Chapter 10 Maxwell's Equations and Electromagnetic Waves

10.1 Introduction

In 1864, James Clerk Maxwell (1831-1879) took all of the then known equations of electricity and magnetism, and with the addition of a new term to one of the equations, combined them into only four equations that could be used to derive all the results of electromagnetic theory. These four equations came to be known as **Maxwell's equations.** The four Maxwell's equations are (1) Gauss's law for electricity, (2) Gauss's law for magnetism, (3) Ampere's law with the addition of a new term called the *displacement current,* and (4) Faraday's law of electromagnetic induction. With these four equations, Maxwell predicted that waves should exist in the electromagnetic field. Thirteen years later, in 1887, Heinrich Hertz (1857-1894) produced and detected these electromagnetic waves. Maxwell also predicted that the speed of these electromagnetic waves should be 3×10^8 m/s. Observing that this is also the speed of light, Maxwell declared that light itself is an electromagnetic wave. In fact it eventually became known that there was an entire spectrum of these electromagnetic waves. They differed only in frequency and wavelength. Finally, it was found that these electromagnetic waves are capable of transmitting energy from one place to another, even through the vacuum of space.

10.2 The Displacement Current and Ampere's Law

In the study of a capacitor in chapter 6 (where we assumed that the current was conventional current, that is a flow of positive charges) we saw that when the switch in the circuit is closed, charge flows from the positive terminal of the battery to one plate of the capacitor, called the positive plate, and charge also flows from the negative plate of the capacitor back to the negative terminal of the battery. This is shown in figure 10.1(a). Until the plates are completely charged, there is a current into the positive plate, and a current out of the negative plate, yet there seems to be no current between the plates. There is thus a discontinuity in the current in the circuit because of the capacitor.

As charge is placed on the plates of the capacitor an electric field is set up between the plates. The electric field between the plates of a capacitor was found by Gauss's law as

$$E = \frac{q}{\varepsilon_o A}$$

(10.1)

As additional charge dq is added to the positive plate, it causes an additional electric field between the plates given by
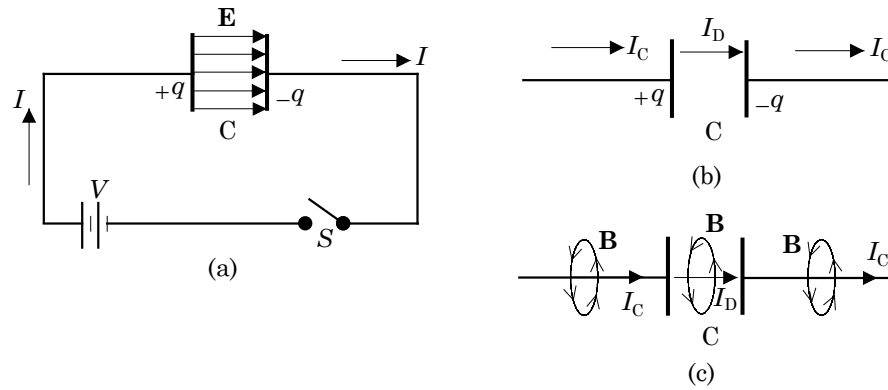
$$dE = \frac{dq}{\varepsilon_o A}$$

(10.2)

Figure 10.1 The Displacement Current.

The additional charge dq just added to the plate came from the current from the battery, and since the current is defined as $I = dq/dt$, we can write the additional charge as

$$dq = Idt \tag{10.3}$$

Substituting equation 10.3 back into equation 10.2, we get

$$dE = \frac{Idt}{\varepsilon_o A} \tag{10.4}$$

Solving equation 10.4 for the current I, we get

$$I_D = \varepsilon_o A \frac{dE}{dt} \tag{10.5}$$

where dE/dt is the rate at which the electric field between the plates changes with time, and we see, from equation 10.5, that it is related to the current entering or leaving the capacitor. *Maxwell said that this changing electric field within the capacitor is equivalent to a current through the capacitor and he called this current the* **displacement current** I_D. *With the concept of the displacement current, there is no discontinuity in the current in the circuit. The usual current in the conducting wires is now called the* **conduction current** I_C. The continuity of current is shown in figure 10.1(b) as the conduction current I_C entering the capacitor, the displacement current I_D through the capacitor, and the conduction current I_C leaving the capacitor.

Example 10.1

The displacement current. At a certain instant, a parallel plate capacitor, rated at 17.4 μF, has a potential of 50.0 V across its plates. The area of the plate is 5.00 × 10^{-2} m². If it takes a time of 0.500 s to reach this 50.0-V potential, find (a) the charge

deposited on the plates of the capacitor, (b) the average conduction current at that time, (c) the average displacement current at that time, and (d) the rate at which the electric field between the plates is changing at that time.

Solution

(a) The charge deposited on the plates, found from chapter 6, equation 6.15, is

$$q = CV$$
$$q = (17.4 \times 10^{-6} \text{ F})(50.0 \text{ V})$$
$$q = 8.70 \times 10^{-4} \text{ C}$$

(b) The current in the circuit, corresponding to that amount of charge flowing in 0.500 s, found from the definition of the conduction current, is

$$I_C = \frac{dq}{dt}$$
$$I_C = \frac{8.70 \times 10^{-4} \text{ C}}{0.500 \text{ s}}$$
$$I_C = 1.74 \times 10^{-3} \text{ A}$$

(c) The displacement current across the capacitor is equal to the conduction current entering the capacitor, therefore

$$I_D = I_C = 1.74 \times 10^{-3} \text{ A}$$

(d) The rate at which the electric field between the plates is changing with time is given by rearranging equation 10.5 to

$$\frac{dE}{dt} = \frac{I_D}{\varepsilon_o A}$$
$$\frac{dE}{dt} = \frac{1.74 \times 10^{-3} \text{ A}}{(8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2)(5.00 \times 10^{-2} \text{ m}^2)}\left(\frac{\text{C/s}}{\text{A}}\right)$$
$$\frac{dE}{dt} = 3.93 \times 10^9 \frac{\text{N/C}}{\text{s}}$$

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Just as there is a magnetic field around a long straight wire carrying a conduction current, there is a magnetic field around the capacitor associated with a displacement current. This is shown in figure 10.1(c).

Ampère's law was stated in chapter 8 as: Along any arbitrary path encircling a total current I_{total}, the integral of the scalar product of the magnetic field **B** with the element of length $d\mathbf{l}$ of the path, is equal to the permeability μ_o times the total

current I_{total} enclosed by the path. *Maxwell reinterpreted Ampère's law to mean that the total current must be the sum of the conduction current and the displacement current. Thus, Maxwell rewrote Ampère's law as*

$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_o(I_C + I_D) \tag{10.6}$$

With even deeper insight, Maxwell felt that the magnetic field that he associated with the displacement current is more likely associated with the changing electric field with time. *In Faraday's law, it was shown that a changing magnetic field induces an electric field, it is therefore reasonable to assume that the inverse situation also occurs in nature; that is, that a changing electric field can produce a magnetic field.* Thus, Maxwell rewrote Ampère's law in the form of equation 10.6 but then he added his result for the displacement current found in equation 10.5. With these modifications, *Ampère's law becomes*

$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_o I_C + \mu_o \varepsilon_o A \frac{dE}{dt} \tag{10.7}$$

Ampère's law, *equation 10.7, says that a magnetic field can be produced by a conduction current or a changing electric field with time.*

As a still further generalization of Ampère's law, notice that the term AdE/dt in equation 10.7 is equal to the change in the electric flux with time. That is, since

$$\Phi_E = EA$$

then

$$\frac{d\Phi_E}{dt} = A\frac{dE}{dt}$$

Hence, Ampère's law can be written in the general form

$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_o I_C + \mu_o \varepsilon_o \frac{d\Phi_E}{dt} \tag{10.8}$$

As an example of the use of Ampère's law, let us determine the magnetic field that exists around a parallel plate capacitor that is caused by the changing electric field within the space between the parallel plates. This is shown in figure 10.2. The parallel plates are circular and have a radius R. Let us determine the magnetic field **B** at a distance r from the center of the capacitor. Within the capacitor there is no conduction current (i.e., $I_C = 0$). Therefore, Ampère's law, equation 10.7, becomes

$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_o \varepsilon_o A \frac{dE}{dt} \tag{10.9}$$

