Chapter 9: Electromagnetic Induction

9.1 Introduction

Since a current in a wire produces a magnetic field, it is logical to ask if the reverse process is possible. That is, is there any way in which a magnetic field can produce a current? The answer to the question is yes, and the two men who were responsible for the discovery are Michael Faraday, (1791-1867) an English physicist, and Joseph Henry, (1797-1878) an American physicist. *The process whereby a magnetic field can produce a current is called electromagnetic induction.*

9.2 Motional Emf and Faraday's Law of Electromagnetic Induction

Let us consider the following experiment as outlined in figure 9.1(a). Two parallel metal rails are separated by a distance l. A metal wire rests on the two rails. A uniform magnetic field **B** is applied such that its direction is into the paper as

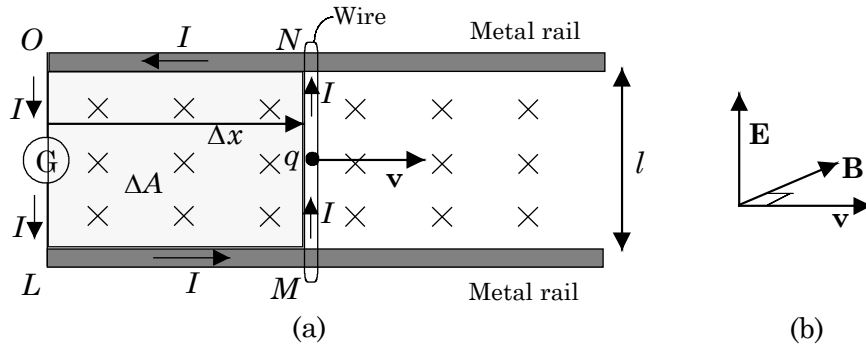**Figure 9.1** Motional emf.

shown. (Recall that the ×'s in the figure represent the tail of the arrow representing the vector as it goes into the plane.) A galvanometer G is connected across the two rails. The galvanometer reads zero indicating that there is no current in the circuit which consists of the rails, and wires, i.e., the circuit is the electrical path designated *LMNOL* in figure 9.1(a). *The metal wire MN is now pulled along the rails at a velocity **v** to the right. The galvanometer now indicates that a current is flowing in the circuit. Somehow, the motion of the wire through the magnetic field generated an electric current.* Let us now analyze the cause of this current.

As the wire *MN* is moved to the right, any charge q within the wire experiences the force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{8.6}$$

as was shown in section 8.2. If both sides of equation 8.6 are divided by q, we have

$$\frac{\mathbf{F}}{q} = \mathbf{v} \times \mathbf{B} \tag{9.1}$$

But an electrostatic field was originally defined as

$$\mathbf{E} = \frac{\mathbf{F}}{q_0} \tag{3.1}$$

where $\mathbf{F}$ was the force acting on a test charge q_0 placed at rest in the electrostatic field. It is therefore reasonable to now *define an induced electric field $\mathbf{E}$ by equation 9.1 as*

$$\mathbf{E} = \frac{\mathbf{F}}{q} = \mathbf{v} \times \mathbf{B} \tag{9.2}$$

This is a different type of field than the electrostatic field. *The induced electric field exists only when the charge is in motion at a velocity $\mathbf{v}$. When $\mathbf{v} = 0$, the induced electric field will also be zero as can be seen by equation 9.2. The induced electric field is the cause of the current in the wire. The cross product $\mathbf{v} \times \mathbf{B}$ shows the direction of the induced electric field in figure 9.1(b), and hence the direction that a positive charge q, within the wire, will move.* Hence, the direction of the current will be in the direction $M \Rightarrow N \Rightarrow O \Rightarrow L \Rightarrow M$, as shown in figure 9.1(a). The magnitude of the induced electric field is found from equation 9.2 and the definition of the cross product as

$$E = vB \sin\theta \tag{9.3}$$

For the particular problem considered here, the angle between $\mathbf{v}$ and $\mathbf{B}$ is 90^0, and the $\sin 90^0 = 1$. Therefore the induced electric field is

$$E = vB \tag{9.4}$$

It was shown in chapter 5 that for a uniform electric field

$$E = V/d$$

where V is the potential difference between two points and d is the distance between them. For the connecting wire MN, the induced electric field within the wire can be assumed to be uniform, and the induced potential V between M and N will be designated by $\mathcal{E}$, and called an induced emf. The distance d between M and N is the length l of the wire. Therefore, the induced electric field can be written as

$$E = \frac{\mathcal{E}}{l} \tag{9.5}$$

Equating 9.5 to 9.4 gives

$$\frac{\mathcal{E}}{l} = vB$$

The induced emf in the wire $\mathcal{E}$ is therefore

$$\mathcal{E} = vBl \qquad\qquad (9.6)$$

If the circuit has a resistance R, then there is an induced current in the circuit given by Ohm's law as

$$I = \frac{\mathcal{E}}{R}$$

This is the current that is recorded by the galvanometer.

Example 9.1

Motional emf. The wire *MN* in figure 9.1(a) moves with a velocity of 50.0 cm/s to the right. If l = 25.0 cm, B = 0.250 T, and the total electric resistance of the circuit is 35.0 Ω, find (a) the induced emf in the circuit, (b) the current in the circuit, and (c) the direction of the current.

Solution

(a) The induced emf in the circuit is found from equation 9.6 to be

$$\mathcal{E} = vBl = (0.500\ \underline{\text{m}})(0.250\text{T})(0.250\ \text{m})\frac{(\text{N}/(\text{A m}))}{(\text{T})}$$
$$\mathcal{E} = 3.13 \times 10^{-2}\ \frac{\text{m}}{\text{s}}\ \frac{\text{N}}{\text{A}}\frac{(\text{J})}{(\text{N m})}$$
$$\mathcal{E} = 3.13 \times 10^{-2}\ \frac{\text{J/s}}{\text{A}}\ \frac{(\text{A V})}{(\text{J/s})}$$
$$\mathcal{E} = 3.13 \times 10^{-2}\ \text{V}$$

Notice how the units are manipulated to give the correct units for the final answer.

(b) The current flowing in the circuit is found from Ohm's law as

$$I = \frac{\mathcal{E}}{R} = \frac{3.13 \times 10^{-2}\ \text{V}}{35.0\ \Omega} = 8.94 \times 10^{-4}\ \text{A}$$

(c) The current in the circuit is counterclockwise as in figure 9.1(a).

To go to this Interactive Example click on this sentence.

Further insight into this induced emf can be obtained by noting that the speed v of the wire is $v = dx/dt$, where dx is the distance the wire moves to the right in the time dt. The product of v and l in equation 9.6 can then be written as

$$vl = \frac{dx}{dt} l \qquad (9.7)$$

But

$$(dx)l = dA \qquad (9.8)$$

is the area of the loop swept out as the wire is moved to the right and is shown in figure 9.1(a). Replacing equation 9.8 into equation 9.7 gives

$$vl = \frac{dA}{dt} \qquad (9.9)$$

And the induced emf, equation 9.6, becomes

$$\mathcal{E} = B \frac{dA}{dt} \qquad (9.10)$$

But recall that the magnetic flux Φ_M was defined as

$$\Phi_M = \mathbf{B} \cdot \mathbf{A} \qquad (8.96)$$

For a constant magnetic field $\mathbf{B}$, the only way for the magnetic flux to change in our problem is for the area $\mathbf{A}$ to change. That is, the change in the magnetic flux is

$$d\Phi_M = \mathbf{B} \cdot d\mathbf{A} \qquad (9.11)$$

The rate at which the magnetic flux changes with time is

$$\frac{d\Phi_M}{dt} = \mathbf{B} \cdot \frac{d\mathbf{A}}{dt} \qquad (9.12)$$

In figure 9.1(a), $\mathbf{B}$ is into the paper, while $d\mathbf{A}$, the change in area vector, is out of the paper, hence the angle between $\mathbf{B}$ and $d\mathbf{A}$ is 180^0. Using this fact in equation 9.12 we get

$$\frac{d\Phi_M}{dt} = B \frac{dA}{dt} \cos 180^0 = - B \frac{dA}{dt} \qquad (9.13)$$

Therefore,

$$B \frac{dA}{dt} = - \frac{d\Phi_M}{dt} \tag{9.14}$$

Combining equation 9.10 and 9.14 gives an extremely important relationship known as Faraday's law, namely

$$\mathcal{E} = - \frac{d\Phi_m}{dt} \tag{9.15}$$

Faraday's law of electromagnetic induction, equation 9.15, states that whenever the magnetic flux changes with time, there will be an induced emf.

In the case considered, **B** was a constant and the area changed with time. It is also possible to keep the area a constant but change the magnetic field **B** with time. That is, if wire *MN* remains fixed in figure 9.1(a), but the magnetic field is changed with time, a current will be indicated on the galvanometer. The reason for this is that the magnetic flux Φ_M given by equation 8.96, can also change if **B** changes. That is,

$$d\Phi_M = d\mathbf{B} \cdot \mathbf{A} \tag{9.16}$$

In general, the change in the magnetic flux caused by a change in area $d\mathbf{A}$ and a change in the magnetic field $d\mathbf{B}$ is

$$d\Phi_M = \mathbf{B} \cdot d\mathbf{A} + \mathbf{A} \cdot d\mathbf{B} \tag{9.17}$$

and Faraday's law, equation 9.15, can also be written as

$$\mathcal{E} = - \frac{d\Phi_M}{dt} = - \mathbf{B} \cdot \frac{d\mathbf{A}}{dt} - \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} \tag{9.18}$$

Faraday's law, in this form, says that an emf can be induced either by changing the area of the loop of wire with time, while the magnetic field remains constant or by keeping the area of the loop of wire constant and changing the magnetic field with time. Because the flux is given by a scalar product, it is also possible to keep the magnitudes *B* and *A* constant, but vary the angle between them. This will also cause a change in the magnetic flux and hence an induced emf. This case will be considered later in the chapter.

