4月7日. 周一. 第14次课

Topic 1: 三角函数有理式的不定积分.(续).

$\boxed{\text{Try it!}}$ $\int \tan^8 x\, dx = \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} - \tan x + x + C$

$\int \frac{1}{1+\sin^2 x} dx = \frac{\sqrt{2}}{2} \arctan(\sqrt{2}\cdot\tan x) + C$

Topic 2: 某些特殊无理式的不定积分:

[中文课本: 5.5. 无理函数的积分. P219 ~ P220 ;

Calculus: P500 : Rationalizing Substitution]

一般来说, 无理式 (带根号的式子) 都需要通过换元消除根号再进行积分.

如果根号下出现 x^2, 如 $\sqrt{x^2+a^2}$, $\sqrt{x^2-a^2}$, $\sqrt{a^2-x^2}$,

可以用之前讲的三角换元;

如果根号下是关于 x 的一次函数或分式. e.g. $\sqrt[n]{g(x)}$.

可以尝试把根号整体作为新的变量. ($t=\sqrt[n]{g(x)}$).

(20 min)

(1). 只含有 $\sqrt[n]{ax+b}$. $(a\neq 0)$ 的式子

令 $t=\sqrt[n]{ax+b}$, 则 $t^n=ax+b$, $x=\frac{t^n-b}{a}$, $dx=\frac{n}{a}\cdot t^{n-1}\cdot dt$.

可以转化成关于 t 的有理函数的积分.

e.g.1. $\int\frac{\sqrt{1+x}}{x}dx$.

令 $t=\sqrt{1+x}$. $x\in[-1,0)\cup(0,+\infty)$. $t\in[0,1)\cup(1,+\infty)$.

则 $x=t^2-1$. $dx=2t\cdot dt$

$$\int\frac{\sqrt{1+x}}{x}dx \xlongequal[t\in[0,1)\cup(1,+\infty)]{x=t^2-1} \int\frac{t}{t^2-1}\cdot 2t\,dt=\int\frac{2t^2}{t^2-1}dt=\int\frac{(2t^2-2)+2}{t^2-1}dt$$

$$=\int\left(2+\frac{2}{t^2-1}\right)dt=\int\left(2+\frac{1}{t-1}-\frac{1}{t+1}\right)dt=2t+\ln|t-1|-\ln|t+1|+C$$

$$=2t+\ln\left|\frac{t-1}{t+1}\right|+C=2\sqrt{1+x}+\ln\left|\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right|+C$$

e.g.2. $$\int\frac{1}{1+\sqrt[3]{x+1}}dx \xlongequal[t\neq -1]{t=\sqrt[3]{x+1}} \int\frac{1}{1+t}\cdot 3t^2dt=\int\frac{(3t^2-3)+3}{1+t}dt$$

$t^3=x+1$, $x=t^3-1$ $\qquad t^2-1=(t+1)(t-1)$.

$= \int \left[3(t-1) + \frac{3}{1+t}\right] dt = \frac{3}{2}t^2 - 3t + 3\ln|1+t| + C$

$= \frac{3}{2}(x+1)^{\frac{2}{3}} - 3\sqrt[3]{x+1} + 3\cdot\ln\left|1+\sqrt[3]{x+1}\right| + C.$

e.g.3. $\int \frac{1-\sqrt{x-1}}{1+\sqrt[6]{x-1}} dx$

看上去有两个不同的根号 $\sqrt{x-1}$, $\sqrt[6]{x-1}$.

但是 $\sqrt{x-1} = (x-1)^{\frac{1}{2}} = (x-1)^{\frac{1}{6}\times 3} = \left(\sqrt[6]{x-1}\right)^3$.

所以这个函数可以写成只含 $\sqrt[6]{x-1}$ 的式子.

$\sim \xrightarrow[x=t^6+1,\ t\geq 1]{t=\sqrt[6]{x-1}} \int \frac{1-t^3}{1+t}(6\cdot t^5)\cdot dt$

$= 6\int \frac{t^5-t^8}{1+t} dt$.(完整过程在结尾的 (4).①).

e.g. 3^+. 如果一个式子里含有 $\sqrt{x+1}$ 和 $\sqrt[3]{x+1}$

做换元 $t = \sqrt[6]{x+1}$. $\quad \sqrt{x+1} = t^3 \quad \sqrt[3]{x+1} = t^2$.

