Chapter 2: Spacetime and General Relativity

> The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth, space by itself, and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.
>
> H. Minkowski - "Space and Time"

2.1 Spacetime Diagrams

Shortly after Einstein published his special theory of relativity, Hermann Minkowski (1864-1909), a former instructor of Einstein, set about to geometrize relativity. He said that time and space are inseparable. In his words, "Nobody has ever noticed a place except at a time, or a time except at a place.… A point of space at a point of time, that is, a system of values of *x,y,z,t,* I will call a world-point. The multiplicity of all thinkable *x,y,z,t,* systems of values we will christen the world."[1]

To simplify the discussion, we will consider only one space dimension, namely the *x*-coordinate. *Any occurrence in spacetime will be called an event,* and is represented in the **spacetime diagram** of figure 2.1(a). This event might be the explosion of a firecracker, let us say. The location of this event is the *world point,* and it has the coordinates x and t. (Many authors of more advanced relativity books interchange the coordinates, showing the time axis in the vertical direction to emphasize that this is a different graph than a conventional plot of distance versus time. However, we will use the conventional graphical format in this book because it is already familiar to the student and will therefore make spacetime concepts easier to understand.)

Figure 2.1(b) is a picture of a **world line** of a particle at rest at the position x. *The graph shows that even though the particle is at rest in space, it is still moving through time.* Its *x*-coordinate is a constant because it is not moving through space, but its time coordinate is continually increasing showing its motion through time. Figure 2.1(c) represents a rod at rest in spacetime. The top line represents the world line of the end of the rod at x_2, whereas the bottom line represents the world line of the opposite end of the rod at x_1. Notice that the stationary rod sweeps out an area in spacetime. Figure 2.1(d) shows the world line of particle A moving at a constant velocity v_A and the world line of particle B moving at the constant velocity v_B. The slope of a straight line on an x versus t graph represents the velocity of the particle. The greater the slope, the greater the velocity. Since particle A has the greater slope it has the greater velocity, that is, $v_A > v_B$. If the velocity of a particle changes with time, its world line is no longer a straight line, but becomes curved, as shown in figure 2.1(e). *Thus, the world line of an accelerated particle is curved in spacetime.* Figure 2.1(f) is the world line of a mass attached to a spring that is executing simple harmonic motion. Note that the world line is curved everywhere

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[1] "Space and Time," by H. Minkowski in *The Principle of Relativity,* Dover Publications.

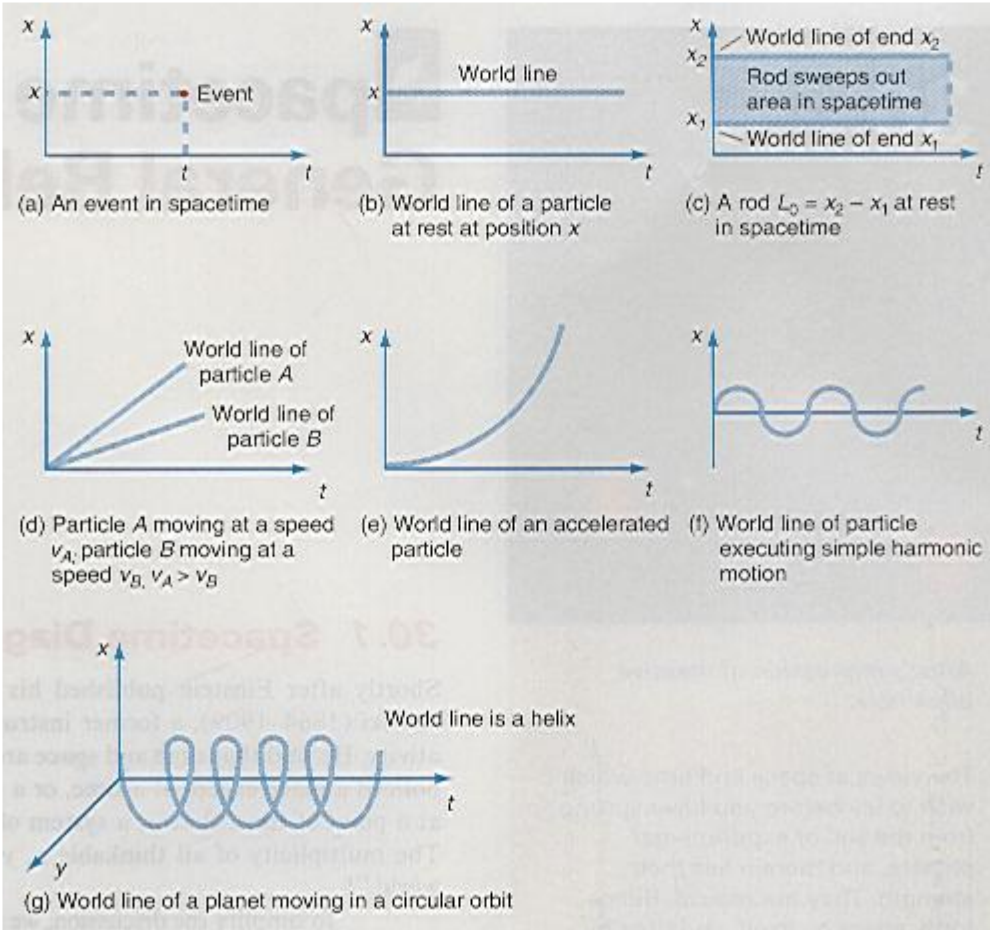

Figure 2.1 Spacetime diagrams.

indicating that this is accelerated motion. Figure 2.1(g) is a two-space dimensional picture of a planet in its orbit about the sun. The motion of the planet is in the x,y plane but since the planet is also moving in time, its world line comes out of the plane and becomes a helix. Thus, when the planet moves from position x, goes once around the orbit, and returns to the same space point x, it is not at the same position in spacetime. It has moved forward through time.

A further convenient representation in spacetime diagrams is attained by changing the time axis to τ, where

$$\tau = ct \tag{2.1}$$

In this representation, τ is actually a length. (The product of a velocity times the time is equal to a length.) The length τ is the distance that light travels in a particular time. If t is measured in seconds, then τ becomes a light second, which is the distance that light travels in 1 s, namely,

$$\tau = ct = \left(3.00\times10^{8}\ \frac{\text{m}}{\text{s}}\right)(1.00\ \text{s}) = 3.00\times10^{8}\ \text{m}$$

If t is measured in years, then τ becomes a light year, the distance that light travels in a period of time of 1 yr, namely,

$$\tau = ct = \left(3.00\times10^{8}\ \frac{\text{m}}{\text{s}}\right)(1\ \text{yr})\left(\frac{365\ \text{days}}{1\ \text{yr}}\right)\left(\frac{24\ \text{hr}}{1\ \text{day}}\right)\left(\frac{3600\ \text{s}}{1\ \text{hr}}\right)$$
$$= 9.47 \times 10^{15}\ \text{m} = 9.47 \times 10^{12}\ \text{km}$$

The light year is a unit of distance routinely used in astronomy.

With this new notation, we draw the spacetime diagram as shown in figure 2.2. A straight line on this diagram can still represent a velocity. However, since a

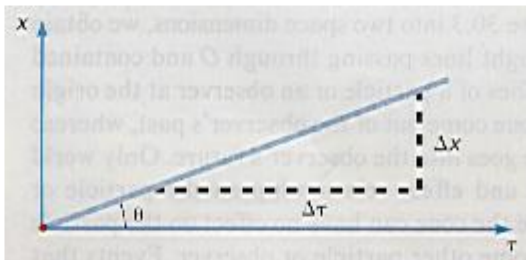

Figure 2.2 Changing the t-axis to a τ-axis.

velocity is given as

$$v = \frac{dx}{dt}$$

and since $\tau = ct$,

$$c\,dt = d\tau$$

or

$$dt = \frac{d\tau}{c}$$

Thus, the velocity becomes

$$v = \frac{dx}{dt} = \frac{dx}{d\tau/c} = \frac{c\,dx}{d\tau}$$

but $dx/d\tau$ is the slope of the line and is given by

$$\frac{dx}{d\tau} = \text{slope of line} = \tan\theta$$

Then the velocity on such a diagram is given by

$$v = c\tan\theta \tag{2.2}$$

As a special case in such a diagram, if $\theta = 45^0$, the $\tan 45^0 = 1$ and equation 2.2 becomes

$$v = c$$

Thus, on a spacetime diagram of x versus τ, a straight line at an angle of 45^0 represents the world line of a light signal.

If a source of light at the origin emits a ray of light simultaneously toward the right and toward the left, we represent it on a spacetime diagram as shown in figure 2.3. Line *OL* is the world line of the light ray emitted toward the right,

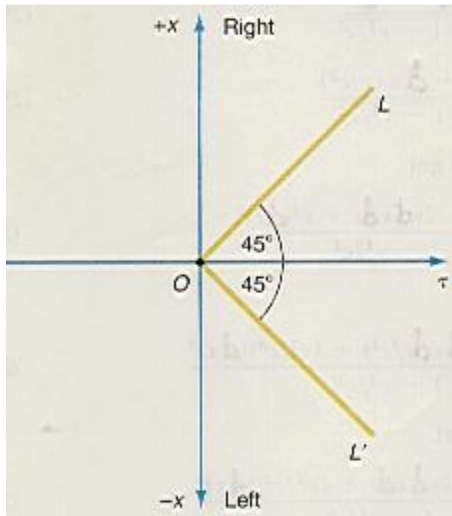

Figure 2.3 World lines of rays of light.

whereas *OL*' is the world line of the light ray emitted toward the left. Since the velocity of a particle must be less than c, the world line of any particle situated at *O* must have a slope less than 45^0 and is contained within the two light world lines *OL* and *OL*'. If the particle at *O* is at rest its world line is the τ-axis.

Example 2.1

The angle that a particle's world line makes as the particle moves through spacetime. If a particle moves to the right at a constant velocity of $c/2$, find the angle that its world line makes with the τ-axis.

Solution

Because the particle moves at a constant velocity through spacetime, its world line is a straight line. The angle that the world line makes with the τ-axis, found from equation 2.2, is

$$\theta = \tan^{-1}\frac{v}{c} \tag{2.3}$$

$$= \tan^{-1}\frac{c/2}{c} = \tan^{-1} 0.500$$

$$= 26.6^0$$

Notice that the world line for this particle is contained between the lines *OL* and *OL*'.

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If we extend the diagram of figure 2.3 into two space dimensions, we obtain the **light cone** shown in figure 2.4. Straight lines passing through O and contained within the light cone are possible world lines of a particle or an observer

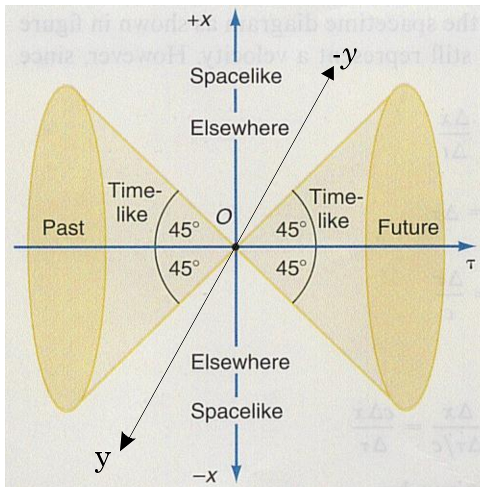

Figure 2.4 The light cone.

at the origin O. Any world lines inside the left-hand cone come out of the observer's past, whereas any world line inside the right-hand cone goes into the observer's future. Only world lines within the cone can have a cause and effect relationship on the particle or observer at O. World lines that lie outside the cone can have no effect on the particle or observer at O and are world lines of some other particle or observer. Events that we actually "see," lie on the light cone because we see these events by light rays. *World lines within the cone are sometimes called timelike because they are accessible to us in time. Events outside the cone are called spacelike because they occur in another part of space that is not accessible to us and hence is called elsewhere.*

2.2 The Invariant Interval

From what has been said so far, it seems as if everything is relative. *In the varying world of spacetime is there anything that remains a constant?* Is there some one single thing that all observers, regardless of their state of motion, can agree on? In the field of physics, we are always looking for some characteristic constants of motion. Recall from General Physics that when we studied the projectile motion of a particle in one dimension and saw that even though the projectile's position and velocity continually changed with time, there was one thing that always remained a constant, namely, the total energy of the projectile. In the same way we ask, isn't there a constant of the motion in spacetime? The answer is yes. *The constant value that all observers agree on, regardless of their state of motion, is called the* ***invariant interval.***

Let us now take the Lorentz transformation for the x-coordinate, equation 1.49

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

The differential dx' becomes

$$dx' = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}} \tag{2.4}$$

Similarly, let us now take the Lorentz transformation for the t-coordinate, equation 1.50

$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}}$$

Taking the time differential dt' we get

$$dt' = \frac{dt - dx(v/c^2)}{\sqrt{1 - v^2/c^2}} \tag{2.5}$$

Let us square each of these transformation equations to get

$$(dx')^2 = \frac{(dx)^2 - 2vdxdt + v^2(dt)^2}{1 - v^2/c^2} \tag{2.6}$$

and

$$(dt')^2 = \frac{(dt)^2 - (2vdxdt/c^2) + (v^2/c^4)(dx)^2}{1 - v^2/c^2} \tag{2.7}$$

Let us multiply equation 2.7 by c^2 to get

$$c^2(dt')^2 = \frac{c^2(dt)^2 - 2vdxdt + (v^2/c^2)(dx)^2}{1 - v^2/c^2} \tag{2.8}$$