Let us consider four possible cases associated with this induced emf.
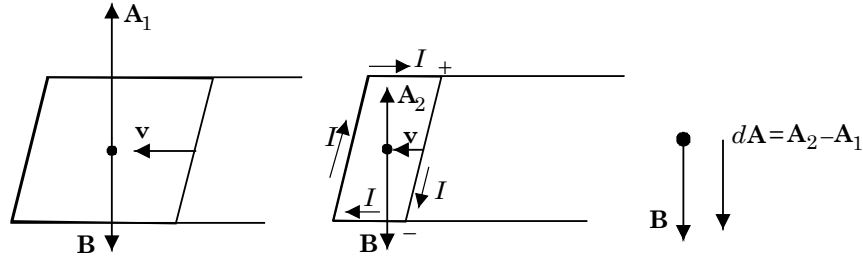
Case I: *The area is increasing with time.*

This is the case that has just been considered and is illustrated in figure 9.2(a). Notice that as **A** increases from $\mathbf{A}_1$ to $\mathbf{A}_2$, the change in area, $d\mathbf{A} = \mathbf{A}_2 - \mathbf{A}_1$, is positive and points upward in the figure. Since **B** points downward, the angle between the vectors **B** and $d\mathbf{A}$ is 180^0 and the induced emf is
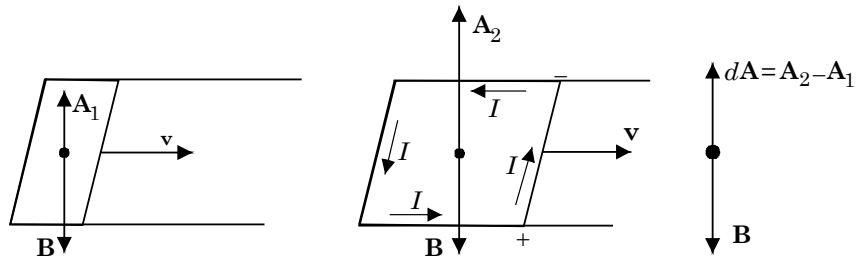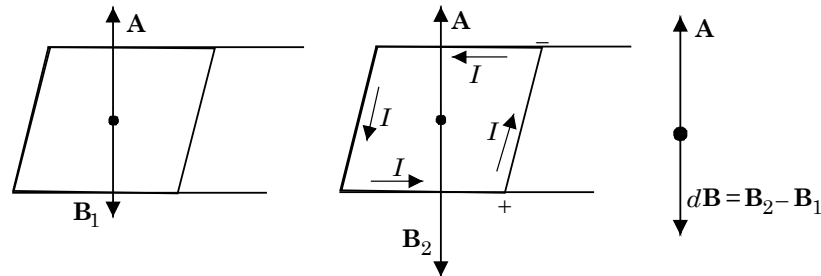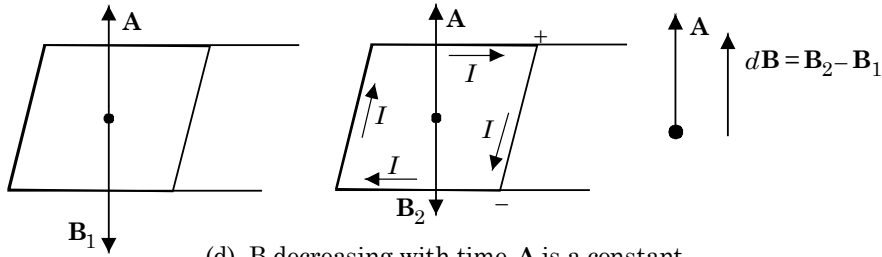
(a) Area increasing with time, **B** is a constant

(b) Area decreasing with time, **B** is a constant

(c) B increasing with time, **A** is a constant

(d) B decreasing with time, **A** is a constant

Figure 9.2 Changing the magnetic flux with time.

$$\mathcal{E} = -\, \mathbf{B} \cdot \frac{d\mathbf{A}}{dt} = -\, B\, \frac{d\mathbf{A}}{dt} \cos 180^0 = B\, \frac{d\mathbf{A}}{dt}$$

just as found before. The diagram also shows the polarity of the induced emf $\mathcal{E}$ and the direction of the current.

Case II: *The area is decreasing with time.*

This case is shown in figure 9.2(b). Note that **A** decreases from $\mathbf{A}_1$ to $\mathbf{A}_2$. Therefore, the change in area, $d\mathbf{A} = \mathbf{A}_2 - \mathbf{A}_1$, is negative and points downward in the figure. The angle between **B** and $d\mathbf{A}$, is 0^0 and the induced emf is then

$$\mathcal{E} = - \mathbf{B} \cdot \frac{d\mathbf{A}}{dt} = - B \frac{dA}{dt} \cos 0^0 = - B \frac{dA}{dt}$$

In this case the emf has a negative sign which means the polarity of $\mathcal{E}$ has been re-versed and the current now flows in a clockwise direction. This reversal of current could also have been found by noticing that the velocity vector $\mathbf{v}$ is now to the left in the figure. Hence $\mathbf{v} \times \mathbf{B}$, the induced electric field, is now in the opposite direction of that found in case I, and therefore, the current should be reversed from case I.

Case III: *The magnetic field is increasing with time.*

This case is shown in figure 9.2(c). The area $\mathbf{A}$ remains a constant, but the magnetic field increases from $\mathbf{B}_1$ to $\mathbf{B}_2$. Therefore, the change in the magnetic field, $d\mathbf{B} = \mathbf{B}_2 - \mathbf{B}_1$, points in the same direction as $\mathbf{B}_1$ and $\mathbf{B}_2$, which is opposite to the di-rection of $\mathbf{A}$. Thus, the angle between $\mathbf{A}$ and $d\mathbf{B}$, is 180^0 and the induced emf is

$$\mathcal{E} = - \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} = - A \frac{dB}{dt} \cos 180^0 = A \frac{dB}{dt}$$

But this is a positive emf as in case I. Hence an increasing magnetic field causes a counterclockwise current, which is in the same direction as that caused by an in-creasing area.

Case IV: *The magnetic field is decreasing with time.*

This case is shown in figure 9.2(d). Since $\mathbf{B}$ is decreasing, the change in the magnetic field, $d\mathbf{B} = \mathbf{B}_2 - \mathbf{B}_1$, is in the opposite direction of $\mathbf{B}$ and points upward in the same direction as $\mathbf{A}$. Thus, the angle between $\mathbf{A}$ and $d\mathbf{B}$ is 0^0, and the induced emf is

$$\mathcal{E} = - \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} = - A \frac{dB}{dt} \cos 0^0 = - A \frac{dB}{dt}$$

Note that this is the opposite result of that found in Case III, and hence the emf is reversed and the current in the circuit will now be clockwise.

One should note here that although Faraday's law was derived on the basis of motional emf, it is true in general. For the general case where there may be N loops of wire, Faraday's law of induction becomes

$$\mathcal{E} = - N \frac{d\Phi_m}{dt} \qquad (9.19)$$

Example 9.2

Changing the magnetic field with time. The wire *MN* in figure 9.1(a) is fixed 10.0 cm away from the galvanometer wire *OL*. The magnetic field varies from 0 to 0.500 T in a time of 2.00×10^{-3} s. If the resistance of the circuit is 35.0 Ω, find (a) the induced

emf, (b) the current in the circuit while the magnetic field is changing with time and, (c) the induced emf if the magnetic field remains at a constant 0.500 T.

Solution

(a) The induced emf is found from Faraday's law, equation 9.18 as

$$\mathcal{E} = - \frac{d\Phi_M}{dt} = - \mathbf{B} \bullet \frac{d\mathbf{A}}{dt} - \mathbf{A} \bullet \frac{d\mathbf{B}}{dt} \qquad (9.18)$$

Because the wire *MN* is fixed, the area of the loop does not change with time, i.e., $d\mathbf{A} = 0$. However, **B** is changing with time, and the induced emf is therefore,

$$\mathcal{E} = - \mathbf{A} \bullet \frac{d\mathbf{B}}{dt} \qquad (9.20)$$

The changing magnetic field is

$$d\mathbf{B} = \mathbf{B}_f - \mathbf{B}_i$$

Since the initial magnetic field $\mathbf{B}_i$ is zero, $d\mathbf{B}$ has the direction of $\mathbf{B}_f$ which is perpendicular to the paper and into the paper. The angle between **A** and $d\mathbf{B}$ is thus 180^0. Therefore

$$\mathcal{E} = - A \frac{dB}{dt} \cos 180^0 = A \frac{dB}{dt} \qquad (9.21)$$

The induced emf is found from equation 9.21 as

$$\mathcal{E} = A \frac{dB}{dt} = (0.100 \text{ m})(0.250 \text{ m}) \frac{(0.500 \text{ T} - 0 \text{ T})}{2.00 \times 10^{-3} \text{ s}}$$

$$\mathcal{E} = 6.25 \text{ T } \frac{\text{m}^2}{\text{s}} \frac{\text{(N/(A m))}}{\text{(T)}}$$

$$\mathcal{E} = 6.25 \frac{\text{N m}}{\text{s A}} \frac{\text{(J)}}{\text{(N m)}} \frac{\text{(A)}}{\text{(C/s)}}$$

$$\mathcal{E} = 6.25 \frac{\text{J}}{\text{C}} \frac{\text{(V)}}{\text{(J/C)}}$$

$$\mathcal{E} = 6.25 \text{ V}$$

Notice how the units have been converted.

(b) The induced current is found from Ohm's law as

$$I = \frac{\mathcal{E}}{R} = \frac{6.25 \text{ V}}{35.0 \text{ }\Omega} = 0.179 \text{ A}$$

(c) When the magnetic field remains constant at 0.500 T, there is no changing magnetic field, $dB = 0$, and there is no induced emf. That is,

$$\mathcal{E} = A \frac{dB}{dt} = 0$$

Thus, for a loop of wire of constant area, the only induced emf occurs when there is a changing magnetic field with time.

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9.3 Lenz's Law

Faraday's law of electromagnetic induction was derived in section 9.2 as

$$\mathcal{E} = - \frac{d\Phi_M}{dt} \tag{9.15}$$

The minus sign in Faraday's law is very important. It was obtained automatically in the derivation by using the angle between the direction of the vectors **B** and $d\mathbf{A}$. The effect of that minus sign gives rise to a relation known as Lenz's law and is stated as: *The direction of an induced emf is such that any current it produces, always opposes, through the magnetic field of the induced current, the change inducing the emf.* As an example, consider figure 9.2(c). This is an example of the magnetic field **B** increasing with time. The induced emf and current flow is as shown. Look at the induced current and recall that whenever a current flows in a wire a magnetic field will be found around that wire. Therefore, there will be an induced magnetic field inside the loop that points upward, opposing the increasing magnetic field that points downward. The induced magnetic field upward opposes the increasing magnetic field downward, which was the cause of the induced emf, the induced current, and the induced magnetic field. Hence, the effect - the induced magnetic field upward, opposes the cause - the increasing magnetic field pointing downward.

Lenz's law is also a statement of the law of conservation of energy. If the induced magnetic field is in the same direction as the changing magnetic field, the total magnetic field will increase even more, which would in turn induce a larger magnetic field, which would again increase the total magnetic field. The process would continue forever, gaining energy all the time without any work being done by a source, which is of course impossible by the law of conservation of energy.

