m_0^2$$
$$m^2 - \frac{m^2v^2}{c^2} = m_o^2 \tag{3.17}$$

Multiplying both sides of equation 3.17 by c^4, we obtain

$$m^2c^4 - m^2v^2c^2 = m_o^2c^4$$

But $m^2c^4 = E^2$, $m_o^2c^4 = E_0^2$, and $m^2v^2 = p^2$, thus,

$$E^2 - p^2c^2 = E_0^2 \tag{3.18}$$

Hence, we find the momentum of any particle from equation 3.18 as

$$p = \frac{\sqrt{E^2 - E_0^2}}{c} \tag{3.19}$$

For the special case of a particle of zero rest mass, $E_0 = m_0c^2 = 0$, and *the momentum of a photon,* found from equation 3.19, is

Momentum of Photon
$$p = \frac{E}{c} \tag{3.20}$$

Using equation 3.6, we can write the momentum of a photon in terms of its frequency as

$$p = \frac{E}{c} = \frac{h\nu}{c}$$

Since $\nu/c = 1/\lambda$, this is also written as

Momentum of Photon
$$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} \tag{3.21}$$

Example 3.5

The momentum of a photon. Find the momentum of visible light for (a) λ = 380.0 nm and (b) λ = 720.0 nm.

Solution

a. The momentum of the photon, found from equation 3.21, is

$$p = \frac{h}{\lambda} = \left(\frac{(6.625\times10^{-34}\text{ J s})}{(380.0\text{ nm})}\right)\left(\frac{1\text{ nm}}{10^{-9}\text{ m}}\right)\left[\frac{(\text{kg m/s}^2)\text{ m}}{\text{J}}\right]$$
$$= 1.74 \times 10^{-27}\text{ kg m/s}$$

b. The momentum of the second photon is found similarly

$$p = \frac{h}{\lambda} = \frac{6.625 \times 10^{-34}\text{ J s}}{720.0\text{ nm}}$$
$$= 9.20 \times 10^{-28}\text{ kg m/s}$$

To go to this Interactive Example click on this sentence.

According to this quantum theory of light, light spreads out from a source in small bundles of energy called quanta or photons. *Although the photon is treated as a particle, its properties of mass, energy, and momentum are described in terms of frequency or wavelength, strictly a wave concept.*

Thus, we say that *light has a dual nature. It can act as a wave or it can act as a particle, but never both at the same time.* To answer the question posed at the beginning of this section, is light a wave or a particle, the answer is that light is both a wave and a particle. This dual nature of light is stated in the ***principle of complementarity:*** *The wave theory of light and the quantum theory of light complement each other. In a specific event, light exhibits either a wave nature or a particle nature, but never both at the same time.*

When the wavelength of an electromagnetic wave is long, its frequency and hence its photon energy ($E = h\nu$) are small and we are usually concerned with the wave characteristics of the electromagnetic wave. For example, radio and television waves have relatively long wavelengths and they are usually treated as waves. When the wavelength of the electromagnetic wave is small, its frequency and hence its photon energy are large. The electromagnetic wave is then usually considered as a particle. For example, X rays have very small wavelengths and are usually treated as particles. However, this does not mean that X rays cannot also act as waves. In fact they do. When X rays are scattered from a crystal, they behave like waves, exhibiting the usual diffraction patterns associated with waves. The important thing is that light can act either as a wave or a particle, but never both at the same time.

Let us *summarize the characteristics of the photon:*

Rest Mass $$m_0 = 0 \tag{3.14}$$

Energy $$E = h\nu \tag{3.6}$$

Mass $$m = \frac{E}{c^2} = \frac{h\nu}{c^2} \tag{3.16}$$

Momentum of Photon $$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} \tag{3.21}$$

Although the two examples considered were for photons of visible light, do not forget that the photon is a particle in the entire electromagnetic spectrum.

Example 3.6

The mass of an X ray and a gamma ray. Find the mass of a photon for (a) an X ray of 100.0-nm wavelength and (b) for a gamma ray of 0.0500 nm.

Solution

a. The mass of an X-ray photon, found from equation 3.16, is

$$m = \frac{h\nu}{c^2} = \frac{h}{c\lambda}$$

$$= \frac{(6.625\times10^{-34}\text{ J s})}{(3.00\times10^{8}\text{ m/s})(100.0\text{ nm})}\left(\frac{1\text{ nm}}{10^{-9}\text{ m}}\right)$$

$$= 2.20 \times 10^{-35}\text{ kg}$$

b. The mass of the gamma ray is

$$m = \frac{h\nu}{c^2} = \frac{h}{c\lambda}$$

$$= \frac{(6.625\times10^{-34}\text{ J s})}{(3.00\times10^{8}\text{ m/s})(0.0500\text{ nm})}\left(\frac{1\text{ nm}}{10^{-9}\text{ m}}\right)$$

$$= 4.42 \times 10^{-32}\text{ kg}$$

To go to this Interactive Example click on this sentence.

Comparing the mass of a photon for red light, violet light, X rays, and gamma rays we see

$$m_{\textbf{red}} = 3.07 \times 10^{-36}\text{ kg}$$
$$m_{\text{violet}} = 5.81 \times 10^{-36}\text{ kg}$$
$$m_{\text{X ray}} = 22.0 \times 10^{-36}\text{ kg}$$
$$m_{\text{gamma ray}} = 44{,}200 \times 10^{-36}\text{ kg}$$

Thus, as the frequency of the electromagnetic spectrum increases (wavelength decreases), the mass of the photon increases.

3.5 The Compton Effect

If light sometimes behaves like a particle, the photon, why not consider the collision of a photon with a free electron from the same point of view as the collision of two billiard balls? Such a collision between a photon and a free electron is called Compton scattering, or the **Compton effect,** in honor of Arthur Holly Compton (1892-1962). In order to get a massive photon for the collision, X rays are used. (Recall that X rays have a high frequency ν, and therefore the energy of the X ray, $E = h\nu$, is large, and thus its mass, $m = E/c^2$, is also large.) In order to get a free electron, a target made of carbon is used. The outer electrons of the carbon atom are very loosely bound, so compared with the initial energy of the photon, the electron looks like a free electron. Thus, the collision between the photon and the electron can be pictured as shown in figure 3.7. We assume that the electron is initially at

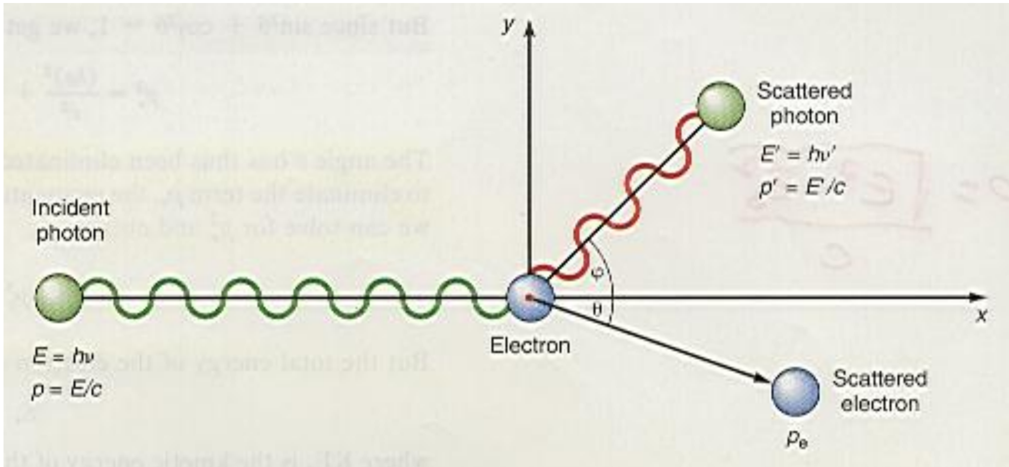


Figure 3.7 Compton scattering.

rest and that the incident photon has an energy ($E = h\nu$) and a momentum ($p = E/c$). After the collision, the electron is found to be scattered at an angle θ from the original direction of the photon. *Because the electron has moved after the collision, some energy must have been imparted to it. But where could this energy come from? It must have come from the incident photon. But if that is true, then the scattered photon must have less energy than the incident photon, and therefore, its wavelength should also have changed.* Let us call the energy of the scattered photon E', where

$$E' = h\nu'$$

and hence, its final momentum is

$$p' = \frac{E'}{c} = \frac{h\nu'}{c}$$

Because momentum is conserved in all collisions, the law of conservation of momentum is applied to the collision of figure 3.7. First however, notice that the collision is two dimensional. Because the vector momentum is conserved, the x-component of the momentum and the y-component of momentum must also be conserved. The law of conservation for the x-component of momentum can be written as

$$p_\mathbf{p} + 0 = p_\mathrm{p}' \cos\phi + p_\mathbf{e} \cos\theta$$

and for the y-component,

$$0 + 0 = p_\mathrm{p}' \sin\phi - p_\mathbf{e} \sin\theta$$

where $p_\mathbf{p}$ is the momentum of the incident photon, p_p' the momentum of the scattered photon, and $p_\mathbf{e}$ the momentum of the scattered electron. Substituting the values for the energy and momentum of the photon, these equations become

$$\frac{h\nu}{c} = \frac{h\nu'}{c} \cos\phi + p_e \cos\theta$$
$$0 = \frac{h\nu'}{c} \sin\phi - p_e \sin\theta$$

There are more unknowns (ν', θ, ϕ, p_e) than we can handle at this moment, so let us eliminate θ from these two equations by rearranging, squaring, and adding them. That is,

$$p_e \cos\theta = \frac{h\nu}{c} - \frac{h\nu'}{c} \cos\phi$$
$$p_e \sin\theta = \frac{h\nu'}{c} \sin\phi$$
$$p_e^2 \cos^2\theta = \frac{(h\nu)^2}{c^2} - \frac{2h\nu h\nu' \cos\phi}{c^2} + \frac{(h\nu')^2}{c^2} \cos^2\phi$$
$$p_e^2 \sin^2\theta = \frac{(h\nu')^2}{c^2} \sin^2\phi$$
$$p_e^2 (\sin^2\theta + \cos^2\theta) = \frac{(h\nu)^2}{c^2} + \frac{(h\nu')^2}{c^2} (\sin^2\phi + \cos^2\phi) - \frac{2(h\nu)(h\nu')}{c^2} \cos\phi$$

But since $\sin^2\theta + \cos^2\theta = 1$, we get

$$p_e^2 = \frac{(h\nu)^2}{c^2} + \frac{(h\nu')^2}{c^2} - \frac{2(h\nu)(h\nu')}{c^2} \cos\phi \tag{3.22}$$