Because the magnetic field around a long straight wire carrying a current I is circular, from the point of view of symmetry, it is reasonable to expect that the magnetic field around a displacement current should also be circular, a result that can be
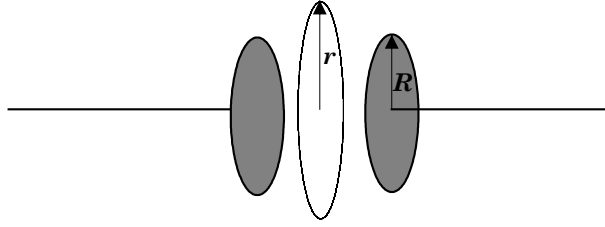
Figure 10.2 The magnetic field caused by the changing electric field.

proven by experiment. Thus, the magnetic field **B** is parallel to $\Delta \boldsymbol{l}$ along the entire circular path shown in figure 10.2. Hence,

$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \oint B dl \cos 0^0 = \oint B dl = B \oint dl$$

But the sum of the path elements $\oint dl$ is the circumference of the circle, namely $2\pi r$. Therefore,

$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = B(2\pi r)$$

Substituting this into Ampère's law, equation 10.9, we get

$$B(2\pi r) = \mu_o \varepsilon_o A \frac{dE}{dt}$$

But A is the area of the parallel plates and is πR^2. Thus,

$$B(2\pi r) = \mu_o \varepsilon_o \pi R^2 \frac{dE}{dt}$$

The magnetic field around, and at a distance r from, the capacitor is thus

$$B = \frac{\mu_o \varepsilon_o R^2}{2r} \frac{dE}{dt} \tag{10.10}$$

Notice that just as the magnetic field around a long straight wire varied as $1/r$, so also the magnetic field around the capacitor varies as $1/r$. Since μ_0 and ε_0 are constants of free space, R is the constant radius of the capacitor plate, and for a fixed value of r from the center of the capacitor to the point where the magnetic field is to be determined, we can write the magnetic field at that point as

$$B = (\text{constant}) \frac{dE}{dt} \tag{10.11}$$

That is, the changing electric field dE/dt is capable of producing a magnetic field B. If the changing electric field within the plates of a capacitor can produce a magnetic

field, should not every changing electric field produce a magnetic field? The answer is yes. Hence, there is a symmetry in nature. *Just as a changing magnetic field can produce an electric field (Faraday's law), a changing electric field can produce a magnetic field (Ampère's law as modified by Maxwell).*

Example 10.2

A changing electric field with time creates a magnetic field. Find the magnetic field a distance of 20.0 cm from the center of the parallel plate capacitor in example 10.1.

Solution

The area of the plates of the capacitor ($A = \pi R^2$) was given as 5.00×10^{-2} m², hence the radius of the plate is

$$R = \sqrt{\frac{A}{\pi}}$$
$$R = \sqrt{\frac{5.00 \times 10^{-2}\ m^2}{\pi}}$$
$$= 0.126 \text{ m}$$

The changing electric field, found in example 10.1, is $dE/dt = 3.93 \times 10^9$ (N/C)/s. Hence, the magnetic field at a distance of 20.0 cm from the center of the capacitor, found from equation 10.10, is

$$B = \frac{\mu_o \varepsilon_o R^2}{2r} \frac{dE}{dt}$$
$$B = \frac{(4\pi \times 10^{-7}\text{ T m/A})(8.85 \times 10^{-12}\text{ C}^2/\text{N m}^2)(0.126\text{ m})^2}{2(0.0200\text{ m})}[3.93 \times 10^9\text{ (N/C)/s}]$$
$$B = 1.74 \times 10^{-9} \text{ T}$$

To go to this Interactive Example click on this sentence.

Example 10.3

The magnetic field outside the long straight line. Find the magnetic field a distance of 20.0 cm from the long straight wire that is carrying the conduction current, $I_C = 1.74 \times 10^{-3}$ A, into the plate of the capacitor.

Solution

The magnetic field around a long straight wire was given by equation 8.48 as

$$B = \frac{\mu_o I_C}{2\pi r}$$

$$B = \frac{(4\pi \times 10^{-7} \text{ T m/A})(1.74 \times 10^{-3} \text{ A})}{2\pi(0.200)}$$
$$B = 1.74 \times 10^{-9} \text{ T}$$

Notice that the magnetic field outside the current-carrying wire is the same as the magnetic field caused by the changing electric field. Of course, this should come as no great surprise because the changing electric field is equivalent to a displacement current and the displacement current is the same as the conduction current. The importance of looking at the problem from the point of view of a changing electric field rather than a displacement current lies in the production and propagation of electromagnetic waves, which we will study shortly.

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10.3 Ampere's Law in Differential Form with the Displacement Current Term

We found Ampere's law in integral form, taking the conduction current into account, in equation 10.8 as

$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_o I_C + \mu_o \varepsilon_o \frac{d\Phi_E}{dt} \qquad (10.8)$$

We would now like to write the conduction current term in a different form. Consider the current I_C flowing through the wire as seen in figure 10.3. Since a

Figure 10.3 Conduction current in a wire.

current is a flow of charges through the cross-sectional area of the wire, *we define a current density **J** as a current per unit area*. With this definition, we can write the conduction current in the form

$$I = \int \boldsymbol{J} \bullet d\boldsymbol{A} \qquad (10.12)$$

Replacing equation 10.12 into equation 10.8 we get

$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_o \int \boldsymbol{J} \bullet d\boldsymbol{A} + \mu_o \varepsilon_o \frac{d\Phi_E}{dt} \qquad (10.13)$$

Equation 10.13 gives Ampere's law in terms of the current density, rather than the conduction current directly. Recall that the electric flux Φ_E was given by equation 4.14 as

$$\Phi_E = \int \boldsymbol{E} \bullet d\boldsymbol{A} \tag{4.14}$$

We now replace equation 4.14 into equation 10.13 to obtain

$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_o \int \boldsymbol{J} \bullet d\boldsymbol{A} + \mu_o \varepsilon_o \frac{d}{dt} \int \boldsymbol{E} \bullet d\boldsymbol{A} \tag{10.14}$$

We can now apply Stokes' theorem, equation 9.80, to the term $\oint \boldsymbol{B} \bullet d\boldsymbol{l}$ in equation 10.14 to obtain

$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \int (\nabla \times \boldsymbol{B}) \bullet d\boldsymbol{A} \tag{10.15}$$

Equating equation 10.14 to equation 10.15 we get

$$\int (\nabla \times \boldsymbol{B}) \bullet d\boldsymbol{A} = \mu_o \int \boldsymbol{J} \bullet d\boldsymbol{A} + \mu_o \varepsilon_o \frac{d}{dt} \int \boldsymbol{E} \bullet d\boldsymbol{A}$$
$$\int (\nabla \times \boldsymbol{B}) \bullet d\boldsymbol{A} - \mu_o \int \boldsymbol{J} \bullet d\boldsymbol{A} - \mu_o \varepsilon_o \frac{d}{dt} \int \boldsymbol{E} \bullet d\boldsymbol{A} = 0 \tag{10.16}$$

Since all three integrations are over the same $d\mathbf{A}$, we can rewrite this as

$$\int [\nabla \times \boldsymbol{B} - \mu_o \boldsymbol{J} - \mu_o \varepsilon_o \frac{d\boldsymbol{E}}{dt}] \bullet d\boldsymbol{A} = 0 \tag{10.17}$$

The only way for this integration to be zero for all values of $\mathbf{B}$, $\mathbf{J}$, and $\mathbf{E}$ is for the entire bracketed term to be zero, that is ,

$$\nabla \times \boldsymbol{B} - \mu_o \boldsymbol{J} - \mu_o \varepsilon_o \frac{d\boldsymbol{E}}{dt} = 0 \tag{10.18}$$

or

$$\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J} + \mu_o \varepsilon_o \frac{d\boldsymbol{E}}{dt} \tag{10.19}$$

Equation 10.19 is the generalized Ampere's law in differential form and is one of Maxwell's equations.