Lenz's law is a convenient tool because it allows us to determine the direction of induced current flow rather simply when the configuration of the changing magnetic field is not simple. As an example, consider the loop of wire in figure 9.3(a). It is connected to the galvanometer and when the bar magnet near the coil is statio-

nary, the galvanometer reads zero. When the north pole of the bar magnet is pushed through the loop, the magnetic field within the loop changes with time and by Faraday's law an emf is induced within the loop which causes a current. The direction of this current flow is easily determined by Lenz's law. Because the cause of the induced emf is the motion of a north magnetic pole toward the loop, the induced current within the loop will cause a magnetic field that will tend to oppose that north magnetic pole. Hence, the induced magnetic field will look like the field of a north pole to oppose the north pole of the bar magnet. (Recall that opposite magnetic poles attract while like magnetic poles repel.) Since a magnetic field emanates from a north pole and enters a south pole, the induced magnetic field $\mathbf{B_i}$ must come out of the loop toward the bar magnet. If the right hand is wrapped around any portion of the loop of wire, such that the fingers point in the direction of the induced magnetic field, the thumb will point in the direction of the current. Thus the direction of the induced current in the loop is as shown in the diagram of figure 9.3(a).
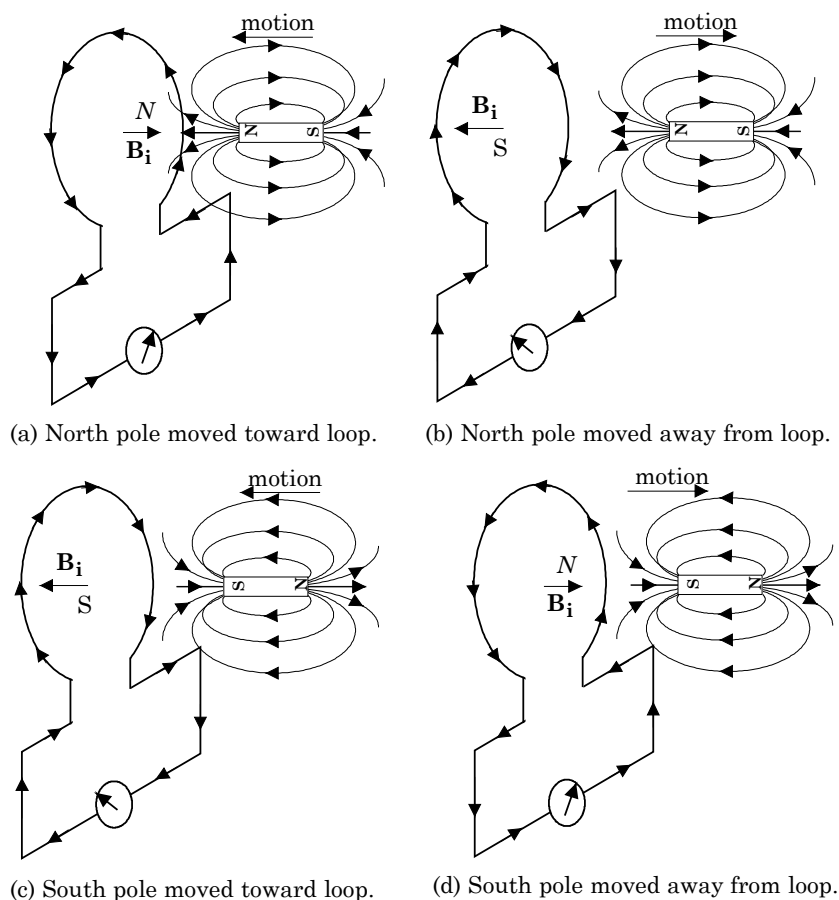

(a) North pole moved toward loop.   (b) North pole moved away from loop.

(c) South pole moved toward loop.   (d) South pole moved away from loop.

Figure 9.3 The determination of the direction of an induced current by Lenz's law.

If the bar magnet is pulled away from the loop, as in figure 9.3(b), then the induced magnetic field should tend to oppose the pulling away of the magnet. Therefore, the face of the loop will behave like a south magnetic pole in order to at-

tract the bar magnet. The magnetic field of the induced south pole, $\mathbf{B}_i$, will point into the loop. Using the right hand rule, and wrapping the hand around the wire with the fingers pointing in the direction of the induced magnetic field $\mathbf{B}_i$ the thumb points in the direction of the current and is shown in figure 9.3(b). Notice that when the magnet is pushed toward the loop, the direction of the induced current is clockwise in the loop; when the magnet is pulled away from the loop, the direction of the current is counterclockwise.

The case of moving a south magnetic pole toward the loop is shown in figure 9.3(c), and pulling it away from the loop is shown in figure 9.3(d). The direction of the induced current is easily found by this analysis of Lenz's law.

9.4 The Induced Emf in a Rotating Loop of Wire in a Magnetic Field - Alternating Emf's and the AC Generator

It was shown in chapter 8 that when a current-carrying loop of wire is placed in an external magnetic field, a torque is produced on the loop, causing it to rotate in the magnetic field. This, as you recall, was the essence of the electric motor. Let us now look at the inverse problem and ask *what happens if a loop of wire is mechanically rotated in an external magnetic field? Can a current be induced in the loop?* The answer is, of course, yes, and the device so described will be an electric generator.

Consider the loop of wire shown in figure 9.4(a) and (b). The loop is made to
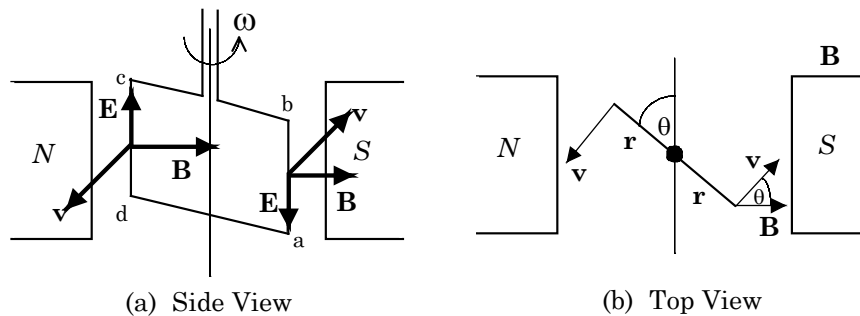

(a) Side View   (b) Top View

Figure 9.4 Rotating Loop of wire in an external magnetic field.

rotate with an angular speed ω. The portion of the wire loop ab is moving with a velocity $\mathbf{v}$ through the magnetic field $\mathbf{B}$. There will be an electric field induced in the wire ab given by equation 9.2 as

$$\mathbf{E} = \mathbf{v} \times \mathbf{B} \tag{9.2}$$

The direction of $\mathbf{E}$ is determined by the cross product of $\mathbf{v} \times \mathbf{B}$ and is shown pointing downward in figure 9.4(a). The induced current will flow in this direction.

The magnitude of the induced electric field is given by equation 9.3 as

$$E = vB \sin\theta \qquad (9.3)$$

where θ is the angle between the velocity vector **v** and the magnetic field **B** and is seen in figure 9.4(b). Notice that θ is also the angle that the loop turns through. The induced electric field in wire ab is given by equation 9.5 as

$$E = \frac{\mathcal{E}}{l} \qquad (9.5)$$

where l is the length of wire from a to b. Combining equations 9.5 and 9.3 gives for the induced emf in wire ab

$$\mathcal{E}_{ab} = vBl \sin\theta \qquad (9.22)$$

Wire cd will have an induced electric field that points upward and is seen in figure 9.4(a). Therefore, an induced current in wire cd will flow upward in the direction of the electric field. The induced emf in wire cd is similar to wire ab and is given by

$$\mathcal{E}_{cd} = vBl \sin\theta \qquad (9.23)$$

Along wires bc and da the induced electric field, $\mathbf{E} = \mathbf{v} \times \mathbf{B}$, is perpendicular to the wire itself and can not cause any current to flow along the length of the wires bc or da. Thus, the emf $\mathcal{E}_{bc} = \mathcal{E}_{da} = 0$. The total emf around the loop is, therefore, the sum of the induced emf along ab, $\mathcal{E}_{ab}$, and the induced emf along cd, $\mathcal{E}_{cd}$. That is,

$$\mathcal{E} = \mathcal{E}_{ab} + \mathcal{E}_{cd} \qquad (9.24)$$

Replacing equations 9.22 and 9.23 into equation 9.24 gives

$$\mathcal{E} = vBl \sin\theta + vBl \sin\theta$$

or

$$\mathcal{E} = 2vBl \sin\theta \qquad (9.25)$$

The coil is rotating at an angular speed ω, and the linear speed v of wire ab and cd is given by

$$v = \omega r \qquad (9.26)$$

Substituting v from equation 9.26 back into equation 9.25 gives for the induced emf in the coil

$$\mathcal{E} = 2(\omega r)Bl \sin\theta \qquad (9.27)$$

But $2r$ is the length of wire bc, and since l is the length of wire ab, the product $2rl$ is the area of the loop, i.e.

$$A = 2rl \qquad (9.28)$$

The total induced emf in the coil is thus

$$\mathcal{E} = \omega AB \sin\theta \qquad (9.29)$$

Because the coil is rotating at an angular speed ω the angle θ turned through at any time is

$$\theta = \omega t$$

The total emf induced in the coil at any instant of time is

$$\mathcal{E} = \omega AB \sin\omega t \qquad (9.30)$$

Equation 9.30 says that to get a large induced emf in a coil: the coil should move at a high angular speed ω; have a relatively large area A; and be placed in a large magnetic field B.

Equation 9.30 also points out that the induced emf $\mathcal{E}$ varies sinusoidally with time. The maximum emf occurs when $\sin\omega t$ is equal to 1 and is denoted by $\mathcal{E}_{max}$, where

$$\mathcal{E}_{max} = \omega AB \qquad (9.31)$$

The induced emf can then be written as

$$\mathcal{E} = \mathcal{E}_{max} \sin\omega t \qquad (9.32)$$

If the coil is connected to a circuit that has a resistance R, the current that will flow from the coil to the circuit is given by Ohm's law as

$$i = \frac{\mathcal{E}}{R} = \frac{\mathcal{E}_{max}}{R} \sin\omega t \qquad (9.33)$$

Equation 9.33 indicates that the current in the circuit will also vary sinusoidally with time. The current starts at zero and increases to a maximum value given by

$$I_{max} = \frac{\mathcal{E}_{max}}{R} \qquad (9.34)$$

The variation of current with time is thus given by

$$i = i_{max} \sin\omega t \qquad (9.35)$$

This rotating loop of wire in an external magnetic field is called an AC Generator. Notice that in the *AC* generator, the magnitudes of the vectors **B** and **A** are

constant, but the direction between the two vectors is constantly changing with time. Because the magnetic flux is equal to $BA \cos\theta$, if the angle θ changes with time, there is also a changing magnetic flux, and hence, an induced emf.

The results for the *AC* generator could have been found as a purely mathematical result by noting that, from Faraday's law, the induced emf is given by

$$\mathcal{E} = -\frac{d\Phi_M}{dt} \tag{9.15}$$

or

$$\mathcal{E} = -\frac{d(BA\cos\theta)}{dt}$$

Since B and A are constants

$$\mathcal{E} = -BA\frac{d(\cos\theta)}{dt} = -BA(-\sin\theta)\frac{d\theta}{dt}$$

But $\frac{d\theta}{dt} = \omega$, the angular speed of the rotating coil. Therefore, the induced emf in the rotating coil becomes

$$\mathcal{E} = \omega AB \sin\theta$$

which is the same result as found in equation 9.30. As you can see this is a very simple derivation of the induced emf for an AC generator, but some of the physical picture is lost in this purely mathematical process.

From a practical standpoint it is usually desirable to obtain an even larger emf from the *AC* generator than that obtained from equation 9.32. This can be accomplished by increasing the number of turns constituting the coil, to N turns. As an example, if two turns are used, the voltage and current will be doubled, if ten turns are used, the voltage and current will increase tenfold. For a coil of N turns, equations 9.32 and 9.33 are multiplied by the factor N, and the voltage and current will increase N times.

Example 9.3

An alternating generator. A coil of 100 turns of wire, with a cross sectional area of 100 cm^2, is rotated at an angular speed of 377 rad/s in an external magnetic field of 0.450 T. Find (a) the maximum induced emf, (b) the frequency of the alternating emf, (c) if the resistance of the circuit is 100 Ω, find the current flow and the maximum current.

