3月13日. 周四. 第7次课.

1. 签到. 2. 提交作业截止时间(deadline): 周六晚上 24:00.

关于作业: ① 交了就有分! 证明题或别的题目不会写的话在作业里说明一下就可以!

② . 作业文件命名: "第x次作业-学号-姓名".

邮件命名也一样. 方便孙老师集中批改.

③ . 不会写作业的话请尽量参加习题课. 课上会讲.

④ . 之前的作业提交情况: HW1 ×17 ; HW2 ×14.

Topic 1: 换元积分法: 第二换元法 (代入法, inverse substitution)

[中文课本: $P_{202} \sim 205$; 英文课本: $P486 \sim 491$)

(1). 第二换元法的内容:

对 $\int f(x)\,dx$, 令 $x = g(t)$. (g 可逆). 则 $dx = g'(t) \cdot dt$.

$\therefore \int f(x)\,dx = \int f(g(t)) \cdot g'(t) \cdot dt$.

右边计算出来后是关于 t 的函数. 再将 $t = g^{-1}(x)$ 代入, 即可得 $\int f(x)\,dx$.

留作习题.

证明方法: 令 $G(t) = \int f(g(t)) \cdot g'(t)\,dt$. 证明 $\frac{d}{dx} G(g^{-1}(x)) = f(x)$ 即可.

(2). 第二换元法的使用:

① . 三角换元消除根号.

第二积分法通常用在含根号($\sqrt{\ }$)的式子, 利用三角函数的等式消除根号, 这种积分方式也称为三角换元. (Trigonometric Substitution).

公式如下表:

$x\in[-a,a]$ $\quad =\sqrt{a^2-a^2\sin^2\theta}=\sqrt{a^2\cdot\cos^2\theta}=|a\cdot\cos\theta|=a\cdot\cos\theta$

① $\sqrt{a^2-x^2}$: $x=a\cdot\sin\theta$, $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$. 利用公式 $1-\sin^2\theta=\cos^2\theta$.

$x\in R$

② $\sqrt{a^2+x^2}$: $x=a\cdot\tan\theta$, $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$. 利用公式 $1+\tan^2\theta=\frac{1}{\cos^2\theta}$

③ $\sqrt{x^2-a^2}$: $x=\frac{a}{\cos\theta}$. $\theta\in[0,\frac{\pi}{2})\cup[\pi,\frac{3}{2}\pi)$. 利用公式 $\frac{1}{\cos^2\theta}-1=\tan^2\theta$.

取值范围的意义: ①.覆盖x的取值范围 & 可逆

②使平方根能不带符号地算出.

② $\sqrt{a^2+x^2}$: $x\in R$. 取 $x=a\tan\theta$, $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$.

$$\sqrt{a^2+x^2}=\sqrt{a^2(1+\tan^2\theta)}=\sqrt{a^2\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)}=\sqrt{a^2\cdot\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}=\sqrt{\frac{a^2}{\cos^2\theta}}$$

$$=\left|\frac{a}{\cos\theta}\right|=\frac{a}{\cos\theta}\quad\left(\theta\in(-\frac{\pi}{2},\frac{\pi}{2})\right)$$

③ $\sqrt{x^2-a^2}$: $x\in(-\infty,-a]\cup[a,+\infty)$. 取 $x=\frac{a}{\cos\theta}$. $\theta\in[0,\frac{\pi}{2})\cup[\pi,\frac{3}{2}\pi)$.

$$\sqrt{x^2-a^2}=\sqrt{\frac{a^2}{\cos^2\theta}-a^2}=\sqrt{a^2\left(\frac{1}{\cos^2\theta}-1\right)}=\sqrt{a^2\cdot\frac{\sin^2\theta}{\cos^2\theta}}=a\cdot\tan\theta.$$

eg. $\int\sqrt{a^2-x^2}\cdot dx$ $(a>0)$.