10.4 Faraday's Law Revisited

In chapter 9 we discussed **Faraday's law** and saw that an electric field can be produced by changing a magnetic field with time. However, it is appropriate here to discuss some of the ramifications of Faraday's law as they apply to electromagnetic waves. First, in figure 9.1, repeated here as figure 10.4(a), a wire that was in a uniform magnetic field that pointed into the paper was pulled to the right with a velocity **v.** We then found that an induced electric field existed in the wire whose
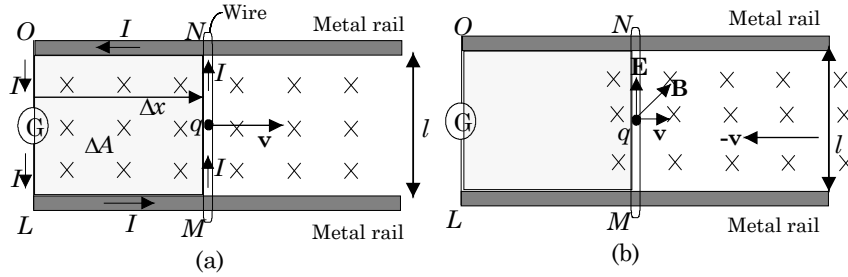
Figure 10.4 (a) Motion of a wire through a uniform magnetic field and (b) motion of a uniform magnetic field past a wire at rest.

magnitude was given by equation 9.3, namely

$$E = vB \sin\theta$$

The induced electric field was the cause of the induced emf given by equation 9.5 as

$$E = \frac{\mathcal{E}}{l}$$

where l was the length of the wire in motion. *Now the important thing about the cause of the induced electric field and the induced emf was the motion of the wire through the magnetic field. However, what is the difference between the wire in motion toward the right through a uniform magnetic field that is stationary and a wire that is at rest while a uniform magnetic field moves toward the left at a velocity $-\mathbf{v}$.* With a little bit of thought, we can see that they must both give the same result. The important thing is the relative motion between the wire and the magnetic field. Hence, if a uniform magnetic field is propagated toward the left, past the stationary wire, then an induced electric field is found in the wire, as shown in figure 9.3(b). The induced electric field causes the induced emf between the points M and N and a current I is observed in the galvanometer, the same as before.

Now if the resistance of the wire MN is increased, the current through it, and hence the current recorded by the galvanometer, decreases. In fact, if the resistance R of wire MN were increased to infinity, no current at all would flow through the wire or the galvanometer. However, the induced electric field would still be present and so would its associated induced emf. If the resistance of MN is increased to infinity, the wire is no longer a conductor, but instead, becomes an insulator. In fact, the wire MN could be replaced by a wooden stick, and the relative motion of the wooden stick with respect to the uniform magnetic field would induce an electric field within the stick. Of course no current would flow through the stick, but the induced electric field would still be there. But what is so special about a wooden stick as an insulator? Suppose the stick were removed entirely and only an air gap for MN is left. The air gap would also act as an insulator. *If, again, the uniform_magnetic field was to move past the air gap, MN, at a speed v toward the left, then there*

must be an induced electric field within the air gap itself, in the same direction as the induced electric field within the conducting wire. As the magnetic field passes the line *MN*, the magnetic field on the line changes with time, *thus a changing magnetic field induces an electric field anywhere, that is, in a conductor, in an insulator, or in an air gap.*

To show this application of Faraday's law let us recall Faraday's law from equation 9.84, that is

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = -\frac{d}{dt} \int \boldsymbol{B} \bullet d\boldsymbol{A}$$

(9.84)

Equation 9.84 is the generalization of Faraday's law and in this form it is the fourth of Maxwell's equations. For our example of the application of Faraday's law in the form of equation 9.84, let us consider the circular loop of wire in the magnetic field of figure 10.5(a). The magnetic field is pointing into the paper and is increasing
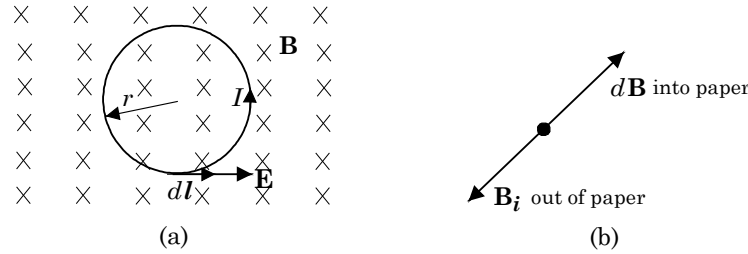

(a)   (b)

Figure 10.5 The induced electric field in a circular loop in an increasing magnetic field.

with time, so that *d***B** also points into the paper. The changing magnetic field induces a current in the coil such as to oppose the changing magnetic field (Lenz's law). Hence, the induced magnetic field **B$_i$** must point outward from the paper, as shown in figure 10.5(b). Therefore, the current in the loop of wire must be counterclockwise. Since charge flows in the direction of the electric field, there must be an induced electric field in the wire, tangential to the wire, as shown in figure 10.5(a). Hence, **E** is always in the direction of *d**l***, and θ is equal to zero. The left hand side of Faraday's law then becomes

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = \oint E dl \cos \theta = \oint E dl \cos 0^0 = \oint E dl$$

(10.20)

But from the symmetry of the problem, the value of E must be the same at every small path dl, and can thus be factored out of the sum in equation 10.20. Therefore,

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = E \oint dl$$

(10.21)

But the sum of dl around the circular loop is just the circumference of the loop itself, that is,

$$\oint dl = 2\pi r$$

(10.22)

Substituting equations 10.21 and 10.22 into equation 9.84, gives

$$E(2\pi r) = -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{A}$$

(10.23)

But

$$E(2\pi r) = -\int \frac{dB}{dt} dA \cos\theta = -\frac{dB}{dt} A \cos\theta$$

since $\int dA = A$, the area of the loop. But **A** points out of the paper while $d\mathbf{B}$ points inward, and therefore $\theta = 180^0$. Thus

$$E(2\pi r) = -\frac{dB}{dt} A \cos 180^0 = -\frac{dB}{dt} A(-1) = +A\frac{dB}{dt}$$

Solving for the induced electric field E,

$$E = \frac{A}{2\pi r}\frac{dB}{dt}$$

(10.24)

The area of the loop is $A = \pi r^2$, thus,

$$E = \frac{\pi r^2}{2\pi r}\frac{dB}{dt}$$

and

$$E = \frac{r}{2}\frac{dB}{dt}$$

(10.25)

Equation 10.25 says that the changing magnetic field with time dB/dt induces an electric field E around the loop.

Because r is the radius of the loop and is a constant, we can also write equation 10.25 as

$$E = (\text{constant})\frac{dB}{dt}$$

(10.26)

Let us now compare equation 10.26 with equation 10.11, namely

$$B = (\text{constant})\frac{dE}{dt}$$

(10.11)

Equations 10.26 and 10.11 show that a changing magnetic field with time induces an electric field, while a changing electric field with time induces a magnetic field.

Example 10.4

A changing magnetic field with time creates an electric field. If the above loop of wire has a radius of 5.00 cm and the magnetic field changes at the rate of 2.50×10^2 T/s, what is the induced electric field in the loop?

Solution

The induced electric field, given by equation 10.25, is

$$E = \frac{r}{2}\frac{dB}{dt}$$
$$E = \frac{5.00 \times 10^{-2} \text{ m}}{2}(2.50 \times 10^2 \text{ T/s})$$
$$E = 6.25 \frac{\text{m T}}{\text{s}}\left(\frac{\text{N/(A m)}}{\text{T}}\right)\left(\frac{\text{A}}{\text{C/s}}\right)$$
$$E = 6.25 \text{ N/C}$$

To go to this Interactive Example click on this sentence.

If, instead of a circular wire loop in figure 10.5(a), we had only air in the region, changing the magnetic field with time would still produce an electric field, only now it would be in the air itself.