The angle θ has thus been eliminated from the equation. Let us now look for a way to eliminate the term p_e, the momentum of the electron. If we square equation 3.19, we can solve for p_e^2 and obtain

$$p_e^2 = \frac{E_e^{\,2} - E_{0e}^2}{c^2} \tag{3.23}$$

But the total energy of the electron E_e, given by equation 1.102, is

$$E_e = \mathrm{KE}_e + E_{0e}$$

where KE_e is the kinetic energy of the electron and E_{0e} is its rest mass. Substituting equation 1.102 back into equation 3.23, gives, for the momentum of the electron,

$$p_e^2 = \frac{(\mathrm{KE}_e + E_{0e})^2 - E_{0e}^2}{c^2}$$
$$= \frac{\mathrm{KE}_e^2 + 2E_{0e}\,\mathrm{KE}_e + E_0^2 - E_0^2}{c^2}$$
$$p_e^2 = \frac{\mathrm{KE}_e^2 + 2E_{0e}\,\mathrm{KE}_e}{c^2} \tag{3.24}$$

But if the law of conservation of energy is applied to the collision of figure 3.7, we get

$$E = E' + \mathrm{KE_e}$$
$$h\nu = h\nu' + \mathrm{KE_e} \tag{3.25}$$

where E is the total energy of the system, E' is the energy of the scattered photon and $\mathrm{KE_e}$ is the kinetic energy imparted to the electron during the collision. Thus, the kinetic energy of the electron, found from equation 3.25, is

$$\mathrm{KE_e} = h\nu - h\nu' \tag{3.26}$$

Substituting the value of the kinetic energy from equation 3.26 and $E_{0e} = m_0c^2$, the rest energy of the electron, back into equation 3.24, we get, for the momentum of the electron,

$$p_e^2 = \frac{(h\nu - h\nu')^2 + 2m_0c^2(h\nu - h\nu')}{c^2}$$
$$p_e^2 = \frac{(h\nu)^2}{c^2} + \frac{(h\nu')^2}{c^2} - \frac{2h\nu h\nu'}{c^2} + 2m_0(h\nu - h\nu') \tag{3.27}$$

Since we now have two separate equations for the momentum of the electron, equations 3.22 and 3.27, we can equate them to eliminate p_e. Therefore,

$$\frac{(h\nu)^2}{c^2} + \frac{(h\nu')^2}{c^2} - \frac{2h\nu h\nu'}{c^2} + 2m_0(h\nu - h\nu') = \frac{(h\nu)^2}{c^2} + \frac{(h\nu')^2}{c^2} - \frac{2(h\nu)(h\nu')}{c^2}\cos\phi$$

Simplifying,

$$2m_0(h\nu - h\nu') = \frac{2h\nu h\nu'}{c^2} - \frac{2(h\nu)(h\nu')}{c^2}\cos\phi$$
$$h\nu - h\nu' = \frac{h\nu h\nu'}{m_0c^2}(1 - \cos\phi)$$
$$\frac{\nu - \nu'}{\nu\nu'} = \frac{h}{m_0c^2}(1 - \cos\phi)$$

However, since $\nu = c/\lambda$ this becomes

$$\frac{c/\lambda - c/\lambda'}{(c/\lambda)(c/\lambda')} = \frac{h}{m_0c^2}(1 - \cos\phi)$$
$$\lambda\lambda'\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{h}{m_0c}\left(1 - \cos\phi\right)$$
$$(\lambda' - \lambda) = \frac{h}{m_0c}\left(1 - \cos\phi\right) \tag{3.28}$$

Equation 3.28 is called the Compton scattering formula. It gives the change in wavelength of the scattered photon as a function of the scattering angle ϕ. The quantity,

$$\frac{h}{m_0c} = 2.426 \times 10^{-12} \text{ m} = 0.002426 \text{ nm}$$

which has the dimensions of a length, is called the *Compton wavelength.*

Thus, in a collision between an energetic photon and an electron, the scattered light shows a different wavelength than the wavelength of the incident light. In 1923, A. H. Compton confirmed the modified wavelength of the scattered photon and received the Nobel Prize in 1927 for his work.

Example 3.7

Compton scattering. A 90.0-KeV X-ray photon is fired at a carbon target and Compton scattering occurs. Find the wavelength of the incident photon and the wavelength of the scattered photon for scattering angles of (a) 30.0^0 and (b) 60.0^0.

Solution

The frequency of the incident photon is found from $E = h\nu$ as

$$\nu = \frac{E}{h} = \left(\frac{90.0\times10^3 \text{ eV}}{6.625\times10^{-34} \text{ J s}}\right)\left(\frac{1.60\times10^{-19} \text{ J s}}{1 \text{ eV}}\right)$$
$$= 2.17 \times 10^{19} \text{ Hz}$$

The wavelength of the incident photon is found from

$$\lambda = \frac{c}{\nu} = \left(\frac{3.00\times10^8 \text{ m/s}}{2.17\times10^{19} \text{ 1/s}}\right)\left(\frac{1 \text{ nm}}{10^{-9} \text{ m}}\right)$$
$$=0.0138 \text{ nm}$$

The modified wavelength is found from the Compton scattering formula, equation 3.28, as

$$\lambda' = \lambda + \frac{h}{m_0c}(1 - \cos\phi)$$

a.

$$\lambda' = 0.0138 \text{ nm} + (0.002426 \text{ nm})(1 - \cos 30.0^0)$$
$$= 0.0141 \text{ nm}$$

b.

$$\lambda' = 0.0138 \text{ nm} + (0.002426 \text{ nm})(1 - \cos 60.0^0)$$
$$= 0.0150 \text{ nm}$$

To go to this Interactive Example click on this sentence.

In an actual experiment both the incident and modified wavelengths are found in the scattered photons. The incident wavelength is found in the scattered photons because some of the incident photons are scattered by the atom. In this case, the rest mass of the electron m_0 must be replaced in equation 3.28 by the mass M of the entire atom. Because M is so much greater than m_0, the Compton wavelength h/MC is so small that the change in wavelength for these photons is too small to be observed. Thus, these incident photons are scattered with the same wavelength.

3.6 The Wave Nature of Particles

We have seen that light displays a dual nature; it acts as a wave and it acts as a particle. Assuming symmetry in nature, the French physicist Louis de Broglie (1892-1987) proposed, in his 1924 doctoral dissertation, that particles should also possess a wave characteristic. Because the momentum of a photon was shown to be

$$p = \frac{h}{\lambda} \tag{3.21}$$

de Broglie assumed that the wavelength of the wave associated with a particle of momentum p, should be given by

$$\lambda = \frac{h}{p} \tag{3.29}$$

Equation 3.29 is called the ***de Broglie relation.*** *Thus, de Broglie assumed that the same wave-particle duality associated with electromagnetic waves should also apply to particles.* Hence, an electron can be considered to be a particle and it can also be considered to be a wave. Instead of solving the problem of the wave-particle duality of electromagnetic waves, de Broglie extended it to include matter as well.

Example 3.8

The wavelength of a particle. Calculate the wavelength of (a) a 0.140-kg baseball moving at a speed of 44.0 m/s, (b) a proton moving at the same speed, and (c) an electron moving at the same speed.

Solution

a. A baseball has an associated wavelength given by equation 3.29 as

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{6.625 \times 10^{-34} \text{ J s}}{(0.140 \text{ kg})(44.0 \text{ m/s})}$$
$$= 1.08 \times 10^{-34} \text{ m}$$

Such a small wavelength cannot be measured and therefore baseballs always appear as particles.

b. The wavelength of the proton, found from equation 3.29, is

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$
$$= \frac{(6.625\times10^{-34} \text{ J s})}{(1.67\times10^{-27} \text{ kg})(44.0 \text{ m/s})}\left(\frac{1 \text{ nm}}{10^{-9} \text{ m}}\right)$$
$$= 9.02 \text{ nm}$$

Although this wavelength is small (it is in the X-ray region of the electromagnetic spectrum), it can be detected.

c. The wavelength of the electron is found from

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$
$$= \frac{(6.625\times10^{-34} \text{ J s})}{(9.11\times10^{-31} \text{ kg})(44.0 \text{ m/s})}\left(\frac{1 \text{ nm}}{10^{-9} \text{ m}}\right)$$
$$= 1.65 \times 10^{4} \text{ nm}$$

which is a very large wavelength and can be easily detected.

To go to this Interactive Example click on this sentence.

Note from example 3.8, that because Planck's constant h is so small, the wave nature of a particle does not manifest itself unless the mass m of the particle is also very small (of the order of an atom or smaller). This is why the wave nature of particles is not part of our everyday experience.

de Broglie's hypothesis was almost immediately confirmed when in 1927 C. J. Davisson and L. H. Germer performed an experiment that showed that electrons could be diffracted by a crystal. G. P. Thomson performed an independent experiment at the same time by scattering electrons from very thin metal foils and obtained the standard diffraction patterns that are usually associated with waves. Since that time diffraction patterns have been observed with protons, neutrons, hydrogen atoms, and helium atoms, thereby giving substantial evidence for the wave nature of particles.

For his work on the dual nature of particles, de Broglie received the 1929 Nobel Prize in physics. Davisson and Thomson shared the Nobel Prize in 1937 for their experimental confirmation of the wave nature of particles.

3.7 The Wave Representation of a Particle

We have just seen that a particle can be represented by a wave. The wave associated with a photon was an electromagnetic wave. *But what kind of wave is associated with a particle?* It is certainly not an electromagnetic wave. de Broglie called the wave a *pilot wave* because he believed that it steered the particle during its motion. The waves have also been called *matter waves* to show that they are associated with matter. Today, the wave is simply referred to as the *wave function* and is represented by Ψ.

Because this wave function refers to the motion of a particle we say that *the value of the wave function Ψ is related to the probability of finding the particle at a specific place and time.* The probability P that something can be somewhere at a certain time, can have any value between 0 and 1. If the probability $P = 0$, then there is an absolute certainty that the particle is absent. If the probability $P = 1$, then there is an absolute certainty that the particle is present. If the probability P lies somewhere between 0 and 1, then that value is the probability of finding the particle there. That is, if the probability $P = 0.20$, there is a 20% probability of finding the particle at the specified place and time.

Because the amplitude of any wave varies between positive and negative values, the wave function Ψ cannot by itself represent the probability of finding the particle at a particular time and place. However, the quantity Ψ^2 is always positive and is called the probability density. *The probability density Ψ^2 is the probability of finding the particle at the position (x, y, z) at the time t.* The new science of wave mechanics, or as it was eventually called, quantum mechanics, has to do with determining the wave function Ψ for any particle or system of particles.