Let us now subtract equation 2.6 from equation 2.8 to get

$$c^2(dt')^2 - (dx')^2 = \frac{c^2(dt)^2 - 2vdxdt + (v^2/c^2)(dx)^2}{1 - v^2/c^2} - \frac{(dx)^2 - 2vdxdt + v^2(dt)^2}{1 - v^2/c^2}$$

$$= \frac{c^2(dt)^2 - v^2(dt)^2 + (v^2/c^2)(dx)^2 - (dx)^2}{1 - v^2/c^2}$$

$$= \frac{(c^2 - v^2)(dt)^2 - (1 - v^2/c^2)(dx)^2}{1 - v^2/c^2}$$

$$c^2(dt')^2 - (dx')^2 = \frac{c^2(1 - v^2/c^2)(dt)^2 - (1 - v^2/c^2)(dx)^2}{1 - v^2/c^2}$$

Dividing each term on the right by $1 - v^2/c^2$ gives

$$c^2(dt')^2 - (dx')^2 = c^2(dt)^2 - (dx)^2 \tag{2.9}$$

Equation 2.9 shows that the quantity $c^2(dt)^2 - (dx)^2$ as measured by the S observer is equal to the same quantity $c^2(dt')^2 - (dx')^2$ as measured by the S' observer. But how can this be? This can be true only if each side of equation 2.9 is equal to a constant. *Thus, the quantity $c^2(dt)^2 - (dx)^2$ is an invariant. That is, it is the same in all inertial systems. This quantity is called the invariant interval and is denoted by $(ds)^2$. Hence the invariant interval is given by*

$$(ds)^2 = c^2(dt)^2 - (dx)^2 \tag{2.10}$$

The invariant interval is thus a constant in spacetime. All observers, regardless of their state of motion, agree on this value in spacetime. If the other two space dimensions are included, the invariant interval in four-dimensional spacetime becomes

$$(ds)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{2.11}$$

The invariant interval of spacetime is something of a strange quantity to us. In ordinary space, not spacetime, an invariant interval is given by the Pythagorean theorem as

$$(ds)^2 = (dx)^2 + (dy)^2 = (dx')^2 + (dy')^2 \tag{2.12}$$

as shown in figure 2.5, where ds is the invariant, and is seen to be nothing more

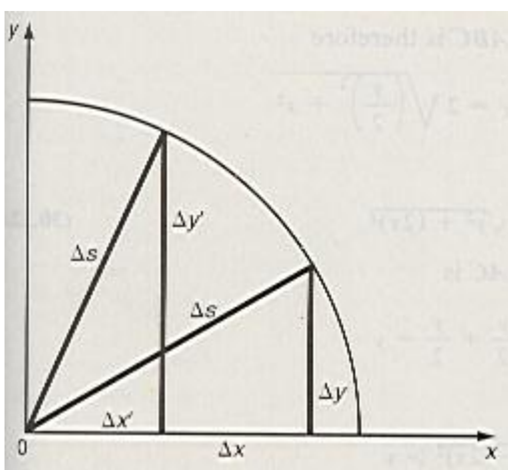

Figure 2.5 The invariant interval of space.

than the radius of the circle shown in figure 2.5 and given by equation 2.12. That is, equation 2.12 is of the form of the equation of a circle $r^2 = x^2 + y^2$. Even though dx

and dx' are different, and dy and dy' are different, the quantity ds is always the same positive quantity.

Now let us look at equation 2.10 for the invariant interval in spacetime. First, however, let $ct = \tau$ as we did previously in equation 2.1. Then we can express the invariant interval, equation 2.10, as

$$(ds)^2 = (d\tau)^2 - (dx)^2 \tag{2.13}$$

Because of the minus sign in front of $(dx)^2$*, the equation is not the equation of a circle* $(x^2 + y^2 = r^2)$*, but is rather the equation of a hyperbola,* $x^2 - y^2 = constant$*.*

The interval between two points in Euclidean geometry is represented by the hypotenuse of a right triangle and is given by the Pythagorean theorem: The square of the hypotenuse is equal to the *sum* of the squares of the other two sides of the triangle. However, *the square of the interval ds in spacetime is not equal to the sum of the squares of the other two sides, but to their difference. Thus, the Pythagorean theorem of Euclidean geometry does not hold in spacetime. Therefore, spacetime is not Euclidean. This new type of geometry described by equation 2.13 is sometimes called flat-hyperbolic geometry.* However, since hyperbolic geometry is another name for the non-Euclidean geometry of the Russian mathematician, Nikolai Ivanovich Lobachevski (1793-1856), rather than calling spacetime hyperbolic, we say that spacetime is non-Euclidean. *Space by itself is Euclidean, but spacetime is not. The fact that spacetime is not Euclidean accounts for the apparently strange characteristics of length contraction and time dilation* as we will see shortly. The minus sign in equation 2.13 is the basis for all the differences between space and spacetime.

Also, because of that minus sign in equation 2.13, $(ds)^2$ can be positive, negative, or zero. When $(d\tau)^2 > (dx)^2$, $(ds)^2$ is positive. Because the time term predominates, the world line in spacetime is called timelike and is found in the future light cone. When $(dx)^2 > (d\tau)^2$, $(ds)^2$ is negative. Because the space term predominates in this case, the world line is called spacelike. A spacelike world line lies outside the light cone in the region called elsewhere, figure 2.4. When $(dx)^2 = (d\tau)^2$, $(ds)^2$ is equal to zero. In this case, $(dx) = d\tau = (cdt)$. Hence, $dx = cdt$, or $dx/dt = c$. But dx/dt is a velocity. For it to equal c, it must be the world line of something moving at the speed of light. Thus $(ds)^2 = 0$ represents a light ray and the world line is called lightlike. Lightlike world lines make up the light cone.

Another characteristic of Euclidean space is that the straight line is the shortest distance between two points. Now we will see that *in non-Euclidean spacetime, the straight line is the longest distance between two points.* Consider the distance traveled along the two space paths of figure 2.6(a). The distance traveled along path AB in Euclidean space is found from the Pythagorean Theorem as

$$s_{AB} = \sqrt{\left(\frac{y}{2}\right)^2 + x^2}$$

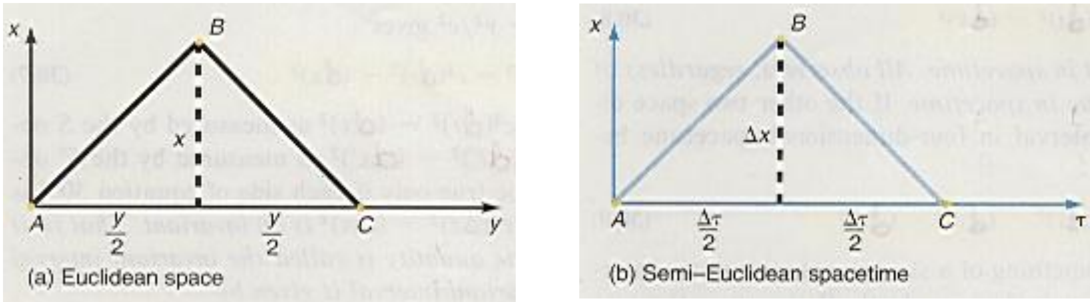

Figure 2.6 Space versus spacetime.

And the distance along path *BC* is similarly

$$s_{BC} = \sqrt{\left(\frac{y}{2}\right)^2 + x^2}$$

The total distance traveled along path *ABC* is therefore

$$s_{ABC} = s_{AB} + s_{BC} = 2\sqrt{\left(\frac{y}{2}\right)^2 + x^2}$$

or

$$s_{ABC} = \sqrt{y^2 + (2x)^2} \tag{2.14}$$

The total distance traveled along path *AC* is

$$s_{AC} = \frac{y}{2} + \frac{y}{2} = y$$

But since

$$\sqrt{y^2 + (2x^2)} > y$$

the round-about path *ABC* is longer than the straight line path *AC*, as expected.

Example 2.2

Path length in Euclidean space. If y = 8.00 and x = 3.00 in figure 2.6(a), find the path lengths s_{ABC} and s_{AC}.

Solution

The length of the path along *ABC*, found from equation 2.14, is

$$s_{ABC} = \sqrt{y^2 + (2x^2)} = \sqrt{(8.00)^2 + (2(3.00))^2}$$
$$= 10.0$$

The length of path AC is simply

$$s_{AC} = y = 8.00$$

Thus, the straight line path in space is shorter than the round-about path.

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Let us now look at the same problem in spacetime, as shown in figure 2.6(b). The distance traveled through spacetime along path AB is found by the invariant interval, equation 2.13, as

$$ds_{AB} = \sqrt{\left(\frac{d\tau}{2}\right)^2 - (dx)^2}$$

Whereas the distance traveled through spacetime along path BC is

$$ds_{BC} = \sqrt{\left(\frac{d\tau}{2}\right)^2 - (dx)^2}$$

The total distance traveled through spacetime along path ABC is thus,

$$ds_{ABC} = ds_{AB} + ds_{BC}$$
$$= 2\sqrt{\left(\frac{d\tau}{2}\right)^2 - (dx)^2}$$
$$ds_{ABC} = \sqrt{(d\tau)^2 - (2dx)^2} \tag{2.15}$$

Whereas the distance traveled through spacetime along the path AC is

$$ds_{AC} = \frac{d\tau}{2} + \frac{d\tau}{2} = d\tau$$

But comparing these two paths, ABC and AC, we see that

$$\sqrt{(d\tau)^2 - (2dx)^2} < d\tau \tag{2.16}$$

Therefore, the distance through spacetime along the round-about path ABC is less than the straight line path AC through spacetime. Thus, the shortest distance *between two points in spacetime is not the straight line. In fact the straight line is the longest distance between two points in spacetime.* These apparently strange effects of relativity occur because spacetime is non-Euclidean. (It is that minus sign again!)

Example 2.3

Path length in non-Euclidean spacetime. If $d\tau = 8.00$ and $dx = 3.00$ in figure 2.6(b), find the path lengths ds_{ABC} and ds_{AC}.

Solution

The interval along path ABC, found from equation 2.15, is

$$ds_{ABC} = \sqrt{(d\tau)^2 - (2dx)^2} = \sqrt{(8.00)^2 - (2(3.00))^2}$$
$$= 5.29$$

The interval along path AC is

$$ds_{AC} = d\tau = 8.00$$

Hence,

$$ds_{ABC} < ds_{AC}$$

and the straight line through spacetime is greater than the round-about line through spacetime.

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The straight line AC in spacetime is the world line of an object or clock at rest at the origin of the coordinate system. The spacetime interval for a clock at rest ($dx = 0$) is therefore

$$(ds)^2 = (d\tau)^2 - (dx)^2 = (d\tau_0)^2$$

or

$$ds = d\tau_0 \tag{2.17}$$

The subscript 0 has been used on τ to indicate that this is the time when the clock is at rest. *The time read by a clock at rest is called its proper time. But since this proper time is also equal to the spacetime interval, equation 2.17, and this spacetime interval is an invariant, it follows that the interval measured along any timelike world line is equal to its proper time.* If a clock is carried along with a body from A to B, ds_{AB} is the time that elapses on that clock as it moves from A to B, and ds_{BC} is the time that elapses along path BC. *Hence, from equation 2.16, the time elapsed along path ABC is less than the time elapsed along path AC. Thus, if two clocks started out synchronized at A, they read different times when they come together at* *point C. It is therefore sometimes said that time, like distance, is a route-dependent quantity.* The path *ABC* represents an accelerated path. (Actually the acceleration occurs almost instantaneously at the point *B*.) Hence *the lapse of proper time for an accelerated observer is less than the proper time for an observer at rest. Thus, time must slow down during an acceleration,* a result that we will confirm in our study of general relativity.

In chapter 1 we discussed the twin paradox, whereby one twin became an astronaut and traveled into outer space while the second twin remained home on earth. The Lorentz time dilation equation showed that the traveling astronaut, on his return, would be younger than his stay-at-home twin. Figure 2.6(b) is essentially a spacetime diagram of the twin paradox. The world line through spacetime for the stay-at-home twin is shown as path *AC*, whereas the world line for the astronaut is given by path *ABC*. Path *ABC* through spacetime is curved because the astronaut went through an acceleration phase in order to turn around to return to earth. Hence, the astronaut can no longer be considered as an inertial observer. Since the stay-at-home twin's path *AC* is a straight line in spacetime, she is an inertial observer. As we have just seen in the last paragraph, the time elapsed along path *ABC*, the astronaut's path, is less than the time elapsed along path *AC*, the stay-at-home's path. Thus the astronaut does indeed return home younger than his stay-at-home twin.

Perhaps one of the most important characteristics of the invariant interval is that it allows us to draw a good geometrical picture of spacetime as it is seen by different observers. For example, a portion of spacetime for a stationary observer *S* is shown in figure 2.7. The x and τ coordinates of *S* are shown as the orthogonal axes. The light lines *OL* and *OL'* are drawn at angles of 45^0. The interval, equation 2.13, is drawn for a series of values of x and τ and appear as the family of hyperbolas in the figure. (We might note that if spacetime were Euclidean the intervals would have been a family of concentric circles around the origin *O* instead of these hyperbolas.) The hyperbolas drawn about the τ-axis lie in the light cone future, while the hyperbolas drawn about the x-axis lie elsewhere. The interval has positive values within the light cone and negative values elsewhere.