10.5 Maxwell's Equations in Integral Form

The four Maxwell's equations that completely describe all electromagnetic phenomena, have now been developed. They are summarized below:

I. Gauss's Law for Electricity

$$\Phi_E = \oint \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{q}{\varepsilon_o} \tag{4.14}$$

II. Gauss's Law for Magnetism

$$\Phi_M = \oint \boldsymbol{B} \bullet d\boldsymbol{A} = 0 \tag{8.99}$$

III. Ampère's Law

$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_o I_C + \mu_o \varepsilon_o \frac{d\Phi_E}{dt} \tag{10.8}$$

IV. Faraday's Law

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = -\frac{d\Phi_M}{dt} \tag{10.14}$$

10.6 Maxwell's Equations in Differential Form

The four Maxwell's equations in their differential form that completely describe all electromagnetic phenomena, are summarized below:

I. Gauss's Law for Electricity

$$\nabla \bullet \boldsymbol{E} = \frac{\rho}{\varepsilon_o} \tag{4.87}$$

II. Gauss's Law for Magnetism

$$\triangledown \bullet \boldsymbol{B} = 0 \qquad (8.103)$$

III. Ampère's Law

$$\triangledown \times \boldsymbol{B} = \mu_o \boldsymbol{J} + \mu_o \varepsilon_o \frac{d\boldsymbol{E}}{dt} \qquad (10.19)$$

IV. Faraday's Law

$$\triangledown \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (9.87)$$

10.7 Electromagnetic Waves

The rest of this chapter concerns the propagation of electromagnetic waves in space. Therefore, the charge q in equation 4.14 will be zero and the current density **J** in equation 10.19 will also be zero. Hence, *Maxwell's equations in their differential form for charge-free space can now be written as*

I. Gauss's Law for Electricity

$$\triangledown \bullet \boldsymbol{E} = 0 \qquad (10.27)$$

II. Gauss's Law for Magnetism

$$\triangledown \bullet \boldsymbol{B} = 0 \qquad (10.28)$$

III. Ampère's Law

$$\triangledown \times \boldsymbol{B} = \mu_o \varepsilon_o \frac{d\boldsymbol{E}}{dt} \qquad (10.29)$$

IV. Faraday's Law

$$\triangledown \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (10.30)$$

The implication of equations 10.27 and 10.28 is that all electric and magnetic fields in charge-free space are continuous, that is, the electric fields do not begin or end on any charges. They also imply that the electric and magnetic flux neither converges nor diverges. Ampère's law, equation 10.29, tells us that a changing electric field produces a magnetic field, and Faraday's law, equation 10.30, says that a changing magnetic field produces an electric field. *The fact that a changing electric field with time produces a magnetic field, whereas a changing magnetic field with time produces an electric field, suggests that it should be possible to propagate an electromagnetic wave through empty space.*

Our task now is to solve the Maxwell equations simultaneously. We start by solving equations 10.29 and 10.30 simultaneously by taking the curl of equation 10.30. That is,

$$\triangledown \times [\triangledown \times \boldsymbol{E}] = -\triangledown \times \frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\partial(\triangledown \times \boldsymbol{B})}{\partial t} \qquad (10.31)$$

But the curl of **B** in the right-hand side of equation 10.31 can be replaced by equation 10.29 as

$$\triangledown \times [\triangledown \times \boldsymbol{E}] = -\frac{\partial(\triangledown \times \boldsymbol{B})}{\partial t} = -\frac{\partial(\mu_o\varepsilon_o\frac{\partial \boldsymbol{E}}{\partial t})}{\partial t} = -\mu_o\varepsilon_o\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{10.32}$$

Let us apply the following vector identity

$$\boldsymbol{A} \times [\boldsymbol{B} \times \boldsymbol{C}] = (\boldsymbol{A} \bullet \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \bullet \boldsymbol{B})\boldsymbol{C} \tag{10.33}$$

to the left-hand side of equation 10.32 to obtain

$$\triangledown \times [\triangledown \times \boldsymbol{E}] = (\triangledown \bullet \boldsymbol{E})\triangledown - (\triangledown \bullet \triangledown)\boldsymbol{E} \tag{10.34}$$

But from equation 10.27, $\triangledown \bullet \mathbf{E} = 0$ for the charged free area we are concerned with, while the term $(\triangledown \bullet \triangledown)\mathbf{E}$ is written as

$$(\triangledown \bullet \triangledown)\boldsymbol{E} = \triangledown^2 \boldsymbol{E} \tag{10.35}$$

Replacing equations 10.27 and 10.35 into equation 10.34 we get

$$\triangledown \times [\triangledown \times \boldsymbol{E}] = 0 - \triangledown^2 \boldsymbol{E} \tag{10.36}$$

Replacing equation 10.36 into equation 10.32 we obtain

$$-\triangledown^2 \boldsymbol{E} = -\mu_o\varepsilon_o\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

or

$$\triangledown^2 \boldsymbol{E} = \mu_o\varepsilon_o\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{10.37}$$

Equation 10.37 is called the wave equation in three-dimensions. The term $\triangledown^2\mathbf{E}$ is called the Laplacian of $\mathbf{E}$ and is found by the definition of $\triangledown$ as

$$(\triangledown \bullet \triangledown)\boldsymbol{E} = \triangledown^2 \boldsymbol{E} = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \bullet \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\boldsymbol{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\boldsymbol{E}$$
$$\triangledown^2 \boldsymbol{E} = \frac{\partial^2 \boldsymbol{E}}{\partial x^2} + \frac{\partial^2 \boldsymbol{E}}{\partial y^2} + \frac{\partial^2 \boldsymbol{E}}{\partial z^2} \tag{10.38}$$

Replacing equations 10.38 into equation 10.37 we get

$$\triangledown^2 \boldsymbol{E} = \frac{\partial^2 \boldsymbol{E}}{\partial x^2} + \frac{\partial^2 \boldsymbol{E}}{\partial y^2} + \frac{\partial^2 \boldsymbol{E}}{\partial z^2} = \mu_o\varepsilon_o\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{10.39}$$

which shows the three dimensional character of the equation because $\mathbf{E}$ itself is, in general, a three-dimensional vector given by

$$\mathbf{E} = \mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z$$

and the derivatives are with respect to the three coordinates x, y, and z. To simplify the problem we will assume that the electric field exists only in the z-direction and hence has only one component, namely E_z, while $E_x = E_y = 0$. Equation 10.39 now reduces to

$$\nabla^2 E_z = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 E_z}{\partial t^2}$$

We also assume that E_z is not a function of x and y, which implies that the derivatives

$$\frac{\partial^2 E_z}{\partial y^2} = \frac{\partial^2 E_z}{\partial z^2} = 0$$

With these simplifying assumptions, equation 10.39 becomes

$$\frac{\partial^2 E_z}{\partial x^2} = \mu_o \varepsilon_o \frac{\partial^2 E_z}{\partial t^2} \tag{10.40}$$

Equation 10.40 is a one-dimensional wave equation for the electric field component E_z. Since we are only dealing with one component, we will even drop the subscript z, and call the electric field simply E. Equation 10.40 becomes

$$\frac{\partial^2 E}{\partial x^2} = \mu_o \varepsilon_o \frac{\partial^2 E}{\partial t^2} \tag{10.41}$$

Equation 10.41 is the one-dimensional wave equation for the electric field E. We will come back to equation 10.41 shortly. We should first say something about the wave equation itself and its solution.

　　Equation 10.41 is called the wave equation because it has the same form as the equation found for the mechanical problem of a vibrating string, namely

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \tag{10.42}$$

where y is the displacement of a point on the string at any instant of time and is shown in figure 10.6.
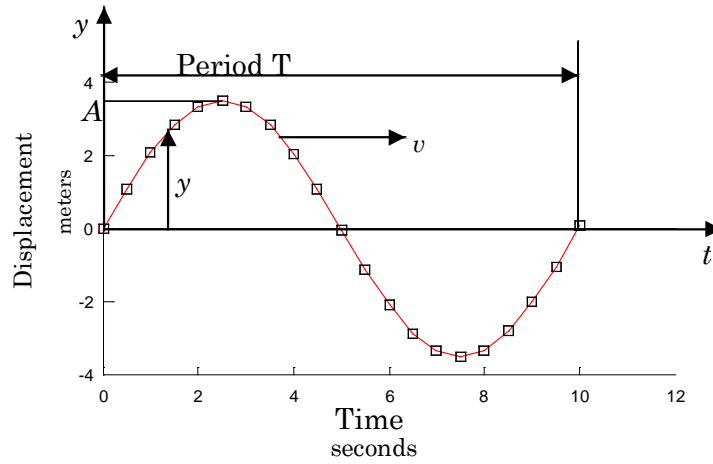
　　The solutions to the wave equation 10.42 are
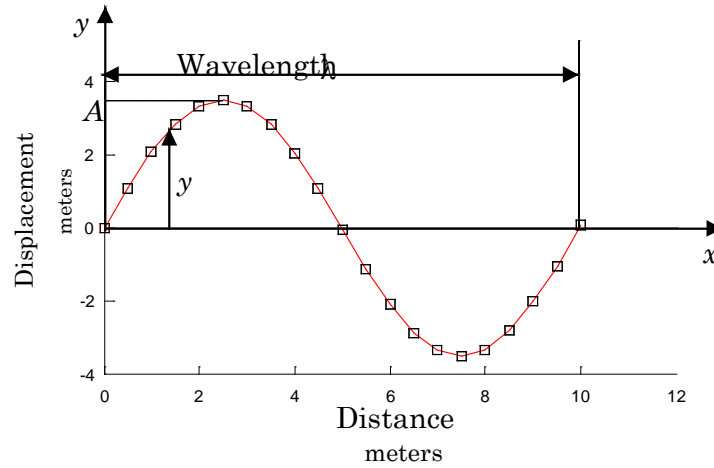
$$y = A \sin(kx - \omega t) \tag{10.43}$$

and

$$y = A \sin(kx + \omega t) \tag{10.44}$$

Equation 10.43 represents a wave traveling to the right, and equation 10.44 represents a wave traveling to the left. The term k is called the wave number and is

(a) Plot of Displacement y of a Wave as a function of time t, for a fixed value of distance x.

