HW2. 证明：1. (1)

$x>0$ $\quad f(x)=e^x$ $\quad \exists\, c\in(0,x)$.

$$e^x = 1+x+\frac{1}{2}x^2+\frac{1}{6}f^{(3)}(c)\cdot x^3$$

$$f^{(3)}(x)=e^x \quad f^{(3)}(c)=e^c.$$

$$e^x = 1+x+\frac{1}{2}x^2+\boxed{\frac{1}{6}e^c\cdot x^3} > 0$$

等式

$e^c>0$, $x>0$

$$\therefore e^x > 1+x+\frac{1}{2}x^2$$

极限 ✗

$$e^x = 1+x+\frac{1}{2}x^2+\boxed{o(x^2)} > 0.$$

$$\geq 1+x+\frac{1}{2}x^2$$

$o(x^2)$ means $\lim\limits_{x\to 0}\frac{o(x^2)}{x^2}=0$

$o(x^2)$ can be x^3, x^4, $-x^3$, $-x^4$

$x<0$ $\quad c\in(x,0)$.

$$e^x = 1+x+\frac{1}{2}x^2+\boxed{\frac{1}{6}e^c x^3} < 0$$

$$e^c>0 \quad x<0 \Rightarrow x^3<0$$

$$e^x < 1+x+\frac{1}{2}x^2.$$

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(2) $x \in [0, \frac{\pi}{2})$, $\exists c \in (0, x)$ $f(x) = \sin x$

$$\sin x = x - \frac{1}{6}x^3 + \boxed{\frac{f^{(5)}(c)}{5!} \cdot x^5} > 0$$

$$f^{(5)}(x) = \cos x \qquad f^{(5)}(c) = \cos c$$

$x > 0$ $c \in (0, \frac{\pi}{2})$, $\underline{\cos c} \in (0,1)$.

$$\sin x \geq \boxed{x - \frac{1}{6}x^3} \geq x - x^3.$$

$$x - \frac{1}{6}x^3 \leq \sin x \leq x$$

$$\underline{x - \sin x \leq \frac{1}{6}x^3}$$

$x = 0.01$. $0.01 - \sin 0.01 \leq \frac{1}{6}(0.01)^3$.

$$\approx 1.67 \times 10^{-7}.$$

$$\sin 0.01 \in (\underline{0.01} - 1.67 \times 10^{-7}, 0.01).$$

$\sin\frac{1}{2}$

$$\sin x = \underbrace{\boxed{x - \frac{1}{6}x^3}}_{\text{估计值}} + \underbrace{\boxed{\frac{f^{(5)}(c)}{5!}x^5}}_{\text{error term}}$$

$$\left|\frac{f^{(5)}(c)}{5!}\cdot x^5\right| = \left|\frac{\cos c}{5!}\cdot x^5\right|$$

$|\cos c| \le 1$

$$\le \frac{1}{120}\cdot x^5$$

$$= \frac{1}{120}\times\frac{1}{2^5}$$

$$= \frac{1}{3840}$$

不够.

$$\frac{1}{100000}$$

$f^{(7)}(x) = -\cos x$

error term

估计值

$f^{(7)}$

$$\sin x = \boxed{x - \frac{1}{6}x^3 + \frac{1}{120}x^5} + \boxed{\frac{f^{(7)}(c)}{7!} \cdot x^7}$$

$$\text{Error term} = \left| \frac{-\cos c}{7!} \cdot \left(\frac{1}{2}\right)^7 \right|$$

$$\leq \frac{1}{7!} \cdot \frac{1}{2^7}$$

$$= \boxed{\frac{1}{645120}}$$

$$\text{估计值} = \frac{1}{2} - \frac{1}{6} \times \left(\frac{1}{2}\right)^3 + \frac{1}{120} \times \left(\frac{1}{2}\right)^5$$

$$= 0.47943.$$

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(2) $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \boxed{\frac{f^{(4)}(c)}{4!} \cdot x^4}$

$f^{(4)}(x) = e^x$

上界

$$\text{error term} = \frac{e^c}{4!} \cdot \left(\frac{1}{2}\right)^4.$$

$c \in (0, \frac{1}{2})$, $e^c \leq e^{\frac{1}{2}}$

$$\text{errorterm} \le \frac{e^{\frac{1}{2}}}{24} \times \frac{1}{2^4}$$

$$= 0.00429 \quad X$$

$x = \frac{1}{2},\ c \in (0, x)$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \boxed{\frac{e^c}{7!} \cdot x^7}$$

$$\text{error term} \le \frac{e^{\frac{1}{2}}}{5040} \cdot \left(\frac{1}{2}\right)^7$$

$$= \frac{e^{\frac{1}{2}}}{645120} < \frac{1}{10^5}$$

$f(x) = \ln(1+x)$, 求 $f(0.1)$

$$\ln(1+x) = \boxed{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4} + \boxed{\frac{f^{(5)}(c)}{5!}x^5}$$

估计值 Error

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$f^{(3)}(x) = \frac{2}{(1+x)^3}$ $\qquad c \in (0, 0.1)$.