思路: $\sqrt{x+1} = (x+1)^{\frac{1}{2}}$, $\sqrt[3]{x+1} = (x+1)^{\frac{1}{3}}$.

换元时肯定是让 $t = (x+1)^{\frac{1}{s}}$. 需要确定数字 s.

那 $x+1 = t^s \quad \therefore \sqrt{x+1} = t^{\frac{1}{2}\cdot s}$, $\sqrt[3]{x+1} = t^{\frac{1}{3}\cdot s}$

我们想消除根号, 就要让 $\frac{1}{2}\cdot s$ 与 $\frac{1}{3}\cdot s$ 都是整数.

$\boxed{\text{Try it!}}$ $\int \frac{dx}{x\cdot\sqrt{x-1}}$

$= 2\cdot\arctan(\sqrt{x-1}) + C$

$\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1} = $.(完整过程在结尾的 (4).②).

($\sqrt{x} = x^{\frac{1}{2}} = t^{\frac{s}{2}}$. $\quad \sqrt[4]{x^3} = x^{\frac{3}{4}} = t^{\frac{3}{4}s}$.

令 $t = x^{\frac{1}{s}}$.

让 $s=4$ 就行.)

(20 min)

(2) 只含有 $\sqrt[n]{\frac{ax+b}{cx+d}}$ $(a\neq 0, c\neq 0)$ 的式子:

令 $t=\sqrt[n]{\frac{ax+b}{cx+d}}$. 则 $t^n=\frac{ax+b}{cx+d}$. $(c\cdot t^n)x+d\cdot t^n=ax+b$

$\therefore (c\cdot t^n-a)x=b-dt^n$ $\therefore x=\frac{b-dt^n}{ct^n-a}$, $\frac{dx}{dt}$ 也是t的有理函数.

可以转化成关于t的有理函数积分.

e.g.4. $\int\sqrt{\frac{x-1}{x+1}}\,dx$

令 $t=\sqrt{\frac{x-1}{x+1}}$. 则 $t^2=\frac{x-1}{x+1}$, $t^2x+t^2=x-1$, $(t^2-1)\cdot x=-1-t^2$

$\therefore x=\frac{1+t^2}{1-t^2}$, $dx=\frac{2t(1-t^2)-(1+t^2)\cdot(-2t)}{(1-t^2)^2}dt=\frac{4t}{(1-t^2)^2}dt$

$\therefore \sim=\int t\cdot\frac{4t}{(1-t^2)^2}\cdot dt=4\cdot\int\frac{t^2\cdot dt}{(1-t)^2\cdot(1+t)^2}$ (完整的过程在结尾的(4)③).

$\boxed{\text{Try it!}}$ $\int\frac{\sqrt{4x+1}-\sqrt{x-1}}{\sqrt{4x+1}+\sqrt{x-1}}\,dx$.

$$=\int\frac{\sqrt{\frac{4x+1}{x-1}}-1}{\sqrt{\frac{4x+1}{x-1}}+1}\,dx \xlongequal[x=\frac{t^2+1}{t^2-4}]{t=\sqrt{\frac{4x+1}{x-1}}} \int\frac{t-1}{t+1}\cdot\frac{-10t}{(t^2-4)^2}\,dt$$

(完整的过程在结尾的(4)④).

(3). 无法用初等函数表示的不定积分. [Calculus: P507].

我们讲的方法无法求出所有的不定积分。

理论上可以证明,一些函数的不定积分无法求出(无法用初等函数表示).

eg.5. $\int \frac{\sin x}{x}dx$; $\int e^{-x^2}dx$; $\int \sin(x^2)\cdot dx$;

$\int \frac{1}{\ln x}dx$; $\int \frac{e^x}{x}dx$; $\int \cos(e^x)dx$;

$\int \sqrt{x^3+1}\,dx$; $\int \sqrt{1-k^2\sin^2 x}\,dx \quad (0<k<1)$.