How can a particle be represented by a wave? Recall from general physics, that a wave moving to the right is defined by the function

$$y = A \sin(kx - \omega t)$$

where the wave number k is

$$k = \frac{2\pi}{\lambda}$$

and the angular frequency ω is given by

$$\omega = 2\pi f$$

or since $f = \nu$, in our new notation,

$$\omega = 2\pi\nu$$

Also recall that the velocity of the wave is given by

$$v = \frac{\omega}{k}$$

We will therefore begin, in our analysis of matter waves, by trying to define the wave function as

$$\Psi = A \sin(kx - \omega t) \tag{3.30}$$

A plot of this wave function for $t = 0$ is shown in figure 3.8(a). The first thing to observe in this picture is that the wave is too spread out to be able to represent a particle. Remember the particle must be found somewhere within the wave. Because the wave extends out to infinity the particle could be anywhere.

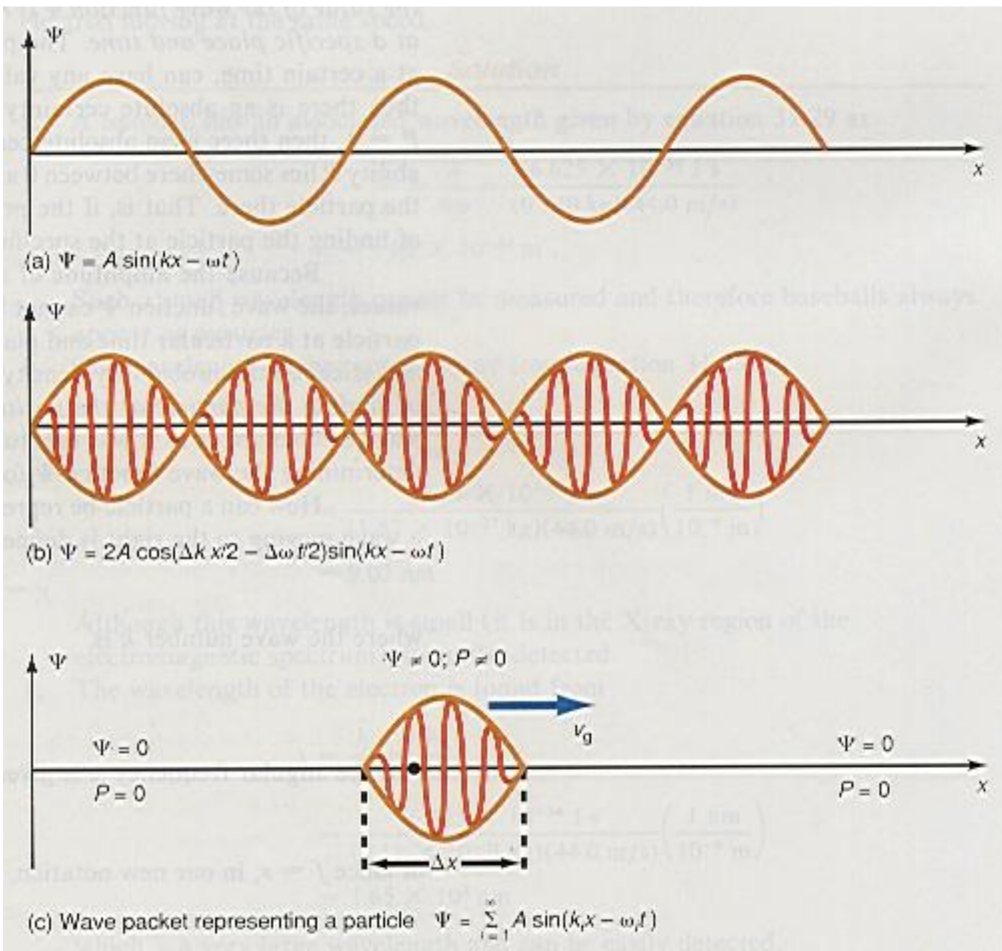


Figure 3.8 Representation of a particle as a wave.

Because one of the characteristics of waves is that they obey the superposition principle, perhaps a wave representation can be found by adding different waves together. As an example, let us add two waves of slightly different wave numbers and slightly different angular frequencies. That is, consider the two waves

$$\Psi_1 = A\sin(k_1x - \omega_1t)$$
$$\Psi_2 = A\sin(k_2x - \omega_2t)$$

where

$$k_2 = k_1 + \Delta k$$

and

$$\omega_2 = \omega_1 + \Delta\omega$$

The addition of these two waves gives

$$\Psi = \Psi_1 + \Psi_2$$
$$= A\sin(k_1x - \omega_1t) + A\sin(k_2x - \omega_2t)$$

The addition of two sine waves is shown in appendix B, to be

$$\sin B + \sin C = 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$

Letting

$$B = k_1x - \omega_1t$$

and

$$C = k_2x - \omega_2t$$

we find

$$\Psi = 2A\sin\left(\frac{k_1x - \omega_1t + k_2x - \omega_2t}{2}\right)\cos\left(\frac{k_1x - \omega_1t - k_2x + \omega_2t}{2}\right)$$

$$= 2A\sin\left[\frac{k_1x - \omega_1t + (k_1 + \Delta k)x - (\omega_1 + \Delta\omega)t}{2}\right]\cos\left(\frac{k_1x - \omega_1t - (k_1 + \Delta k)x + (\omega_1 + \Delta\omega)t}{2}\right)$$

$$= 2A\sin\left[\frac{2kx + (\Delta k)x - 2\omega t - (\Delta\omega)t}{2}\right]\cos\left(\frac{-\Delta k}{2}x + \frac{\Delta\omega}{2}t\right)$$

We have dropped the subscript 1 on k and ω to establish the general case. Now as an approximation

$$2kx + (\Delta k)x \approx 2kx$$

and

$$-2\omega t - (\Delta\omega)t \approx -2\omega t$$

Therefore,

$$\Psi = 2A\sin(kx - \omega t)\cos\left(\frac{-\Delta k}{2}x + \frac{\Delta\omega}{2}t\right)$$

One of the properties of the cosine function is that $\cos(-\theta) = \cos\theta$. Using this relation the wave function becomes

$$\Psi = 2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\sin(kx - \omega t) \tag{3.31}$$

A plot of equation 3.31 is shown in figure 3.8(b). The amplitude of this wave is modulated and is given by the first part of equation 3.31 as

$$A_m = 2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right) \tag{3.32}$$

This wave superposition gives us a closer representation of a particle. Each modulated portion of the wave represents a group of waves and any one group can represent a particle. The velocity of the group of waves represents the velocity of the particle.

Equation 3.31 and figure 3.8(b) approaches a wave representation of the particle. If an infinite number of waves, each differing slightly in wave number and angular frequency, were added together we would get the wave function

$$\Psi = \sum_{i=1}^{\infty} A\sin(k_i x - \omega_i t) \tag{3.33}$$

which is shown in figure 3.8(c) and is called a *wave packet. This wave packet can indeed represent the motion of a particle. Because the wave function* Ψ *is zero everywhere except within the packet, the probability of finding the particle is zero everywhere except within the packet.* The wave packet localizes the particle to be within the region Δx shown in figure 3.8(c), and the wave packet moves with the group velocity of the waves and this is the velocity of the particle. The fundamental object of wave mechanics or quantum mechanics is to find the wave function Ψ associated with a particle or a system of particles.

3.8 The Heisenberg Uncertainty Principle

One of the characteristics of the dual nature of matter is a fundamental limitation in the accuracy of the measurement of the position and momentum of a particle. This can be seen in a very simplified way by looking at the modulated wave of figure 3.8(b) and reproduced in figure 3.9. A particle is shown located in the first group of the modulated wave. Since the particle lies somewhere within the wave packet its exact position is uncertain. The amount of the uncertainty in its position is no greater than Δx, the width of the entire wave packet or wave group. The wavelength of the modulated amplitude λ_m is shown in figure 3.9 and we can see that a wave group is only half that distance. Thus, the uncertainty in the location of the particle is given by

$$\Delta x = \frac{\lambda_m}{2} \tag{3.34}$$

The uncertainty in the momentum can be found by solving the de Broglie relation, equation 3.29, for momentum as

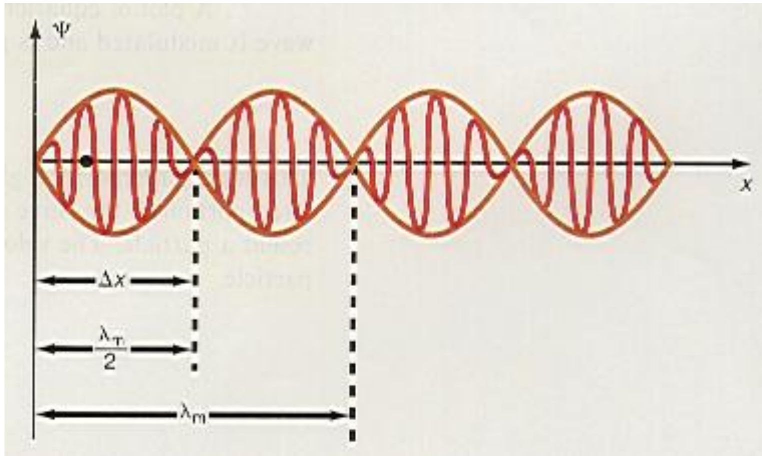


Figure 3.9 Limitations on position and momentum of a particle.

$$p = \frac{h}{\lambda} \tag{3.35}$$

and the fact that the wavelength is given in terms of the wave number by

$$\lambda = \frac{2\pi}{k} \tag{3.36}$$

Substituting equation 3.36 into equation 3.35 gives

$$p = \frac{h}{\lambda} = \frac{h}{2\pi/k} = \frac{h}{2\pi} k \tag{3.37}$$

The uncertainty in the momentum, found from equation 3.37, is

$$\Delta p = \frac{h}{2\pi} \Delta k \tag{3.38}$$

Because the wave packet is made up of many waves, there is a Δk associated with it. This means that in representing a particle as a wave, there is automatically an uncertainty in the wave number, k, which we now see implies an uncertainty in the momentum of that particle. For the special case considered in figure 3.9, the wave number of the modulated wave Δk_m is found from

$$A_m = 2A \cos(k_m x - \omega_m t)$$

and from equation 3.32 as

$$k_m = \frac{\Delta k}{2} \tag{3.39}$$

But from the definition of a wave number

$$k_m = \frac{2\pi}{\lambda_m} \tag{3.40}$$

Substituting equation 3.40 into equation 3.39 gives, for Δk,

$$\Delta k = 2k_m = 2\left(\frac{2\pi}{\lambda_m}\right) \tag{3.41}$$

Substituting the uncertainty for Δk, equation 3.41, into the uncertainty for Δp, equation 3.38, gives

$$\Delta p = \frac{h}{2\pi}\Delta k = \frac{h}{2\pi}2\left(\frac{2\pi}{\lambda_m}\right) = \frac{h}{\lambda_m/2} \tag{3.42}$$