A frame of reference *S*', moving at the velocity v, would have for the world line of its origin of coordinates, a straight line through spacetime inclined at an angle θ given by

$$\theta = \tan^{-1}\frac{v}{c} \tag{2.3}$$

For example, if *S*' is moving at a speed of $c/2$, $\theta = 26.6^0$. This world line is drawn in figure 2.7. But the world line of the origin of coordinates ($x' = 0$) is the time axis τ' of the *S*' frame, and is thus so labeled in the diagram. *Where* τ' *intersects the family of hyperbolas at* $ds = 1, 2, 3, \ldots$, it establishes the time scale along the τ'-*axis as* $\tau' =$

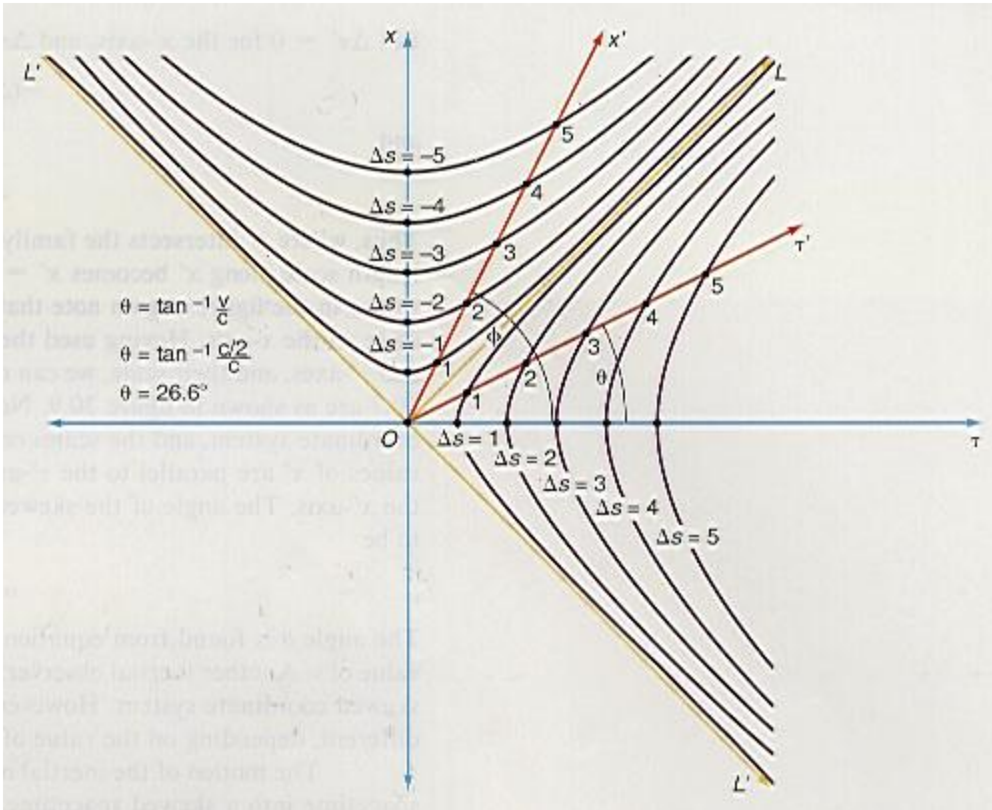

Figure 2.7 The invariant interval on a spacetime map.

1, 2, 3, … (Recall that because $(ds)^2 = (d\tau')^2 - (dx')^2$, and the origin of the coordinate system, $dx' = 0$, hence $ds = d\tau'$.) *Note that the scale on the τ'-axis is not the same as the scale on the τ-axis.*

To draw the x'-axis on this graph, we note that the x'-axis represents all the points for which $t' = 0$. The Lorentz equation for t' was given in chapter 33 by equation 1.50 as

$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}}$$

For $t' = 0$, we must have

$$t = \frac{xv}{c^2}$$

or

$$x = \frac{c^2}{v}t = \frac{c}{v}(ct)$$

$$x = \frac{c}{v}\tau \tag{2.18}$$

Equation 2.18 is the equation of a straight line passing through the origin with the slope c/v. *This line represents the x'-axis because it results from setting t' = 0 in the Lorentz equation.* Because the slope of the τ'-axis was given by $\tan\theta = v/c$, the triangle of figure 2.8 can be drawn. Note that we can write the ratio of c/v, the slope of the x'-axis, as

$$\tan \phi = \frac{c}{v}$$

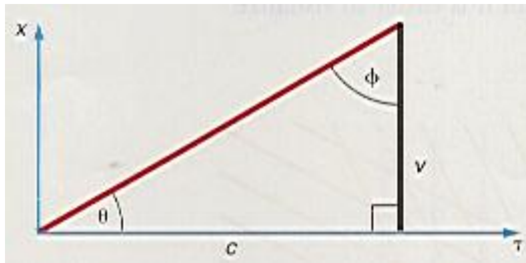

Figure 2.8 Determining the slope of the x'-axis.

But from the figure $\theta + \phi = 90^0$. Hence, the angle for the slope of the x'-axis must be $\phi = 90^0 - \theta$ In our example, $\theta = 26.6^0$, thus $\phi = 63.4^0$. The x'-axis is drawn in figure 2.7 at this angle. Note that the x'-axis makes an angle ϕ with the τ-axis, but an angle θ with the x-axis. The intersection of the x'-axis with the family of hyperbolas establishes the scale for the x'-axis. The interval is

$$(ds)^2 = (d\tau')^2 - (dx')^2$$

But $d\tau' = 0$ for the x'-axis, and ds is a negative quantity elsewhere, hence

$$-(dx')^2 = -(ds)^2$$

and

$$dx' = + ds$$

Thus, where x' intersects the family of hyperbolas at $ds = -1, -2, -3, \ldots$ the length scale along x' becomes $x' = 1, 2, 3, \ldots$ The scale on the x'-axis is now shown in the figure. Again note that the scale on the x'-axis is not the same as the scale on the x-axis. Having used the hyperbolas for the interval to establish the x'- and τ'-axes, and their scale, we can now dispense with them and the results of figure 2.7 are as shown in figure 2.9. Notice that the S' frame of reference is a skewed coordinate system, and the scales on S' are not the same as on S. Lines of constant values of x' are parallel to the τ'-axis, whereas lines of constant τ' are parallel to the x'-axis. The angle of the skewed coordinate system α is found from the figure to be

$$\alpha = 90^0 - 2\theta \tag{2.19}$$

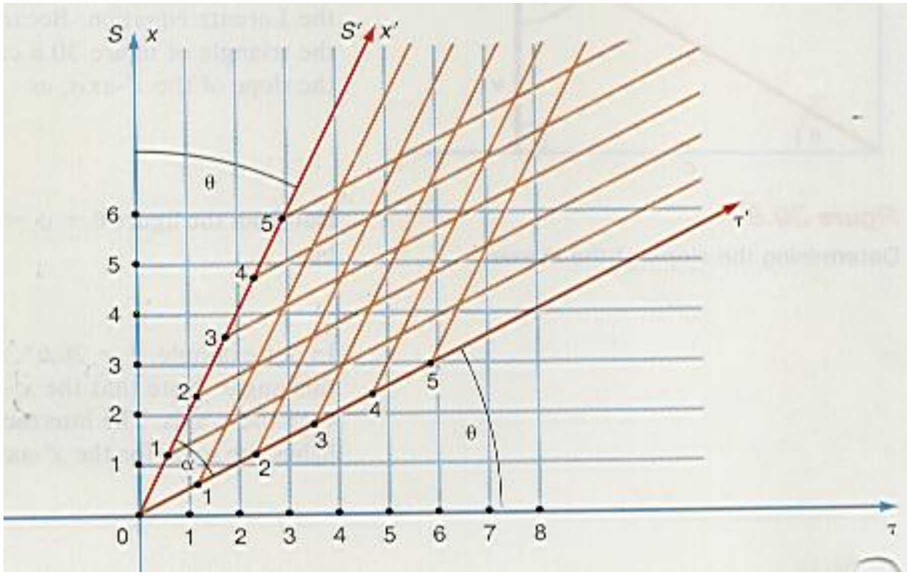

Figure 2.9 Relation of S and S' frame of references.

The angle θ is found from equation 2.3. This *S*' frame is unique for a particular value of *v*. Another inertial observer moving at a different speed would have another skewed coordinate system. However, the angle θ and hence, the angle α, would be different, depending on the value of *v*.

The motion of the inertial observer *S*' seems to warp the simple orthogonal spacetime into a skewed spacetime. The length contraction and time dilation can easily be explained by this skewed spacetime. Figures 2.10 through 2.15 are a series of spacetime diagrams based on the invariant interval, showing length contraction, time dilation, and simultaneity.

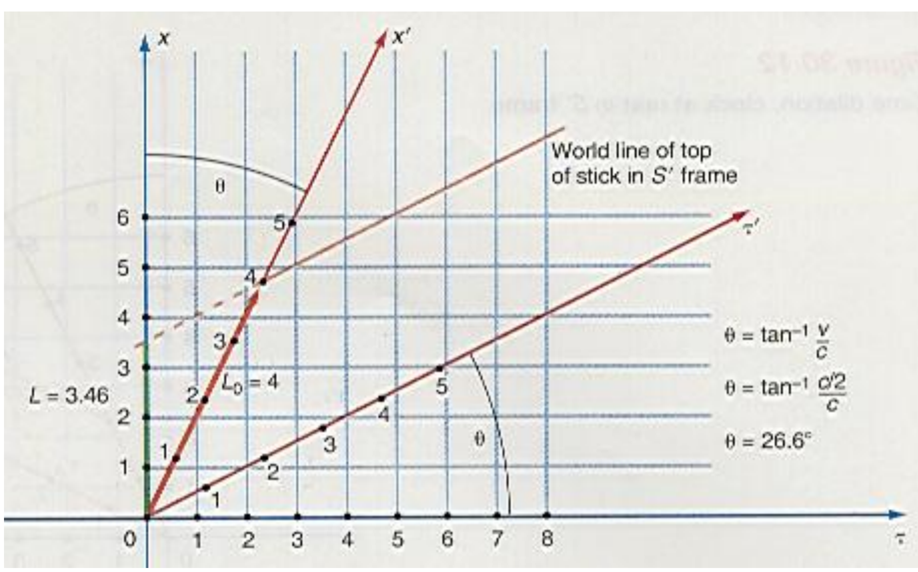

Figure 2.10 Length contraction, rod at rest in S' frame.

Figure 2.10 represents a rod 4.00 units long at rest in a rocket ship S', moving at a speed of $c/2$. The world line of the top of the stick in S' is drawn parallel to the τ'-axis. (Any line parallel to the τ'-axis has one and only one value of x' and thus represents an object at rest in S'.)

If the world line is dashed backward to the x-axis, it intersects the x-axis at $x = 3.46$, which is the length of the rod L, as observed by the S frame observer. Thus, the rod at rest in the moving rocket frame appears contracted to the observer on earth, the S frame. The contraction of the moving rod is, of course, the Lorentz contraction. With the spacetime diagram it is easier to visualize.

Figure 2.11 shows the same Lorentz contraction but as viewed from the S' frame. A rod 4.00 units long L_0 is at rest in the S frame, the earth. An observer in

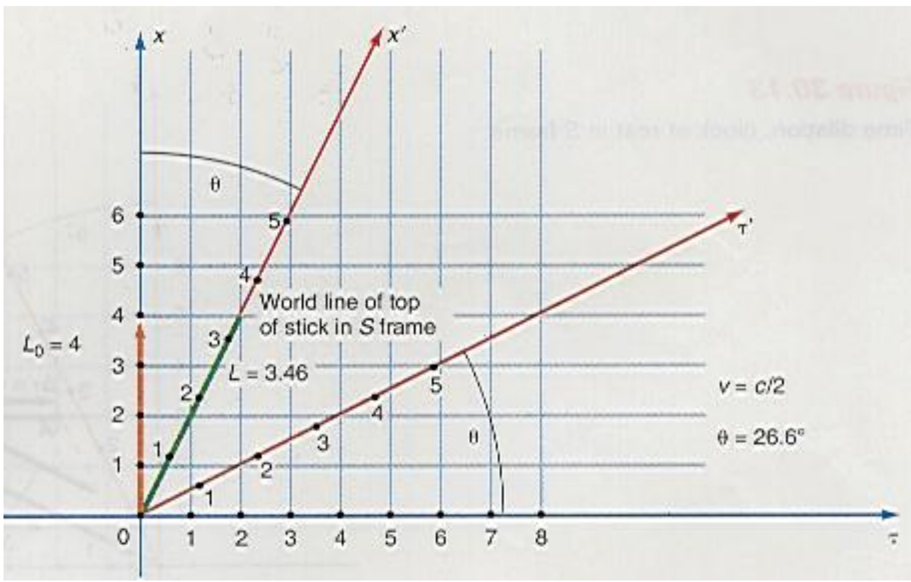

Figure 2.11 Length contraction, rod at rest in S frame.

the rocket ship frame, the S' frame, considers himself to be at rest while the earth is moving away from him at a velocity $-v$. The astronaut sees the world line, which emanates from the top of the rod, as it intersects his coordinate system. The length of the rod that he sees is found by drawing the world line of the top of the rod in the S frame, as shown in the figure. This world line intersects the x'-axis at the position $x' = 3.46$. Hence, the rocket observer measures the rod on earth to be only 3.46 units long, the length L. Thus, the rocket ship observer sees the same length contraction. *The cause of these contractions is the non-Euclidity of spacetime.*

The effect of time dilation is also easily explained by the spacetime diagram, figure 2.12. A clock is at rest in a moving rocket ship at the position $x' = 2$. Its world line is drawn parallel to the τ'-axis, as shown. Between the occurrence of the events A and B a time elapses on the S' clock of $d\tau' = 4.0 - 2.0 = 2.0$, as shown in the figure. This time interval, when observed by the S frame of the earthman, is found by dropping the dashed lines from the events A and B down to the τ-axis. (These lines are parallel to the x-axis, but because S is an orthogonal frame, they are also perpendicular to the τ-axis.) The time interval elapsed on earth is read from the graph as $d\tau = 5.9 - 3.6 = 2.3$. A time lapse of 2 s on the rocket ship clock would

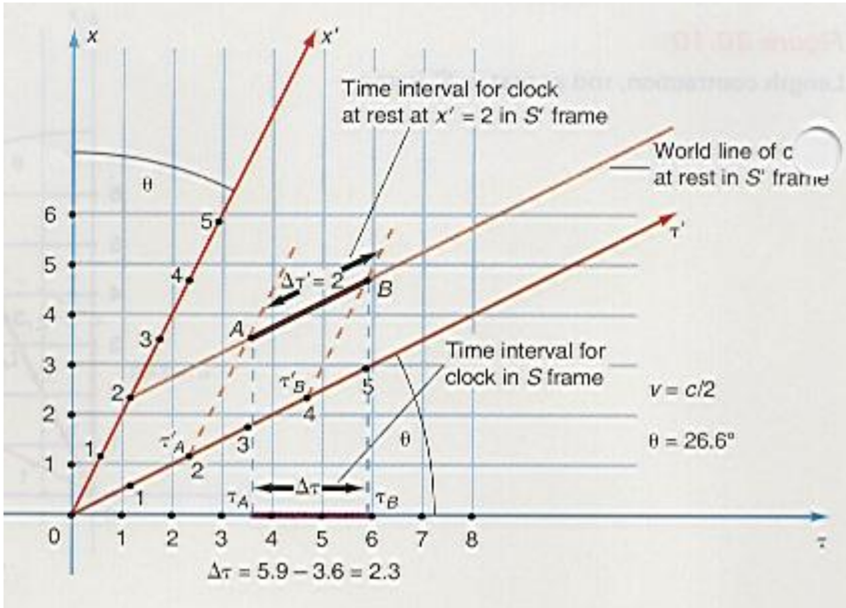

Figure 2.12 Time dilation, clock at rest in S′ frame.

appear as a lapse of 2.3 s on earth. Thus the moving clock in S' is running at a slower rate than a clock in S. Time has slowed down in the moving rocket ship. This is, of course, the Lorentz time dilation effect.

