(1) $\cos x$

(2) $\sin x$

(3) $\ln(1+x)$

(4) $e^x$

(6) $(1+x)^a$.

记忆

(5) $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}\cdot x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + o(x^5)$

$\tan 0 = 0$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x\cdot(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x\cdot(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

① $1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$ ② $\frac{1}{\cos^2 x}$

$f'(x) = 1 + \tan^2 x$ $f'(0) = \boxed{1}$

$f''(x) = \boxed{2\tan x}\cdot(\tan x)' = 2\tan x\,(1+\tan^2 x)$

$= 2\tan x + 2\tan^3 x$

$$\frac{df}{d\tan x}\cdot\frac{d\tan x}{dx}$$

$$f'(0)=0$$

$$f^{(3)}(x)=(2+6\tan^2 x)(1+\tan^2 x)$$

$$=6\tan^4 x+8\tan^2 x+2$$

$$f^{(3)}(0)=\boxed{2}$$

$(\tan x)'$

$(\tan^2 x+1)$

$$f^{(4)}(x)=(24\tan^3 x+16\tan x)$$

$$=24\tan^5 x+40\tan^3 x+16\tan x$$

$$f^{(4)}(0)=0$$

$$f^{(5)}(x)=(120\tan^4 x+120\tan^2 x+16)(1+\tan^2 x)$$

$$f^{(5)}(0)=\boxed{16}$$

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$$\tan x=x+\frac{2}{3!}x^3+\frac{16}{5!}x^5$$

$$=x+\frac{1}{3}x^3+\frac{2}{15}x^5+o(x^5)$$

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$$\tan x = \frac{\sin x}{\cos x}$$

设 $\tan x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + o(x^5)$

$$\sin x = \cos x \cdot \tan x$$

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5) = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^5)\right)\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + o(x^5)\right)$$

(如何简化? $\tan x$ 奇函数 $\rightarrow a_0 = a_2 = a_4 = 0$)

$$= a_0 + a_1 x + \left(a_2 - \frac{1}{2}a_0\right)x^2 + \left(a_3 - \frac{1}{2}a_1\right)x^3$$
$$+ \left(a_4 - \frac{1}{2}a_2 + \frac{1}{24}a_0\right)x^4$$
$$+ \left(a_5 - \frac{1}{2}a_3 + \frac{1}{24}a_1\right)x^5 + o(x^5)$$

① $a_0 = 0$ ② $a_1 = \boxed{1}$ ③ $a_2 - \frac{1}{2}a_0 = 0 \Rightarrow a_2 = 0$

④ $a_3 - \frac{1}{2}a_1 = -\frac{1}{6} \Rightarrow a_3 = \frac{1}{2}a_1 - \frac{1}{6} = \boxed{\frac{1}{3}}$

⑤ $a_4 - \frac{1}{2}a_2 + \frac{1}{24}a_0 = 0 \Rightarrow a_4 = 0$

⑥ $a_5 - \frac{1}{2}a_3 + \frac{1}{24}a_1 = \frac{1}{120}$

$$\Rightarrow a_5 = \frac{1}{2}a_3 - \frac{1}{24}a_1 + \frac{1}{120}$$
$$= \frac{1}{6} - \frac{1}{24} + \frac{1}{120} = \frac{20 - 5 + 1}{120}$$
$$= \boxed{\frac{2}{15}}$$

(7) $\cos x \cdot \ln(1+x)$

$$= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^5)\right)\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + o(x^5)\right)$$

$$= x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \left(\frac{1}{24} - \frac{1}{6} + \frac{1}{5}\right)x^5$$

$11$

$\frac{3}{40}$

(9) $\frac{e^x + e^{-x}}{2}$ (偶函数，$x, x^3, x^5$ 系数 0)