The uncertainty between the position and momentum of the particle is obtained by substituting equation 3.34 for $\lambda_m/2$ into equation 3.42 to get

$$\Delta p = \frac{h}{\Delta x}$$

or

$$\Delta p\Delta x = h \tag{3.43}$$

Because Δp and Δx are the smallest uncertainties that a particle can have, their values are usually greater than this, so their product is usually greater than the value of h. To show this, equation 3.43 is usually written with an inequality sign also, that is,

$$\Delta p\Delta x \geq h$$

The analysis of the wave packet was greatly simplified by using the modulated wave of figure 3.8(b). A more sophisticated analysis applied to the more reasonable wave packet of figure 3.8(c) yields the relation

$$\Delta p\Delta x \geq \hbar \tag{3.44}$$

where the symbol $\hbar$, called h bar, is

$$\hbar = \frac{h}{2\pi} = 1.05\times10^{-34} \text{ J s} \tag{3.45}$$

Equation 3.44 is called the ***Heisenberg uncertainty principle.*** *It says that the position and momentum of a particle cannot both be measured simultaneously with perfect accuracy. There is always a fundamental uncertainty associated with any measurement. This uncertainty is not associated with the measuring instrument. It is a consequence of the wave-particle duality of matter.*

As an example of the application of equation 3.44, if the position of a particle is known exactly, then $\Delta x = 0$ and Δp would have to be infinite in order for the product $\Delta x \Delta p$ to be greater than $\hbar$. If Δp is infinite, the value of the momentum of the particle is completely unknown. A wave packet associated with a very accurate value of position is shown in figure 3.10(a). Although this wave packet

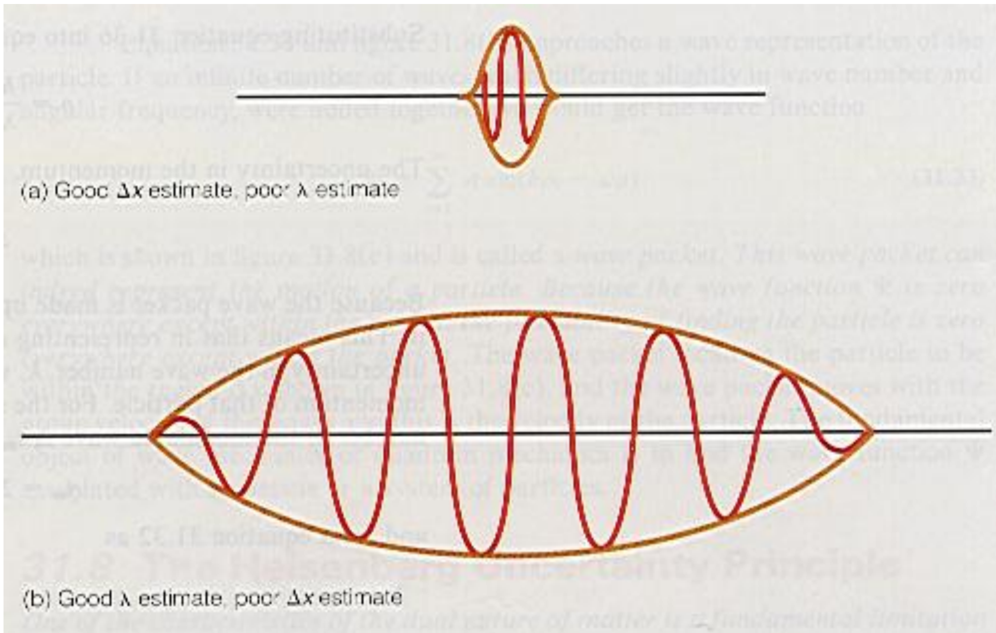


Figure 3.10 Wave packets of different size.

gives a very small value of Δx, it gives an exceedingly poor representation of the wavelength. Because the uncertainty in the wavelength λ is large, the uncertainty in the wave number is also large. Since the uncertainty in the wave number is related to the uncertainty in the momentum of the particle by equation 3.38, there is also a large uncertainty in the momentum of the particle. Thus, a good Δx estimate always gives a poor Δp estimate.

If the momentum of a particle is known exactly, then $\Delta p = 0$, and this implies that Δx must approach infinity. That is, if the momentum of a particle is known exactly, the particle could be located anywhere. A wave packet approximating this case is shown in figure 3.10(b). Because the wave packet is spread out over a large area it is easy to get a good estimate of the de Broglie wavelength, and hence a good estimate of the momentum of the particle. On the other hand, since the wave packet is so spread out, it is very difficult to locate the particle inside the wave packet. Thus a good Δp estimate always gives a poor Δx estimate.

Example 3.9

The uncertainty in the velocity of a baseball. A 0.140-kg baseball is moving along the x-axis. At a particular instant of time it is located at the position $x = 0.500$ m with an uncertainty in the measurement of Δx = 0.001 m. How accurately can the velocity of the baseball be determined?

Solution

The uncertainty in the momentum is found by the Heisenberg uncertainty principle, equation 3.44, as

$$\Delta p \geq \frac{\hbar}{\Delta x}$$
$$\geq \frac{1.05 \times 10^{-34} \text{ J s}}{0.001 \text{ m}}$$
$$\geq 1.05 \times 10^{-31} \text{ kg m/s}$$

Since $p = mv$, the uncertainty in the velocity is

$$\Delta v \geq \frac{\Delta p}{m}$$
$$= \frac{1.05 \times 10^{-31} \text{ kg m/s}}{0.140 \text{ kg}}$$
$$\geq 7.50 \times 10^{-31} \text{ m/s}$$

The error in Δp and Δv caused by the uncertainty principle is so small for macroscopic bodies moving around in the everyday world that it can be neglected.

To go to this Interactive Example click on this sentence.

Example 3.10

The uncertainty in the velocity of an electron confined to a box the size of the nucleus. We want to confine an electron, $m_e = 9.11 \times 10^{-31}$ kg, to a box, 1.00×10^{-14} m long (approximately the size of a nucleus). What would the speed of the electron be if it were so confined?

Solution

Because the electron can be located anywhere within the box, the worst case of locating the electron is for the uncertainty of the location of the electron to be equal to the size of the box itself. That is, $\Delta x = 1.00 \times 10^{-14}$ m.

We also assume that the uncertainty in the velocity is so bad that it is equal to the velocity of the electron itself. The uncertainty in the speed, found from the Heisenberg uncertainty principle, is

$$\Delta p \geq \frac{\hbar}{\Delta x}$$

$$m\Delta v \geq \frac{\hbar}{\Delta x}$$

$$\Delta v \geq \frac{\hbar}{m\Delta x} \tag{3.46}$$

$$\geq \frac{1.05\times10^{-34}\ \text{J s}}{(9.11\times10^{-31}\ \text{kg})(1.00\times10^{-14}\ \text{m})}\left[\frac{(\text{kg m/s}^2)\ \text{m}}{\text{J}}\right]$$

$$\geq 1.15 \times 10^{10}\ \text{m/s}$$

Hence, for the electron to be confined in a box about the size of the nucleus, its speed would have to be greater than the speed of light. Because this is impossible, we must conclude that an electron can never be found inside of a nucleus.

To go to this Interactive Example click on this sentence.

Example 3.11

The uncertainty in the velocity of an electron confined to a box the size of an atom. An electron is placed in a box about the size of an atom, that is, $\Delta x = 1.00 \times 10^{-12}$ m. What is the velocity of the electron?

Solution

We again assume that the velocity of the electron is of the same order as the uncertainty in the velocity, then from equation 3.46, we have

$$\Delta v \geq \frac{\hbar}{m\Delta x}$$

$$\geq \frac{1.05 \times 10^{-34}\ \text{J s}}{(9.11 \times 10^{-31}\ \text{kg})(1.00 \times 10^{-12}\ \text{m})}$$

$$\geq 1.15 \times 10^{8}\ \text{m/s}$$

Because this velocity is less than the velocity of light, an electron can exist in an atom. Notice from these examples that the uncertainty principle is only important on the microscopic level.

To go to this Interactive Example click on this sentence.

Another way to observe the effect of the uncertainty principle from a more physical viewpoint is to see what happens when we "see" a particle in order to locate its position. Figure 3.11(a) shows how we locate a moving baseball. The

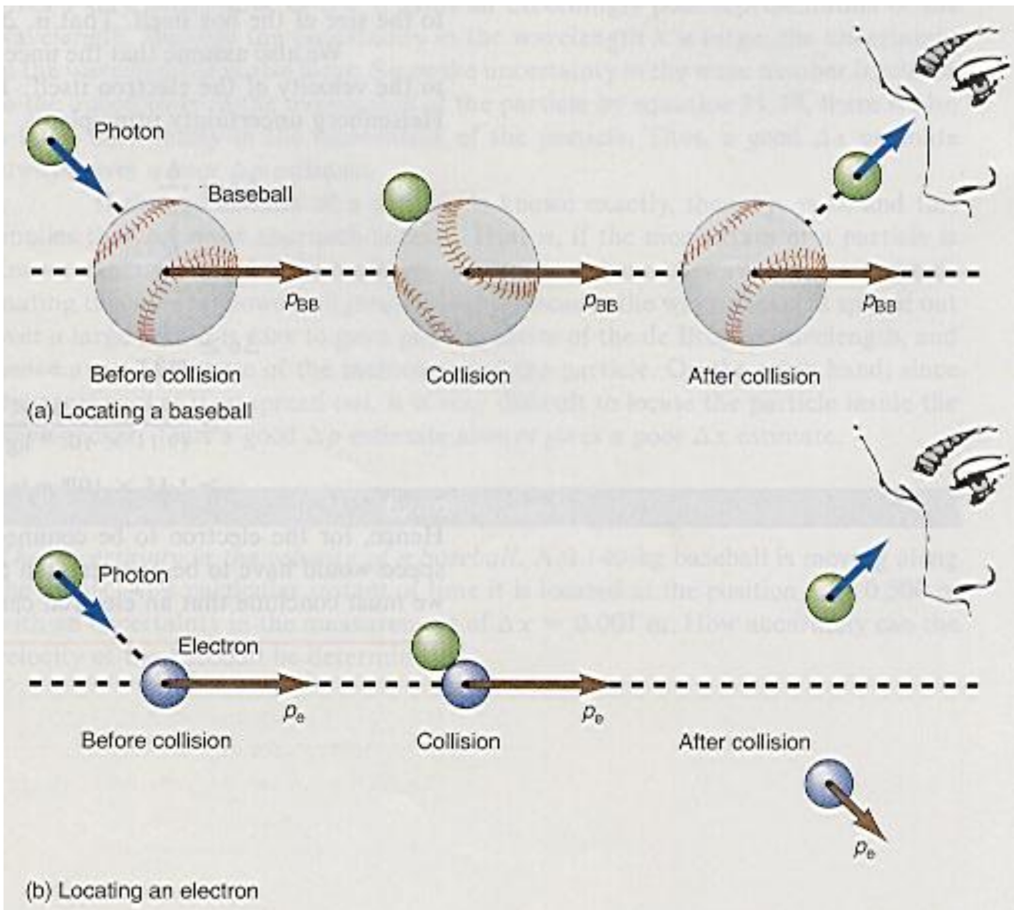


Figure 3.11 The Heisenberg uncertainty principle.

process is basically a collision between the photon of light and the baseball. The photon hits the baseball and then bounces off (is reflected) and proceeds to our eye. We then can say that we saw the baseball at a particular location. Because the mass of the photon is so small compared to the mass of the baseball, the photon bounces off the baseball without disturbing the momentum of the baseball. Thus, in the process of "locating" the baseball, we have done nothing to disturb its momentum.