$f^{(4)}(x) = \frac{-6}{(1+x)^4}$

$f^{(5)}(x) = \frac{24}{(1+x)^5}$ $\qquad \underbrace{f^{(5)}(c) = \frac{24}{(1+c)^5}}$ ≤ 24.

Error term $\leq \frac{24}{5!} \cdot (0.1)^5$

$= \frac{1}{500000}$

2.(1) $\frac{1-x}{1-\sqrt{x}} = 1+\sqrt{x}$

$$\boxed{1-x = 1-(\sqrt{x})^2 = (1-\sqrt{x})(1+\sqrt{x})}.$$

$$\int \frac{1-x}{1-\sqrt{x}}\,dx = \int (1+\sqrt{x})\,dx$$

$$= x + \int x^{\frac{1}{2}}\,dx$$

$$= x + \frac{2}{3}x^{\frac{3}{2}} + C$$

(2) $(1+\cos^2 x)\boxed{\sec^2 x}$ $\frac{1}{\cos^2 x}$

$$= \frac{1}{\cos^2 x} + \cos^2 x \cdot \frac{1}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} + 1$$

$$\int (1+\cos^2 x)\sec^2 x\,dx$$

$$= \int \left(\frac{1}{\cos^2 x} + 1\right) dx$$

$$= \tan x + x + C$$

(c): $\underline{\frac{1}{x^2(1+x^2)} = \frac{1}{x^2} - \frac{1}{1+x^2}}$

裂项.

$$\frac{1+x^2}{x^2(1+x^2)} - \frac{x^2}{x^2(1+x^2)}$$

$$\left(\frac{1}{x^2(x^2+a)} = \frac{1}{a}\left(\frac{1}{x^2} - \frac{1}{x^2+a}\right) \right)$$

x^{-2}

$$\int \left(\frac{1}{x^2} - \frac{1}{1+x^2}\right) dx$$

$$= -\frac{1}{x} - \arctan x + C$$

(d): $x + C$

3. (a) $F(x) = \int_0^x f(t)\,dt$

则 $F'(x) = f(x)$

$F(x) = \int_0^x \boxed{2xt}\,dt$ $\qquad F'(x) = 2x \cdot x = 2x^2$ ✗.

not $f(t)$

option.

$$= x\int_0^x 2t\,dt \; \boxed{= x\cdot x^2 = x^3}$$

$$F'(x) = 1\cdot\int_0^x 2t\,dt + x\cdot(2x)$$

$$= t^2\Big|_0^x + 2x^2$$

$$= x^2 + 2x^2 = 3x^2$$

(b)

$$F'(x) = (x^2)'\cdot(\sin x^2)$$

$$= 2x\sin x^2$$

(c). $F(x) = \underline{\int_x^{x^2} e^x\cdot e^t\,dt}$

$$= e^x\cdot\left(\int_0^{x^2} e^t\,dt - \int_0^x e^t\,dt\right)$$

$$F'(x) = (e^x)'\left(\int_0^{x^2} e^t\,dt - \int_0^x e^t\,dt\right).$$

$$+ e^x\cdot\left((x^2)'\cdot e^{x^2} - e^x\right)$$

$$= e^x\cdot\left((e^t)\Big|_0^{x^2} - (e^t)\Big|_0^x\right)$$

$$+ e^x\cdot\left(2x\cdot e^{x^2} - e^x\right)$$

$$= e^{x}\left(e^{x^2} - e^{0} - (e^{x} - e^{0}) + 2x e^{x^2} - e^{x} \right)$$

$$= e^{x}\left((2x+1) e^{x^2} - 2e^{x} \right).$$

$$F(x) = e^{x} \int_{x}^{x^2} e^{t}\,dt$$

$$= e^{x}\left(e^{t} \Big|_{x}^{x^2} \right)$$

$$= e^{x} \cdot \left(e^{x^2} - e^{x} \right).$$

$$= e^{x^2+x} - e^{2x}.$$

$$F'(x) = (x^2+x)' \cdot e^{x^2+x} - 2 \cdot e^{2x}$$

$$= (2x+1) e^{x^2+x} - 2e^{2x}.$$

4.

(a) $$\lim_{x \to 0} \frac{\int_0^x \sin t^2 \, dt}{x^3} \quad \to 0, \quad \to 0$$

$$\underline{\underline{\text{L'Hosp}}} \quad \lim_{x \to 0} \frac{\sin x^2}{3x^2}$$

$$= \frac{1}{3}$$

(b) $$\lim_{x \to 0} \frac{\int_0^{x^2} t \sin t \, dt}{x^6}$$

$$\underline{\underline{\text{L'Hosp}}} \quad \lim_{x \to 0} \frac{(x^2)' \cdot (x^2 \cdot \sin x^2)}{6x^5}$$

$$= \lim_{x \to 0} \frac{2x \cdot x^2 \cdot \sin x^2}{6x^5}$$

$$= \frac{1}{3}$$

(C) $$\lim_{x \to 0} \frac{\int_0^x \sqrt{1+t^4}\,dt - x}{\sin x - x + \frac{1}{6}x^3}$$