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + o(x^5),$$

$$e^{-x} = 1 + (-x) + \frac{1}{2}(-x)^2 + \frac{1}{6}(-x)^3 + \frac{1}{24}(-x)^4 + \frac{1}{120}(-x)^5 + o(x^5)$$

$$= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + o(x^5)$$

$$\frac{e^x + e^{-x}}{2} = 1 + 0x + \frac{1}{2}x^2 + 0x^3 + \frac{1}{24}x^4 + o(x^5)$$

$$= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^5)$$

(10) $\frac{1}{2}\ln\left(1-\frac{2x}{1+x}\right)=\frac{1}{2}\left[-\frac{2x}{1+x}-\frac{1}{2}\left(\frac{2x}{1+x}\right)^2+\cdots\right]$

$\times$

$f(-x)=\frac{1}{2}\left(\ln(1+x)-\ln(1-x)\right)$

$\frac{1}{2}\left(\ln(1-x)-\ln(1+x)\right)$

$\ln(1-x)=(-x)-\frac{1}{2}(-x)^2+\frac{1}{3}(-x)^3-\frac{1}{4}(-x)^4+\frac{1}{5}(-x)^5+o(x^5)$

$=-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4-\frac{1}{5}x^5+o(x^5).$

$\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5+o(x^5)$

$\frac{1}{2}\left(\ln(1-x)-\ln(1+x)\right)=\frac{1}{2}\left(-2x-\frac{2}{3}x^3-\frac{2}{5}x^5+o(x^5)\right)$

$=-x-\frac{1}{3}x^3-\frac{1}{5}x^5+o(x^5)$

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(11) $\sin^2 x=(\sin x)\cdot(\sin x)$

$=\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5+o(x^5)\right)\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5+o(x^5)\right)$

$=x^2-\frac{1}{3}x^4+o(x^5).$

$\sin 2x=2\sin x\cos x$

$\cos 2x=\cos^2 x-\sin^2 x$

$=2\cos^2 x-1$

$=1-2\sin^2 x.$

② $\sin^2 x=\frac{1-\cos 2x}{2}$

$$\cos 2x = 1 - \frac{1}{2}(2x)^2 + \frac{1}{24}(2x)^4 + o(x^5).$$

$$= 1 - 2x^2 + \frac{2}{3}x^4 + o(x^5).$$

$$\sin^2 x = \frac{1}{2}(1 - \cos x)$$

$$= \frac{1}{2}\left(1 - \left(1 - 2x^2 + \frac{2}{3}x^4 + o(x^5)\right)\right)$$

$$= x^2 - \frac{1}{3}x^4 + o(x^5)$$

$$x^6 \quad \lim_{x \to 0} \frac{\sin^2 x - x^2 + \frac{1}{3}x^4}{x^5} = 0,$$

(12) $\cos x^3 = 1 - \frac{1}{2}\underline{(x^3)^2} +$

$$= 1 + \underline{\boxed{o(x^5).}}$$

what is $o(x^5)$?

$o(x^5)$是一个函数 满足 $\lim_{x \to 0} \frac{o(x^5)}{x^5} = 0$

$$\underline{\cos x^3 - 1} = o(x^5)$$

$\cos x^3 = 1 + o(x^5)$.

means. (equivalent to).

$$\lim_{x \to 0} \frac{\cos x^3 - 1}{x^5} = 0$$

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3.(2) $f(x) = \underline{\sin(e^x - 1)}$

$\underline{\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5)}$ $(x \to 0)$

$$\boxed{\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3 - \frac{1}{120}x^5}{x^5}} = 0$$

$\underline{\sin(e^x)} = e^x - \frac{1}{6}e^{3x} + \frac{1}{120}e^{5x}$

x.