Solution

(a) The induced emf is found from equation 9.30 as

$$\mathcal{E} = N\omega AB \sin\omega t \qquad\qquad (9.30)$$

and the maximum emf is

$$\mathcal{E}_{max} = N\omega AB \qquad\qquad (9.31)$$

$$\mathcal{E}_{max} = (100)(377\underline{rad})(100\text{cm}^2)(\underline{1\text{ m}^2})(0.450\text{ T})$$
$$\phantom{\mathcal{E}_{max} = (100)(377}\text{s}\phantom{rad)(100\text{cm}^2)(}(10^4\text{ cm}^2)$$
$$\mathcal{E}_{max} = 170\ \underline{\text{T m}^2}\ \underline{(\text{N}/(\text{A m}))}(\underline{\text{ A }})\ \underline{(\text{ J })}\quad \underline{(\text{ V })}$$
$$\phantom{\mathcal{E}_{max} = 170\ }\text{s}\quad (\quad\text{T}\quad)\ (\text{C/s})\ (\text{N m})\quad(\text{J/C})$$
$$\mathcal{E}_{max} = 170\text{ V}$$

Notice how the conversion factors of units cancel, leaving the correct unit for the answer.

(b) The frequency of the rotation is related to the angular speed of the loop by

$$\omega = 2\pi f$$
$$f = \frac{\omega}{2\pi}$$

or

$$f = \frac{377\text{ rad/s}}{2\pi} = 60.0\text{ cycles/s} = 60.0\text{ Hz}$$

Thus, a 60.0 Hz frequency is obtained from a generator by rotating the coil at an angular speed of 377 rad/s.

(c) The current is found from equation 9.33 as

$$i = \frac{\mathcal{E}_{max}}{R}\sin\omega t$$
$$i = \frac{170\text{ V}}{100\ \Omega}\sin\omega t$$
$$i = (1.70\text{ A})\sin\omega t$$

and the maximum current is 1.70 A.

To go to this Interactive Example click on this sentence.

9.5 Mutual Induction

Consider the two coaxial solenoids of the same size shown in figure 9.5. (One blue wire and one red wire are wrapped around a hollow roll of cardboard.) If the current i_1, in coil 1, is changed with time, this will change the magnetic field B_1 of solenoid 1 with time. But solenoid 2 is in the magnetic field of coil 1 and any change in B_1 will
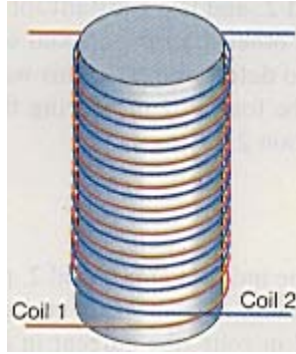
Figure 9.5 Two coaxial solenoids.

cause an induced emf in coil 2. From Faraday's law of induction, this can be stated as

$$\mathcal{E}_2 = - N_2 \frac{d\Phi_M}{dt}$$

$$\mathcal{E}_2 = - N_2 \frac{d(B_1 A)}{dt}$$

$$\mathcal{E}_2 = - N_2 A \frac{dB_1}{dt} \tag{9.36}$$

The magnetic field inside solenoid 1 is found from equation 8.58 as

$$B_1 = \mu_o n_1 i_1$$

$$B_1 = \mu_o \frac{N_1}{l} i_1 \tag{9.37}$$

When i_1 changes with time, B_1 will change as

$$dB_1 = \mu_o \frac{N_1}{l} di_1 \tag{9.38}$$

Replacing equation 9.38 into equation 9.36 gives

$$\mathcal{E}_2 = - N_2 A (\mu_o \frac{N_1}{l} \frac{di_1}{dt})$$

or

$$\mathcal{E}_2 = - \mu_o A \frac{N_1 N_2}{l} \frac{di_1}{dt}$$

Let us *define the quantity*

$$\mu_o A \frac{N_1 N_2}{l} = M \tag{9.39}$$

to be the coefficient of mutual induction for the coaxial solenoids. Faraday's law now becomes

$$\mathcal{E}_2 = -M \frac{di_1}{dt} \qquad (9.40)$$

Equation 9.40 says that changing the current i_1 in coil 1 with time induces an emf in coil 2. Note that M in equation 9.39 is only a function of the geometry of solenoids 1 and 2, and is a constant for the particular set of solenoids chosen. For other configurations than the coaxial solenoids, it is very difficult, sometimes almost impossible to determine M in this way. However, even in these difficult cases, M can always be found by measuring the quantities di_1/dt and $\mathcal{E}_2$. *The coefficient of mutual induction M is then found from equation 9.40 as*

$$M = -\frac{\mathcal{E}_2}{di_1/dt} \qquad (9.41)$$

Hence, equation 9.40 can be used to determine the induced emf in coil 2, produced by the changing current in coil 1, regardless of the actual geometrical configuration.

If the emf in equation 9.41 is expressed in volts and the current and time by amperes and seconds, respectively, then the unit for mutual inductance M is called a henry and is given by

$$1 \text{ henry} = \frac{\text{volt}}{\text{ampere/second}}$$

This will be abbreviated as

$$1 \text{ H} = \frac{\text{V}}{\text{A/s}}$$

By using a similar analogy, the current in coil 2 can be varied and an emf will be induced in coil 1 given by

$$\mathcal{E}_1 = -N_1 \frac{d\Phi_M}{dt} = -N_1 A \frac{dB_2}{dt}$$

where

$$dB_2 = \mu_o \frac{N_2}{l} di_2$$

and

$$\mathcal{E}_1 = -N_1 A \mu_o \frac{N_2}{l} \frac{di_2}{dt}$$

$$\mathcal{E}_1 = -\mu_o A N_1 \frac{N_2}{l} \frac{di_2}{dt} \qquad (9.42)$$

Notice that the coefficient of di_2/dt in equation 9.42 is the coefficient of mutual induction found in equation 9.39. Therefore, *the induced emf in coil 1 caused by a changing current in coil 2 is given by*

$$\mathcal{E}_1 = -M \frac{di_2}{dt} \qquad (9.43)$$

Example 9.4

Mutual inductance. Find the mutual inductance of two coaxial solenoids of 5.00 cm radius and 30.0 cm long if one coil has 10 turns and the second has 1000 turns.

Solution

The area of the solenoid is found as

$$A = \pi r^2 = \pi(0.0500 \text{ m})^2 = 7.85 \times 10^{-3} \text{ m}^2$$

The mutual inductance is found from equation 9.39 as

$$M = \frac{\mu_o A \, N_1 N_2}{l}$$

$$M = (4\pi \times 10^{-7} \, \underline{\frac{\text{T m}}{\text{A}}})\frac{(7.85 \times 10^{-3} \text{ m}^2)(10)(1000)}{0.300 \text{ m}}$$

and

$$M = 3.29 \times 10^{-4} \, \frac{\text{T m}^2}{\text{A}} \, \frac{(\text{N}/(\text{C m/s}))}{(\text{T})}$$

$$M = 3.29 \times 10^{-4} \, \frac{\text{N m}}{\text{A C/s}} \, \frac{(\text{J})}{(\text{N m})} \frac{(\text{V})}{(\text{J/C})}$$

$$M = 3.29 \times 10^{-4} \, \frac{\text{V}}{\text{A/s}} \, \frac{(\text{H})}{(\text{V}/(\text{A/s}))}$$

$$M = 3.29 \times 10^{-4} \text{ H}$$

Notice how the unit conversion factors are used to show the correct unit for the answer.

To go to this Interactive Example click on this sentence.

Example 9.5

Mutual Induction. If the current in the first coil of example 9.4 changes by 2.00 A in 0.001 seconds, what is the induced emf in the second coil?

Solution

The induced emf in the second coil is found from equation 9.40 to be

$$\mathcal{E}_2 = -M\frac{di_1}{dt} = -(3.29 \times 10^{-4}\text{ H})\frac{(2.00\text{ A})}{0.001\text{ s}}$$

$$\mathcal{E}_2 = -6.58 \times 10^{-1}\text{ H}\frac{(V/(A/s))}{(H)}\frac{A}{s}$$

$$\mathcal{E}_2 = -0.658\text{ V}$$

To go to this Interactive Example click on this sentence.

9.6 Self-Induction

Not only will a changing magnetic flux in one coil induce an emf in an adjacent coil, it will even induce an emf in itself. That is, if the current is varied in a single solenoid, the changing current will produce a changing magnetic field and the changing magnetic field will induce an emf in the single solenoid itself. This process is called self-induction and can be explained by Faraday's law of electromagnetic induction. A single solenoid is shown in figure 9.6. If the two ends of the solenoid
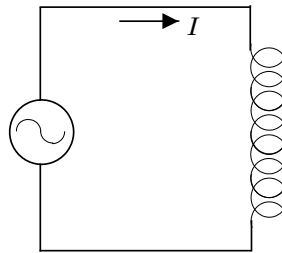

Figure 9.6 Self-induction.

coil are attached to the AC Generator of section 9.4 (shown as a circle with a sine wave in it in figure 9.6), the varying current in the coil will cause a varying magnetic field to exist in the coil. Because the magnetic field of a solenoid is given by

$$B = \mu_o n i \qquad (9.44)$$

changing the current i causes the magnetic field to change by

$$dB = \mu_o n\, di \qquad (9.45)$$

The induced emf is given by Faraday's law as

$$\mathcal{E} = -N \frac{d\Phi_M}{dt} = -N \frac{d(B\,A)}{dt}$$

Because the area A of the coil does not change, this becomes

$$\mathcal{E} = -NA \frac{dB}{dt} \qquad (9.46)$$

The changing magnetic field within the solenoid is given in equation 9.45 and substituting it into equation 9.46 gives

$$\mathcal{E} = -NA\mu_o n \frac{di}{dt} \qquad (9.47)$$

The total number of turns N is related to n the number of turns per unit length by

$$N = nl$$

where l is the length of the solenoid. Replacing this in equation 9.47 gives

$$\mathcal{E} = -(\mu_o A l n^2) \frac{di}{dt}$$

Note that the coefficient of di/dt is a constant which depends only upon the geometry of the solenoid coil. This constant is called the self-inductance of the solenoid coil and is designated by L, where

$$L = \mu_o A l n^2 \qquad (9.48)$$

The self-induced emf is now given by

$$\mathcal{E} = -L \frac{di}{dt} \qquad (9.49)$$

Equation 9.49 says that changing the current in a coil will induce an emf in the coil, and the minus sign says that the induced emf will act to oppose the cause of the induced emf. That is, if the applied emf is positive, the self-induced emf is negative; if the applied emf is negative, the self-induced emf is positive. Or stated another way, if the current is increasing in the coil, the direction of the induced emf is opposite to that of the current. If the current is decreasing, the induced emf is in the same direction as the original current.

In many cases it will be very difficult, if not impossible, to derive L as we did for the special case of the solenoid in equation 9.48. *However, in all cases equation 9.49 holds and can be used to define the self-inductance as*

$$L = -\frac{\mathcal{E}}{di/dt} \qquad (9.50)$$

$\mathcal{E}$ and *di/dt* can be measured experimentally for any coil configuration and *L* can be determined from equation 9.50. *L*, the self-inductance, is usually called the inductance of the coil, and is measured in henries, just as the mutual inductance was, i.e.

$$1\ H = 1\ \frac{V}{A/s}$$

A circuit element in which a self-induced emf accompanies a changing current is called an inductor. L is the inductance of the inductor and is represented in a circuit diagram by the symbol for a coil.

Example 9.6

Self induction. A battery is connected through a switch to a solenoid coil. The coil has 50 turns per centimeter, has a diameter of 10.0 cm, and is 50.0 cm long. When the switch is closed, the current goes from 0 to its maximum value of 3.00 A in 0.002 seconds. Find the inductance of the coil and the induced emf in the coil during this period.