The inverse problem of time dilation is shown in figure 2.13. Here a clock is at rest on the earth, the S frame, at the position $x = 3$. The world line of the clock is

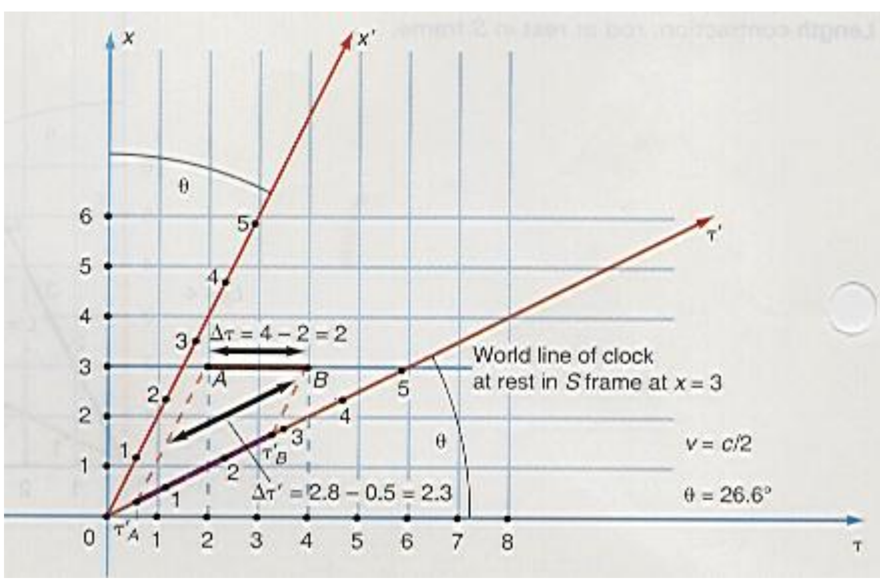

Figure 2.13 Time dilation, clock at rest in S frame.

drawn parallel to the τ-axis. The occurrence of two events, *A* and *B*, are noted and the time interval elapsed between these two events on earth is $d\tau = 4.0 - 2.0 = 2.0$. The same events *A* and *B* are observed in the rocket ship, and the time of these events as observed on the rocket ship is found by drawing the dashed lines parallel to the *x*'-axis to where they intersect the τ'-axis. Thus, event *A* occurs at $\tau'_A = 0.5$, and event *B* occurs at $\tau'_B = 2.8$. The elapsed time on the rocket ship is thus

$$d\tau' = \tau_B' - \tau_A' = 2.8 - 0.5 = 2.3$$

From the point of view of the rocket observer, he is at rest, and the earth is moving away from him at a velocity $-v$. Hence, he sees an elapsed time on the moving earth of 2 s while his own clock records a time interval of 2.3 s. He therefore concludes that time has slowed down on the moving earth.

Another explanation for this time dilation can be found in the concept of *simultaneity*. If we look back at figure 2.12 we see that the same event *A* occurs at the times $\tau_A = 3.6$ and $\tau'_A = 2.0$, whereas event *B* occurs at the times $\tau_B = 5.9$ and $\tau'_B = 4$. *The same event does not occur at the same time in the different coordinate systems.* Because the events occur at different times their time intervals should be expected to be different also. In fact, a more detailed picture of simultaneity can be found in figures 2.14 and 2.15.

Figure 2.14 shows two events *A* and *B* that occur simultaneously at the time τ' = 2 on the moving rocket ship. However, the earth observer sees the two events

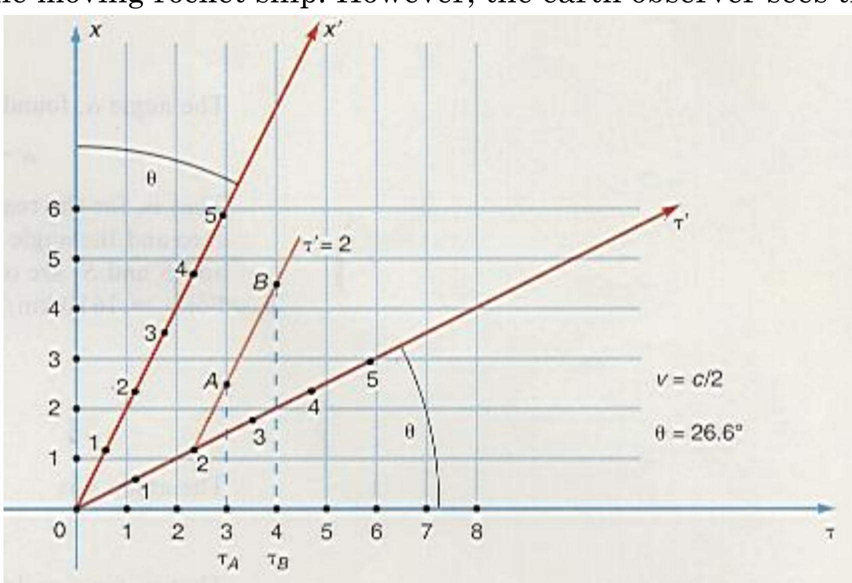

Figure 2.14 Simultaneity, two events simultaneous in S' frame.

occurring, not simultaneously, but rather at the two times $\tau_A = 3$ and $\tau_B = 4$. That is, the earth observer sees event *A* happen before event *B*. This same type of effect is shown in figure 2.15, where the two events *A* and *B* now occur simultaneously at τ = 4 for the earth observer. However the rocket ship observer sees event *B* occurring at

$\tau'_B = 1.6$ and event *A* at $\tau'_A = 2.7$. Thus, to the rocket ship observer events *A* and *B* are not simultaneous, but rather event *B* occurs before event *A*.

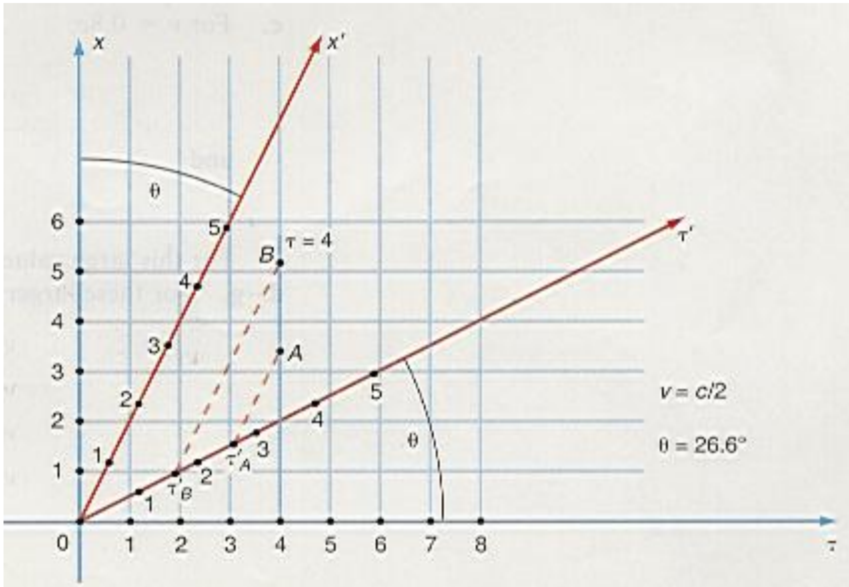

Figure 2.15 Simultaneity, two events simultaneous in S frame.

In summary, these spacetime diagrams are based on the invariant interval. Because the invariant interval is based on hyperbolas, spacetime is non-Euclidean. The S' frame of reference becomes a skewed coordinate system and the scales of the S' frame are not the same as the scales on the S frame.

Example 2.4

The skewing of the spacetime diagram with speed. Find the angles θ and α for a spacetime diagram if (a) v = 1610 km/hr = 1000 mph = 477 m/s, (b) v = 1610 km/s = 1000 miles/s, (c) $v = 0.8c$, (d) $v = 0.9c$, (e) $v = 0.99c$, (f) $v = 0.999c$, and (g) $v = c$.

Solution

a. The angle θ of the spacetime diagram, found from equation 2.3, is

$$\theta = \tan^{-1}\frac{v}{c}$$
$$= \tan^{-1}\frac{477 \text{ m/s}}{3.00 \times 10^8 \text{ m/s}}$$
$$= (8.54 \times 10^{-5})^0$$

The angle α, found from equation 2.19, is

$$\alpha = 90^0 - 2\theta = 90^0 - 2(8.54 \times 10^{-5})^0 = 90^0$$

That is, for the reasonably large speed of 1000 mph, the angle θ is effectively zero and the angle $\alpha = 90^0$. There is no skewing of the coordinate system and *S* and *S'* are orthogonal coordinate systems.

b. For $v = 1610$ km/s, the angle θ is

$$\theta = \tan^{-1}\frac{v}{c} = \tan^{-1}\frac{1.61 \times 10^6 \text{ m/s}}{3.00 \times 10^8 \text{ m/s}}$$
$$= 0.31^0$$

The angle α is

$$\alpha = 90^0 - 2\theta = 90^0 - 2(0.31^0) = 89.4^0$$

That is, for $v = 1610$ km/s = 3,600,000 mph, the τ'- and *x'*-axes are just barely skewed.

c. For $v = 0.8c$,

$$\theta = \tan^{-1}\frac{v}{c} = \tan^{-1}\frac{0.8c}{c} = 38.7^0$$

and

$$\alpha = 90^0 - 2\theta = 90^0 - 2(38.7) = 12.6^0$$

For this large value of *v*, the axes are even more skewed than in figure 2.8.

d.-g. For these larger values of *v*, equations 2.3 and 2.19 give

$v = 0.9c$;	$\theta = 41.9^0$;	$\alpha = 6.2^0$
$v = 0.99c$;	$\theta = 44.7^0$;	$\alpha = 0.576^0$
$v = 0.999c$;	$\theta = 44.97^0$;	$\alpha = 0.057^0$
$v = c$;	$\theta = 45^0$;	$\alpha = 0^0$

Hence, as *v* gets larger and larger the angle θ between the coordinate axes becomes larger and larger, eventually approaching 45^0. The angle α gets smaller until at $v = c$, α has been reduced to zero and the entire *S'* frame of reference has been reduced to a line.

To go to this Interactive Example click on this sentence.

2.3 The General Theory of Relativity

We saw in the special theory of relativity that the laws of physics must be the same in all inertial reference systems. *But what is so special about an inertial reference* *system? The inertial reference frames are, in a sense, playing the same role as Newton's absolute space.* That is, absolute space has been abolished only to replace it by absolute inertial reference frames. Shouldn't the laws of physics be the same in all coordinate systems, whether inertial or noninertial? The inertial frame should not be such a privileged frame. But clearly, accelerations can be easily detected, whereas constant velocities cannot. How can this very obvious difference be reconciled? That is, we must show that even all accelerated motions are relative. How can this be done?