$$\overset{x=a\sin\theta}{=\!=}\int a\cdot\cos\theta\cdot a\cdot\cos\theta\cdot d\theta=a^2\cdot\int\cos^2\theta\, d\theta.\quad(\cos2\theta=2\cos^2\theta-1)$$

$$=a^2\cdot\int\frac{\cos2\theta+1}{2}\cdot d\theta=a^2\cdot\left[\underline{\int\frac{\cos2\theta}{2}d\theta}+\underline{\int\frac{1}{2}\cdot d\theta}\right]$$

$$\int\frac{\cos2\theta}{2}d\theta\overset{u=2\theta}{=\!=}\int\frac{\cos u\cdot du}{4}=\frac{1}{4}\sin u+C=\frac{1}{4}\sin2\theta+C.\qquad \int\frac{1}{2}\cdot d\theta=\frac{\theta}{2}+C$$

$$\therefore\int\sqrt{a^2-x^2}\,dx=a^2\cdot\left[\frac{1}{4}\cdot\sin2\theta+\frac{\theta}{2}\right]+C$$

$$\overset{\theta=\arcsin\frac{x}{a}}{=\!=}a^2\left[\underline{\frac{1}{4}\cdot\sin\left(2\cdot\arcsin\frac{x}{a}\right)}+\frac{1}{2}\arcsin\frac{x}{a}\right]+C$$

$$\parallel\quad(\sin2\theta=2\sin\theta\cdot\cos\theta).$$

$$\frac{1}{2}\cdot\sin\left(\arcsin\frac{x}{a}\right)\cdot\cos\left(\arcsin\frac{x}{a}\right)=\frac{1}{2}\cdot\frac{x}{a}\cdot\sqrt{1-\left(\frac{x}{a}\right)^2}.$$

$$=\frac{1}{2}\cdot x\cdot\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot\arcsin\frac{x}{a}+C.$$

e.g. $\int \sqrt{a^2+x^2}\,dx.\ (a>0).$

$$\overset{x=a\cdot\tan\theta}{=\!=}\int \frac{a}{\cos\theta}\cdot\frac{a}{\cos^2\theta}d\theta = a^2\int\frac{d\theta}{\cos^3\theta} = a^2\cdot\int\frac{\cos\theta\cdot d\theta}{\cos^4\theta}$$

$$= a^2\int\frac{d(\sin\theta)}{(1-\sin^2\theta)^2} \overset{u=\sin\theta}{=\!=} a^2\cdot\int\frac{du}{(1-u^2)^2}$$

$\frac{1}{1-u^2} = \frac{1}{(1-u)(1+u)}$

$= \frac{1}{2}\left(\frac{1}{1-u}+\frac{1}{1+u}\right)$

$$= a^2\int\frac{1}{4}\cdot\left(\frac{1}{1-u}+\frac{1}{1+u}\right)^2\cdot du$$

$$= a^2\cdot\int\left[\frac{1}{4}\cdot\frac{1}{(1-u)^2}+\frac{1}{2}\cdot\frac{1}{1-u}\cdot\frac{1}{1+u}+\frac{1}{4}\cdot\frac{1}{(1+u)^2}\right]du.$$

$$= \frac{a^2}{4}\cdot\underbrace{\int\frac{du}{(1-u)^2}}_{\overset{t=1-u}{=}\ \frac{1}{1-u}} + \frac{a^2}{2}\underbrace{\int\frac{du}{1-u^2}}_{=\int\frac{1}{2}\left(\frac{1}{1-u}+\frac{1}{1+u}\right)du} + \frac{a^2}{4}\cdot\underbrace{\int\frac{du}{(1+u)^2}}_{\overset{t=1+u}{=}\ \frac{-1}{1+u}.}$$

$= -\frac{1}{2}\ln|u-1| + \frac{1}{2}\ln|1+u|$

$$= \frac{a^2}{4}\left[\frac{1}{1-u}-\frac{1}{1+u}+\ln\left|\frac{1+u}{1-u}\right|\right]$$

$$= \frac{a^2}{4}\cdot\left(\frac{1}{1-\sin\theta}-\frac{1}{1+\sin\theta}+\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|\right).$$

$$= \frac{a^2}{4}\cdot\left(\frac{2\sin\theta}{\cos^2\theta}+\ln\cdot\left|\frac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta}\right|\right).$$