Solution

The cross sectional area of the coil is

$$A = \frac{\pi d^2}{4} = \frac{\pi (0.100\ m)^2}{4} = 7.85 \times 10^{-3}\ m^2$$

The inductance of the solenoid coil is found from equation 9.48 as

$$L = \mu_o A l n^2$$
$$L = (4\pi \times 10^{-7}\ \frac{T\ m}{A})(7.85 \times 10^{-3}\ m^2)(0.500\ m)(\frac{(50)(100\ cm)}{(cm)(\ 1\ m\ )})^2$$
$$L = 0.19\ H$$

The induced emf in the coil is found from equation 9.49 as

$$\mathcal{E} = - L \frac{di}{dt}$$
$$\mathcal{E} = - (0.19\ H)\frac{(3.00\ A - 0\ A)}{0.002\ s}$$
$$\mathcal{E} = - 185\ V$$

The minus sign on $\mathcal{E}$, indicates that it is opposing the battery voltage V.

To go to this Interactive Example click on this sentence.

Example 9.7

Induction when the current is constant. What is the induced emf in example 9.6 after 0.002 seconds?

Solution

Since the maximum current in the circuit, 3.00 A, occurs at 0.002 s, there is no longer a change in current with time, i.e., $di/dt = 0$, after 0.002 s. Therefore, the induced emf is given by

$$\mathcal{E} = - L \frac{di}{dt} = - (0.19 \text{ H})(0) = 0$$

When the current is steady there is no longer a self-induced emf in the circuit.

Example 9.8

More induction. The switch in the above examples is now opened. If the current decays at the same rate at which it rose in the circuit, find the induced emf in the circuit.

Solution

The induced emf is given by equation 9.49 as

$$\mathcal{E} = - L \frac{di}{dt}$$

$$\mathcal{E} = - (0.19 \text{ H})\frac{(0 - 3.00 \text{ A})}{0.002 \text{ s}}$$

$$\mathcal{E} = + 185 \text{ V}$$

Note that $\mathcal{E}$ is now positive because the current is going from 3.00 A to 0 A and hence *di* is negative. This positive induced emf is opposing the decay of the current in the circuit.

To go to this Interactive Example click on this sentence.

9.7 The Energy Stored in the Magnetic Field of an Inductor

It was shown in section 6.3 that energy can be stored in the electric field between the plates of a capacitor. In a similar manner energy can be stored in the magnetic field of the coils of an inductor. When the switch in figure 9.7(a) is closed, the total applied voltage V is impressed across the ends of the solenoid coil. The initial current i is zero. In order for the current to increase an amount of charge dq must be taken out of the positive side of the battery and moved against the induced emf $\mathcal{E}$ in the coil. The amount of work done by the battery in moving this small amount of charge is

$$dW = (dq)\, \mathcal{E} \tag{9.51}$$

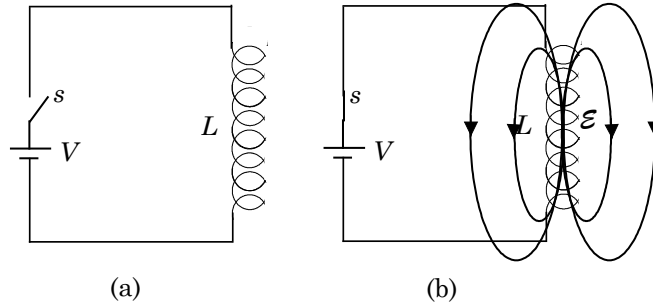But the magnitude of the induced emf is given by equation 9.49 as

(a)   (b)

Figure 9.7 Energy stored in the magnetic field of an inductor.

$$\mathcal{E} = L\, \frac{di}{dt} \tag{9.49}$$

Therefore, the small amount of work done is

$$dW = dq\, L\, \frac{di}{dt}$$

Upon rearranging terms

$$dW = \frac{dq}{dt}\, L\, di$$

$$dW = iL\, di \tag{9.52}$$

where use has been made of the fact that $dq/dt = i$, the current. This current i is not, however, a constant, but varies with time. Therefore, the small amount of work done is not a constant, but varies with the current i in the circuit. The total work done is equal to the sum or integral of all these dW's, that is

$$W = \int dW = \int_0^I Li\,di = \tfrac{1}{2}Li^2 \,\big|_0^I \qquad (9.53)$$

$$\boxed{W = \tfrac{1}{2}LI^2} \qquad (9.54)$$

This work W done by the battery on the charge, shows up as potential energy of the charge. This energy is said to reside in the magnetic field of the coil, figure 9.7(b), and is designated as U_M. *Thus, the energy stored in the inductor is given by*

$$\boxed{U_M = \tfrac{1}{2}LI^2} \qquad (9.55)$$

This stored energy can also be expressed in terms of the magnetic field B by recalling that for a solenoid

$$L = \mu_o A l n^2 \qquad (9.48)$$

and

$$B = \mu_o n I \qquad (9.44)$$

Solving equation 9.44 for the current I, we get

$$I = \frac{B}{\mu_o n} \qquad (9.56)$$

Replacing equations 9.48 and 9.56 into equation 9.55 gives

$$U_M = 1/2\ LI^2 = 1/2\ (\mu_o A l n^2)\frac{(B)^2}{(\mu_o n)^2}$$

$$U_M = 1/2\ \frac{B^2 A l}{\mu_o} \qquad (9.57)$$

Equation 9.57 gives the energy stored in the magnetic field of a solenoid.

The energy density is defined as the energy per unit volume and can be represented as

$$u_M = \frac{U_M}{V} = 1/2\ \frac{(B^2 A l)/\mu_o}{V}$$

But the volume of the solenoid is

$$V = A l$$

Therefore,

$$\boxed{u_M = \tfrac{1}{2}\frac{B^2}{\mu_o}} \qquad (9.58)$$

Equation 9.58 gives the magnetic energy density, or the energy per unit volume that is stored in the magnetic field. Although equations 9.54 and 9.58 were derived for a solenoid, they are perfectly general and apply to any inductor. Note the similarity with the energy density for an electric field.

$$u_E = 1/2\ \varepsilon_o E^2 \qquad (6.57)$$

Example 9.9

Energy in the magnetic field. What is the energy stored in the magnetic field in example 9.8?

Solution

In that example the inductance was found to be $L = 0.19$ H and the final current was 3.00 A. Therefore, the energy stored in the magnetic field is found from equation 9.55 as

$$U_M = 1/2 \; LI^2$$
$$U_M = 1/2 \; (0.19 \text{ H})(3.00 \text{ A})^2$$
$$U_M = 0.554 \text{ H A}^2 \; \frac{(V/(A/s))}{(H)}$$
$$U_M = 0.554 \; \frac{C \; V \; s}{s} \; \frac{(J/C)}{(V)}$$
$$U_M = 0.554 \text{ J}$$

To go to this Interactive Example click on this sentence.

9.8 Comparison of the Electrostatic Field and the Induced Electric Field

In studying the potential of a point charge in chapter 5 we showed that the potential difference between two points A and B in an electric field was given by

$$V_B - V_A = -\int \boldsymbol{E} \bullet d\boldsymbol{l} \tag{5.25}$$

Equation 5.25 says that if we start at the position A, where the potential is V_A, and move to the position B, where the potential is V_B, then the difference in potential $V_A - V_B$ is equal to the integral of $\mathbf{E} \bullet d\mathbf{l}$. But what happens if after we reach B we return to A, what is the difference in potential then? We see from equation 5.25 that the difference in potential in this case will be

$$V_A - V_A = 0 = -\oint \boldsymbol{E} \bullet d\boldsymbol{l} \tag{9.59}$$

Notice that the integral sign now has a circle around it to indicate that the integration is around a closed path, we returned to where we started from. We write equation 9.59 in the standard form

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = 0 \tag{9.60}$$

Equation 9.60 is a characteristic of all electrostatic fields. It says that the electrostatic field is a conservative field. This means that the work done in moving a charge in an electrostatic field is independent of the path taken. A characteristic of a conservative field is that a potential exists and we studied that electrostatic potential in chapter 5. We will see in a moment that this is not the same for an induced electric field.

Let us now reconsider the problem of motional emf studied in section 9.1 and illustrated in figure 9.1. Let us compute the integral of **E** • d**l** for the loop *MNOLM* in figure 9.1. Since the integral is around the entire loop we can break the integral into two parts, the first is over the path *MN* and the second is over the path *NOLM*, i.e.,

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = \int_{MN} \boldsymbol{E} \bullet d\boldsymbol{l} + \int_{NOLM} \boldsymbol{E} \bullet d\boldsymbol{l} \tag{9.61}$$

But there is no electric field **E** along the path *NOLM*, while the electric field **E** along the path *MN* is the induced electric field given by equation 9.2 as

$$\mathbf{E} = \mathbf{v} \times \mathbf{B} \tag{9.2}$$

which is caused by the motion of the wire in the magnetic field **B**. Equation 9.61 becomes

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = \int_{MN} (\boldsymbol{v} \times \boldsymbol{B}) \bullet d\boldsymbol{l} + 0 \tag{9.62}$$

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = \int_{MN} (\boldsymbol{v} \times \boldsymbol{B}) \bullet d\boldsymbol{l} = \int_{MN} vB \sin 90^0 dl \cos 0^0 = vBl \tag{9.63}$$

but we showed in equation 9.6 that

$$\mathcal{E} = vBl \tag{9.6}$$

Combining equations 9.6 with 9.63 gives

$$\mathcal{E} = \oint \boldsymbol{E} \bullet d\boldsymbol{l} \tag{9.64}$$

*Equation 9.64 says that the integral of **E** • d**l** around the closed loop is equal to the induced emf $\mathcal{E}$ in the loop.*

*Compare equations 9.60 and 9.64, they show the basic difference between the electrostatic field and the induced electric field. For the electrostatic field, the integral of **E** • d**l** around the closed loop is equal to zero, while for an induced electric field the integral of **E** • d**l** around the closed loop is equal to the induce emf $\mathcal{E}$ in the loop. Hence electrostatic fields are very different from induced electric fields. As we shall show shortly, the electrostatic field is constant in time whereas the induced electric field varies with time.*

9.9 Stokes' Theorem

Stokes' theorem is a theorem in vector analysis that relates a line integral of a vector quantity along a closed path surrounding an area, to the curl of that vector quantity throughout the area. Figure 9.8 shows a nonuniform electric field $\mathbf{E}$ in the x,y-plane. Let us calculate the quantity $\oint \mathbf{E} \cdot d\mathbf{l}$ around the rectangular path $abcd$ shown.