(b) Plot of Displacement y of a Wave as a function of distance x for a fixed value of time.

Figure 10.6   Characteristics of waves.

defined as

$$k = \frac{2\pi}{\lambda}$$

(10.45)

while λ is called the wavelength of the wave and is the distance in which the wave repeats itself, and is seen in figure 10.6(b). The term ω in equation 10.43 is called the angular frequency of the wave and is given by

$$\omega = kv = 2\pi f$$

(10.46)

where v is the velocity of the wave and f is the frequency of the wave. The variables λ, f, and v, are not independent but are related by the fundamental equation of wave propagation as

$$v = \lambda f \qquad\qquad (10.47)$$

Since equation 10.41 has the same form as equation 10.42, the wave equation for the vibrating string, a mechanical wave, we assume that the solution to the electrical wave, equation 10.41, should be similar to the solution of the mechanical wave. We therefore assume that the solutions of equation 10.41 for the electric wave should be

$$E = E_o \sin(kx - \omega t) \qquad\qquad (10.46)$$

and

$$E = E_o \sin(kx + \omega t) \qquad\qquad (10.47)$$

where E_o is the amplitude of the electric wave, k is the wave number, and ω is the angular frequency of the wave. Equation 10.46 should represent an electric wave traveling to the right and equation 10.47 should represent an electric wave traveling to the left. If equation 10.46 is a solution to the electrical wave equation, equation 10.41, we can place it in equation 10.41 and operate on it. If indeed it is a solution, the right-hand-side of the equation must be equal to the left-hand-side of the equation. That is,

$$\begin{aligned}
E &= E_o \sin(kx - \omega t) \\
\frac{\partial E}{\partial x} &= E_o k \cos(kx - \omega t) \\
\frac{\partial^2 E}{\partial x^2} &= -E_o k^2 \sin(kx - \omega t)
\end{aligned} \qquad\qquad (10.48)$$

while

$$\begin{aligned}
\frac{\partial E}{\partial t} &= E_o(-\omega) \cos(kx - \omega t) \\
\frac{\partial^2 E}{\partial t^2} &= -E_o \omega^2 \sin(kx - \omega t)
\end{aligned} \qquad\qquad (10.49)$$

Replacing equations 10.48 and 10.49 into the wave equation, equation 10.41, we get

$$\begin{aligned}
\frac{\partial^2 E}{\partial x^2} &= \mu_o \varepsilon_o \frac{\partial^2 E}{\partial t^2} \\
-E_o k^2 \sin(kx - \omega t) &= \mu_o \varepsilon_o [-E_o \omega^2 \sin(kx - \omega t)] \\
k^2 &= \mu_o \varepsilon_o \omega^2 \\
\frac{k^2}{\omega^2} &= \mu_o \varepsilon_o \\
\frac{k}{\omega} &= \sqrt{\mu_o \varepsilon_o}
\end{aligned} \qquad\qquad (10.50)$$

But as can be seen from equation 10.46, $k/\omega = 1/v$. Replacing this into equation 10.50 we obtain

$$\frac{1}{v} = \sqrt{\mu_o \varepsilon_o}$$

or

$$v = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$$

(10.51)

Equation 10.51 gives the velocity of the electric wave in terms of the constants of permeability of free space μ_o *and permittivity of free space* ε_o. Replacing the numerical values of these constants in equation 10.51 gives for the speed of the electric wave

$$v = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = \frac{1}{\sqrt{(4\pi \times 10^{-7} \text{ T m/A})(8.854 \times 10^{-12} \text{ C}^2/\text{N m}^2)}}$$

$$v = 3.00 \times 10^8 \text{ m/s}$$

But a speed of 3.00 × 10⁸ m/s is the speed of light, usually designated by the letter c. Hence the electric wave, given by equation 10.46, is indeed a solution to the wave equation, and that electric wave must move at the speed of light. This result led Maxwell to declare that light itself must be an electromagnetic wave of some appropriate wavelength and frequency, a prediction since confirmed many times over. Also note that equation 10.51 should now be written as

$$c = \sqrt{\frac{1}{\mu_o \varepsilon_o}}$$

(10.52)

for the speed of an electromagnetic wave.

We have solved Maxwell's equations 10.29 and 10.30 simultaneously by eliminating the magnetic field **B** and upon solving for the equation of the electric field, we obtained the wave equation 10.41 for the electric wave. But what about the magnetic field? We can similarly eliminate the electric field **E** between equations 10.29 and 10.30 to obtain a wave equation for the magnetic field **B**. We will however try a slightly different approach which will impart a more physical picture of the waves.

Let us start with Faraday's law, equation 10.30.

$$\nabla \times \boldsymbol{E} = - \frac{\partial \boldsymbol{B}}{\partial t}$$

(10.30)

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} = -\left[\boldsymbol{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \boldsymbol{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) + \boldsymbol{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \right]$$

(10.53)

But we have let $E_x = E_y = 0$, so that equation 10.53 reduces to

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\left[\boldsymbol{i}\left(\frac{\partial E_z}{\partial y} - 0 \right) + \boldsymbol{j}\left(0 - \frac{\partial E_z}{\partial x} \right) + \boldsymbol{k}(0 - 0) \right]$$

(10.54)

But for a plane wave E_z does not vary with the coordinate y, that is, $\partial E_z/\partial y = 0$. Therefore, equation 10.54 becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\left[\boldsymbol{i}(0-0) + \boldsymbol{j}\left(0 - \frac{\partial E_z}{\partial x}\right) + \boldsymbol{k}(0-0)\right] = -\boldsymbol{j}\left(-\frac{\partial E_z}{\partial x}\right) = \boldsymbol{j}\frac{\partial E_z}{\partial x}$$

which says that as the wave moves in the x-direction, the electric field intensity is changing in the z-direction, E_z, and the magnetic field is also changing with time, $\partial\boldsymbol{B}/\partial t$. Notice that since the left-hand-side of the equation must equal the right-hand-side of the equation, the term $\partial\boldsymbol{B}/\partial t$ must also point in the $\boldsymbol{j}$-direction. We can see this as

$$\frac{\partial(\boldsymbol{i}B_x + \boldsymbol{j}B_y + \boldsymbol{k}B_z)}{\partial t} = \boldsymbol{i}\frac{\partial B_x}{\partial t} + \boldsymbol{j}\frac{\partial B_y}{\partial t} + \boldsymbol{k}\frac{\partial B_z}{\partial t} = \boldsymbol{j}\frac{\partial E_z}{\partial x}$$

or

$$\boldsymbol{j}\frac{\partial B_y}{\partial t} = \boldsymbol{j}\frac{\partial E_z}{\partial x} \tag{10.55}$$

Equation 10.55 says that when the electric field is changing with x, $\partial E_z/\partial x$, B is changing with time. So just as the electric field varies from positive to negative values, the magnetic field does also, and it would appear that maybe there is also a magnetic wave propagating with the electric wave. Since the electric wave is in the x-z-plane, and because of the unit vector $\boldsymbol{j}$ for the direction of the changing magnetic field, this implies that the changing magnetic field B is perpendicular to the electric wave.