$$= \frac{a^2}{4}\cdot\left(2\frac{x}{a}\cdot\sqrt{\left(\frac{x}{a}\right)^2+1}+\ln\cdot\left|2\left(\frac{x}{a}\right)^2+1+2\frac{x}{a}\cdot\sqrt{\left(\frac{x}{a}\right)^2+1}\right|\right)+C$$

$$= \frac{x}{2}\cdot\sqrt{x^2+a^2}+\frac{a^2}{4}\cdot\ln\cdot\left|\underbrace{2x^2+a^2+2x\cdot\sqrt{x^2+a^2}}_{=(x+\sqrt{x^2+a^2})^2}\right|+C'$$

$$= \frac{x}{2}\cdot\sqrt{x^2+a^2}+\frac{a^2}{2}\cdot\ln(x+\sqrt{x^2+a^2})+C'.$$

e.g. $\int \frac{1}{\sqrt{a^2+x^2}}\,dx$. $(a>0)$.

$\underline{\underline{x=a\cdot\tan\theta}}$ $\int \frac{1}{\frac{a}{\cos\theta}}\cdot\frac{a}{\cos^2\theta}\cdot d\theta = \int\frac{d\theta}{\cos\theta} = \int\frac{\cos\theta\cdot d\theta}{\cos^2\theta}$

$= \int\frac{d(\sin\theta)}{1-\sin^2\theta} \overset{t=\sin\theta}{=\!=} \int\frac{dt}{1-t^2} = \int\cdot\frac{1}{2}\cdot\left(\frac{1}{1-t}+\frac{1}{1+t}\right)\cdot dt$

$= -\frac{1}{2}\cdot\ln|1-t| + \frac{1}{2}\cdot\ln|1+t| + C = \frac{1}{2}\cdot\ln\left|\frac{1+t}{1-t}\right| + C = \frac{1}{2}\cdot\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + C$

$= \frac{1}{2}\ln\left|\frac{(1+\sin\theta)^2}{1-\sin^2\theta}\right| + C = \frac{1}{2}\cdot\ln\left|\frac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta}\right| + C$

$= \frac{1}{2}\cdot\ln\left|\frac{\sin^2\theta+\cos^2\theta+2\sin\theta+\sin^2\theta}{\cos^2\theta}\right| + C = \frac{1}{2}\cdot\ln\left|2\cdot\tan^2\theta+1+2\tan\theta\frac{1}{\cos\theta}\right| + C$

$= \frac{1}{2}\cdot\ln\left|2\cdot\tan^2\theta+1+2\cdot\tan\theta\sqrt{\tan^2\theta+1}\right| + C$

$= \frac{1}{2}\cdot\ln\left|2\cdot\frac{x^2}{a^2}+1+2\cdot\frac{x}{a}\cdot\sqrt{\frac{x^2}{a^2}+1}\right| + C$

$= \frac{1}{2}\cdot\ln\left|\frac{1}{a^2}\cdot\left(2x^2+a^2+2x\sqrt{x^2+a^2}\right)\right| + C$

$= \frac{1}{2}\cdot\ln\left|\frac{1}{a^2}\cdot\left(x+\sqrt{x^2+a^2}\right)^2\right| + C.$

$= \ln\left|x+\sqrt{x^2+a^2}\right|. \quad + \underbrace{C-\frac{1}{2}\ln a^2}$

$= \ln\left(x+\sqrt{x^2+a^2}\right) + C'.$

e.g. $\int\frac{\sqrt{9-x^2}}{x^2}\cdot dx \overset{x=3\cdot\sin\theta.}{=\!=} \int\frac{3\cdot\cos\theta}{9\cdot\sin^2\theta}\cdot 3\cdot\cos\theta\, d\theta = \int\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$

$= \int\frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta = \int\frac{d\theta}{\sin^2\theta} - \int 1\cdot d\theta = -\frac{\cos\theta}{\sin\theta} - \theta + C$

$= -\frac{\sqrt{1-(x/3)^2}}{x/3} - \arcsin\frac{x}{3} + C = -\frac{\sqrt{9-x^2}}{x} - \arcsin\frac{x}{3} + C.$

e.g. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$. $\underline{\underline{x=\frac{a}{\cos\theta}}}$ $\int \frac{1}{a\tan\theta}\cdot a\cdot\frac{\sin\theta}{\cos^2\theta}\cdot d\theta = \int \frac{1}{\cos\theta}\cdot d\theta$

$\underline{\underline{\text{用上面的结论}}}$ $\frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + C$.