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_{ab} \mathbf{E} \cdot d\mathbf{l} + \int_{bc} \mathbf{E} \cdot d\mathbf{l} + \int_{cd} \mathbf{E} \cdot d\mathbf{l} + \int_{da} \mathbf{E} \cdot d\mathbf{l} \tag{9.65}$$

Since

$$\mathbf{E} = \mathbf{i}E_x + \mathbf{j}E_y \tag{9.66}$$

we can evaluate each of these integrals as

$$\int_{ab} \mathbf{E} \cdot d\mathbf{l} = \int_{ab} (\mathbf{i}E_x + \mathbf{j}E_y) \cdot d\mathbf{l} = \int_{ab} (\mathbf{i}E_x + \mathbf{j}E_y) \cdot \mathbf{i}dx = \int_{ab} E_x(y = y_1)dx \tag{9.67}$$
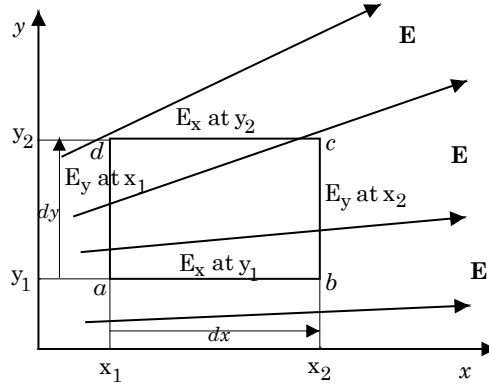
Figure 9.8 Stokes' Theorem.

where use has been made of the fact that $d\mathbf{l} = \mathbf{i}dx$ along path ab, and $\mathbf{i} \cdot \mathbf{i} = 1$ and $\mathbf{i} \cdot \mathbf{j} = 0$. We have also used the notation $E_x(y = y_1)$ to indicate that the E_x is evaluated everywhere along the path where $y = y_1$.

Performing the integration along path bc we obtain

$$\int_{bc} \mathbf{E} \cdot d\mathbf{l} = \int_{bc} (\mathbf{i}E_x + \mathbf{j}E_y) \cdot d\mathbf{l} = \int_{bc} (\mathbf{i}E_x + \mathbf{j}E_y) \cdot \mathbf{j}dy = \int_{bc} E_y(x = x_2)dy \tag{9.68}$$

where now $d\mathbf{l} = \mathbf{j}dy$ along path bc, and $E_y(x = x_2)$ means that E_y is evaluated everywhere along the path where $x = x_2$.

Performing the integration along path cd we obtain

$$\int_{cd} \mathbf{E} \cdot d\mathbf{l} = \int_{cd} (\mathbf{i}E_x + \mathbf{j}E_y) \cdot d\mathbf{l} = \int_{cd} (\mathbf{i}E_x + \mathbf{j}E_y) \cdot (-\mathbf{i}dx) = -\int_{cd} E_x(y = y_2)dx \tag{9.69}$$

where now $d\mathbf{l} = -\mathbf{i}dx$ along path cd, and $E_x(y = y_2)$ means that E_x is evaluated everywhere along the path where $y = y_2$.

Performing the final integration along path da we obtain

$$\int_{da} \mathbf{E} \cdot d\mathbf{l} = \int_{da} (\mathbf{i}E_x + \mathbf{j}E_y) \cdot d\mathbf{l} = \int_{da} (\mathbf{i}E_x + \mathbf{j}E_y) \cdot (-\mathbf{j}dy) = -\int_{da} E_y(x = x_1)dy \tag{9.70}$$

and for this final case $d\mathbf{l} = -\mathbf{j}dy$ along path da, and $E_y(x = x_1)$ means that E_y is eva-luated everywhere along the path where $x = x_1$.

Replacing the integrals from equations 9.67 to 9.70 in equation 9.65 we obtain

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_{ab} E_x(y = y_1)dx + \int_{bc} E_y(x = x_2)dy - \int_{cd} E_x(y = y_2)dx - \int_{da} E_y(x = x_1)dy$$
$$\oint \mathbf{E} \cdot d\mathbf{l} = \int [E_x(y = y_1)dx - E_x(y = y_2)dx + E_y(x = x_2)dy - E_y(x = x_1)dy]$$
$$\oint \mathbf{E} \cdot d\mathbf{l} = \int [E_x(y = y_1) - E_x(y = y_2)]dx + [E_y(x = x_2) - E_y(x = x_1)]dy \tag{9.71}$$

Let us now look at the meaning of some of these terms. The term $E_x(y = y_2)$ can be written as

$$E_x(y = y_2) = E_x(y = y_1) + \frac{\partial E_x}{\partial y}dy \tag{9.72}$$

Equation 9.72 says that the value of E_x at $y = y_2$ is equal to the value of E_x at $y = y_1$ plus the change in E_x as you move in the y-direction. But this change in E_x is equal to the rate at which E_x changes in the y-direction ($\partial E_x / \partial y$) times the total change in y, namely dy. Rearranging terms in equation 9.72 we get

$$E_x(y = y_1) - E_x(y = y_2) = -\frac{\partial E_x}{\partial y}dy \tag{9.73}$$

We will come back to this equation shortly. The term $E_y(x = x_2)$ can be written as

$$E_y(x = x_2) = E_y(x = x_1) + \frac{\partial E_y}{\partial x}dx \tag{9.74}$$

Equation 9.74 says that the value of E_y at $x = x_2$ is equal to the value of E_y at $x = x_1$ plus the change in E_y as you move in the x-direction. But this change in E_y is equal to the rate at which E_y changes in the x-direction ($\partial E_y / \partial x$) times the total change in x, namely dx. Rearranging terms in equation 9.74 we obtain

$$E_y(x = x_2) - E_y(x = x_1) = +\frac{\partial E_y}{\partial x}dx \tag{9.75}$$

Replacing equations 9.73 and 9.75 into equation 9.71 yields

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int [-\frac{\partial E_x}{\partial y}dy]dx + [+\frac{\partial E_y}{\partial x}dx]dy \tag{9.76}$$

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = \int [\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}] dx dy \tag{9.77}$$

But

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = \boldsymbol{k} \bullet [\nabla \times \boldsymbol{E}] \tag{9.78}$$

That is, the term $(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y})$ is equal to the *z*-component of the curl of the vector **E**. (As an exercise for the student, prove equation 9.78 by taking the curl of **E**, using equation 1.69 for the curl of **E**, and dot multiplying it by the unit vector **k**.) Replacing equation 9.78 into equation 9.77 gives

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = \int \boldsymbol{k} \bullet [\nabla \times \boldsymbol{E}] dx dy \tag{9.79}$$

But the product $dxdy$ is equal to the area dA in figure 9.8, and since an area can be represented as a vector, $\boldsymbol{k}dA = d\boldsymbol{A}$ is the vector area whose magnitude is equal to dA and whose direction is perpendicular to the *x,y*-plane. Incorporating this into equation 9.79 yields

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = \int [\nabla \times \boldsymbol{E}] \bullet d\boldsymbol{A} \tag{9.80}$$

*Equation 9.80 is Stokes' theorem. It relates the line integral of $\oint \boldsymbol{E} \bullet d\boldsymbol{l}$, sometimes called the circulation, around a closed path to the curl of **E** throughout the area bounded by the path. It is a very convenient tool that allows us to go from a surface integral to a line integral.* We will show the use of Stokes' theorem in the next section.

9.10 Generalization of Faraday's Law of Electromagnetic Induction

We have shown that Faraday's law can be written in the form

$$\mathcal{E} = -\frac{d\Phi_M}{dt} \tag{9.15}$$

We have used for the magnetic flux, the relation

$$\Phi_M = \boldsymbol{B} \bullet \boldsymbol{A} \tag{8.96}$$

Equation 8.96 was used to describe the amount of magnetic flux that passed through a flat surface area **A**. But in general, the surface could be any shape. To handle this case the surface is broken up into a large number of infinitesimal surface areas $d\boldsymbol{A}$, and the infinitesimal amount of flux $d\Phi_M$ through each of these little areas is computed, i.e.,

$$d\Phi_M = \mathbf{B} \bullet d\mathbf{A} \tag{9.81}$$

The total flux through the surface becomes the sum or integral of all the infinitesimal fluxes $d\Phi_M$, through all the infinitesimal areas $d\mathbf{A}$, that is

$$\Phi_M = \int d\Phi_M = \int \boldsymbol{B} \bullet d\boldsymbol{A} \tag{9.82}$$

We can substitute equation 9.82 into equation 9.15 to yield

$$\mathcal{E} = -\frac{d\Phi_M}{dt} = -\frac{d}{dt}\int B \bullet dA \tag{9.83}$$

But we also found in equation 9.64 that the induced emf could also be given as

$$\mathcal{E} = \oint \mathbf{E} \bullet d\mathbf{l} \tag{9.64}$$

Setting equation 9.64 equal to equation 9.83 gives

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = -\frac{d}{dt}\int \boldsymbol{B} \bullet d\boldsymbol{A} \tag{9.84}$$

Equation 9.84 is the integral form of the generalization of Faraday's law of electromagnetic induction.

Just as we used Gauss's divergence theorem to reduce an integral form of Gauss's law to a differential form, we now use Stokes' theorem to reduce Faraday's law in integral form to a differential form. Recall that Stokes' theorem was given by equation 9.80 as

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = \int [\nabla \times \boldsymbol{E}] \bullet d\boldsymbol{A} \tag{9.80}$$

Since the left-hand sides of equations 9.84 and 9.80 are the same, we can equate them to give

$$\int [\nabla \times \boldsymbol{E}] \bullet d\boldsymbol{A} = -\frac{d}{dt}\int \boldsymbol{B} \bullet d\boldsymbol{A} \tag{9.85}$$

Since the integrations are over the same area we can place each term under the same integral to obtain

$$\int [\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t}] \bullet d\boldsymbol{A} = 0 \tag{9.86}$$

For equation 9.86 to be equal to zero for all values of $\mathbf{E}$ and $\mathbf{B}$, the term in brackets must also equal zero. Therefore,

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0$$

and

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{9.87}$$

*Equation 9.87 is **Faraday's law of electromagnetic induction in differential form**. It is the fourth of Maxwell's equations. It says that changing the magnetic field* **B** *with time will produce an electric field* **E**. We will say much more about this equation in the next chapter.

It is interesting to also apply Stokes' theorem to an electrostatic field. We showed in equation 9.60 that for an electrostatic field

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = 0 \tag{9.60}$$

Applying Stokes' theorem, equation 9.80, we get

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = 0 = \int [\nabla \times \boldsymbol{E}] \bullet d\boldsymbol{A}$$

Therefore

$$\nabla \times \boldsymbol{E} = 0 \tag{9.88}$$

*Equation 9.88 says that the **electrostatic field has no curl**.*

9.11 The Curl of the Electric Field

It was shown in section 1.12 that if you take the cross product of the del operator with any vector **E**, the result is given by equation 1.69 as

$$\nabla \times \boldsymbol{E} = \boldsymbol{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \tag{1.69}$$

Equation 1.69 is called the curl of the vector **E**. Using the notation for the cross product of two vectors given in equation 1.64, the curl of a vector can also be written as

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \tag{9.89}$$

In this section we will discuss the physical significance of the curl of a vector. As discussed in section 4.13, the earliest use of vector analysis was in the application of the motion of fluids. Such terminology as divergence and curl were used to describe the fluid. *We will continue to describe fluid motion in order to simplify the concept of the curl. However, the basic concept belongs to the description of any vector field and as such they will apply to the electric and magnetic fields that we will study in this course.* Everywhere that we use the velocity vector **v**, you can also use the electric field vector **E**, and the magnetic field vector **B**.