Now let us look at the problem of "seeing" an electron, figure 3.11(b). The process of "seeing" again implies a collision between the photon of light and the object we wish to see; in this case, the electron. However, the momentum of the photon is now of the same order of magnitude as the momentum of the electron. Hence, as the photon hits the electron, the electron's momentum is changed just as in the Compton effect. Thus, we have located the electron by "seeing" it, but in the process of "seeing" it, we have disturbed or changed its momentum. *Hence, in the process of determining its position, we have caused an uncertainty in its momentum. The uncertainty occurs because the mass of the photon is of the same order of magnitude as the mass of the electron. Thus, the uncertainty always occurs when dealing with microscopic objects.*

The classical picture of being able to predict the exact position and velocity of a particle by Newton's second law and the kinematic equations obviously does not hold in the microscopic region of atoms because of the uncertainty principle. The exact positions and velocities are replaced by a probabilistic determination of position and velocity. That is, we now speak of the probability of finding a particle at a particular position, and the probability that its velocity is a particular value.

On the macroscopic level, the mass of the photon is totally insignificant with respect to the mass of the macroscopic body we wish to see and there is, therefore, no intrinsic uncertainty in measuring the position and velocity of the particle. This is why we are not concerned with the uncertainty principle in classical mechanics.

3.9 Different Forms of the Uncertainty Principle

The limitation on simultaneous measurements is limited not only to the position and momentum of a particle but also to its angular position and angular momentum, and also to its energy and the time in which the measurement of the energy is made.

One of the ways that the angular momentum of a particle is defined is

$$L = rp \sin \theta \tag{3.47}$$

For a particle moving in a circle of radius r, the velocity, and hence the momentum is perpendicular to the radius. Hence, $\theta = 90^0$, and $\sin 90^0 = 1$. Thus, we can also write the angular momentum of a particle as the product of the radius of the circle and the linear momentum of the particle. That is,

$$L = rp \tag{3.48}$$

With this definition of angular momentum, we can easily see the effect of the uncertainty principle on a particle in rotational motion.

Calling x the displacement of a particle along the arc of the circle, when the particle moves through the angle θ, we have

$$x = r\theta$$

The uncertainty Δx in terms of the uncertainty $\Delta\theta$ in angle, becomes

$$\Delta x = r\Delta\theta$$

Substituting this uncertainty into Heisenberg's uncertainty relation, we get

$$\Delta x \Delta p \geq \hbar$$
$$r\Delta\theta\Delta p \geq \hbar$$
$$(\Delta\theta)(r\Delta p) \geq \hbar \tag{3.49}$$

But equation 3.48, which gave us the angular momentum of the particle, also gives us the uncertainty in this angular momentum as

$$\Delta L = r\Delta p \tag{3.50}$$

But this is exactly one of the terms in equation 3.49. Therefore, substituting equation 3.50 into equation 3.49 gives the Heisenberg uncertainty principle for rotational motion as

$$\Delta\theta\ \Delta L \geq \hbar \tag{3.51}$$

Heisenberg's uncertainty principle in this form says that the product of the uncertainty in the angular position and the uncertainty in the angular momentum of the particle is always equal to or greater than the value $\hbar$. Thus, if the angular position of a particle is known exactly, $\Delta\theta = 0$, then the uncertainty in the angular momentum is infinite. On the other hand, if the angular momentum is known exactly, $\Delta L = 0$, then we have no idea where the particle is located in the circle.

The relationship between the uncertainty in the energy of a particle and the uncertainty in the time of its measurement is found as follows. Because the velocity of a particle is given by $v = \Delta x/\Delta t$, the distance that the particle moves during the measurement process is

$$\Delta x = v\Delta t \tag{3.52}$$

The momentum of the particle is given by the de Broglie relation as

$$p = \frac{h}{\lambda} = \frac{h\nu}{v} = \frac{E}{v} \tag{3.53}$$

because $1/\lambda = \nu/v$ and $h\nu = E$. The uncertainty of momentum in terms of the uncertainty in its energy, found from equation 3.53, is

$$\Delta p = \frac{\Delta E}{v} \tag{3.54}$$

Substituting equations 3.52 and 3.54 into the Heisenberg uncertainty relation, gives

$$\Delta x \Delta p \geq \hbar$$

$$(v\Delta t)\left(\frac{\Delta E}{v}\right) \geq \hbar$$

or

$$\Delta E \Delta t \geq \hbar \tag{3.55}$$

Equation 3.55 says that the product of the uncertainty in the measurement of the energy of a particle and the uncertainty in the time of the measurement of the particle is always equal to or greater than $\hbar$. Thus, in order to measure the energy of a particle exactly, $\Delta E = 0$, it would take an infinite time for the measurement. To measure the particle at an exact instant of time, $\Delta t = 0$, we will have no idea of the energy of that particle (ΔE would be infinite).

Example 3.12

The uncertainty in the energy of an electron in an excited state. The lifetime of an electron in an excited state is about 10^{-8} s. (This is the time it takes for the electron to stay in the excited state before it jumps back to the ground state.) What is its uncertainty in energy during this time?

Solution

The energy uncertainty, found from equation 3.55, is

$$\Delta E \Delta t \geq \hbar$$

$$\Delta E \geq \frac{\hbar}{\Delta t}$$

$$\geq \frac{1.05 \times 10^{-34} \text{ J s}}{1.00 \times 10^{-8} \text{ s}}$$

$$\geq 1.05 \times 10^{-26} \text{ J}$$

To go to this Interactive Example click on this sentence.

3.10 The Heisenberg Uncertainty Principle and Virtual Particles

It is a truly amazing result of the uncertainty principle that it is possible to violate the law of conservation of energy by borrowing an amount of energy ΔE*, just as long as it is paid back before the time* Δt*, required by the uncertainty principle, equation 3.55, has elapsed.* That is, the energy ΔE can be borrowed if it is paid back before the time

$$\Delta t = \frac{\hbar}{\Delta E} \tag{3.56}$$

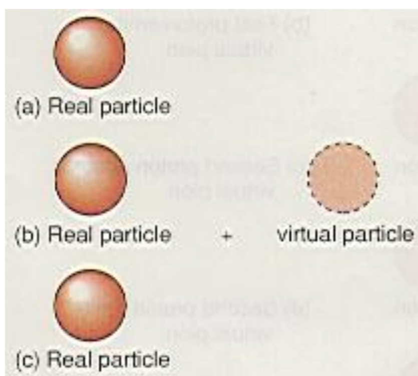
This borrowed energy can be used to create particles. The borrowed energy ΔE is converted to a mass Δm, given by Einstein's mass-energy relation

$$\Delta E = (\Delta m)c^2$$

The payback time thus becomes

$$\Delta t = \frac{\hbar}{(\Delta m)c^2} \tag{3.57}$$

These particles must have very short lifetimes because the energy must be repaid before the elapsed time Δt. These ghostlike particles are called **virtual particles.** Around any real particle there exists a host of these virtual particles. We can visualize virtual particles with the help of figure 3.12. The real particle is shown in figure 3.12(a). In the short period of time Δt, another particle, the virtual particle, materializes as in figure 3.12(b). Before the time Δt is over, the virtual particle returns to the original particle, repaying its energy, and leaving only the real particle, figure 3.12(c). The original particle continues to fluctuate into the two particles.



(a) Real particle (b) Real particle + virtual particle (c) Real particle
Figure 3.12 The virtual particle.

We can determine approximately how far the virtual particle moves away from the real particle by assuming that the maximum speed at which it could possibly move is the speed of light. The distance that the virtual particle can move and then return is then found from

$$d = c\,\frac{\Delta t}{2} \tag{3.58}$$

As an example, suppose the real particle is a proton. Let us assume that we borrow enough energy from the proton to create a particle called the pi-meson (*pion* for short). The mass of the pion is about 2.48×10^{-28} kg. How long can this virtual pion live? From equation 3.57, we have

$$\Delta t = \frac{\hbar}{(\Delta m)c^2}$$

$$= \frac{1.05 \times 10^{-34} \text{ J s}}{(2.48 \times 10^{-28} \text{ kg})(3.00 \times 108 \text{ m/s})^2}$$

$$= 4.70 \times 10^{-24} \text{ s}$$

The approximate distance that the pion can move in this time and return, found from equation 3.58, is

$$d = c\frac{\Delta t}{2}$$

$$= (3.00 \times 10^8 \text{ m/s})\frac{(4.70 \times 10^{-24} \text{ s})}{2}$$

$$= 0.705 \times 10^{-15} \text{ m}$$

This distance is, of course, only approximate because the pion does not move at the speed of light. However, the calculation does give us the order of magnitude of the distance. What is interesting is that the radius of the nucleus of hydrogen is 1.41×10^{-15} m and for uranium it is 8.69×10^{-15} m. Thus, the distance that a virtual particle can move is of the order of the size of the nucleus.