Let us consider the very simple case of a mass m on the floor of a rocket ship that is at rest in a uniform gravitational field on the surface of the earth, as depicted in figure 2.16(a). The force acting on the mass is its weight w, which we write as

$$F = w = mg \tag{2.20}$$

Let us now consider the case of the same rocket ship in interstellar space far removed from all gravitational fields. Let the rocket ship now accelerate upward, as in figure 2.16(b), with an acceleration a that is numerically equal to the acceleration due to gravity g, that is, $a = g = 9.80\ \text{m/s}^2$. The mass m that is sitting on the floor of the rocket now experiences the force, given by Newton's second law as

$$F = ma = mg = w \tag{2.21}$$

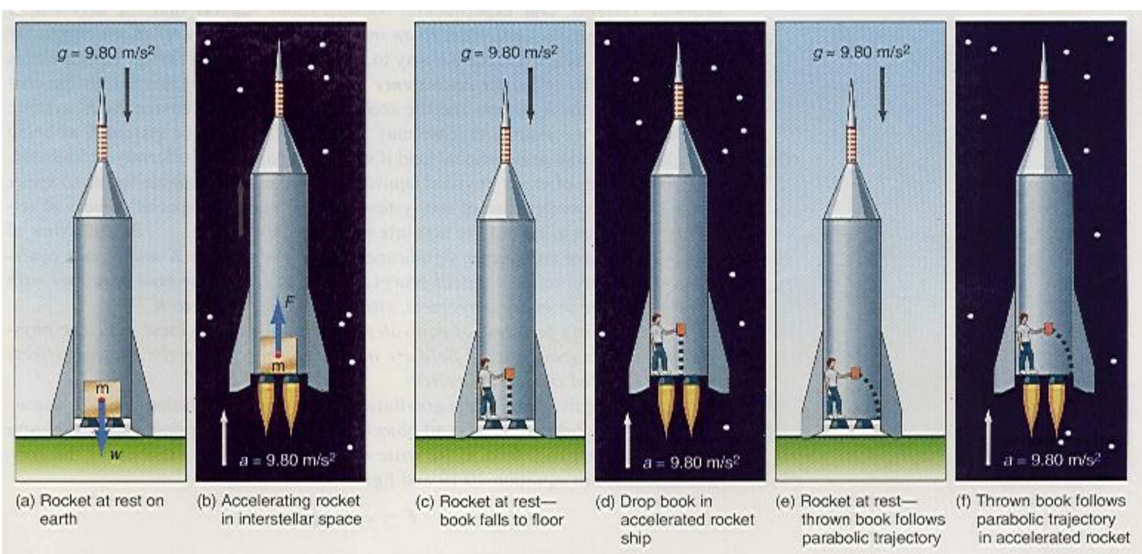

Figure 2.16 An accelerated frame of reference is equivalent to an inertial frame of reference plus gravity.

That is, the mass m sitting on the floor of the accelerated rocket experiences the same force as the mass m sitting on the floor of the rocket ship when it is at rest in the uniform gravitational field of the earth. Therefore, there seems to be some relation between accelerations and gravity.

Let us experiment a little further in the rocket ship at rest by holding a book out in front of us and then dropping it, as in figure 2.16(c). The book falls to the floor and if we measured the acceleration we would, of course, find it to be the acceleration due to gravity, g = 9.80 m/s^2. Now let us take the same book in the accelerated rocket ship and again drop it, as in figure 2.16(d). An inertial observer outside the rocket would see the book stay in one place but would see the floor accelerating upward toward the book at the rate of a = 9.80 m/s^2. The astronaut in the accelerated rocket ship sees the book fall to the floor with the acceleration of 9.80 m/s^2 just as the astronaut at rest on the earth observed.

The astronaut in the rocket at rest on the earth now throws the book across the room of the rocket ship. He observes that the book follows the familiar parabolic trajectory of the projectile and that is again shown in figure 2.16(e). Similarly, the astronaut in the accelerated rocket also throws the book across the room. An outside inertial observer would observe the book moving across the room in a straight line and would also see the floor accelerating upward toward the book. The accelerated astronaut would simply see the book following the familiar parabolic trajectory it followed on earth, figure 2.16(f).

Hence, the same results are obtained in the accelerated rocket ship as are found in the rocket ship at rest in the gravitational field of the earth. Thus, *the effects of gravity can be either created or eliminated by the proper choice of coordinate systems*. Our experimental considerations suggest that *the accelerated frame of reference is equivalent to an inertial frame of reference in which gravity is present*. Einstein, thus found a way to make accelerations relative. He stated his results in what he called the ***equivalence principle.*** Calling the inertial system containing gravity the K system and the accelerated frame of reference the K' system, Einstein said, “we assume that we may just as well regard the system K as being in a space free from gravitational field if we then regard K as uniformly accelerated. This assumption of exact physical equivalence makes it impossible for us to speak of the absolute acceleration of the system, just as the usual (special) theory of relativity forbids us to talk of the absolute velocity of a system… But this view of ours will not have any deeper significance unless *the systems K and K' are equivalent with respect to all physical processes, that is, unless the laws of nature with respect to K are in entire agreement with those with respect to K'*”[2]

Einstein’s principle of equivalence is stated as: on a local scale the physical effects of a gravitational field are indistinguishable from the physical effects of an accelerated coordinate system.

The equivalence of the gravitational field and acceleration “fields” also accounts for the observation that all objects, regardless of their size, fall at the same rate in a gravitational field. If we write m_g for the mass that experiences the gravitational force in equation 2.20 and figure 2.16(a), then

$$F = w = m_g g$$

2[2]“On the Influence of Gravitation on the Propagation of Light,” from A. Einstein, *Annalen der Physik* 35, 1911, in *The Principle of Relativity,* Dover Publishing Co.

And if we write m_i for the inertial mass that resists the motion of the rocket in figure 2.16(b) and equation 2.21, then

$$F = m_i a = m_i g$$

Since we have already seen that the two forces are equal, by the equivalence principle, it follows that

$$m_g = m_i$$

That is, the gravitational mass is in fact equal to the inertial mass. Thus, the equivalence principle implies the equality of inertial and gravitational mass and this is the reason why all objects of any size fall at the same rate in a gravitational field.

As a final example of the equivalence of a gravitational field and an acceleration let us consider an observer in a closed room, such as a nonrotating space station in interstellar space, far removed from all gravitating matter. This space station is truly an inertial coordinate system. Let the observer place a book in front of him and then release it, as shown in figure 2.17(a). Since there are no forces

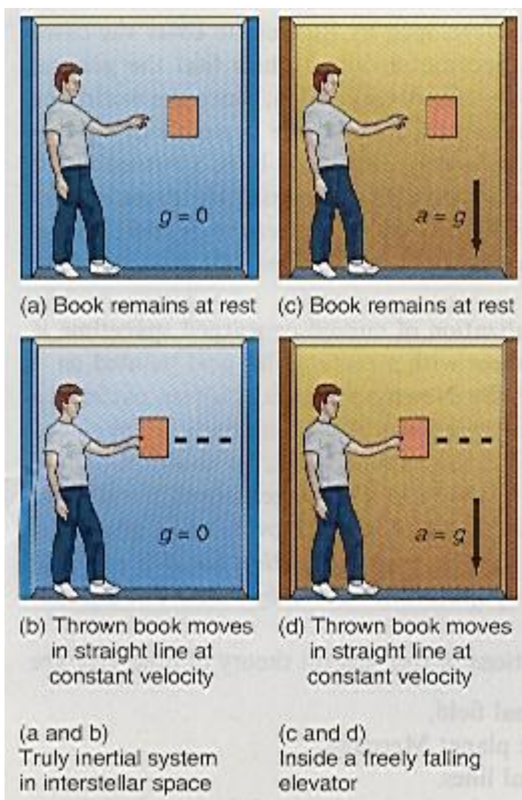

Figure 2.17 A freely falling frame of reference is locally the same as an inertial frame of reference.

present, not even gravity, the book stays suspended in space, at rest, exactly where the observer placed it. If the observer then took the book and threw it across the room, he would observe the book moving in a straight line at constant velocity, as shown in figure 2.17(b).

Let us now consider an elevator on earth where the supporting cables have broken and the elevator goes into free-fall. An observer inside the freely falling elevator places a book in front of himself and then releases it. The book appears to that freely falling observer to be at rest exactly where the observer placed it, figure 2.17(c). (Of course, an observer outside the freely falling elevator would observe both the man and the book in free-fall but with no relative motion with respect to each other.) If the freely falling observer now takes the book and throws it across the elevator room he would observe that the book travels in a straight line at constant velocity, figure 2.17(d).

Because an inertial frame is defined by Newton's first law as a frame in which a body at rest, remains at rest, and a body in motion at some constant velocity continues in motion at that same constant velocity, we must conclude from the illustration of figure 2.17 that the freely falling frame of reference acts exactly as an inertial coordinate system to anyone inside of it. *Thus, the acceleration due to gravity has been transformed away by accelerating the coordinate system by the same amount as the acceleration due to gravity.* If the elevator were completely closed, the observer could not tell whether he was in a freely falling elevator or in a space station in interstellar space.

The equivalence principle allows us to treat an accelerated frame of reference as equivalent to an inertial frame of reference with gravity present, figure 2.16, or to consider an inertial frame as equivalent to an accelerated frame in which gravity is absent, figure 2.17. By placing all frames of reference on the same footing, Einstein was then able to *postulate* ***the general theory of relativity, namely, the laws of physics are the same in all frames of reference.***

A complete analysis of the general theory of relativity requires the use of very advanced mathematics, called tensor analysis. However, many of the results of the general theory can be explained in terms of the equivalence principle, and this is the path that we will follow in the rest of this chapter.

From his general theory of relativity, Einstein was quick to see its relation to gravitation when he said, "It will be seen from these reflections that in pursuing the General Theory of Relativity we shall be led to a theory of gravitation, since we are able to produce a gravitational field merely by changing the system of coordinates. *It will also be obvious that the principle of the constancy of the velocity of light in vacuo must be modified."*[3]

Although the general theory was developed by Einstein to cover the cases of accelerated reference frames, it soon became obvious to him that the general theory had something quite significant to say about gravitation. Since the world line of an accelerated particle in spacetime is curved, then by the principle of equivalence, a particle moving under the effect of gravity must also have a curved world line in spacetime. *Hence, the mass that is responsible for causing the gravitational field,*

3[3]"The Foundation of the General Theory of Relativity" from A. Einstein, *Annalen der Physik* 49, 1916 in *The Principle of Relativity,* Dover Publishing Co.

must warp spacetime to make the world lines of spacetime curved. This is sometimes expressed as, matter warps spacetime and spacetime tells matter how to move.

A familiar example of the visualization of curved or **warped spacetime** is the rubber sheet analogy. A flat rubber sheet with a rectangular grid painted on it is stretched, as shown in figure 2.18(a). By Newton's first law, a free particle, a small rolling ball m moves in a straight line as shown. A bowling ball is then placed on the rubber sheet distorting or warping the rubber sheet, as shown in figure 2.18(b). When the small ball m is rolled on the sheet it no longer moves in a straight line path but it now curves around the bowling ball M, as shown. *Thus,*

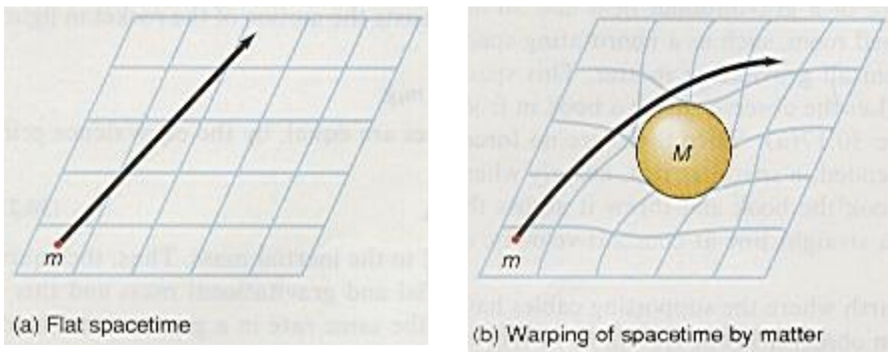

Figure 2.18 Flat and curved spacetime

gravity is no longer to be thought of as a force in the Newtonian tradition but it is rather a consequence of the warping or curvature of spacetime caused by mass. The amount of warping is a function of the mass.

The four experimental confirmations of the general theory of relativity are

1. The bending of light in a gravitational field.
2. The advance of the perihelion of the planet Mercury.
3. The gravitational red shift of spectral lines.
4. The Shapiro experiment, which shows the slowing down of the speed of light near a large mass.

Let us now look at each of these confirmations.

2.4 The Bending of Light in a Gravitational Field

Let us consider a ray of light that shines through a window in an elevator at rest, as shown in figure 2.19(a). The ray of light follows a straight line path and hits the opposite wall of the elevator at the point P. Let us now repeat the experiment, but let the elevator accelerate upward very rapidly, as shown in figure 2.19(b). The ray of light enters the window as before, but before it can cross the room to the opposite wall the elevator is displaced upward because of the acceleration. Instead of the ray of light hitting the wall at the point P, it hits at some lower point Q because of the upward acceleration of the elevator. To an observer in the elevator, the ray of light follows the parabolic path, as shown in figure 2.19(c). Thus, *in the accelerated*

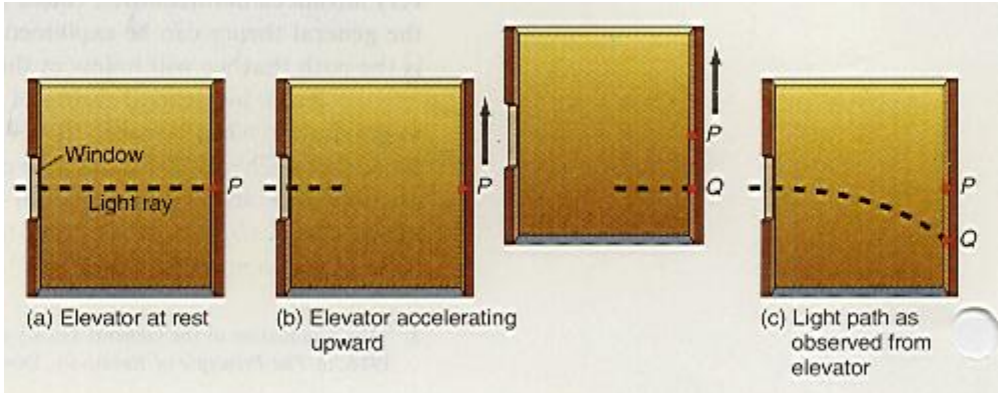

Figure 2.19 The bending of light in an accelerated elevator.

coordinate system of the elevator, light does not travel in a straight line, but instead follows a curved path. But by the principle of equivalence the accelerated elevator can be replaced by a gravitational field. Therefore light should be bent from a straight line path in the presence of a gravitational field. The gravitational field of the earth is relatively small and the bending cannot be measured on earth. However, the gravitational field of the sun is much larger and Einstein predicted in 1916 that rays of light that pass close to the sun should be bent by the gravitational field of the sun.