(*)(4). 部分习题的完整过程:

①.(e.g.3).

$$\int \frac{1-\sqrt{x-1}}{1+\sqrt[6]{x-1}}dx \xlongequal[x=t^6+1,\ (t\geq 1)]{t=\sqrt[6]{x-1}} \int \frac{1-t^3}{1+t}(6\cdot t^5)\cdot dt$$

$$=6\int \frac{t^5-t^8}{1+t}dt$$

$$\begin{array}{r|l} & -t^7+t^6-t^5+2t^4-2t^3+2t^2-2t+2 \\ \hline t+1 & -t^8+t^5 \\ & -t^8-t^7 \\ \hline & t^7+t^5 \\ & t^7+t^6 \\ \hline & -t^6+t^5 \\ & -t^6-t^5 \\ \hline & 2t^5 \\ & 2t^5+2t^4 \\ \hline & -2t^4 \\ & -2t^4-2t^3 \\ \hline & 2t^3 \\ & 2t^3+2t^2 \\ \hline & -2t^2 \\ & -2t^2-2t \\ \hline & 2t \\ & 2t+2 \\ \hline & -2 \end{array}$$

$$=6\cdot\int\left(-t^7+t^6-t^5+2t^4-2t^3+2t^2-2t+2-\frac{2}{1+t}\right)dt$$

$$=6\left[-\frac{t^8}{8}+\frac{t^7}{7}-\frac{t^6}{6}+\frac{2t^5}{5}-\frac{2}{4}t^4+\frac{2}{3}t^3-t^2+2t-2\ln|1+t|\right]+C.$$

$$=-\frac{3}{4}(x-1)^{\frac{4}{3}}+\frac{6}{7}\cdot(x-1)^{\frac{7}{6}}-(x-1)+\frac{12}{5}(x-1)^{\frac{5}{6}}$$

$$-3\cdot(x-1)^{\frac{2}{3}}+4(x-1)^{\frac{1}{2}}-6\cdot(x-1)^{\frac{1}{3}}+12\cdot(x-1)^{\frac{1}{6}}$$

$$-12\cdot\ln\left(1+\sqrt[6]{x-1}\right)+C.$$

② $\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1}dx \xlongequal[x=t^4,(t\geq 0)]{t=\sqrt[4]{x}} \int \frac{t^2}{t^3+1}\cdot 4\cdot t^3 dt = \int \frac{4t^3}{t^3+1}\cdot\frac{dt^3}{3}.$

$\xlongequal{u=t^3} \int \frac{4u}{u+1}\cdot\frac{du}{3} = \frac{4}{3}\int\frac{u}{u+1}du = \frac{4}{3}\int\left(1-\frac{1}{u+1}\right)du$

$= \frac{4}{3}\left(u-\ln|u+1|\right)+C$

$= \frac{4}{3}\left[\sqrt[4]{x^3}-\ln\left(\sqrt[4]{x^3}+1\right)\right]+C.$

③ $\int\sqrt{\frac{x-1}{x+1}}dx \xlongequal[\substack{x=\frac{1+t^2}{1-t^2}\\(t\geq 0,\, t\neq 1).}]{t=\sqrt{\frac{x-1}{x+1}}} \int t\cdot\frac{4t}{(1-t^2)^2}dt = \int\frac{4t^2}{(1-t)^2(1+t)^2}dt$

设 $\frac{4t^2}{(1-t)^2\cdot(1+t)^2} = \frac{a}{1-t}+\frac{b}{(1-t)^2}+\frac{c}{1+t}+\frac{d}{(1+t)^2}$

$\therefore 4t^2 = a\cdot(1-t)(1+t)^2 + b\cdot(1+t)^2 + c\cdot(1-t)^2(1+t) + d\cdot(1-t)^2$

取 $t=1$. $4=4b$ $\therefore b=1$.

取 $t=-1$. $4=4d$ $\therefore d=1$.

$\therefore 4t^2 = a(1-t)\cdot(1+t)^2 + (1+t)^2 + c\cdot(1-t)^2(1+t) + (1-t)^2.$

$\therefore 2\cdot(t^2-1) = a\cdot(1-t)(1+t)^2 + c\cdot(1-t)^2\cdot(1+t)$

$\therefore -2 = a\cdot(1+t) + c\cdot(1-t) = (a-c)\cdot t + a+c$

$\therefore \begin{cases} a-c=0 \\ a+c=-2 \end{cases}$ $\therefore a=c=-1.$

$\therefore \sim = \int\left(\frac{-1}{1-t}+\frac{1}{(1-t)^2}-\frac{1}{1+t}+\frac{1}{(1+t)^2}\right)dt$