If there are two real protons relatively close together as in the nucleus of an atom as shown in figure 3.13(a), then one proton can emit a virtual pion that can travel to the second proton, figure 3.13(b). The second proton can absorb the virtual

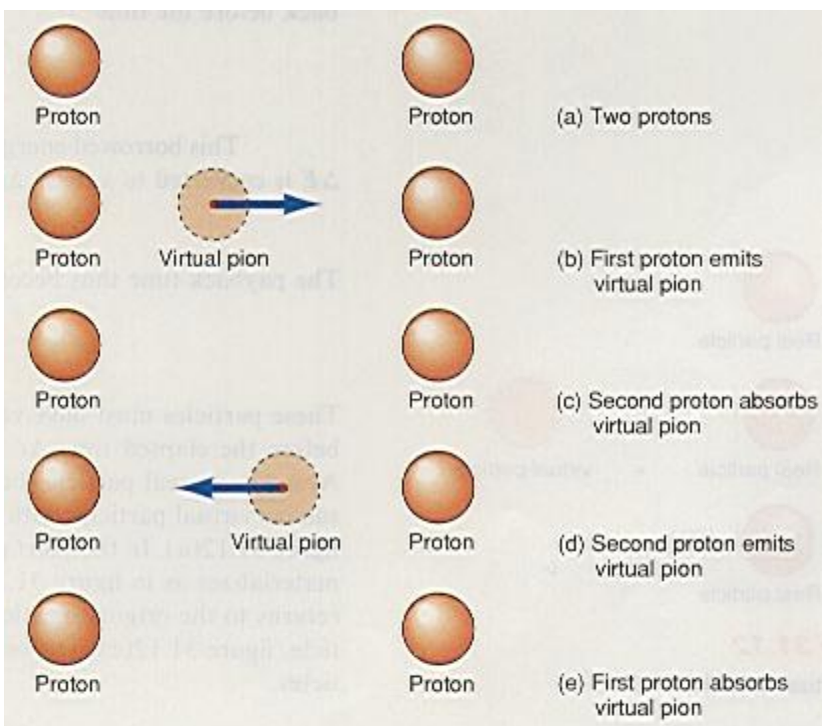


Figure 3.13 The exchange of a virtual pion.

pion, figure 3.13(c). The second pion then emits a virtual pion that can travel to the first pion, figure 3.13(d). The first proton then absorbs the virtual pion, figure 3.13(e). Thus, the protons can exchange virtual pions with one another. *In 1934, the Japanese physicist, Hideki Yukawa (1907-1981), proposed that if two protons exchanged virtual mesons, the result of the exchange would be a very strong attractive force between the protons.* The exchange of virtual mesons between neutrons would also cause a strong attractive force between the neutrons. This exchange force must be a very short-ranged force because it is not observed anywhere outside of the nucleus. The predicted pi-meson was found in cosmic rays by Cecil F. Powell in 1947. Yukawa won the Nobel Prize in physics in 1949, and Powell in 1950.

The concept of a force caused by the exchange of particles is a quantum mechanical concept that is not found in classical physics. The best way to try to describe it classically is to imagine two boys approaching each other on roller skates, as shown in figure 3.14(a). Each boy is moving in a straight line as they approach each other. When the boys are relatively close, the first boy throws a

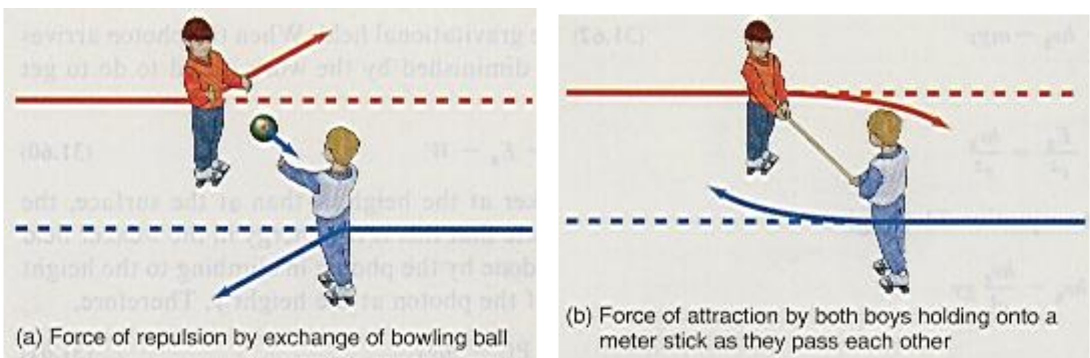


Figure 3.14 Classical analogy of exchange force.

bowling ball to the second boy. By the law of conservation of momentum the first boy recoils after he throws the ball, whereas the second boy recoils after he catches it. The net effect of throwing the bowling ball is to deviate the boys from their straight line motion as though a force of repulsion acted on the two boys. In this way we can say that the exchange of the bowling ball caused a repulsive force between the two boys.

A force of attraction can be similarly analyzed. Suppose, again, that the two boys are approaching each other on roller skates in a straight line motion. When the boys are relatively close the first boy holds out a meterstick for the second boy to grab, figure 3.14(b). As both boys hold on to the meterstick as they pass, they exert a force on each other through the meterstick. The force pulls each boy toward the other boy and deviates the straight line motion into the curved motion toward each other. When the first boy lets go of the meterstick, the attractive force disappears and the boys move in a new straight line motion. Thus, the exchange of the meterstick acted like an attractive force.

The exchange of the virtual pions between the protons in the nucleus cause a very large attractive force that is able to overcome the electrostatic force of repulsion between the protons. The virtual pions can be thought of as a *nuclear glue* that holds the nucleus together. *The tremendous importance of the concept of borrowing energy to form virtual particles, a concept that comes from the Heisenberg uncertainty principle, allows us to think of all forces as being caused by the exchange of virtual particles. Thus, the electrical force can be thought of as caused by the exchange of virtual photons and the gravitational force by the exchange of virtual gravitons (a particle not yet discovered).*

3.11 The Gravitational Red Shift by the Theory of Quanta

The relation for the gravitational red shift was derived in chapter 2 by observing how a clock slows down in a gravitational field. A remarkably simple derivation of this red shift can be obtained by treating light as a particle.

Let an atom at the surface of the earth emit a photon of light of frequency ν_g. This photon has the energy

$$E_g = h\nu_g \tag{3.59}$$

The subscript g is to remind us that this is a photon in the gravitational field. Let us assume that the light source was pointing upward so that the photon travels upward against the gravitational field of the earth until it arrives at a height y above the surface, as shown in figure 3.15. (We have used y for the height instead of h, as used previously, so as not to confuse the height with Planck's constant h.) As the photon rises it must do work against the gravitational field. When the photon arrives at the height y, its energy E_f must be diminished by the work it had to do to get there. Thus

$$E_f = E_g - W \tag{3.60}$$

Because the gravitational field is weaker at the height y than at the surface, the subscript f has been used on E to indicate that this is the energy in the weaker field

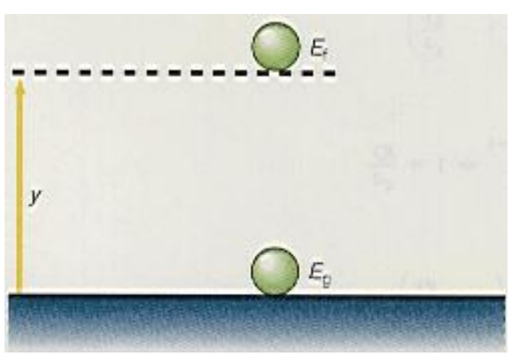


Figure 3.15 A photon in a gravitational field.

or even in a field-free space. The work done by the photon in climbing to the height y is the same as the potential energy of the photon at the height y. Therefore,

$$W = \mathrm{PE} = mgy \tag{3.61}$$

Substituting equation 3.61 and the values of the energies back into equation 3.60, gives

$$h\nu_{\mathbf{f}} = h\nu_{\mathbf{g}} - mgy \tag{3.62}$$

But the mass of the emitted photon is

$$m = \frac{E_{\mathbf{g}}}{c^2} = \frac{h\nu_{\mathbf{g}}}{c^2}$$

Placing this value of the mass back into equation 3.62, gives

$$h\nu_{\mathbf{f}} = h\nu_{\mathbf{g}} - \frac{h\nu_{\mathbf{g}}}{c^2}gy$$

or

$$\nu_f = \nu_g\left(1 - \frac{gy}{c^2}\right) \tag{3.63}$$

Equation 3.63 says that the frequency of a photon associated with a spectral line that is observed away from the gravitational field is less than the frequency of the spectral line emitted by the atom in the gravitational field itself. Since the frequency ν is related to the wavelength λ by $c = \lambda\nu$, the observed wavelength in the field-free space $\lambda_{\mathbf{f}}$ is longer than the wavelength emitted by the atom in the gravitational field $\lambda_{\mathbf{g}}$. Therefore, the observed wavelength is shifted toward the red end of the spectrum. Note the equation 3.63 is the same as equation 2.37. *The slowing down of a clock in a gravitational field follows directly from equation (3.63)* by noting that the frequency ν is related to the period of time T by $\nu = 1/T$. Hence

$$\frac{1}{T_f} = \frac{1}{T_g}\left(1 - \frac{gy}{c^2}\right)$$

$$T_{\mathbf{f}} = \frac{T_{\mathbf{g}}}{1 - gy/c^2}$$

$$T_f = T_g\left(1 - \frac{gy}{c^2}\right)^{-1}$$

But by the binomial theorem,

$$\left(1 - \frac{gy}{c^2}\right)^{-1} = 1 + \frac{gy}{c^2}$$

Thus,

$$T_f = T_g\left(1 + \frac{gy}{c^2}\right) \tag{3.64}$$

Equation 3.64 is identical to equation 2.34. Finally calling the period of time T an elapsed time, Δt, we have

$$\Delta t_f = \Delta t_g\left(1 + \frac{gy}{c^2}\right) \tag{3.65}$$

which is identical to equation 2.31, which shows the slowing down of a clock in a gravitational field.