Using equation 10.46, let us now solve equation 10.55 for B_y.

$$\frac{\partial B_y}{\partial t} = \frac{\partial E_z}{\partial x} = \frac{\partial(E_o \sin(kx - \omega t)}{\partial x} = kE_o \cos(kx - \omega t)$$

$$\frac{\partial B_y}{\partial t} = kE_o \cos(kx - \omega t) \tag{10.56}$$

$$dB_y = kE_o \cos(kx - \omega t)dt \tag{10.57}$$

We now integrate equation 10.57 to solve for B_y.

$$\int_0^{B_y} dB_y = \int kE_o \cos(kx - \omega t)dt$$

$$\int_0^{B_y} dB_y = \int \frac{kE_o}{(-\omega)} \cos(kx - \omega t)dt(-\omega) = = \frac{kE_o}{(-\omega)} \int \cos(kx - \omega t)(-\omega dt)$$

$$B_y = -\frac{kE_o}{\omega} \sin(kx - \omega t) \tag{10.58}$$

Equation 10.58 says that the magnetic field changes sinusoidally with time just as the electric field changes with time. Notice that B_y is negative when E is positive, equation 10.58, indicating that when E is in the positive z-direction, B is in the negative y-direction. The electric and magnetic waves are identical in form but are perpendicular to each other. Hence there is not just an electric wave nor just a magnetic wave, but rather there exists an electromagnetic wave propagating through space as shown in figure 10.7. Notice how the electric and magnetic field vectors are perpendicular to each other.

Equation 10.58, for the magnetic wave, can also be written in the form
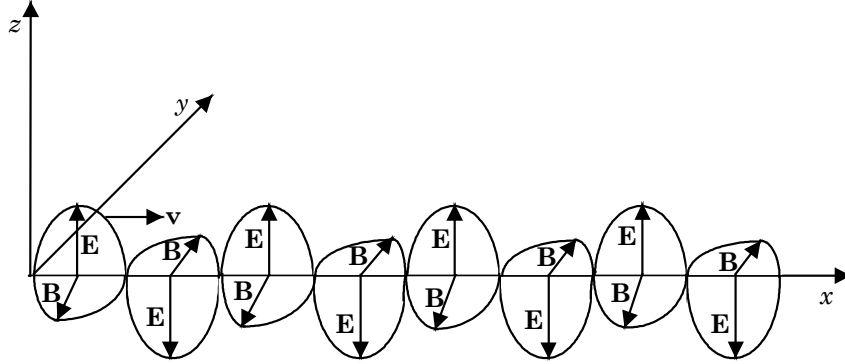
$$B = -B_0 \sin(kx - \omega t) \tag{10.59}$$

Figure 10.7 An electromagnetic wave.

with

$$B_o = \frac{kE_0}{\omega}$$

where B_0 is the maximum value of the magnetic field and B is the magnitude of the magnetic field at any position x and time t. As we showed earlier, k is the wave number, given by

$$k = \frac{2\pi}{\lambda} \tag{10.45}$$

while ω is the angular frequency of the wave, given by

$$\omega = 2\pi f \tag{10.46}$$

Here λ is the wavelength of the wave and f is its frequency. *Hence, it is logical to assume that there should be a large number of electromagnetic waves, differing only in frequency and wavelength.* We will say more about this shortly.

Almost everything said about mechanical waves, also applies to the electromagnetic waves. That is, electromagnetic waves can be reflected and transmitted. The principle of superposition applies so that any number of electromagnetic waves can be added together. Standing electromagnetic waves are produced and a Doppler effect, slightly different than that for sound waves, is also observed.

Finally, using equation 10.47, we can rewrite equation 10.58 in the form

$$B_y = -\frac{kE_o}{\omega} \sin(kx - \omega t) = -\frac{k}{\omega}[E_o \sin(kx - \omega t)]$$
$$B_y = -\frac{k}{\omega}E \tag{10.60}$$

But as can be seen from equation 10.46, $k/\omega = 1/v$. Therefore equation 10.60 becomes

$$B_y = -\frac{1}{v}E$$

or

$$E = vB_y \tag{10.61}$$

With the designation of c as the speed of an electromagnetic wave, equation 10.61 should now be written as

$$E = cB_y \tag{10.62}$$

Example 10.5

Compare the size of the magnetic field with the electric field. What is the relative strength of the magnetic wave as compared to the electric wave in an electromagnetic wave?

Solution

The strength of the magnetic field at any instant, found from equation 10.62, is

$$B = \frac{E}{c}$$
$$= \frac{E}{3.00 \times 10^8 \text{ m/s}}$$
$$= (3.33 \times 10^{-9} \text{ s/m})E$$

Thus, the magnetic field is much smaller numerically than the electric field, although, as we will see later, they both carry the same energy.

To go to this Interactive Example click on this sentence.

10.8 The Electromagnetic Spectrum

We have seen that electromagnetic waves exist and propagate through space at the speed of light. We represent the electric wave by equation 10.43, and the electric field intensity E depends on the wavelength and frequency of the wave. The wavelength and frequency are not independent but are related by the fundamental equation of wave propagation, equation 10.47 with the speed v replaced by c, and the frequency f replaced by the Greek lower case letter ν (nu). Thus,

$$c = \lambda v \tag{10.63}$$

The use of the letter ν for the frequency of the electromagnetic wave rather than the letter *f,* that we used previously when waves were discussed, is customary in physics when dealing with electromagnetic radiation in modern physics.

It is evident that an entire series of electromagnetic waves should exist, differing only in frequency and wavelength. Such a group of electromagnetic waves has been found and they are divided into six main categories: radio waves, infrared waves, visible light waves, ultraviolet light, X rays, and gamma rays. *The entire group of electromagnetic waves is called **the electromagnetic spectrum.*** Let us look at some of the characteristics of these waves.

1. ***Radio Waves****.* Radio waves are usually described in terms of their frequency. AM (amplitude modulated) radio waves are emitted at frequencies from 550 kHz to 1600 kHz. (Recall that the unit kHz is a kilohertz, which is a thousand cycles per second, hence 550 kHz is equal to 550×10^3 cycles per second or 5.50×10^5 cycles/s.) FM (frequency modulated) radio waves, on the other hand, are transmitted in the range of 88 MHz to 108 MHz. (Recall that MHz is a megahertz which is equal to 10^6 Hz.) Television waves are transmitted in the range of 44 MHz to 216 MHz. Ultrahigh frequency (UHF) TV waves are broadcast in the range of 470 MHz to 890 MHz. Microwaves, which are used in radar sets and microwave ovens fall in the range of 1 GHz to 30 GHz. A gigahertz (GHz) is equal to 10^9 cycles/s.

2. ***Infrared Waves****.* Infrared waves are usually described in terms of their wavelength rather than their frequency. The infrared spectrum extends from approximately 720 nm to 50,000 nm. Recall that the unit nm is a nanometer and is equal to 10^{-9} m. Infrared frequencies can be determined from equation 10.63.

3. ***Visible Light****.* Visible light occupies a very small portion of the electromagnetic spectrum, from 380 nm to 720 nm. The wavelength of 380 nm corresponds to a violet color, while 720 nm corresponds to a red color.

4. ***Ultraviolet Light****.* The ultraviolet portion of the spectrum extends from around 10 nm up to about 380 nm. It is this ultraviolet radiation from the sun that causes sunburn and skin cancer.

5. ***X Rays****.* X rays are very energetic electromagnetic waves. They are usually formed when high-speed charged particles are brought to rest on impact with matter. The x-ray portion of the electromagnetic spectrum lies in the range 0.01 nm up to about 150 nm.

6. ***Gamma Rays****.* Gamma rays are the most energetic of all the electromagnetic waves and fall in the range of almost 0 to 0.1 nm overlapping the x-ray region. They differ from X rays principally in origin. They are emitted from the nucleus of an atom, whereas X rays are usually associated with processes occurring in the electron shell structure of the atom.