We will replace the vector **E** in equation 1.69 by the vector **v**, the velocity of water flowing in an ocean or a large river. As an example let us assume that the velocity of the water is in the x-direction, but varies with y. That is, let $v_y = v_z = 0$ and

$$v_x = yv \tag{9.90}$$

The curl of **v** is obtained from equation 1.69 as

$$\nabla \times \boldsymbol{v} = \boldsymbol{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$\nabla \times \boldsymbol{v} = \boldsymbol{i}\left(\frac{\partial 0}{\partial y} - \frac{\partial 0}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial(yv)}{\partial z} - \frac{\partial 0}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial 0}{\partial x} - \frac{\partial(yv)}{\partial y}\right)$$

$$\nabla \times \boldsymbol{v} = \boldsymbol{k}\left(-\frac{\partial(yv)}{\partial y}\right)$$

$$\nabla \times \boldsymbol{v} = -\boldsymbol{k}v \tag{9.91}$$

$$\tag{9.90}$$

A picture of the fluid flow is shown in figure 9.9. Since the velocity increases as a function of y, notice that the length of the velocity vectors increases with y in the figure. This means that the water is flowing more rapidly as the value of y increases. The higher velocity water starts to rotate or curl down toward the x-axis as shown in the figure. That is, the water will start curling or rotating in a clockwise direction. For negative values of y, the flow of water is shown in the lower half of figure 9.9. Again the water will start curling or rotating in a clockwise direction. From this analysis, *we see that curl is a measure of the rotation of the vector field.* The curl is a vector, and as seen in equation 9.91, for this example the curl vector lies along the negative z-axis and points in the $-\boldsymbol{k}$ direction.
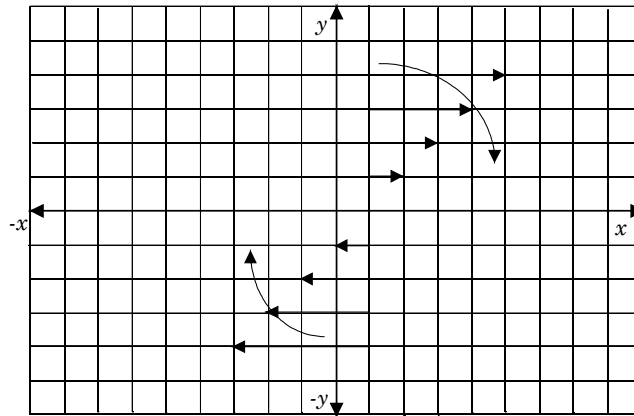

Figure 9.9 An example of negative curl.

An analogy that may prove helpful can be obtained by considering the rotation of a body about an axis at a constant angular velocity ω as studied in your College Physics course. The linear velocity of a point on the rotating body was found to be

$$v = \omega r \tag{9.92}$$

That is, as the distance r from the axis of rotation increased, the linear velocity v also increased. Hence, for a body in rotational motion, points furthest from the axis of rotation have the greatest linear velocity. As a corollary, if the points furthest from the axis are moving faster than the points closest to the axis, then the body is rotating. Thus the velocity distribution in figure 9.9 will cause the water to rotate or curl. For a large value of v in equation 9.91, the curl will be a large number indicating that the rotation of the fluid is very rapid. For a small value of v in equation 9.91, the curl will be a small number indicating that the rotation of the fluid is very slow.

Another way to visualize this curl or rotation is to hold a pencil in the palm of your left hand, such that the axis of the pencil is perpendicular to the length of your hand. Now place your right hand on top of your left hand and move your right hand to the right and your left hand to the left simultaneously. As you do this you will notice that the pencil rotates in a clockwise direction. Your right hand moving toward the right is like the velocity distribution in the top half of figure 9.9, and your left hand moving toward the left is like the velocity distribution in the bottom half of figure 9.9. The overall effect is a curl or rotation of the pencil clockwise, just as the fluid would rotate or curl clockwise.

As a second example, figure 9.10, let the velocity of the water be given by

$$v_x = -yv \tag{9.93}$$

That is, let $v_y = v_z = 0$. Notice that this is almost the same as the first example except for the negative sign. The curl of $\mathbf{v}$ is obtained from equation 9.90 as
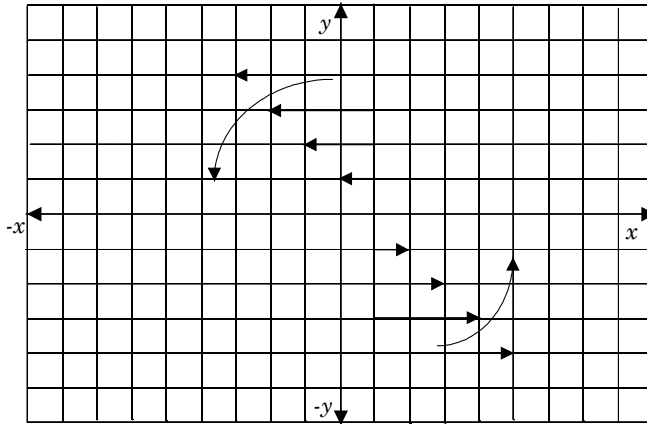

Figure 9.10 An example of positive curl.

$$\nabla \times \boldsymbol{v} = \boldsymbol{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$\nabla \times \boldsymbol{v} = \boldsymbol{i}\left(\frac{\partial 0}{\partial y} - \frac{\partial 0}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial (-yv)}{\partial z} - \frac{\partial 0}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial 0}{\partial x} - \frac{\partial (-yv)}{\partial y}\right) \tag{9.90}$$

$$\nabla \times \boldsymbol{v} = \boldsymbol{k}\left(-\frac{\partial(-yv)}{\partial y}\right)$$
$$\nabla \times \boldsymbol{v} = +\boldsymbol{k}v \tag{9.94}$$

Notice that this is almost the same solution as for the first example except for the positive sign.

The picture of the fluid flow is shown in figure 9.10. As expected the flow is the negative of the flow of figure 9.9. In this case you can see that the water will rotate or curl in a counterclockwise direction, and, from equation 9.94, the curl vector lies along the positive z-axis in the positive $\boldsymbol{k}$ direction.

Example 9.10

The curl of $\boldsymbol{E}$. In a certain region the electric field intensity $\boldsymbol{E}$ has a uniform value of 2 N/C in the x-direction. Find the curl of $\boldsymbol{E}$.

Solution

The electric field vector is given by the function, $\mathbf{E} = 2\mathbf{i}$. Hence, $E_y = E_z = 0$. The curl of the electric field is found from equation 1.69 as

$$\nabla \times \mathbf{E} = \mathbf{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$\nabla \times \boldsymbol{E} = \mathbf{i}\left(\frac{\partial 0}{\partial y} - \frac{\partial 0}{\partial z}\right) + \mathbf{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial 0}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial 0}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$\nabla \times \boldsymbol{E} = \mathbf{j}\frac{\partial E_x}{\partial z} + \boldsymbol{k}\left(-\frac{\partial E_x}{\partial y}\right) = \mathbf{j}\frac{\partial(2)}{\partial z} - \boldsymbol{k}\left(\frac{\partial(2)}{\partial y}\right)$$

$$\nabla \times \boldsymbol{E} = 0$$

Notice that this is the same electric field shown in figure 4.15. As can be seen in that figure, the electric field is uniform in the x-direction and there is no tendency for rotational motion or curl, and substantiates our result that the curl of $\boldsymbol{E}$ is zero. Also note that since the curl of $\boldsymbol{E}$ is zero, this must be an electrostatic field.

Example 9.11

The curl of $\boldsymbol{E}$. In a certain region the electric field intensity $\boldsymbol{E}$ has a distribution given by

$$\mathbf{E} = \mathbf{i}yE_o + \mathbf{j}xE_o$$

Find the curl of $\mathbf{E}$.

Solution

The curl of **E** is found from equation 1.69 as

$$\bigtriangledown \times \boldsymbol{E} = \boldsymbol{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \quad (1.69)$$

$$\bigtriangledown \times \boldsymbol{E} = \boldsymbol{i}\left(\frac{\partial 0}{\partial y} - \frac{\partial (xE_o)}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial (yE_o)}{\partial z} - \frac{\partial 0}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial (xE_o)}{\partial x} - \frac{\partial (yE_o)}{\partial y}\right)$$

$$\bigtriangledown \times \boldsymbol{E} = +\boldsymbol{k}(E_o - E_o) = 0 \quad (9.95)$$

Notice that in this case the curl of **E** is equal to zero but the electric field is not a constant as in example 9.10. We can see from equation 9.95 that there is a curl + **k**E_o in a counterclockwise direction and a curl – **k**E_o in a clockwise direction and they balance each other out to give zero curl.

Summary of Important Concepts

Faraday's law of electromagnetic induction - whenever the magnetic flux through a coil changes with time, an emf will be induced in the coil. The magnetic flux can be changed by changing the magnetic field *B*, the area *A* of the loop, or the direction between the magnetic field and the area vector.

Lenz's law - The direction of an induced emf is such that any current it produces, always opposes, through the magnetic field of the induced current, the change inducing the emf.

AC Generator - a device in which a coil of wire is manually rotated in an external magnetic field. The rotating coil has an alternating, sinusoidally varying emf and current induced in the coil. The *AC* generator is a source of alternating current.

Mutual Induction - Changing the magnetic flux in one coil induces an emf in an adjacent coil.

Self-Induction - Changing the magnetic flux in a coil will induce an emf in that coil. The induced emf opposes the changing magnetic flux.

Inductor - A circuit element in which a self-induced emf accompanies a changing current.

The curl ($\bigtriangledown \times$ E) of a vector - a measure of the amount of rotation in the vector field. When the curl is positive the vector field will start curling or rotating in a counterclockwise direction. When the curl is negative the vector field will start curl-

ing or rotating in a clockwise direction. When the curl is equal to zero there is no tendency for the field to rotate in either a clockwise or counterclockwise direction.

Summary of Important Equations

Induced electric field $\qquad\qquad\qquad$ $\mathbf{E} = \mathbf{F}/q = \mathbf{v} \times \mathbf{B}$ $\qquad\qquad$ (9.2)

Magnitude of induced electric field $\qquad\qquad$ $E = vB \sin\theta$ $\qquad\qquad$ (9.3)

Faraday's law of induction $\qquad$ $\mathcal{E} = -\dfrac{d\Phi_M}{dt} = -\mathbf{B} \bullet \dfrac{d\mathbf{A}}{dt} - \mathbf{A} \bullet \dfrac{d\mathbf{B}}{dt}$ $\qquad$ (9.18)

Faraday's law for N loops $\qquad\qquad$ $\mathcal{E} = -N\dfrac{d\Phi_M}{dt}$ $\qquad\qquad$ (9.19)

Induced emf in a rotating coil $\qquad\qquad$ $\mathcal{E} = \omega AB \sin\omega t$ $\qquad\qquad$ (9.30)

An alternating emf $\qquad\qquad\qquad$ $\mathcal{E} = \mathcal{E}_{max} \sin\omega t$ $\qquad\qquad$ (9.32)

An alternating current $\qquad\qquad\qquad$ $i = i_{max} \sin\omega t$ $\qquad\qquad$ (9.35)

emf induced in coil 2 by changing current in coil 1 $\qquad$ $\mathcal{E}_2 = -M\dfrac{di_1}{dt}$ $\qquad$ (9.40)

Mutual inductance $\qquad\qquad$ $M = -\dfrac{\mathcal{E}_2}{di_1/dt}$ $\qquad\qquad$ (9.41)

Inductance of a solenoid $\qquad\qquad$ $L = \mu_o A l n^2$ $\qquad\qquad$ (9.48)