3.12 An Accelerated Clock

An extremely interesting consequence of the gravitational red shift can be formulated by invoking Einstein's principle of equivalence discussed in chapter 2. Calling the inertial system containing gravity the K system and the accelerated frame of reference the K' system, Einstein stated, "we assume that we may just as well regard the system K as being in a space free from a gravitational field if we then regard K as uniformly accelerated." *Einstein's principle of equivalence was thus stated as: on a local scale the physical effects of a gravitational field are indistinguishable from the physical effects of an accelerated coordinate system. "Hence the systems K and K ' are equivalent with respect to all physical processes, that is, the laws of nature with respect to K are in entire agreement with those with respect to K '.* " Einstein then *postulated his theory of general relativity, as: The laws of physics are the same in all frames of reference.*

Since a clock slows down in a gravitational field, equation 3.65, using the equivalence principle, an accelerated clock should also slow down. Replacing the acceleration due to gravity g by the acceleration of the clock a, equation 3.65 becomes

$$\Delta t_f = \Delta t_a\left(1 + \frac{ay}{c^2}\right) \tag{3.66}$$

Note that the subscript g on $\Delta t_{\mathbf{g}}$ in equation 3.65 has now been replaced by the subscript a, giving $\Delta t_{\mathbf{a}}$, to indicate that this is the time elapsed on the accelerated clock. Notice from equation 3.66 that

$$\Delta t_{\mathbf{f}} > \Delta t_{\mathbf{a}}$$

indicating that time slows down on the accelerated clock. *That is, an accelerated clock runs more slowly than a clock at rest.* In section 1.8 we saw, using the Lorentz transformation equations, that a clock at rest in a moving coordinate system slows down, and called the result the Lorentz time dilation. However, nothing was said at that time to show how the coordinate system attained its velocity. *Except for zero* *velocity, all bodies or reference systems must be accelerated to attain a velocity. Thus, there should be a relation between the Lorentz time dilation and the slowing down of an accelerated clock.* Let us change our notation slightly and call Δt_f the time Δt in a stationary coordinate system and Δt_a the time interval on a clock that is at rest in a coordinate system that is accelerating to the velocity v. Assuming that the acceleration is constant, we can use the kinematic equation

$$v^2 = v_0^2 + 2ay$$

Further assuming that the initial velocity v_0 is equal to zero and solving for the quantity ay we obtain

$$ay = \frac{v^2}{2} \tag{3.67}$$

Substituting equation 3.67 into equation 3.66, yields

$$\Delta t = \Delta t_a \left(1 + \frac{v^2}{2c^2}\right) \tag{3.68}$$

Using the binomial theorem in reverse

$$1 - nx = (1 - x)^n$$

with $x = v^2/c^2$ and $n = -1/2$, we get

$$\left(1 + \frac{v^2}{2c^2}\right) = \left[1 - \left(-\frac{1}{2}\right)\frac{v^2}{c^2}\right] = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{3.69}$$

Equation 3.68 becomes

$$\Delta t = \frac{\Delta t_a}{\sqrt{1 - v^2/c^2}} \tag{3.70}$$

But this is exactly the time dilation formula, equation 1.64, found by the Lorentz transformation. Thus the Lorentz time dilation is a special case of the slowing down of an accelerated clock. This is a very important result. *Therefore, it is more reasonable to take the slowing down of a clock in a gravitational field, and thus by the principle of equivalence, the slowing down of an accelerated clock as the more basic physical principle. The Lorentz transformation for time dilation can then be derived as a special case of a clock that is accelerated from rest to the velocity v.*

Just as the slowing down of a clock in a gravitational field can be attributed to the warping of spacetime by the mass, it is reasonable to assume that the slowing down of the accelerated clock can also be thought of as the warping of spacetime by the increased mass, due to the increase in the velocity of the accelerating mass.

The Lorentz length contraction can also be derived from this model by the following considerations. Consider the emission of a light wave in a gravitational field. We will designate the wavelength of the emitted light by $\lambda_{\mathbf{g}}$, and the period of the light by $T_{\mathbf{g}}$. The velocity of the light emitted in the gravitational field is given by

$$c_{\mathbf{g}} = \frac{\lambda_{\mathbf{g}}}{T_{\mathbf{g}}} \tag{3.71}$$

We will designate the velocity of light in a region far removed from the gravitational field as $c_{\mathbf{f}}$ for the velocity in a field-free region. The velocity of light in the field-free region is given by

$$c_{\mathbf{f}} = \frac{\lambda_{\mathbf{f}}}{T_{\mathbf{f}}} \tag{3.72}$$

where $\lambda_{\mathbf{f}}$ is the wavelength of light, and $T_{\mathbf{f}}$ is the period of the light as observed in the field-free region. If the gravitating mass is not too large, then we can make the reasonable assumption that the velocity of light is the same in the gravitational field region and the field-free region, that is, $c_{\mathbf{g}} = c_{\mathbf{f}}$. We can then equate equation 3.71 to equation 3.72 to obtain

$$\frac{\lambda_{\mathbf{g}}}{T_{\mathbf{g}}} = \frac{\lambda_{\mathbf{f}}}{T_{\mathbf{f}}}$$

Solving for the wavelength of light in the field-free region, we get

$$\lambda_{\mathbf{f}} = \frac{T_{\mathbf{f}}}{T_{\mathbf{g}}}\lambda_{\mathbf{g}}$$

Substituting the value of $T_{\mathbf{f}}$ from equation 3.64 into this we get

$$\lambda_f = \frac{T_g}{T_g}\left(1 + \frac{gy}{c^2}\right)\lambda_g$$

$$\boxed{\lambda_f = \left(1 + \frac{gy}{c^2}\right)\lambda_g} \tag{3.73}$$

Equation 3.73 gives the wavelength of light $\lambda_{\mathbf{f}}$ in the gravitational-field-free region. By the principle of equivalence, the wavelength of light emitted from an accelerated observer, accelerating with the constant acceleration a through a distance y is obtained from equation 3.73 as

$$\boxed{\lambda_0 = \left(1 + \frac{ay}{c^2}\right)\lambda_a} \tag{3.74}$$

where λ_0 is the wavelength of light that is observed in the region that is not accelerating, that is, the wavelength observed by an observer who is at rest. This result can be related to the velocity v that the accelerated observer attained during the constant acceleration by the kinematic equation

$$v^2 = v_0^2 + 2ay$$

Further assuming that the initial velocity v_0 is equal to zero and solving for the quantity ay we obtain

$$ay = \frac{v^2}{2}$$

Substituting this result into equation 3.74 we obtain

$$\lambda_0 = \left(1 + \frac{v_2}{2c^2}\right)\lambda_a \tag{3.75}$$

Using the binomial theorem in reverse as in equation 3.69,

$$\left(1 + \frac{v^2}{2c^2}\right) = \frac{1}{\sqrt{1 - v^2/c^2}}$$

equation 3.75 becomes

$$\lambda_0 = \frac{\lambda_a}{\sqrt{1 - v^2/c^2}}$$

Solving for $\lambda_{\mathbf{a}}$ we get

$$\lambda_a = \lambda_0\sqrt{1 - v^2/c^2} \tag{3.76}$$

But λ is a length, in particular $\lambda_{\mathbf{a}}$ is a length that is observed by the observer who has accelerated from 0 up to the velocity v and is usually referred to as L, whereas λ_0 is a length that is observed by an observer who is at rest relative to the measurement and is usually referred to as L_0. Hence, we can write equation 3.76 as

$$L = L_0\sqrt{1 - v^2/c^2} \tag{3.77}$$

But equation 3.77 is the Lorentz contraction of special relativity. *Hence, the Lorentz contraction is a special case of contraction of a length in a gravitational field, and by the principle of equivalence, a rod L_0 that is accelerated to the velocity v is contracted to the length L.* (That is, if a rod of length L_0 is at rest in a stationary spaceship, and the spaceship accelerates up to the velocity v, then the stationary observer on the earth would observe the contracted length L.) ***Hence, the acceleration of the rod is the basic physical principle underlying the length contraction***.

Thus, both the time dilation and length contraction of special relativity should be attributed to the warping of spacetime by the accelerating mass.

The warping of spacetime by the accelerating mass can be likened to the Doppler effect for sound. Recall from general physics that if a source of a sound wave is stationary, the sound wave propagates outward in concentric circles. When the sound source is moving, the waves are no longer circular but tend to bunch up in advance of the moving source. Since light does not require a medium for propagation, the Doppler effect for light is very much different. However, we can speculate that the warping of spacetime by the accelerating mass is comparable to the bunching up of sound waves in air. In fact, if we return to equation 3.63, for the gravitational red shift, and again, using the principle of equivalence, let $g = a$, and dropping the subscript f, this becomes

$$\nu = \nu_a \left(1 - \frac{ay}{c^2} \right) \tag{3.78}$$

Using the kinematic equation for constant acceleration, $ay = v^2/2$. Hence equation 3.78 becomes

$$\nu = \nu_a \left(1 - \frac{v^2}{2c^2} \right) \tag{3.79}$$

Again using the binomial theorem

$$\left(1 - \frac{v^2}{2c^2} \right) = \sqrt{1 - v^2/c^2}$$

Equation 3.78 becomes

$$\nu = \nu_a \sqrt{1 - v^2/c^2} \tag{3.80}$$

Equation 3.80 is called the transverse Doppler effect. It is a strictly relativistic result and has no counterpart in classical physics. The frequency $\nu_\mathbf{a}$ is the frequency of light emitted by a light source that is at rest in a coordinate system that is accelerating past a stationary observer, whereas ν is the frequency of light observed by the stationary observer. *Notice that the transverse Doppler effect comes directly from the gravitational red shift by using the equivalence principle.* Thus the transverse Doppler effect should be looked on as a frequency shift caused by accelerating a light source to the velocity v.

It is important to notice here that this entire derivation started with the gravitational red shift by the theory of the quanta, then the equivalence principle was used to obtain the results for an accelerating system. The Lorentz time dilation and length contraction came out of this derivation as a special case. Thus, the Lorentz equations should be thought of as kinematic equations, whereas the gravitational and acceleration results should be thought of as a dynamical result.

Time dilation and length contraction have always been thought of as only depending upon the velocity of the moving body and not upon its acceleration. As an example, in Wolfgang Rindler's book *Essential Relativity*,[1] he quotes results of experiments at the CERN laboratory where muons were accelerated. He states "that accelerations up to 10^{19} g (!) do not contribute to the muon time dilation." The only time dilation that could be found came from the Lorentz time dilation formula. They could not find the effect of the acceleration because they had it all the time. The Lorentz time dilation formula itself is a result of the acceleration. Remember, it is impossible to get a nonzero velocity without an acceleration.

In our study in chapter 2 we discussed how a very large collapsing star could become a black hole. Pursuing the equivalence principle further, if gravitational mass can warp spacetime into a black hole, can the singularity that would occur if a body could be accelerated to the velocity c, be considered as an accelerating black hole, and if so what implications would this have?

The Language of Physics

Photon
A small bundle of electromagnetic energy that acts as a particle of light. The photon has zero rest mass and its energy and momentum are determined in terms of the wavelength and frequency of the light wave (p.).

Photoelectric effect
Light falling on a metallic surface produces electrical charges. The photoelectric effect cannot be explained by classical electromagnetic theory. Einstein used the quantum theory to successfully explain this effect and won the Nobel Prize in physics. He said that a photon of light collides with an electron and imparts enough energy to it to remove it from its position in the metal (p.).