$\cos\theta = \frac{a}{x}$. $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{a^2}{x^2}}$

$$= \frac{1}{2}\cdot\ln\left|\frac{1+\sqrt{1-\frac{a^2}{x^2}}}{1-\sqrt{1-a^2/x^2}}\right| + C$$

$$= \frac{1}{2}\cdot\ln\left|\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}\right| + C$$

$$= \frac{1}{2}\cdot\ln\left|\frac{(x+\sqrt{x^2-a^2})^2}{x^2-(x^2-a^2)}\right| + C$$

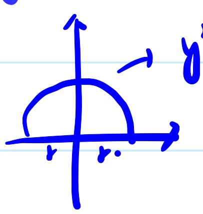
$$= \ln\left|x+\sqrt{x^2-a^2}\right| + \underbrace{C-\frac{1}{2}\ln a^2}$$

$$= \ln\left|x+\sqrt{x^2-a^2}\right| + C'.$$

note: $\int\frac{1}{\sqrt{x^2\pm a^2}} = \ln\left|x+\sqrt{x^2\pm a^2}\right| + C$.

比较推荐大家记住这个式子.

e.g. 求证半径为r的圆面积为$\pi\cdot r^2$.



$y^2+x^2=r^2\ (y\geq 0)$

$y=\sqrt{r^2-x^2}$.

$S_{\text{半圆}} = \int_{-r}^{r}\sqrt{r^2-x^2}\cdot dx$.

取 $F(x) = \frac{x}{2}\sqrt{r^2-x^2} + \frac{r^2}{2}\cdot\arcsin\frac{x}{r}$.

由上方例子可知, $F(x)$ 是 $\sqrt{r^2-x^2}$ 的一个原函数

$\therefore S_{\text{半圆}} = F(r)-F(-r) = (0+\frac{r^2}{2}\cdot\arcsin 1) - (0+\frac{r^2}{2}\cdot\arcsin(-1))$

$= \frac{r^2}{2}\cdot\frac{\pi}{2} - \frac{r^2}{2}\cdot(-\frac{\pi}{2}) = \frac{\pi}{2}r^2$

e.g. $\int \frac{dx}{x^2\sqrt{x^2+4}} \xlongequal[]{x=2\tan\theta} \int \frac{1}{4\tan^2\theta \cdot 2\frac{1}{\cos\theta}} \cdot \frac{2}{\cos^2\theta} \cdot d\theta = \int \frac{\cos\theta \, d\theta}{4\sin^2\theta}$

$= \int \frac{d(\sin\theta)}{4\sin^2\theta} = -\frac{1}{4\sin\theta} = -\frac{1}{4}\cdot\sqrt{\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta}}$

$= -\frac{1}{4}\sqrt{1+\frac{1}{\tan^2\theta}} = -\frac{1}{4}\cdot\sqrt{1+\frac{1}{(x/2)^2}} = -\frac{1}{4}\cdot\sqrt{\frac{x^2+4}{x^2}}$

e.g. $\int \frac{x}{\sqrt{3-2x-x^2}} \cdot dx.$

对根号下的式子凑平方. $3-2x-x^2 = -(x+1)^2+4$

$= \int \frac{x}{\sqrt{4-(x+1)^2}} dx \xlongequal[]{x+1=2\sin\theta} \int \frac{2\sin\theta-1}{2\cdot\cos\theta} \cdot 2\cos\theta \cdot d\theta$

$= \int (2\sin\theta-1) \cdot d\theta = -2\cos\theta - \theta = -2\cdot\sqrt{1-\sin^2\theta} - \theta$

$= -2\cdot\sqrt{1-\left(\frac{x+1}{2}\right)^2} - \arcsin\frac{x+1}{2} = -\sqrt{4-(x+1)^2} - \arcsin\frac{x+1}{2}$