10.9 Energy Transmitted by an Electromagnetic Wave

In chapter 6, we saw that the energy density in the electric field is

$$u_E = \frac{1}{2} \varepsilon_0 E^2 \tag{6.57}$$

whereas the energy density in a magnetic field was found in chapter 9 to be

$$u_M = \frac{1}{2} \frac{B^2}{\mu_0} \tag{9.58}$$

Hence, the total energy density residing in the electromagnetic field is the sum of the electric energy density and the magnetic energy density, which is simply the sum of equations 6.57 and 9.58. That is,

$$u = u_E + u_M \tag{10.64}$$

$$u = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2}\frac{B^2}{\mu_o} \tag{10.65}$$

To show how this energy is distributed, we use equation 9.58 for B, and substitute it into equation 10.65 to get

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2} \frac{(E/c)^2}{\mu_0}$$

But, substituting for c^2 from equation 10.52, we get

$$u = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \frac{(\mu_o \varepsilon_o)E^2}{\mu_0}$$

Hence,

$$u = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \varepsilon_o E^2 \tag{10.66}$$

The second term on the right-hand side of equation 10.66 represents the magnetic energy density, and since it is equal to the first term, which represents the electric energy density, it is clear that *the total energy of the electromagnetic wave is divided evenly between the electric wave and the magnetic wave.* That is, one half of the total energy of the electromagnetic wave is contained in the electric wave while the other half of the total energy is contained in the magnetic wave. *The total energy density of the electromagnetic field can be written from equation 10.66 as*

$$u = \varepsilon_o E^2 \tag{10.67}$$

Another quantity that is of great interest is the *intensity of the electromagnetic rad-iation. The **intensity of radiation** is defined as the total energy per unit area per unit time.* Because the total energy per unit time is power, *the intensity of the radia-tion can also be defined as the power of the electromagnetic wave falling on a unit area.* Thus,

$$\text{Intensity} = \frac{\text{Total energy}}{(\text{area})(\text{time})} \qquad (10.68)$$

Equation 10.67 represents the energy density, that is, the energy per unit volume. To obtain the total energy, the energy density u must be multiplied by a volume V of the field. This can be seen more clearly by referring to figure 10.8. The total energy that falls on a unit area A of the surface in a time Δt is all the energy contained in the imaginary cylindrical surface shown in the figure. The volume of the cylinder is
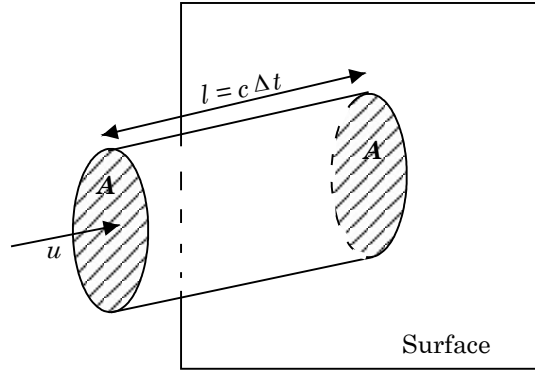
$$V = Al = Ac\Delta t$$

Figure 10.8 Energy Intensity.

Hence, the intensity becomes

$$\text{Intensity} = \frac{uV}{A\Delta t}$$
$$= \frac{uAc\Delta t}{A\Delta t}$$
$$= uc$$

$$\text{Intensity} = \varepsilon_o cE^2 \qquad (10.69)$$

Again using the fact that $E = cB$, we get

$$\text{Intensity} = \varepsilon_0 cE(cB) = \varepsilon_0 c^2 EB$$

We now also use the result that $c^2 = 1/(\mu_0\varepsilon_0)$ to get

$$\text{Intensity} = \frac{\varepsilon_0 EB}{(\mu_0 \varepsilon_0)}$$

$$\text{Intensity} = \frac{EB}{\mu_o} \tag{10.70}$$

Example 10.6

Determining the values of E and B for the sun's radiation received at the earth. The solar constant, 1.38×10^3 J/(m² s), is the average intensity of radiation from the sun falling on the top of the earth's atmosphere. What are the average values of the E and B fields associated with this intensity?

Solution

The average value of the electric field is found from equation 10.69 as

$$\text{Intensity} = \varepsilon_0 c E^2$$

$$E = \sqrt{\frac{\text{Intensity}}{\varepsilon_o c}}$$

$$E = \sqrt{\frac{1.38 \times 10^3 \, \text{J/(m}^2\text{s})}{[8.85 \times 10^{-12} \, \text{C}^2/(\text{N m}^2)](3.00 \times 10^8 \, \text{m/s})} \left(\frac{\text{N m}}{\text{J}}\right)}$$

$$= 721 \, \text{N/C} = 721 \, \text{V/m}$$

The average value of B is found from

$$B = \frac{E}{c} = \frac{721 \, \text{N/C}}{3.00 \times 10^8 \, \text{m/s}} = 2.40 \times 10^{-6} \, \frac{\text{N}}{\text{A s}}$$

$$= 2.40 \times 10^{-6} \, \text{T}$$

Note that the value of B is very much smaller than E, yet each wave contains one half of the total energy of the electromagnetic wave.

To go to this Interactive Example click on this sentence.

Summary of Important Concepts

Maxwell's equations
A set of four equations that completely describe all electromagnetic phenomena. They include Gauss's law for electricity, Gauss's law for magnetism, Ampere's law with a correction for the displacement current, and Faraday's law of Electromagnetic Induction.

Displacement current
A changing electric field in a capacitor is equivalent to a current through the capacitor. This current is called the displacement current.

Conduction current
Ordinary current in conducting wires.

Ampère's law
A magnetic field can be produced by a conduction current or a changing electric field with time.

Faraday's law
An electric field can be produced by changing a magnetic field with time.

Electromagnetic waves
Waves that are characterized by a changing electric field and a changing magnetic field. They propagate through space at the speed of light. The electric wave and the magnetic wave are always perpendicular to each other.

The electromagnetic spectrum
The complete range of electromagnetic waves, from the longest radio waves down to infrared rays, visible light, ultraviolet light, X rays, and the shortest waves, the gamma rays.

Intensity of radiation
The total energy of an electromagnetic wave impinging on a unit area in a unit period of time. It is also represented as the power per unit area.

Summary of Important Equations

The displacement current
$$I_D = \varepsilon_o A \frac{dE}{dt} \tag{10.5}$$

Ampere's law
$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_o (I_C + I_D) \tag{10.6}$$

$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_o I_C + \mu_o \varepsilon_o A \frac{dE}{dt} \tag{10.7}$$

$$\oint \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_o I_C + \mu_o \varepsilon_o \frac{d\Phi_E}{dt} \tag{10.8}$$

Ampere's law in differential form
$$\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J} + \mu_o \varepsilon_o \frac{d\boldsymbol{E}}{dt} \tag{10.19}$$

A changing magnetic field produces an electric field
$$E = (\text{constant}) \frac{dB}{dt} \tag{10.26}$$

A changing electric field produces a magnetic field $\qquad B = (\text{constant})\dfrac{dE}{dt}$ (10.11)

Maxwell's equations in integrable form

I. Gauss's law for electricity $\qquad\qquad \Phi_E = \oint \boldsymbol{E} \cdot d\boldsymbol{A} = \dfrac{q}{\varepsilon_o}$ (4.14)

II. Gauss's law for magnetism $\qquad\qquad \Phi_M = \oint \boldsymbol{B} \cdot d\boldsymbol{A} = 0$ (8.99)

III. Ampère's law $\qquad\qquad \oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_o I_C + \mu_o \varepsilon_o \dfrac{d\Phi_E}{dt}$ (10.8)

IV. Faraday's law $\qquad\qquad \oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\dfrac{d\Phi_M}{dt}$ (10.14)

Maxwell's equations in differential form

I. Gauss's Law for electricity $\qquad\qquad \nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\varepsilon_o}$ (4.87)

II. Gauss's Law for magnetism $\qquad\qquad \nabla \cdot \boldsymbol{B} = 0$ (8.103)

III. Ampère's law $\qquad\qquad \nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J} + \mu_o \varepsilon_o \dfrac{d\boldsymbol{E}}{dt}$ (10.19)

