3月17日、周一、第8次课.

1. 签到.　　2. 作业 HW4截止时间: 周日晚上24:00.

Topic 1: 简单三角函数的积分. [英文课本 7.2 Trigonometric Integrals, P479~P484]

(1). $\int \sin^m x \cdot \cos^n x\, dx\ (m,n\in Z)$ 类型:

e.g. $\int \cos^3 x \cdot dx = \int \cos^2 x \cdot \cos x \cdot dx = \int \cos^2 x \cdot d(\sin x)$

$= \int (1-\sin^2 x)\cdot d(\sin x) \xlongequal{t=\sin x} \int (1-t^2)\cdot dt$

$= t - \frac{1}{3}t^3 + C = \sin x - \frac{1}{3}\cdot \sin^3 x + C$

我们主要用到的三角函数公式:

$\cos^2 x + \sin^2 x = 1$; $\cos x \cdot dx = d(\sin x)$; $\sin x \cdot dx = -d\cos x$.

e.g. $\sin^5 x \cdot \cos^2 x \cdot dx = \int \sin^4 x \cdot \cos^2 x \cdot \sin x\, dx = \int \sin^4 \cdot \cos^2 x \cdot (-d\cos x)$

$= -\int (1-\cos^2 x)^2 \cdot \cos^2 x \cdot d(\cos x) \xlongequal{t=\cos x} -\int (1-t^2)^2 t^2 \cdot dt$

$= -\int (t^2 - 2t^4 + t^6) dt = -(\frac{1}{3}t^3 - \frac{2}{5}t^5 + \frac{1}{7}t^7 + C) = -\frac{1}{3}\cdot\cos^3 x + \frac{2}{5}\cdot\cos^5 x - \frac{1}{7}\cos^7 x + C'$

e.g. $\int \sin x \cdot \cos x\, dx = \int \sin x \cdot d(\sin x) = \frac{1}{2}\sin^2 x + C$

$= \int \frac{1}{2}\cdot \sin 2x \cdot dx = \frac{1}{4}\cdot \int \sin 2x \cdot d(2x) = \frac{1}{4}(-\cos 2x) + C = -\frac{\cos 2x}{4} + C$

e.g. $\int \frac{1}{\sin x} dx = \int \frac{1}{\sin^2 x}\cdot \sin x\, dx = \int \frac{1}{1-\cos^2 x}(-d\cos x)$

$\xlongequal{t=\cos x} -\int \frac{1}{1-t^2}\cdot dt = -\int \frac{1}{2}\cdot(\frac{1}{1-t} + \frac{1}{1+t}) dt$

$= -\frac{1}{2}\cdot\int \frac{dt}{1-t} - \frac{1}{2}\cdot\int \frac{dt}{1+t} = -\frac{1}{2}[-\ln|1-t|] - \frac{1}{2}\cdot \ln|1+t| + C$

$= \frac{1}{2}\cdot\ln\left|\frac{1-t}{1+t}\right| + C = \frac{1}{2}\cdot \ln\left|\frac{1-\cos x}{1+\cos x}\right| + C$. $(= \ln|\tan\frac{x}{2}| + C)$

以上出现的例子中, m、n 中至少一个为奇数. 这时可以从奇数的那一项中抽一次进行换元, 剩下的部分利用 $\sin^2 x + \cos^2 x = 1$ 进行转化.

e.g. $\int \sin^3 x \cdot \cos^4 x\, dx = \int \sin^2 x \cdot \cos^4 x \cdot (\sin x\, dx)$

奇数.　　→ 只剩下 cosx.

$= \int (1-\cos^2 x)\cdot \cos^4 x \cdot (-d\cdot\cos x)$

$\xlongequal{t=\cos x} \int (1-t^2)\cdot t^4 (-dt)$

如果都是偶数，如何换元？需要借用一些三角函数公式：

二倍角公式 $\begin{cases} \cos 2x = \cos^2 x - \sin^2 x \\ \sin 2x = 2\sin x \cdot \cos x \end{cases} \Rightarrow \begin{aligned} &\sin x \cdot \cos x = \frac{1}{2}\sin 2x. \\ &\cos^2 x = \frac{1+\cos 2x}{2} \\ &\sin^2 x = \frac{1-\cos 2x}{2} \end{aligned}$

通过这些公式，可以将 sin 与 cos 的次数降低。(但 x 会变成 2x)。

可以一直降低它们的次数，直到出现奇数。

之后就可以用之前的换元方法了。

e.g. $\int \sin^4 x \cdot dx = \int (\sin^2 x)^2 \cdot dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 \cdot dx$

$= \int \frac{1}{4}(1 - 2\cdot\cos 2x + \cos^2 2x)\, dx = \int \frac{dx}{4} - \frac{1}{2}\cdot\int \cos 2x\, dx + \frac{1}{4}\cdot\int \cos^2 2x\, dx$

$= \frac{x}{4} - \frac{1}{2}\cdot\int \cos 2x \cdot \frac{d(2x)}{2} + \frac{1}{4}\cdot\int \frac{1+\cos 4x}{2}\cdot dx$

$= \frac{x}{4} - \frac{1}{4}\cdot\sin 2x + \frac{1}{4}\cdot\int \frac{dx}{2} + \frac{1}{8}\cdot\int \cos 4x \cdot \frac{d(4x)}{4}$

$= \frac{x}{4} - \frac{1}{4}\cdot\sin 2x + \frac{x}{8} + \frac{1}{32}\cdot\sin 4x = \frac{3}{8}x - \frac{1}{4}\cdot\sin 2x + \frac{1}{32}\cdot\sin 4x$

e.g. $\int \sin^2 x \cdot \cos^2 x\, dx$

(way 1): $= \int (\sin x \cos x)^2\, dx = \int \left(\frac{1}{2}\cdot\sin 2x\right)^2 dx = \frac{1}{4}\cdot\int \sin^2 2x\, dx.$