Another way of considering this bending of light is to say that light has energy and energy can be equated to mass, thus the light-mass should be attracted to the sun. Finally, we can think of this bending of light in terms of the curvature of spacetime caused by the mass of the sun. Light follows the shortest path, called a *geodesic,* and is thus bent by the curvature of spacetime.

Regardless of which conceptual picture we pick, Einstein predicted that a ray of light should be deflected by the sun by the angle of 1.75 seconds of arc. In order to observe this deflection it was necessary to measure the angular deviation between two stars when they are far removed from the sun, and then measure the deflection again when they are close to the sun (see figure 2.20). Of course when they are close to the sun, there is too much light from the sun to be able to see the stars. Hence, to test out Einstein's prediction it was necessary to measure the separation during a total eclipse of the sun. Sir Arthur Eddington led an expedition to the west coast of Africa for the solar eclipse of May 29, 1919, and measured the deflection. On November 6, 1919, the confirmation of Einstein's prediction of the bending of light was announced to the world.

More modern techniques used today measure radio waves from the two quasars, 3c273 and 3c279 in the constellation of Virgo. A quasar is a quasi-stellar object, a star that emits very large quantities of radio waves. Because the sun is very dim in the emission of radio waves, radio astronomers do not have to wait for an eclipse to measure the angular separation but can measure it at any time. On October 8, 1972, when the quasars were close to the sun, radio astronomers measured the angular separation between 3c273 and 3c279 in radio waves and found that the change in the angular separation caused by the bending of the radio waves around the sun was 1.73 seconds of arc, in agreement with the general theory of relativity.

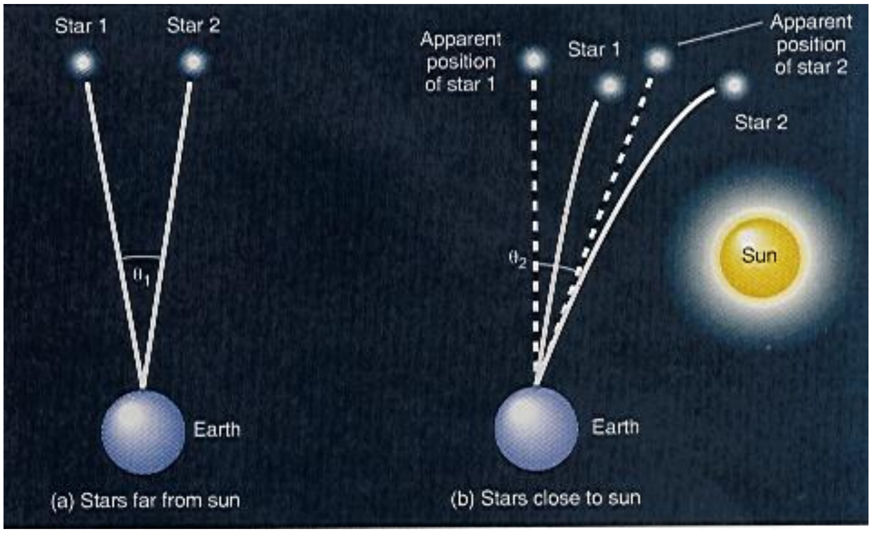

Figure 9.3 Bending of light by the Sun.

2.5 The Advance of the Perihelion of the Planet Mercury

According to Newton's laws of motion and his law of universal gravitation, each planet revolves around the sun in an elliptic orbit, as shown in figure 2.21. The

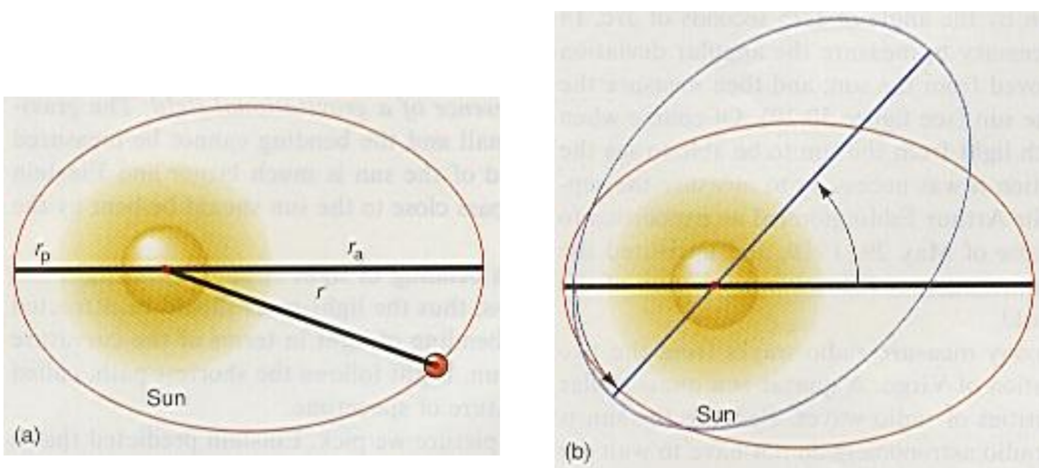

Figure 2.21 Advance of the perihelion of the planet Mercury.

closest approach of the planet to the sun is called its *perihelion distance* r_p, whereas its furthest distance is called its *aphelion distance* r_a. If there were only one planet in the solar system, the elliptical orbit would stay exactly as it is in figure 2.21(a). However, there are other planets in the solar system and each of these planets exerts forces on every other planet. Because the masses of each of these planets is small compared to the mass of the sun, their gravitational effects are also relatively small. These extra gravitational forces cause a perturbation of the elliptical orbit. In particular, they cause the elliptical orbit to rotate in its plane, as shown in figure 2.21(b). The total precession of the perihelion of the planet Mercury is 574 seconds of arc in a century. The perturbation of all the other planets can only explain 531 seconds of arc by the Newtonian theory of gravitation, leaving a discrepancy of 43 seconds of arc per century of the advance of the perihelion of Mercury. Einstein, using the full power of his tensor equations, predicted an advance of the perihelion by 43 seconds of arc per century in agreement with the known observational discrepancy.

2.6 The Gravitational Red Shift

Let us consider the two clocks A and B located at the top and bottom of the rocket, respectively, in figure 2.22(a). The rocket is in interstellar space where we assume

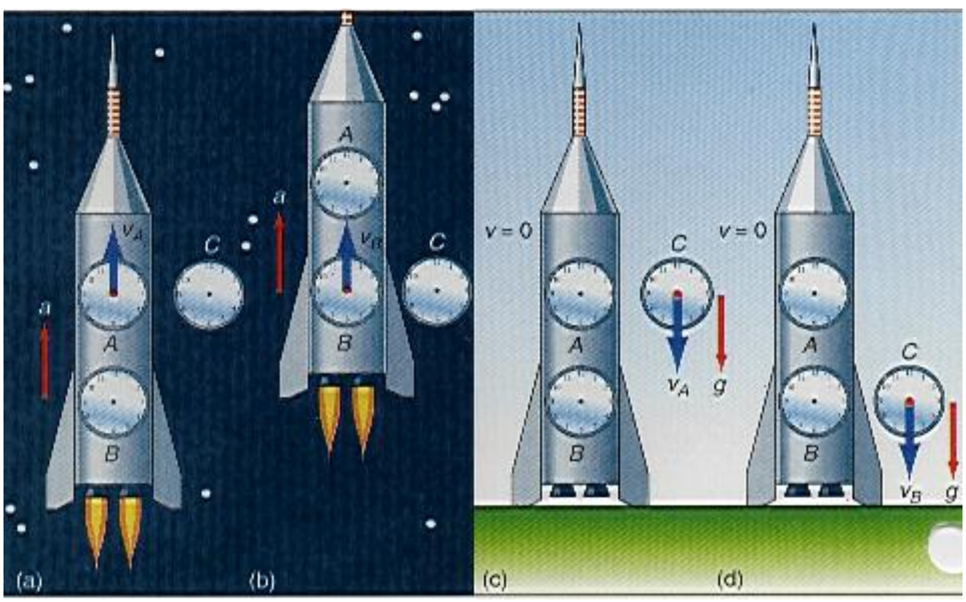

Figure 2.22 A clock in a gravitational field.

that all gravitational fields, if any, are effectively zero. The rocket is accelerating uniformly, as shown. Located in this interstellar space is a clock C, which is at rest. At the instant that the top of the rocket accelerates past clock C, clock A passes clock C at the speed v_A. Clock A, the moving clock, when observed from clock C, the stationary clock, shows an elapsed time Δt_A, given by the time dilation equation 1.64 as

$$\Delta t_C = \frac{\Delta t_A}{\sqrt{1 - v_A^2 / c^2}} \tag{2.22}$$

And since $\sqrt{1-v_A^2/c^2}$ is less than 1, then $\Delta t_C > \Delta t_A$, and the moving clock A runs slow compared to the stationary clock C.

A few moments later, clock B passes clock C at the speed v_B, as in figure 2.22(b). The speed v_B is greater than v_A because of the acceleration of the rocket. Let us read the same time interval Δt_C on clock C when clock B passes as we did for clock A so the two clocks can be compared. The difference in the time interval between the two clocks, B and C, is again given by the time dilation equation 1.64 as

$$\Delta t_C = \frac{\Delta t_B}{\sqrt{1-v_B^2/c^2}} \tag{2.23}$$

Because the time interval Δt_C was set up to be the same in both equations 2.22 and 2.23, the two equations can be equated to give a relation between clocks A and B. Thus,

$$\frac{\Delta t_A}{\sqrt{1-v_A^2/c^2}} = \frac{\Delta t_B}{\sqrt{1-v_B^2/c^2}}$$

Rearranging terms, we get

$$\frac{\Delta t_A}{\Delta t_B} = \frac{\left(1-v_A^2/c^2\right)^{1/2}}{\left(1-v_B^2/c^2\right)^{1/2}}$$

$$\frac{\Delta t_A}{\Delta t_B} = \left(1-v_A^2/c^2\right)^{1/2}\left(1-v_B^2/c^2\right)^{-1/2} \tag{2.24}$$

But the two terms on the right-hand side of equation 2.24 can be expanded by the binomial theorem, equation 1.33, as

$$(1-x)^n = 1 - nx + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n-2)x^3}{3!} + \ldots$$

This is a valid series expansion for $(1 - x)^n$ as long as x is less than 1. In this particular case,

$$x = v^2/c^2$$

which is much less than 1, and therefore.

$$(1-x)^n = 1 - nx$$

$$\left(1-v_A^2/c^2\right)^{1/2} = 1-\left(\frac{1}{2}\right)\frac{v_A^2}{c^2} = 1-\frac{v_A^2}{2c^2}$$

and

$$\left(1-v_B^2/c^2\right)^{-1/2} = 1-\left(\frac{-1}{2}\right)\frac{v_B^2}{c^2} = 1+\frac{v_B^2}{2c^2}$$

where again the assumption is made that v is small enough compared to c, to allow us to neglect the terms x^2 and higher in the expansion. Thus, equation 2.24 becomes

$$\frac{\Delta t_A}{\Delta t_B} = \left(1 - \frac{v_A^2}{2c^2}\right)\left(1 + \frac{v_B^2}{2c^2}\right)$$
$$= 1 + \frac{v_B^2}{2c^2} - \frac{v_A^2}{2c^2} - \frac{1}{4}\frac{v_B^2 v_A^2}{c^4}$$

The last term is set equal to zero on the same assumption that the speeds v are much less than c. Finally, rearranging terms,

$$\frac{\Delta t_A}{\Delta t_B} = 1 + \left(\frac{v_B^2}{2} - \frac{v_A^2}{2}\right)\frac{1}{c^2} \tag{2.25}$$

But by Einstein's principle of equivalence, we can equally well say that the rocket is at rest in the gravitational field of the earth, whereas the clock C is accelerating toward the earth in free-fall. When the clock C passes clock A it has the instantaneous velocity v_A, figure 2.22(c), and when it passes clock B it has the instantaneous velocity v_B, figure 2.22(b). We can obtain the velocities v_A and v_B by the law of conservation of energy, that is,

$$\frac{1}{2}mv^2 + \text{PE} = E_0 = \text{Constant} = \text{Total energy} \tag{2.26}$$

The total energy per unit mass, found by dividing equation 2.26 by m, is

$$\frac{v^2}{2} + \frac{\text{PE}}{m} = \frac{E_0}{m}$$

The conservation of energy per unit mass when clock C is next to clock A, obtained with the aid of figure 2.23, is

$$\frac{v_A^2}{2} + \frac{mgh_A}{m} = \frac{E_0}{m}$$

or

$$\frac{v_A^2}{2} + gh_A = \frac{E_0}{m} \tag{2.27}$$

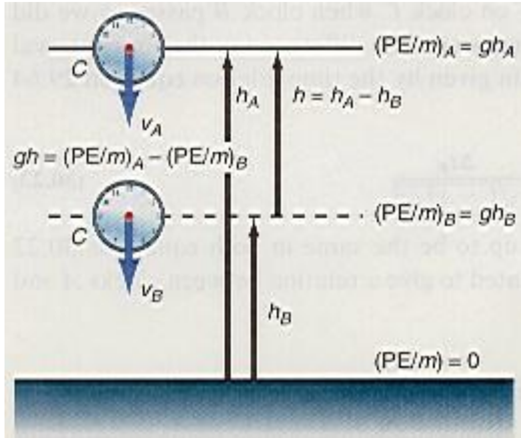

Figure 2.23 Freely falling clock C.