$= \ln|t-1| + \frac{1}{1-t} - \ln|1+t| - \frac{1}{1+t} + C.$

$= \ln\left|\frac{t-1}{t+1}\right| + \frac{2t}{1-t^2} + C$

$$= \ln\left|\frac{\sqrt{\frac{x-1}{x+1}}-1}{\sqrt{\frac{x-1}{x+1}}+1}\right| + \frac{2\cdot\sqrt{\frac{x-1}{x+1}}}{1-\frac{x-1}{x+1}} + C$$

$$= \ln\left|\frac{\sqrt{\frac{x-1}{x+1}}-1}{\sqrt{\frac{x-1}{x+1}}+1}\right| + (x+1)\cdot\sqrt{\frac{x-1}{x+1}} + C$$

根据 $x<-1$ or $x\geq 1$，可以进一步化简．

$x<-1$ 时，$(x+1) = -\sqrt{(x+1)^2}$． $\therefore (x+1)\cdot\sqrt{\frac{x-1}{x+1}} = -\sqrt{x^2-1}$；

$x\geq 1$ 时，$x+1 = \sqrt{(x+1)^2}$ $\therefore (x+1)\cdot\sqrt{\frac{x-1}{x+1}} = \sqrt{x^2-1}$．

$\ln$ 内部的项也可以类似地化简．

④ $$\int \frac{\sqrt{4x+1}-\sqrt{x-1}}{\sqrt{4x+1}+\sqrt{x-1}}\,dx = \int \frac{\sqrt{\frac{4x+1}{x-1}}-1}{\sqrt{\frac{4x+1}{x-1}}+1}\,dx$$

$$\xlongequal[x=\frac{t^2+1}{t^2-4}]{t=\sqrt{\frac{4x+1}{x-1}}} \int \frac{t-1}{t+1}\cdot\frac{-10t}{(t^2-4)^2}\,dt = -10\cdot\int\frac{t(t-1)}{(t+1)(t-2)^2(t+2)^2}\,dt.$$

设 $$\frac{t(t-1)}{(t+1)(t-2)^2(t+2)^2} = \frac{a}{t+1}+\frac{b}{t-2}+\frac{c}{(t-2)^2}+\frac{d}{t+2}+\frac{e}{(t+2)^2}$$

$$\therefore t(t-1) = a\cdot(t-2)^2(t+2)^2 + b\cdot(t+1)\cdot(t-2)(t+2)^2 + c\cdot(t+1)(t+2)^2$$
$$+ d\cdot(t+1)\cdot(t-2)^2(t+2) + e\cdot(t+1)\cdot(t-2)^2.$$

令 $t=-1$： $2 = a\cdot 9$ $\therefore a=\frac{2}{9}$.

令 $t=2$： $2 = c\cdot 3\cdot 16$ $\therefore c=\frac{1}{24}$

令 $t=-2$： $6 = e\cdot(-1)\cdot 16$ $\therefore e=-\frac{3}{8}$

令 $t=0$： $0 = 16a-8b+4c+8d+4e = \frac{32}{9}+\frac{1}{6}-\frac{3}{2}-8b+8d$.

令 $t=1$： $0 = 9a-18b+18c+6d+2e = 2+\frac{3}{4}-\frac{3}{4}-18b+6d$

$\therefore b=\frac{1}{36}$，$d=-\frac{1}{4}$.

$$\therefore \sim = -10 \cdot \int \left[\frac{2}{9} \cdot \frac{1}{t+1} + \frac{1}{36} \cdot \frac{1}{t-2} + \frac{1}{24} \cdot \frac{1}{(t-2)^2} - \frac{1}{4} \cdot \frac{1}{t+2} - \frac{3}{8} \cdot \frac{1}{(t+2)^2} \right] dt$$

$$= -\frac{20}{9} \cdot \ln|t+1| - \frac{5}{18} \cdot \ln|t-2| + \frac{5}{12} \cdot \frac{1}{t-2} + \frac{5}{2} \cdot \ln|t+2| - \frac{15}{4} \cdot \frac{1}{t+2} + C$$

$$= -\frac{20}{9} \cdot \ln\left(\sqrt{\frac{4x+1}{x-1}} + 1\right) - \frac{5}{18} \cdot \ln\left|\sqrt{\frac{4x+1}{x-1}} - 2\right| + \frac{5}{2} \cdot \ln\left(\sqrt{\frac{4x+1}{x-1}} + 2\right)$$

$$+ \frac{5}{12} \cdot \frac{1}{\sqrt{\frac{4x+1}{x-1}} - 2} - \frac{15}{4} \cdot \frac{1}{\sqrt{\frac{4x+1}{x-1}} + 2} + C.$$