Self-induced emf of a coil $\qquad\qquad$ $\mathcal{E} = -L\, di/dt$ $\qquad\qquad$ (9.49)

Inductance $\qquad\qquad\qquad$ $L = -\dfrac{\mathcal{E}}{di/dt}$ $\qquad\qquad$ (9.50)

Energy stored in magnetic field of an inductor $\qquad$ $U_M = 1/2\, LI^2$ $\qquad$ (9.55)

Energy stored in magnetic field of a solenoid $\qquad$ $U_M = \dfrac{1/2\, B^2 Al}{\mu_o}$ $\qquad$ (9.57)

Magnetic energy density $\qquad\qquad$ $u_M = 1/2\,\dfrac{B^2}{\mu_o}$ $\qquad\qquad$ (9.58)

Characteristic of all electrostatic fields $\qquad$ $\oint \boldsymbol{E} \bullet d\boldsymbol{l} = 0$ $\qquad$ (9.60)

Characteristic of induced electric fields $\qquad$ $\mathcal{E} = \oint \boldsymbol{E} \bullet d\boldsymbol{l}$ $\qquad$ (9.64)

Stokes' theorem
$$\oint \mathbf{E} \cdot d\mathbf{l} = \int [\nabla \times \mathbf{E}] \cdot d\mathbf{A} \qquad (9.80)$$

Integral form of Faraday's law of electromagnetic induction
$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A} \qquad (9.84)$$

Differential form of Faraday's law of electromagnetic induction
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (9.87)$$

The curl of the electric field
$$\nabla \times \mathbf{E} = \mathbf{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \qquad (1.69)$$

Questions for Chapter 9

1. If changing the magnetic flux with time induces an electric field, does changing the electric flux with time induce a magnetic field?

2. If the metal wire *MN* in figure 9-1 were replaced with a wooden stick, how would this affect the experiment?

3. Show that if the area vector in figure 9-2 were defined in the opposite direction, the analysis would not be consistent with Lenz's law. Therefore, show that the choice of direction for the area vector **A** cannot be arbitrary.

4. Is it possible to change both the area of a loop and the magnetic field passing through the loop and still not have an induced emf in the loop?

5. Discuss Lenz's law.

6. Can an electric motor be used to drive an AC generator, with the output from the generator being used to operate the motor?

7. Discuss how energy is stored in the magnetic field of a coil. Compare this to the way energy is stored in the electric field of a capacitor.

8. If changing an electric field with time produces a magnetic field, and changing a magnetic field with time produces an electric field, is it possible for these changing fields to couple together to propagate through space?
 charge?

Problems for Chapter 9

1. The magnetic flux through a coil of 10 turns, changes from 5.00×10^{-4} Wb to 5.00×10^{-3} Wb in 1.00×10^{-2} s. Find the induced emf in the coil.

2. A circular coil 6.00 cm in diameter is placed in a uniform magnetic field of 0.500 T. If *B* drops to 0 in 0.002 s find the maximum induced emf in the coil.

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3. A 25-turn circular coil 6.00 cm in diameter is placed in, and perpendicular to, a magnetic field that is changing at 2.50×10^{-2} T/s. (a) Find the induced emf in the coil. (b) If the resistance of the coil is 25.0 Ω, find the induced current in the coil.

4. A circular coil 5.00 cm in diameter has a resistance of 2.00 Ω. If a current of 4.00 A is to flow in the coil, at what rate should the magnetic field change with time if (a) the coil is perpendicular to the magnetic field and (b) if the coil makes an angle of 30^0 with the field?

5. The wire *MN* in the diagram of figure 9-1(a) moves to the right with a velocity of 25.0 cm/s. If l = 20.0 cm, *B* = 0.300 T, and the total resistance of the circuit is 50.0 Ω, find (a) the induced emf in wire *MN*, (b) the current flowing in the circuit, and (c) the direction of the current.

6. Repeat problem 5 if the wire *MN* is moving to the left with a velocity of 25.0 cm/s.

7. If wire *MN* of problem 5 moves with a velocity of 50.0 m/s to the right, at what rate should **B** change when the wire is at a distance of 5.00 cm from the left end of the loop, such that there will be no induced emf in the circuit?

8. Wire *MN* (l = 25.0 cm) of figure 9-1(a) is fixed 20.0 cm away from the galvanometer wire *OL*. The resistance of the circuit is 50.0 Ω. (a) If the magnetic field varies from 0 to 0.350 T in a time of 0.030 s, find the induced emf and current in the circuit during this time. (b) If the magnetic field remains constant at 0.350 T, find the induced emf and current in the coil. (c) If the magnetic field decays from 0.350 T to 0 T in 0.0200 s, find the induced emf, the current, and its direction in the wire.

9. An airplane is flying at 200 knots through an area where the vertical component of the earth's magnetic field is 3.50×10^{-5} T. If the wing span is 12.0 m, what is the difference in potential from wing tip to wing tip? Could this potential difference be used as a source of current to operate the aircraft's equipment?

10. A train is traveling at 40.0 km/hr in a location where the vertical component of the earth's magnetic field is 3.00×10^{-5} T. If the axle of the wheels is 1.50 m apart, what is the induced potential difference across the axle?

11. A coil of 10 turns and 35.0-cm² area is in a perpendicular magnetic field of 0.0500 T. The coil is then pulled completely out of the field in 0.100 s. Find the induced emf in the coil as it is pulled out of the field.

12. It is desired to determine the magnitude of the magnetic field between the poles of a large magnet. A rectangular coil of 10 turns, with dimensions of 5.00 cm by 8.00 cm, is placed perpendicular to the magnetic field. The coil is then pulled out of the field in 0.0500 s and an induced emf of 0.0250 V is observed in the coil. What is the value of the magnetic field between the poles?

13. It is desired to determine the magnetic field of a bar magnet. The bar magnet is pushed through a 10.0-cm diameter circular coil in 2.50×10^{-3} s and an emf of 0.750 V is obtained. Find the magnetic field of the bar magnet.

14. The flux through a 20-turn coil of 20.0 Ω resistance changes by 5.00 Wb/m² in a time of 0.0200 s. Find the induced current in the coil.

15. Show that whenever the flux through a coil of resistance R changes by $\Delta\Phi_M$, there will always be an induced charge in the loop given by

$$\Delta q = -\frac{N\Delta\Phi_M}{R}$$

This induced charge becomes the induced current in the coil.

16. A circular coil in a generator has 50 turns, a diameter of 10.0 cm, and rotates in a field of 0.500 T. What must be the angular velocity of the coil if it is to generate a maximum voltage of 50.0 V?

17. A 50.0-Ω circular coil of wire of 20 turns, 5.00 cm in diameter, is rotated at an angular velocity of 377 rad/s in an external magnetic field of 2.00 T. Find (a) the maximum induced emf, (b) the frequency of the alternating emf, (c) the maximum current in the coil, and (d) the instantaneous current at 5.00 s.

18. An AC generator consists of a coil of 200 turns, 10.0 cm in diameter. If the coil rotates at 500 rpm in a magnetic field of 0.250 T, find the maximum induced emf.

19. You are asked to design an AC generator that will give an output voltage of 156 V maximum at a frequency of 60.0 Hz. (a) Find the product of N, the number of turns in the coil, B, the magnetic field, and A, the area of the coil, that will give this value of voltage. (b) Pick a reasonable set of values for N, B, and A to satisfy the requirement.

20. At what angular speed should a coil of 1.00-m^2 area be rotated in the earth's magnetic field in a region where B is 3.50×10^{-5} T in order to generate a maximum emf of 1.50 V?

21. A circular coil of wire of 25.0-cm^2 area is placed in a magnetic field of 0.0500 T. What will the induced emf in the coil be if the coil is rotated at 20.0 rad/s (a) about an axis that is aligned with the magnetic field and (b) about an axis that is perpendicular to the magnetic field?

22. An AC generator is designed with a circular cross section of radius 1.25 cm to produce 120 V. What would the radius of the generator coil have to be if it is to produce 240 V, assuming that the magnetic field, the frequency, and the number of turns remains constant?

23. An AC generator produces 10.0 V when the coil rotates at 500 rpm. Find the emf when the coil is made to rotate at 1500 rpm.

24. An emf of 0.800 V is observed in a circuit when a nearby circuit has a current change at the rate of 200 A/s. What is the mutual inductance of the circuits?

25. Find the mutual inductance of two coaxial solenoids of 10.0-cm radius, 20.0 cm in length, with 120 turns in coil 1 and 200 turns in coil 2.

26. Two coils have a mutual inductance of 5.00 mH. If the current in the first coil changes by 3.00 A in 0.0200 s, what is the induced emf in the second coil?

27. What is the self-induced emf in a coil of 5.00 H if the current through it is changing at the rate of 150 A/s?

28. A coil has an inductance of 5.00 mH. At what rate should current change in the coil to give an induced emf of 100 V?

29. The current through a 10.0-mH inductor changes at the rate of 250 A/s. Find the change of flux in the coil.

30. How much energy is stored in the magnetic field of a coil of 5.00-mH inductance, carrying a current of 5.00 A?

31. A solenoid has 2500 turns/m, a diameter of 10.0 cm, and a length of 20.0 cm. If the current varies from 0 to 10.0 A, how much energy is stored in the magnetic field of the solenoid?

32. If 80.0 J of energy are stored in an inductor when the current changes from 5.00 A to 15.0 A, find the inductance of the inductor.

33. A rod 25.0 cm long is rotated at an angular velocity of 20.0 rad/s in a uniform magnetic field of 4.5×10^{-3} T that points into the page. Find the induced emf from end to end of the rod.

34. A square loop of wire, 2.00 mm on each side, is placed at the center of a flat coil of 10 turns and 10.0 cm radius. The current in the coil increases from 0 to 1.00 A in 2.50 s. If the resistance of the loop is 0.15 Ω, find the emf induced in the square loop.

35. A 7.50-cm diameter circular loop of wire is initially oriented so that an external magnetic field of 4.00 T is normal to the plane of the loop. In 2.00 s, the loop rotates about an axis in its own plane so that it ultimately makes an angle of 45^0 with the field. During the *same* 2.00 s, the magnitude of the external field rises to 5.66 T. Determine the emf induced in the loop.

36. The current through a coil changes from 300 mA to 150 mA in 5.00×10^{-3} s. An induced emf of 2.00×10^{-2} V is obtained. Find (a) the inductance of the coil, (b) the initial energy in the field, and (c) the final energy in the field.

37. An air solenoid has an inductance of 5.00 mH. Find its inductance if a piece of iron ($\mu = 800 \, \mu_0$) is placed within the solenoid.

38. A 5.00-mH inductor and a 50.0-Ω resistor are connected in series to a 24.0-V battery. Find (a) the final current in the circuit and (b) the energy stored in the magnetic field of the inductor.

39. In the derivation of equation 23.18, the induced emf in wire *cd* was given by $\mathcal{E}_{cd} = vBl \sin\theta$. Using figure 23.5, find the angle between **v** and **B** for wire *cd*. Show that the sine of that angle reduces to the sine of the angle θ.

40. It was shown in figure 9-1(a) that when the wire *MN* is moved to the right a current is induced in the wire from *M* to *N*. Show that this wire is a current-carrying wire in an external magnetic field, and as such experiences a force. Show that this force opposes the original motion and tends to slow down the motion of the wire and is a manifestation of Lenz's law.