Similarly, when the clock C is next to clock B, the conservation of energy per unit mass becomes

$$\frac{v_B^2}{2} + gh_B = \frac{E_0}{m} \tag{2.28}$$

Subtracting equation 2.27 from equation 2.28, gives

$$\frac{v_B^2}{2} + gh_B - \frac{v_A^2}{2} - gh_A = \frac{E_0}{m} - \frac{E_0}{m} = 0$$

Hence,

$$\frac{v_B^2}{2} - \frac{v_A^2}{2} = gh_A - gh_B = gh \tag{2.29}$$

where h is the distance between A and B, and gh is the gravitational potential energy per unit mass, which is sometimes called the *gravitational potential.* Substituting equation 2.29 back into equation 2.25, gives

$$\frac{\Delta t_A}{\Delta t_B} = 1 + \frac{gh}{c^2} \tag{2.30}$$

For a clearer interpretation of equation 2.30, let us change the notation slightly. Because clock B is closer to the surface of the earth where there is a stronger gravitational field than there is at a height h above the surface where the gravitational field is weaker, we will let

$$\Delta t_B = \Delta t_g$$

and

$$\Delta t_A = \Delta t_f$$

where Δt_g is the elapsed time on a clock in a strong gravitational field and Δt_f is the elapsed time on a clock in a weaker gravitational field. If we are far enough away from the gravitational mass, we can say that Δt_f is the elapsed time in a gravitational-field-**f**ree space. With this new notation equation 2.30 becomes

$$\frac{\Delta t_f}{\Delta t_g} = 1 + \frac{gh}{c^2} \tag{2.31}$$

or

$$\Delta t_f = \Delta t_g \left(1 + \frac{gh}{c^2}\right) \tag{2.31}$$

Since $(1 + gh/c^2) > 0$, *the elapsed time on the clock in the gravitational-field-free space* Δt_f *is greater than the elapsed time on a clock in a gravitational field* Δt_g. *Thus, the time elapsed on a clock in a gravitational field is less than the time elapsed on a clock in a gravity-free space. Hence, a clock in a gravitational field runs slower than a clock in a field-free space.*

We can find the effect of the slowing down of a clock in a gravitational field by placing an excited atom in a gravitational field, and then observing a spectral line from that atom far away from the gravitational field. The speed of the light from that spectral line is, of course, given by

$$c = \lambda\nu = \frac{\lambda}{T} \tag{2.32}$$

where λ is the wavelength of the spectral line, ν is its frequency, and T is the period or time interval associated with that frequency. Hence, if the time interval $\Delta t = T$ changes, then the wavelength of that light must also change. Solving for the period or time interval from equation 2.32, we get

$$T = \frac{\lambda}{c} \tag{2.33}$$

Substituting T from 2.33 for Δt in equation 2.31, we get

$$T_f = T_g \left(1 + \frac{gh}{c^2}\right) \tag{2.34}$$

$$\frac{\lambda_f}{c} = \frac{\lambda_g}{c}\left(1 + \frac{gh}{c^2}\right)$$

$$\lambda_f = \lambda_g \left(1 + \frac{gh}{c^2}\right) \tag{2.35}$$

where λ_g is the wavelength of the emitted spectral line in the gravitational field and λ_f is the wavelength of the observed spectral line in gravity-free space, or at least farther from where the atom is located in the gravitational field. Because the term $(1 + gh/c^2)$ is a positive number, it follows that

$$\lambda_f > \lambda_g \tag{2.36}$$

That is, *the wavelength observed in the gravity-free space is greater than the wavelength emitted from the atom in the gravitational field.* Recall that the visible portion of the electromagnetic spectrum runs from violet light at around 380.0 nm to red light at 720.0 nm. Thus, red light is associated with longer wavelengths. *Hence, since $\lambda_f > \lambda_g$, the wavelength of the spectral line increases toward the red end of the spectrum, and the entire process of the slowing down of clocks in a gravitational field is referred to as the* ***gravitational red shift.***

A similar analysis in terms of frequency can be obtained from equations 2.32, 2.34, and the binomial theorem equation 1.34, to yield

$$\nu_f = \nu_g\left(1 - \frac{gh}{c^2}\right) \tag{2.37}$$

Where now the frequency observed in the gravitational-free space is less than the frequency emitted in the gravitational field because the term $\left(1 - \frac{gh}{c^2}\right)$ is less than one. The change in frequency per unit frequency emitted, found from equation 2.37, is

$$\nu_f - \nu_g = -\frac{gh}{c^2}\nu_g$$

$$\frac{\nu_g - \nu_f}{\nu_g} = \frac{gh}{c^2}$$

$$\frac{\Delta\nu}{\nu_g} = \frac{gh}{c^2} \tag{2.38}$$

The gravitational red shift was confirmed on the earth by an experiment by R. V. Pound and G. A. Rebka at Harvard University in 1959 using a technique called the *Mossbauer effect.* Gamma rays were emitted from radioactive cobalt in the basement of the Jefferson Physical Laboratory at Harvard University. These gamma rays traveled 22.5 m, through holes in the floors, up to the top floor. The difference between the emitted and absorbed frequency of the gamma ray was found to agree with equation 2.38.

Example 2.5

Gravitational frequency shift. Find the change in frequency per unit frequency for a γ-ray traveling from the basement, where there is a large gravitational field, to the roof of the building, which is 22.5 m higher, where the gravitational field is weaker.

Solution

The change in frequency per unit frequency, found from equation 2.38, is

$$\frac{\Delta\nu}{\nu_g} = \frac{gh}{c^2}$$

$$= \left(9.80\ \frac{\text{m}}{\text{s}^2}\right)\left[\frac{22.5\ \text{m}}{\left(3\times 10^8\ \text{m/s}\right)^2}\right]$$

$$= 2.45 \times 10^{-15}$$

To go to this Interactive Example click on this sentence.

The experiment was repeated by Pound and J. L. Snider in 1965, with another confirmation. Since then the experiment has been repeated many times, giving an accuracy to the gravitational red shift to within 1%.

Further confirmation of the gravitational red shift came from an experiment by Joseph Hafele and Richard Keating. Carrying four atomic clocks, previously synchronized with a reference clock in Washington, D.C., Hafele and Keating flew around the world in 1971. On their return they compared their airborne clocks to the clock on the ground and found the time differences associated with the time dilation effect and the gravitational effect exactly as predicted. Further tests with atomic clocks in airplanes and rockets have added to the confirmation of the gravitational red shift.

2.7 The Shapiro Experiment

Einstein's theory of general relativity not only predicts the slowing down of clocks in a gravitational field but it also predicts a contraction of the length of a rod in a gravitational field. The shrinking of rods and slowing down of clocks in a gravitational field can also be represented as a curvature of spacetime caused by mass. The slowing down of clocks and gravitational length contraction result in a reduction in the speed of light near a large massive body such as the sun. I. I. Shapiro performed an experiment in 1970 where he measured the time it takes for a radar signal (a light wave) to bounce off the planet Venus and return to earth at a time when Venus is close to the sun. The slowing down of light as it passes the sun causes the radar signal to be delayed by about 240×10^{-6} s. Shapiro's results agree with Einstein's theory to an accuracy of about 3%.

As an additional confirmation the delay in the travel time of radio signals to the spacecraft *Mariner 6* and *Mariner 7* showed the same kind of results.

Have You Ever Wondered?...
An Essay on the Application of Physics
The Black Hole

Have you ever wondered, while watching those science fiction movies, why the astronauts were afraid of a black hole? They certainly make them seem very sinister. Are they really that dangerous? What is a black hole? How is it formed? What are its characteristics? What would happen if you went into one? Is it possible to go space traveling through a black hole?

The simplest way to describe the black hole is to start with a classical analogue. Suppose we wished to launch a rocket from the earth to a far distant place in outer space. How fast must the rocket travel to escape the gravitational pull of the earth? When we launch the rocket it has a velocity v, and hence, a kinetic energy. As the rocket proceeds into space, its velocity decreases but its potential energy increases. The potential energy of an object when it is a distance r away from the center of the earth is found from

$$\text{PE} = -\frac{GM_e m}{r}$$

where G is the universal gravitational constant, M_e is the mass of the earth, and m is the mass of the object. Let us now apply this potential energy term to a rocket that is trying to escape from the gravitational pull of the earth. The total energy of the rocket at any time is equal to the sum of its potential energy and its kinetic energy, that is,

$$E = \text{KE} + \text{PE} = \tfrac{1}{2}mv^2 - \left[GM_e m\left(\frac{1}{r}\right)\right] \tag{2H.1}$$

When the rocket is fired from the surface of the earth, $r = R$, at an escape velocity v_e its total energy will be

$$E = \frac{1}{2}mv_e^2 - \left[GM_e m\left(\frac{1}{R}\right)\right]$$

By the law of conservation of energy, the total energy of the rocket remains a constant. Hence, we can equate the total energy at the surface of the earth to the total energy when the rocket is far removed from the earth. That is,

$$\frac{1}{2}mv_e^2 - \left[GM_e m\left(\frac{1}{R}\right)\right] = \frac{1}{2}mv^2 - \left[GM_e m\left(\frac{1}{r}\right)\right] \tag{2H.2}$$

When the rocket escapes the pull of the earth it has effectively traveled to infinity, that is, $r = \infty$, and its velocity at that time is reduced to zero, that is, $v = 0$. Hence, equation 2H.2 reduces to

$$\frac{1}{2}mv_e^2 - \left[GM_e m\left(\frac{1}{R}\right)\right] = 0 - \left[GM_e m\left(\frac{1}{\infty}\right)\right] = 0$$
$$\frac{1}{2}mv_e^2 = \frac{GM_e m}{R}$$
$$v_e^2 = \frac{2GM_e}{R}$$
$$v_e = \sqrt{\frac{2GM_e}{R}} \tag{2H.3}$$

Equation 2H.3 is the *escape velocity of the earth.* This is the velocity that an object must have if it is to escape the gravitational field of the earth. Now it was first observed by a British amateur astronomer, the Rev. John Michell, in 1783, and then 15 years later by Marquis Pierre de Laplace, that if light were a particle, as originally proposed by Sir Isaac Newton, then there was a limit to the size the earth could be and still have light escape from it. That is, if we solve equation 2H.3 for R, and replace the velocity of escape v_e by the velocity of light c, we get

$$R_S = \frac{2GM_e}{c^2} \tag{2H.4}$$

For reasons that will be explained later, this value of R is called the *Schwarzschild radius,* and is designated as R_S. Solving equation 2H.4 for the Schwarzschild radius of the earth we get 8.85×10^{-3} m, which means that if the earth were contracted to a sphere of radius smaller than 8.85×10^{-3} m, then the escape velocity from the earth would be greater than the velocity of light. That is, nothing, not even light could escape from the earth if it were this small. The earth would then be called a black hole because we could not see anything coming from it.

The reason for the name, black hole, comes from the idea that if we look at an object in space, such as a star, we see light coming from that star. If the star became a black hole, no light could come from that star. Hence, when we look into space we would no longer see a bright star at that location, but rather nothing but the blackness of space. There seems to be a hole in space where the star used to be and therefore we say that there is a black hole there.

Solving equation 2H.4 for the Schwarzschild radius of the sun, by replacing the mass of the earth by the mass of the sun, we get 2.95×10^3 m. Thus, if the sun were to contract to a radius below 2.95×10^3 m the gravitational force would become so great that no light could escape from the sun, and the sun would become a black hole.

Up to this point the arguments have been strictly classical. Since Einstein's theory of general relativity is a theory of gravitation, what does it say about black holes? As we have seen, Einstein's theory of general relativity says that mass warps spacetime and we saw this in the rubber sheet analogy in figure 2.18. The greater the mass of the gravitating body the greater the warping of spacetime. Figure 1(a) shows the warping of spacetime by a star. Figure 1(b) shows the warping for a much more massive star. As the radius of the star becomes much smaller, the warping becomes more pronounced as the star approaches the size of a black hole, figure 1(c).

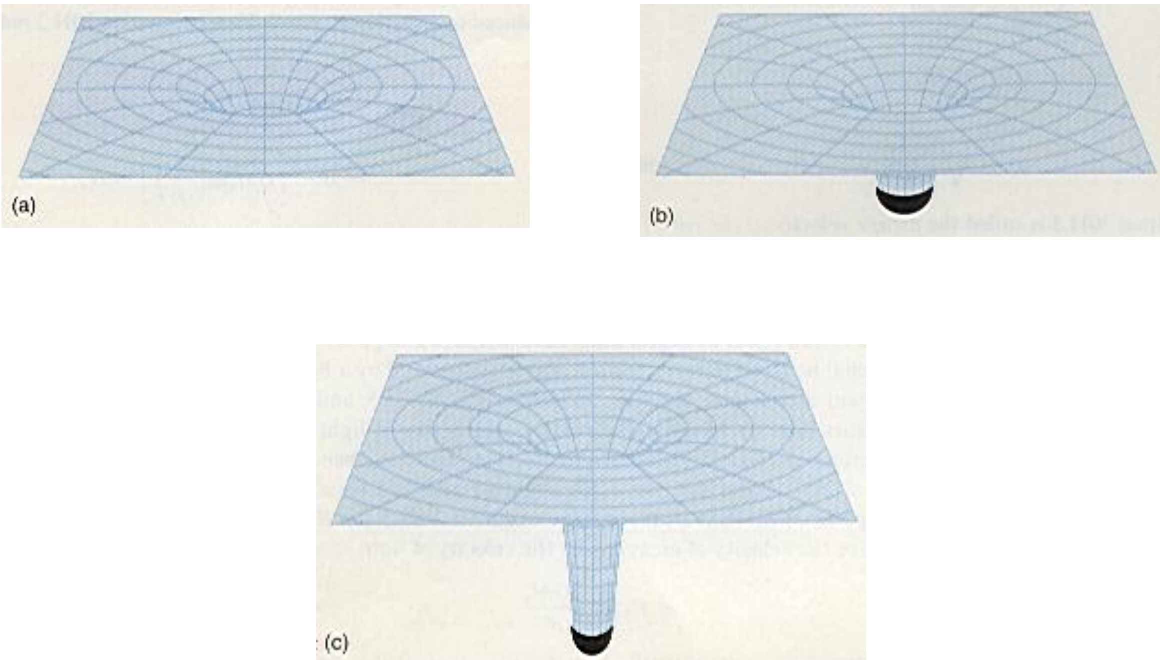

Figure 1 The warping of spacetime.