IV. Faraday's law $\qquad\qquad \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$ (9.87)

Maxwell's equations in differential form for charge-free space

I. Gauss's law for electricity $\qquad\qquad \nabla \cdot \boldsymbol{E} = 0$ (10.27)

II. Gauss's law for magnetism $\qquad\qquad \nabla \cdot \boldsymbol{B} = 0$ (10.28)

III. Ampère's law $\qquad\qquad \nabla \times \boldsymbol{B} = \mu_o \varepsilon_o \dfrac{d\boldsymbol{E}}{dt}$ (10.29)

IV. Faraday's law $\qquad\qquad \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$ (10.30)

The wave equation in three-dimensions. The term $\nabla^2 \mathbf{E}$ is called the Laplacian of $\mathbf{E}$

$$\nabla^2 \boldsymbol{E} = \mu_o \varepsilon_o \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$ (10.37)

$$\nabla^2 \boldsymbol{E} = \frac{\partial^2 \boldsymbol{E}}{\partial x^2} + \frac{\partial^2 \boldsymbol{E}}{\partial y^2} + \frac{\partial^2 \boldsymbol{E}}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$ (10.39)

One-dimensional wave equation for the electric field E

$$\frac{\partial^2 E}{\partial x^2} = \mu_o \varepsilon_o \frac{\partial^2 E}{\partial t^2}$$ (10.41)

The wave equation for the mechanical problem of a vibrating string

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

(10.42)

Electric plane wave $\qquad E = E_0 \sin(kx - \omega t)$ (10.46)

Magnetic plane wave $\qquad B = B_0 \sin(kx - \omega t)$ (10.59)

Wave number $\qquad k = \frac{2\pi}{\lambda}$ (10.45)

Angular frequency $\qquad \omega = kc = 2\pi f$ (10.46)

Speed of light $\qquad c = \sqrt{\frac{1}{\mu_o \varepsilon_o}}$ (10.52)

Relation of electric field to magnetic field $\qquad E = cB$ (10.62)

Fundamental equation of wave propagation $\qquad c = \lambda v$ (10.63)

Electric energy density $\qquad u_E = \frac{1}{2}\varepsilon_0 E^2$ (6.57)

Magnetic energy density $\qquad u_M = \frac{1}{2}\frac{B^2}{\mu_0}$ (9.58)

Energy density of electromagnetic field

$$u = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2}\frac{B^2}{\mu_o}$$
(10.65)
$$u = \varepsilon_o E^2$$
(10.67)

Intensity of radiation

$$\text{Intensity} = \frac{\text{Total energy}}{(\text{area})(\text{time})}$$
(10.68)
$$\text{Intensity} = \varepsilon_0 c E^2$$
(10.69)
$$\text{Intensity} = \frac{EB}{\mu_o}$$
(10.70)

Questions for Chapter 10

1. If an electromagnetic wave has energy, should it also have momentum?
2. How does an antenna receive electromagnetic waves?
3. If a radio wave is 1 km long does the radio antenna have to be this long?

4. A student's automobile antenna was stolen from her car. She then took a metal coat hanger and placed it into the empty antenna mount. Would this work to operate her car radio?

5. How can you take a picture of people at night with an infrared camera?

6. Most people are concerned about receiving too many X rays, but are not concerned about receiving too much visible radiation. Since both radiations are electromagnetic waves, what is the difference?

8. You have no antenna for your FM radio. Will connecting a 1-m length of TV wire to the FM set act as an antenna?

9. There is growing concern that the earth may be losing its ozone layer. Why should this concern us?

10. What is the difference between a whip antenna and a loop antenna?

Problems for Chapter 10

1. A displacement current of 5.00 A exists in a parallel plate capacitor that has an area of 7.50 cm². Find the rate at which the electric field changes within the capacitor.

2. A potential of 100 V is placed across the plates of a parallel plate capacitor rated at 8.50 μF. If it took 0.800 s for this potential to be reached, and if the plates have an area of 25.0×10^{-3} m², find (a) the charge deposited on the plates, (b) the conduction current, (c) the displacement current, (d) the rate at which the electric field changed with time.

3. Show that the displacement current given by equation 10-5 can also be written as

$$I_D = C\frac{\Delta V}{\Delta t}$$

where C is the capacitance of the capacitor and $\Delta V/\Delta t$ is the rate of change of the voltage across the capacitor.

4. For a parallel plate capacitor of 6.00 μF, what should the value of $\Delta V/\Delta t$ be in order that the displacement current be 3.00 mA?

5. A parallel plate capacitor of 6.00 μF has its applied voltage across the plates changing at the rate of 10,000 V/s. What is its displacement current?

6. If the electric field between the plates of a circular parallel plate capacitor changes at the rate of 4.00×10^8 (V/m)/s, and if the radius of the capacitor is 10.0 cm, find the magnetic field at (a) $r = 10.0$ cm, (b) $r = 50.0$ cm, and (c) $r = 100$ cm.

7. Show that the magnetic field at the distance r from the center of a parallel plate capacitor, equation 10-10, can also be written as

$$B = \mu_0\frac{C\Delta V/\Delta t}{2\pi r}$$

where C is the capacitance of the capacitor and $\Delta V/\Delta t$ is the rate at which the voltage changes across the capacitor.

8. If the voltage that is applied to the parallel plates of a capacitor varies at the rate of 0.500 V/s, find the magnetic field at a distance of 20.0 cm from the center of a 5.00-µF capacitor.

9. An electric plane wave has a frequency of 90.0 MHz and an amplitude of 0.85 V/m. Write the equation for the electric wave and the magnetic wave.

10. A radar pulse is sent to the moon when the moon is at its mean distance from the earth, 3.84×10^8 m. How long does it take the pulse to get to the moon and be reflected back to earth?

11. How long does it take to transmit and receive a reflected signal from a satellite that is orbiting Mars when earth and Mars are aligned, a separation distance of 7.8×10^{10} m ?

12. A radar set picks up an aircraft in a time of 3.33×10^{-3} s. How far away is the aircraft?

13. What is the range of frequencies for visible light of wavelengths 380 nm to 720 nm?

14. What is the frequency of a 0.100-nm gamma ray?

15. What is the range of frequencies for infrared radiation lying between 720 nm and 50,000 nm?

16. A diathermy machine generates an electromagnetic wave of 6.00-m wavelength. What frequency does this correspond to?

17. An FM radio station broadcasts at 93.4 MHz. What wavelength is associated with this wave?

18. Channel 2 TV operates in a frequency range of 54 to 60 MHz. What range of wavelengths does this represent?

19. Approximately 60.0% of the solar radiation that impinges on the top of the atmosphere makes it to the surface of the earth. How much energy per square meter hits the surface in 8.00 hr?

20. Find the intensity of a 100-W incandescent light bulb at a distance of (a) 20.0 cm, (b) 40.0 cm, (c) 60.0 cm, (d) 80.0 cm, and (e) 100.0 cm from the source.

21. Using the results of problem 20, find the energy density at (a) 20.0 cm, (b) 40.0 cm, (c) 60.0 cm, (d) 80.0 cm, and (e) 100.0 cm.

22. Using the results of problems 20 and 21 find the average value of the electric field and the magnetic field at (a) 20.0 cm, (b) 40.0 cm, (c) 60.0 cm, (d) 80.0 cm, and (e) 100.0 cm.

23. Show that if the distance from the source doubles, the intensity of the radiation decreases by a fourth.

24. Find the average value of the electric and magnetic field a distance of 20.0 m from a 100-W incandescent lamp bulb.

25. What is the maximum intensity of an electromagnetic wave whose maximum electric field is 200 N/C?

26. A radio station transmits at 1000 W. Find the value of the electric field at a distance of 10.0 km.

27. Find the intensity associated with an electric wave that has a value of 63.0 V/m.

28. The displacement current in a parallel plate capacitor is 50.0 mA when the electric field changes at the rate of 16.0 V/(m s). Find the area of each plate.

29. If the intensity of a source of electromagnetic waves is 6.38 W/m² at a distance of 20.0 cm, find the power output of the source.

30. A radio station emits a power of 50,000 W. Assuming that this power is emitted uniformly in all directions, (a) what would be the power received at a radio antenna of 0.0900 m² area, 10.0 miles away? (b) What is the maximum value of the E field picked up by the radio?

31. A ray of light of 400-nm wavelength is traveling in air. It then enters a pool of water where its speed is reduced to 2.26×10^8 m/s. What is the wavelength of the light in the water?

32. The speed of light in a vacuum was given by equation 10-52. The speed of light in a medium of permittivity ε is also given by equation 10-52, but with ε_0 replaced by ε. Show that the index of refraction, which is defined as the ratio of the speed of light in vacuum to the speed of light in the medium is given by

$$n = \frac{c}{v} = \sqrt{\kappa}$$

where κ is the dielectric constant of the medium.

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