$(1+x^4)^{\frac{1}{2}}$

$$\overset{\text{L'Hosp}}{=} \lim_{x \to 0} \frac{\sqrt{1+x^4} - 1}{\cos x - 1 + \frac{1}{2}x^2}$$

$(1+x)^{\alpha}$

$(1+x^4)^{\frac{1}{2}}$

$$\overset{\text{Taylor}}{=} \lim_{x \to 0} \frac{\underline{1 + \frac{1}{2}x^4 + o(x^4)} - 1.}{\underbrace{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)}_{\cos x} - 1 + \frac{1}{2}x^2}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2}x^4 + o(x^4)}{\frac{1}{24}x^4 + o(x^4)}$$

同除x^4.

$$= \lim_{x \to 0} \frac{\frac{1}{2} + o(1)}{\frac{1}{24} + o(1)}$$

$$= \frac{\frac{1}{2}}{\frac{1}{24}} = 12$$

5.

$$\lim_{x \to 0} \frac{\int_0^x f(t)\,dt}{x^2}$$

$$\overset{\text{L'Hosp}}{=} \lim_{x \to 0} \frac{f(x)}{2x}$$

$$= \frac{1}{2} \boxed{\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}} \checkmark$$

$$= \frac{1}{2} f'(0).$$

$$\lim_{x \to 0} \frac{f(x)}{2x}$$

$$\underline{\underline{\text{L'Hosp}}} \lim_{x \to 0} \frac{f'(x)}{2}$$

不好!

$$= \frac{f'(0)}{2}$$

默认了 $f'(x)$ 连续.

证明题:

1. $\forall x_1, x_2 \in I$, $H(x_1) = H(x_2)$

$\Leftrightarrow$ $H(x)$在I上恒为常数

eg: $H(x)=0$, $H(x)=1$, ⋯ —

回顾:

$\boxed{f'(x)=0} \Rightarrow f(x)$是常数.

拉格朗日中值

证明:

$\forall x_1 < x_2 \in I$, $\exists c \in (x_1, x_2)$, s.t.

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

条件

已知 $\forall x,\ f'(x)=0$ 条件.

$\Rightarrow f'(c)=0$

$\Rightarrow \boxed{f(x_1)=f(x_2)}$.

$\Rightarrow f(x)$ 是常数.

回到原题 F, G 都是 f 的原函数

$F'(x)=f(x)$

$G'(x)=f(x)$

$$H(x)=F(x)-G(x)$$

$\Rightarrow H'(x)=f(x)-f(x)=0,\ \forall x\in I$

$\Rightarrow H(x)$ 是常数.

2. 要证什么？

$F(x)+G(x)+C$ 的导数是

$f(x)+g(x)$

$$(F(x)+G(x)+C)' = f(x)+g(x)+0$$

$$= f(x)+g(x)$$

$y=f(x)$

$$\frac{dy}{dx}=f'(x)$$

$dy=f'(x)\,dx$

$y=x^2 \quad dy=2x\,dx$

第一换元：$\longleftrightarrow$ 复合函数求导

例1：$(e^{x^2})'=(x^2)'\cdot e^{x^2}$

$y=e^{x^2} \quad u=x^2$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=e^u\cdot(x^2)'$$
$$=e^{x^2}\cdot(x^2)'$$

$dy=e^{x^2}\cdot 2x\,dx$ 找 u

$$\int 2x\, e^{x^2}\, dx \qquad du = 2x\, dx$$

$$\underline{\underline{u=x^2}}\ \int e^u\, du$$

$$= e^u + C = e^{x^2} + C$$

特征：含有一些"复合"的结构

$$\int \frac{x}{1+x^2}\, dx$$

拆成

$u = f(x)$

$\to du$

$$\int \sim\sim \boxed{\_\_\ dx}$$

$$\frac{1}{2}\int \frac{1}{1+x^2} \cdot 2x\, dx$$

du, $u = x^2$

$$\frac{1}{2}\int \frac{1}{1+u}\,du$$

$$= \frac{1}{2}\ln(1+u).$$

$$\int \underline{(y+1)}^2\,dy$$

$u = y+1$

$du = dy$

$$\int u^2\,du = \frac{1}{3}u^3 = \frac{1}{3}(y+1)^3$$

第二: $x = \underline{f(t)}$ $\quad u = f(x)$

$dx = f'(t)\,dt$ $\quad du = f'(x)\,dx$

$$\int g(x)\,dx = \int h(t)\,f'(t)\,dt$$

$$\int \sqrt{1-x^2}\, dx \qquad x = \cos\theta$$

$$dx = -\sin\theta\, d\theta$$

$$\int \sqrt{1-\cos^2\theta} \cdot (-\sin\theta)\, d\theta$$

$$= \int \sin\theta\,(-\sin\theta)\, d\theta$$

$$= \int -\sin^2\theta \quad d\theta$$

$$= -\int \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \cdot - -$$