Shortly after Einstein stated his principle of general relativity, K. Schwarzschild solved Einstein's equations for the gravitational field of a point mass. For the radial portion of the solution he obtained

$$(ds)^2 = \frac{(dr)^2}{1 - 2GM/rc^2} - (1 - 2GM/rc^2)c^2(dt)^2 \tag{2H.5}$$

Equation 2H.5 is called the *radial portion of the Schwarzschild metric* and is the radial portion of the invariant interval of spacetime curved by the presence of a point mass. The invariant interval found previously in equation 2.11 is the metric for a flat spacetime, that is, one in which there is no mass to warp spacetime. That is, for flat spacetime

$$(ds)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{2.11}$$

and in only one space dimension by

$$(ds)^2 = c^2(dt)^2 - (dx)^2 \tag{2.10}$$

We saw there that if $ds = 0$, then $dx/dt = c$, the velocity of light, and it is a constant, hence $ds = 0$ represents the world line of a ray of light. Using the same analogy for the radial portion of the Schwarzschild solution we have

$$(ds)^2 = \frac{(dr)^2}{1 - 2GM/rc^2} - \left(1 - \frac{2GM}{rc^2}\right)c^2(dt)^2$$

As we have just seen, $ds = 0$ represents the world line of a ray of light. Applying this to the Schwarzschild solution we get

$$\frac{(dr)^2}{1 - 2GM/rc^2} = \left(1 - \frac{2GM}{rc^2}\right)c^2(dt)^2$$

$$\frac{(dr)^2}{(dt)^2} = \left(1 - \frac{2GM}{rc^2}\right)^2 c^2$$

$$\frac{dr}{dt} = \left(1 - \frac{2GM}{rc^2}\right)c \tag{2H.6}$$

Notice that if $r = 2GM/c^2$, then $dr/dt = 0$. This means that the velocity of light dr/dt is then zero, and no light is able to leave the gravitating body. But notice that this quantity is exactly what we already called the Schwarzschild radius. The Schwarzschild radius is also called the *event horizon of the black hole.* We can generalize equation 2H.6 to the form

$$\frac{dr}{dt} = \left(1 - \frac{R_s}{r}\right)c \tag{2H.7}$$

The solution of equation 2H.7 for various values of r is shown in table 2H.1. Notice that the velocity of light is not a constant near the black hole, but in a

Table 2H.1 Variation of the Velocity of Light as a Function of the Schwarzschild Radius	
r	dr/dt
$R_s/10$	$-9c$
$R_s/5$	$-4c$
$R_s/2$	$-c$
R_s	0
$2R_s$	$0.5c$
$10R_s$	$0.9c$
$100R_s$	$0.99c$
$1000R_s$	$0.999c$

distance of only 1000 times the radius of the black hole, the velocity of light approaches the constant value c. *Note that the constancy of the velocity of light is not a postulate of general relativity as it is for special relativity*. Also note that as we get far away from the black hole, $r \gg R_s$, we enter the region of flat spacetime and the velocity of light has the constant value c of special relativity. However, within the event horizon, equation 2H.7 and table 2H.1 show that the velocity of light can be greater than c.

The argument up to now may seem somewhat academic, in that we have described some of the characteristics of black holes, but do they really exist in nature? That is, is it possible for any objects in the universe to become black holes? The answer is yes. In the ordinary evolution of very massive stars, black holes can be formed. A star is essentially a gigantic nuclear reactor converting hydrogen to helium in a process called *nuclear fusion.* Think of the star as millions of hydrogen bombs going off at the same time, thereby producing enormous quantities of energy and enormous forces outward from the star. There is an equilibrium between the gravitational forces inward and the forces outward caused by the exploding gases. Eventually, when all the nuclear fuel is used, there is no longer an equilibrium condition. The gravitational force causes the gas to become very compact. If the star is large enough, it is compressed below its Schwarzschild radius and a black hole is formed. For an evolving star to condense into a black hole it must be approximately 25 times the mass of the sun. When the star condenses to a black hole it does not stop at the event horizon but continues to reduce in size until it becomes a singularity, a point mass. That is, the entire mass of the star has condensed to the size of a point.

There is experimental evidence that a black hole has been found as a companion of the star Cygnus X-1 and more are looked for every day.

Since time slows down in a gravitational field, the effect becomes much more pronounced in the vicinity of the black hole. If a person were to fall into the black hole he would eventually be crushed due to the enormous gravitational forces. Time would slow down for him as he approached the event horizon. At the event horizon, time would stand still for him.

The Schwarzschild black hole is an example of a nonrotating massive body. However, just as the sun and planets rotate about their axes, a more general solution of a black hole should also be concerned with the rotation of the massive body. The solution to the rotating black hole is called a *Kerr black hole,* after Roy Kerr, a New Zealand mathematician. The rotating black hole[4] (essentially an accelerating black hole) drags spacetime around with it, forming a second event horizon, thus leaving a space between the first event horizon and the second event horizon. It has been speculated that it may be possible to enter the first event horizon, but not the second, and exit somewhere else in either another universe or in this universe in another place and/or time.

It has also been speculated that there might also exist white holes in space. That is, mass is drawn into a black hole, but would be spewed out of a white hole. In fact some physicists have speculated that a black hole in one universe is a white hole in another universe.

4[4]See interactive tutorial problem 15.

The Language of Physics

Spacetime diagram
A graph of a particle's space and time coordinates. The time coordinate is usually expressed as τ, which is equal to the product of the speed of light and the time (p.).

World line
A line in a spacetime diagram that shows the motion of a particle through spacetime. A world line of a particle at rest or moving at a constant velocity is a straight line in spacetime. The world line of a light ray makes an angle of 45^0 with the τ-axis in spacetime. The world line of an accelerated particle is a curve in spacetime (p.).

Light cone
A cone that is drawn in spacetime showing the relation between the past and the future of a particle in spacetime. World lines within the cone are called timelike because they are accessible to us in time. Events outside the cone are called spacelike because they occur in another part of space that is not accessible to us and hence is called elsewhere (p.).

Invariant interval
A constant value in spacetime that all observers agree on, regardless of their state of motion. The equation of the invariant interval is in the form of a hyperbola in spacetime. Because of the hyperbolic form of the invariant interval, Euclidean geometry does not hold in spacetime. The reason for length contraction and time dilation is the fact that spacetime is non-Euclidean. The longest distance in spacetime is the straight line (p.).

Equivalence principle
On a local scale, the physical effects of a gravitational field are indistinguishable from the physical effects of an accelerated coordinate system. Hence, an accelerated frame of reference is equivalent to an inertial frame of reference in which gravity is present, and an inertial frame is equivalent to an accelerated frame in which gravity is absent (p.).

The general theory of relativity
The laws of physics are the same in all frames of reference (note that there is no statement about the constancy of the velocity of light as in the special theory of relativity) (p.).

Warped spacetime
Matter causes spacetime to be warped so that the world lines of particles in spacetime are curved. Hence, matter warps spacetime and spacetime tells matter how to move. Gravity is a consequence of the warping of spacetime by matter (p.).

Gravitational red shift
Time elapsed on a clock in a gravitational field is less than the time elapsed on a clock in a gravity-free space. This effect of the slowing down of a clock in a gravitational field is manifested by observing a spectral line from an excited atom in a gravitational field. The wavelength of the spectral line of that atom is shifted toward the red end of the electromagnetic spectrum (p.).

Summary of Important Equations

Tau in spacetime $\tau = ct$ (2.1)

Velocity in a spacetime diagram $v = c \tan \theta$ (2.2)

The square of the invariant interval

$$(ds)^2 = c^2(dt)^2 - (dx)^2 \tag{2.10}$$

$$(ds)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{2.11}$$

$$(ds)^2 = (d\tau)^2 - (dx)^2 \tag{2.13}$$

Slowing down of a clock in a gravitational field

$$\Delta t_f = \Delta t_g \left(1 + \frac{gh}{c^2}\right) \tag{2.31}$$

Gravitational red shift of wavelength

$$\lambda_f = \lambda_g \left(1 + \frac{gh}{c^2}\right) \tag{2.35}$$

Gravitational red shift of frequency

$$\nu_f = \nu_g \left(1 - \frac{gh}{c^2}\right) \tag{2.37}$$

Change in frequency per unit frequency

$$\frac{\Delta \nu}{\nu_g} = \frac{gh}{c^2} \tag{2.38}$$

Questions for Chapter 2

1. Discuss the concept of spacetime. How is it like space and how is it different?
2. How many light cones are there in your classroom?
3. Why can't a person communicate with another person who is elsewhere?
*4. Discuss the twin paradox on the basis of figure 2.6(b).
5. Using figure 2.7, discuss why the scales in the S' system are not the same as the scales in the S system.
*6. Considering some of the characteristics of spacetime, that is, it can be warped, and so forth, could spacetime be the elusive ether?

7. What does it mean to say that spacetime is warped?
8. Describe length contraction by a spacetime diagram.
9. Describe time dilation by a spacetime diagram.
10. Discuss simultaneity with the aid of a spacetime diagram.

Problems for Chapter 2

2.1 Spacetime Diagrams

1. Draw the world line in spacetime for a particle moving in (a) an elliptical orbit, (b) a parabolic orbit, and (c) a hyperbolic orbit.

2.2 The Invariant Interval

2. Find the angle that the world line of a particle moving at a speed of $c/4$ makes with the τ-axis in spacetime.

3. The world line of a particle is a straight line making an angle of 30^0 below the τ-axis. Determine the speed of the particle.

4. The world line of a particle is a straight line of length 150 m. Find the value of dx if $d\tau = 200$ m.

5. (a) On a sheet of graph paper draw the hyperbolas representing the invariant interval of spacetime as shown in figure 2.7. (b) Draw the S'-axes on this diagram for a particle moving at a speed of $c/4$.

6. Using the graph of problem 5, draw a rod 1.50 units long at rest in the S frame of reference. (a) From the graph determine the length of the rod in the S' frame of reference. (b) Determine the length of the rod using the Lorentz contraction equation.

7. Using the graph of problem 5, draw a rod 1.50 units long at rest in the S' frame of reference. (a) From the graph determine the length of the rod in the S frame of reference. (b) Determine the length of the rod using the Lorentz contraction equation.

2.6 The Gravitational Red Shift

8. One twin lives on the ground floor of a very tall apartment building, whereas the second twin lives 61.0 m above the ground floor. What is the difference in their age after 50 years?

9. The lifetime of a subatomic particle is 6.25×10^{-7} s on the earth's surface. Find its lifetime at a height of 500 km above the earth's surface.

10. An atom on the surface of Jupiter ($g = 23.1$ m/s^2) emits a ray of light of wavelength 528.0 nm. What wavelength would be observed at a height of 10,000 m above the surface of Jupiter?

Additional Problems

*11. Using the principle of equivalence, show that the difference in time between a clock at rest and an accelerated clock should be given by

$$\Delta t_R = \Delta t_A\left(1 + \frac{ax}{c^2}\right)$$

where Δt_R is the time elapsed on a clock at rest, Δt_A is the time elapsed on the accelerated clock, a is the acceleration of the clock, and x is the distance that the clock moves during the acceleration.

*12. A particle is moving in a circle of 1.00-m radius and undergoes a centripetal acceleration of 9.80 m/s^2. Using the results of problem 11, determine how many revolutions the particle must go through in order to show a 10% variation in time.

13. The pendulum of a grandfather clock has a period of 0.500 s on the surface of the earth. Find its period at an altitude of 200 km. *Hint:* Note that the change in the period is due to two effects. The acceleration due to gravity is smaller at this height even in classical physics, since

$$g = \frac{GM}{(R + h)^2}$$

To solve this problem, use the fact that the average acceleration is

$$g = \frac{GM}{R(R + h)}$$

and assume that

$$\Delta t_f = \Delta t_g\left(1 + \frac{gh}{c^2}\right)$$

14. Compute the fractional change in frequency of a spectral line that occurs between atomic emission on the earth's surface and that at a height of 325 km.

Interactive Tutorials

15. *A rotating black hole.* Assume the sun were to collapse to a black hole as described in the "Have you ever wondered … ?" section. (a) Calculate the radius of the black hole, which is called the Schwarzschild radius R_S. Since the sun is also rotating, angular momentum must be conserved. Therefore as the sun collapses the angular velocity of the sun must increase, and hence the tangential velocity of a point on the surface of the sun must also increase. (b) Find the radius of the sun during the collapse such that the tangential velocity of a point on the equator is equal to the velocity of light c. Compare this radius to the Schwarzschild radius. Some characteristics of the sun are radius, $r_0 = 6.96 \times 10^8$ m, mass of sun $M = 1.99 \times 10^{30}$ kg, and the angular velocity of the sun $\omega_0 = 2.86 \times 10^{-6}$ rad/s.

16. *Gravitational red shift.* An atom on the surface of the earth emits a ray of light of wavelength $\lambda_g = 528.0$ nm, straight upward. (a) What wavelength λ_f would be observed at a height $y = 10{,}000$ m? (b) What frequency ν_f would be observed at this height? (c) What change in time would this correspond to?

To go to these Interactive Tutorials click on this sentence.

To go to another chapter, return to the table of contents by clicking on this sentence.