Principle of complementarity
The wave theory of light and the quantum theory of light complement each other. In a specific case, light exhibits either a wave nature or a particle nature, but never both at the same time (p.).

Compton effect
Compton bombarded electrons with photons and found that the scattered photon has a different wavelength than the incident light. The photon lost energy to the electron in the collision (p.).

de Broglie relation
de Broglie assumed that the same wave-particle duality associated with electromagnetic waves should also apply to particles. Thus, particles should also act

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1[1] Springer-Verlag, New York, 1979, Revised 2nd edition, p. 44.

as waves. The wave was first called a pilot wave, and then a matter wave. Today, it is simply called the wave function (p.).

Heisenberg uncertainty principle
The position and momentum of a particle cannot both be measured simultaneously with perfect accuracy. There is always a fundamental uncertainty associated with any measurement. This uncertainty is not associated with the measuring instrument. It is a consequence of the wave-particle duality of matter (p.).

Virtual particles
Ghostlike particles that exist around true particles. They exist by borrowing energy from the true particle, and converting this energy into mass. The energy must, however, be paid back before the time Δt, determined by the uncertainty principle, elapses. The virtual particles supply the force necessary to keep protons and neutrons together in the nucleus (p.).

Summary of Important Equations

Planck's relation $$E = nh\nu \tag{3.2}$$

Einstein's photoelectric equation $$\mathrm{KE}_{\mathbf{max}} = h\nu - W_0 \tag{3.10}$$

The work function $$W_0 = h\nu_0 \tag{3.11}$$

Properties of the photon

Rest mass $$m_0 = 0 \tag{3.14}$$

Energy $$E = h\nu \tag{3.6}$$

Relativistic mass $$m = \frac{E}{c^2} = \frac{h\nu}{c^2} \tag{3.16}$$

Momentum $$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} \tag{3.21}$$

Momentum of any particle $$p = \frac{\sqrt{E^2 - E_0^2}}{c} \tag{3.19}$$

Compton scattering formula $$(\lambda' - \lambda) = \frac{h}{m_0 c}(1 - \cos\phi) \tag{3.28}$$

de Broglie relation $$\lambda = \frac{h}{p} \tag{3.29}$$

The uncertainty principle

$$\Delta p\, \Delta x \geq \hbar \tag{3.44}$$

$$\Delta\theta\, \Delta L \geq \hbar \tag{3.51}$$

$$\Delta E\,\Delta t \geq \hbar \tag{3.55}$$

Angular momentum of a particle

$$L = rp \sin \theta \tag{3.47}$$

$$L = rp \tag{3.48}$$

Payback time for a virtual particle
$$\Delta t = \frac{\hbar}{(\Delta m)c^2} \tag{3.57}$$

Gravitational red shift

$$\nu_f = \nu_g\left(1 - \frac{gy}{c^2}\right) \tag{3.63}$$

$$T_f = T_g\left(1 + \frac{gy}{c^2}\right) \tag{3.64}$$

Slowing down of a clock in a gravitational field

$$\Delta t_f = \Delta t_g\left(1 + \frac{gy}{c^2}\right) \tag{3.65}$$

Slowing down of an accelerated clock

$$\Delta t_f = \Delta t_a\left(1 + \frac{ay}{c^2}\right) \tag{3.66}$$

$$\Delta t = \frac{\Delta t_a}{\sqrt{1 - v^2/c^2}} \tag{3.70}$$

Length contraction in a gravitational field
$$\lambda_f = \left(1 + \frac{gy}{c^2}\right)\lambda_g \tag{3.73}$$

Length contraction in an acceleration
$$\lambda_0 = \left(1 + \frac{ay}{c^2}\right)\lambda_a \tag{3.74}$$

$$L = L_0\sqrt{1 - v^2/c^2} \tag{3.77}$$

Questions for Chapter 3

*1. How would the world appear if Planck's constant h were very large? Describe some common occurrences and how they would be affected by the quantization of energy.

2. When light shines on a surface, is momentum transferred to the surface?

3. Could photons be used to power a spaceship through interplanetary space?

4. Should the concept of the cessation of all molecular motion at absolute zero be modified in view of the uncertainty principle?

5. Which photon, red, green, or blue, carries the most (a) energy and (b) momentum?

6. Discuss the entire wave-particle duality. That is, is light a wave or a particle, and is an electron a particle or a wave?

*7. Discuss the concept of determinism in terms of the uncertainty principle.

*8. Why isn't the photoelectric effect observed in all metals?

9. Ultraviolet light has a higher frequency than infrared light. What does this say about the energy of each type of light?

*10. Why can red light be used in a photographic dark room when developing pictures, but a blue or white light cannot?

Problems for Chapter 3

3.2 Blackbody Radiation

1. A weightless spring has a spring constant of 18.5 N/m. A 500-g mass is attached to the spring. It is then displaced 10.0 cm and released. Find (a) the total energy of the mass, (b) the frequency of the vibration, (c) the quantum number *n* associated with this energy, and (d) the energy change when the oscillator changes its quantum state by one value.

Diagram for problem 1.

2. Find the energy of a photon of light of 400.0-nm wavelength.

3. A radio station broadcasts at 92.4 MHz. What is the energy of a photon of this electromagnetic wave?

3.3 The Photoelectric Effect

4. The work function of a material is 4.52 eV. What is the threshold wavelength for photoelectronic emission?

5. The threshold wavelength for photoelectronic emission for a particular material is 518 nm. Find the work function for this material.

*6. Light of 546.0-nm wavelength is incident on a cesium surface that has a work function of 1.91 eV. Find (a) the frequency of the incident light, (b) the energy of the incident photon, (c) the maximum kinetic energy of the photoelectron, (d) the stopping potential, and (e) the threshold wavelength.

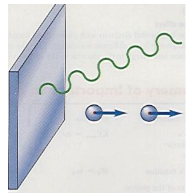

Diagram for problem 6.

3.4 The Properties of the Photon

7. A photon has an energy of 5.00 eV. What is its frequency and wavelength?
8. Find the mass of a photon of light of 500.0-nm wavelength.
9. Find the momentum of a photon of light of 500.0-nm wavelength.
10. Find the wavelength of a photon whose energy is 500 MeV.
11. What is the energy of a 650 nm photon?

3.5 The Compton Effect

12. An 80.0-KeV X ray is fired at a carbon target and Compton scattering occurs. Find the wavelength of the incident photon and the wavelength of the scattered photon for an angle of 40.0^0.

13. If an incident photon has a wavelength of 0.0140 nm, and is found to be scattered at an angle of 50.0^0 in Compton scattering, find the energy of the recoiling electron.

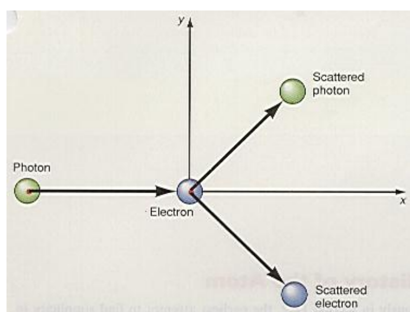

Diagram for problem 13.

14. In a Compton scattering experiment, 0.0400-nm photons are scattered by the target, yielding 0.0420-nm photons. What is the angle at which the 0.0420-nm photons are scattered?

3.6 The Wave Nature of Particles

15. Find the wavelength of a 4.60×10^{-2} kg golf ball moving at a speed of 60.0 m/s.
16. Find the wavelength of a proton moving at 10.0% of the speed of light.
17. Find the wavelength of an electron moving at 10.0% of the speed of light.
18. Find the wavelength of a 5.00-KeV electron.
19. Find the wavelength of an oxygen molecule at room temperature.
20. What is the frequency of the matter wave representing an electron moving at a speed of $2c/3$?

21. (a) Find the total energy of a proton moving at a speed of $c/2$. (b) Compute the wavelength of this proton.

3.8 The Heisenberg Uncertainty Principle

22. A 4.6×10^{-2} kg golf ball is in motion along the x-axis. If it is located at the position $x = 1.00$ m, with an uncertainty of 0.005 m, find the uncertainty in the determination of the momentum and velocity of the golf ball.

23. Find the minimum uncertainty in the determination of the momentum and speed of a 1300-kg car if the position of the car is to be known to a value of 10 nm.

24. The uncertainty in the position of a proton is 100 nm. Find the uncertainty in the kinetic energy of the proton.

3.9 Different Forms of the Uncertainty Principle

*25. The lifetime of an electron in an excited state of an atom is 10^{-8} s. From the uncertainty in the energy of the electron, determine the width of the spectral line centered about 550 nm.

Additional Problems

26. Approximately 5.00% of a 100-W incandescent lamp falls in the visible portion of the electromagnetic spectrum. How many photons of light are emitted from the bulb per second, assuming that the wavelength of the average photon is 550 nm?

Interactive Tutorials

27. *The photoelectric effect*. Light of wavelength $\lambda = 577.0$ nm is incident on a cesium surface. Photoelectrons are observed to flow when the applied voltage $V_0 = 0.250$ V. Find (a) the frequency ν of the incident photon, (b) the initial energy E of the incident photon, (c) the maximum kinetic energy KE_{max} of the photoelectrons, (d) the work function W_0 of cesium, (e) the threshold frequency ν_0, and (f) the corresponding threshold wavelength λ_0.

28. *The photoelectric effect*. Light of wavelength $\lambda = 460$ nm is incident on a cesium surface. The work function of cesium is $W_0 = 3.42 \times 10^{-19}$ J. Find (a) the frequency ν of the incident photon, (b) the initial energy E of the incident photon, (c) the maximum kinetic energy KE_{max} of the emitted photoelectrons, (d) the maximum speed v of the electron, (e) the threshold frequency ν_0, and (f) the corresponding longest wavelength λ_0 that will eject electrons from the metal.

29. *Properties of a photon*. A photon of light has a wavelength $\lambda = 420.0$ nm, find (a) the frequency ν of the photon, (b) the energy E of the photon, (c) the mass m of the photon, and (d) the momentum p of the photon.

30. *The Compton effect*. An x-ray photon of energy $E = 90.0$ KeV is fired at a carbon target and Compton scattering occurs at an angle $\phi = 30.0^0$. Find (a) the frequency ν of the incident photon, (b) the wavelength λ of the incident photon, and (c) the wavelength λ' of the scattered photon.

31. *Wave particle duality*. Using the concept of wave particle duality, calculate the wavelength λ of a golf ball whose mass $m = 4.60 \times 10^{-2}$ kg and is traveling at a speed $v = 60.0$ m/s.

To go to these Interactive Tutorials click on this sentence.

To go to another chapter, return to the table of contents by clicking on this sentence.