$= \frac{1}{4}\cdot\int \frac{1-\cos 4x}{2}\, dx = \frac{1}{4}\int \frac{dx}{2} - \frac{1}{8}\cdot\int \cos 4x\, \frac{d(4x)}{4} = \frac{x}{8} - \frac{1}{32}\cdot\sin 4x + C$

(way 2): $= \int \frac{1-\cos 2x}{2}\cdot\frac{1+\cos 2x}{2}\cdot dx = \frac{1}{4}\cdot\int (1-\cos^2 2x)\, dx$

$= \frac{1}{4}\cdot\int \left(1 - \frac{1+\cos 4x}{2}\right) dx = \frac{x}{8} - \frac{1}{32}\cdot\sin 4x + C$

Do it yourself! $\int \sin^3 x \cdot \cos^5 x \cdot dx = -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + C$; $\int \frac{\cos^2 x}{\sin x}\cdot dx = \cos x + \ln\left|\tan\frac{x}{2}\right| + C$

$\int \sin^2 x \cdot \cos^4 x \cdot dx = \frac{x}{16} - \frac{\sin 4x}{32} + \frac{\sin^3 2x}{24} + C$; $\int \cos^4 x \cdot dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

(2). $\begin{cases} \sin mx \cdot \cos nx \\ \sin mx \cdot \sin nx \\ \cos mx \cdot \cos nx \end{cases}$ $(m,n \in R)$ 类型:
$m \neq n$

使用下列三角函数公式,将乘积化为加减.

$\sin A \cdot \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$.

$\sin A \cdot \sin B = \frac{1}{2} \cdot [\cos(A-B) - \cos(A+B)]$

$\cos A \cdot \cos B = \frac{1}{2} \cdot [\cos(A-B) + \cos(A+B)]$

e.g. $\int \sin 4x \cdot \cos 5x \cdot dx = \int \frac{1}{2}[\sin 9x + \underbrace{\sin(-x)}_{=-\sin x}] dx$

$= \frac{1}{2} \cdot \int \sin 9x \frac{d(9x)}{9} - \frac{1}{2} \cdot \int \sin x \cdot dx.$

$= \frac{1}{18}(-\cos 9x) - \frac{1}{2}(-\cos x) + C = -\frac{1}{18}\cos 9x + \frac{1}{2}\cos x + C$

e.g. $\int \sin 6x \cdot \cos \sqrt{2}x \, dx = \int \frac{1}{2} \cdot [\sin(6+\sqrt{2})x + \sin(6-\sqrt{2})x] dx$

$= \frac{1}{2} \cdot \int \sin(6+\sqrt{2})x \cdot \frac{d(6+\sqrt{2})x}{6+\sqrt{2}} + \frac{1}{2} \cdot \int \sin(6-\sqrt{2})x \cdot \frac{d(6-\sqrt{2})x}{6-\sqrt{2}}$

$= \frac{1}{2} \cdot \frac{-\cos(6+\sqrt{2})x}{6+\sqrt{2}} + \frac{1}{2} \frac{-\cos(6-\sqrt{2})x}{6-\sqrt{2}}.$

Do it yourself! $\int \sin 3x \cdot \cos x \, dx = -\frac{1}{8} \cdot \cos 4x - \frac{1}{4} \cdot \cos 2x + C$

$\int \sin \pi x \cdot \cos(-3x) \cdot dx = -\frac{\cos(\pi-3)x}{2(\pi-3)} - \frac{\cos(\pi+3)x}{2(\pi+3)} + C$

(3). $\frac{\tan^m x}{\cos^n x}$ $(m,n\in N)$ 类型

主要用到的三角函数公式：

$\frac{d\tan x}{dx}=\frac{1}{\cos^2 x}$; $\frac{d(\frac{1}{\cos x})}{dx}=\frac{\tan x}{\cos x}$. $\frac{1}{\cos^2 x}=1+\tan^2 x$;

e.g. $\int \frac{\tan^6 x}{\cos^4 x}\cdot dx = \int \tan^6 x \cdot \frac{1}{\cos^2 x}\cdot\frac{dx}{\cos^2 x}$

$= \int \tan^6 x\,(1+\tan^2 x)\,d(\tan x)$

$\xlongequal{t=\tan x} \int t^6\cdot(1+t^2)\cdot dt = \int (t^6+t^8)\,dt$

$= \frac{1}{7}t^7+\frac{1}{9}t^9+C = \frac{1}{7}\tan^7 x+\frac{1}{9}\tan^9 x + C$

e.g. $\int \frac{\tan^5\theta}{\cos^7\theta}\cdot d\theta = \int \frac{\tan^4\theta}{\cos^6\theta}\cdot\left(\frac{\tan\theta}{\cos\theta}\,d\theta\right)$

$= \int \frac{(\tan^2\theta)^2}{\cos^6\theta}\,d\frac{1}{\cos\theta}$

$= \int \frac{(\frac{1}{\cos^2\theta}-1)^2}{\cos^6\theta}\cdot d\frac{1}{\cos\theta}$

$\xlongequal{t=\frac{1}{\cos\theta}} \int (t^2-1)^2\cdot t^6\cdot dt$

$= \int (t^{10}-2t^8+t^6)\cdot dt$

$= \frac{1}{11}t^{11}-\frac{2}{9}\cdot t^9+\frac{1}{7}\cdot t^7+C$

$= \frac{1}{11}\cdot\frac{1}{\cos^{11}\theta}-\frac{2}{9}\cdot\frac{1}{\cos^9\theta}+\frac{1}{7}\frac{1}{\cos^7\theta}+C.$