$x \to 0 \quad e^x \to 1$

3. (1). $x \to 0$ $\quad \sin x \to 0$

$\sin x \to 0$

$$e^{\sin x} = (\sin x) + \frac{1}{2}(\sin x)^2 + \frac{1}{6}(\sin x)^3 + \frac{1}{24}(\sin x)^4 + \frac{1}{120}(\sin x)^5 + o((\sin x)^5)$$

[redacted]

$$f(x) = o((\sin x)^5) \iff \lim_{x\to 0}\frac{f(x)}{(\sin x)^5} = 0$$

$$\iff \lim_{x\to 0}\frac{f(x)}{x^5} = 0 \iff f(x) = o(x^5)$$

$$e^{\sin x} = \boxed{1 + (x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5))} + \frac{1}{2}(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5))^2$$
$$+ \frac{1}{6}(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5))^3 + \frac{1}{24}(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots)^4$$
$$+ \frac{1}{120}(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots +)^5$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \boxed{\frac{1}{2}x^2}(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4)^2$$

$$(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4)(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4)$$

$$\boxed{\frac{1}{2}x^2(1 - \frac{1}{3}x^2)}$$

$$+ \frac{1}{6}x^3(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4)^3 \qquad x^2, x, 1.$$

$$\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4\right)\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4\right)$$

$$\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4\right)$$

$$\left(1-\frac{1}{3}x^2\right)\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4\right)$$

$$= 1 - \left(\frac{1}{3}+\frac{1}{6}\right)x^2 = 1-\frac{1}{2}x^2$$

$$\boxed{\frac{1}{6}x^3\left(1-\frac{1}{2}x^2\right)}$$

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$$\frac{1}{24}x^4\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4\right)^4.$$

$$\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4\right)\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4\right)$$

$$\cdots$$

$$\boxed{\frac{1}{24}x^4\cdot 1}$$

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$$\frac{1}{120}x^5\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4\right)^5$$

$$= \boxed{\frac{1}{120} x^5}.$$

$$e^{\sin x}$$

$$\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) + \frac{1}{2}x^2\left(1 - \frac{1}{3}x^2\right)$$

$$+ \frac{1}{6}x^3 \cdot \left(1 - \frac{1}{2}x^2\right)$$

$$+ \frac{1}{24}x^4 + \frac{1}{120}x^5$$

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$$x + \frac{1}{2}x^2 + \left(\frac{1}{6} - \frac{1}{6}\right)x^3 + \left(\frac{1}{24} - \frac{1}{6}\right)x^4$$

$$+ \left(\frac{1}{120} + \frac{1}{120} - \frac{1}{12}\right)x^5$$

$$= \cdots + \underline{\left(\frac{1}{60} - \frac{1}{12}\right)}x^5$$

$$\frac{1}{60} - \frac{1}{12} = \frac{f^{(5)}(0)}{5!}$$

$$\Rightarrow f^{(5)}(0) = \frac{5!}{60} - \frac{5!}{12}$$

$$= 2 - \frac{120}{12}$$

$$= -8$$

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$\infty \quad \infty$

$$2.\ \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\ln(1+x)}\right)$$

$$= \lim_{x\to 0}\frac{\ln(1+x) - x}{x\ln(1+x)}$$

$\ln(1+x)$

$\sim x$

$$= \lim_{x \to 0} \frac{\ln(1+x) - x}{x^2}$$ 非多项式 $\Rightarrow$ 多项式

$$= \lim_{x \to 0} \frac{x - \frac{1}{2}x^2 + o(x^2) - x}{x^2}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{2}x^2 + o(x^2)}{x^2}$$

$$= \lim_{x \to 0} \left(-\frac{1}{2} + \frac{o(x^2)}{x^2}\right)$$ $\to 0$

$$= -\frac{1}{2}$$

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(2) $$\lim_{x \to 0} \frac{\left(\cancel{1}+\cancel{x}+\cancel{\frac{1}{2}x^2}+\frac{1}{6}x^3+\frac{1}{24}x^4+o(x^4)\right) - \cancel{1} - \cancel{x} - \cancel{\frac{x}{2}}\left(x - \frac{1}{6}x^3 + o(x^4)\right)}{\cancel{x} - \frac{1}{6}x^3 + o(x^4) - x\cancel{(}1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4))}$$

$$= \lim_{x \to 0} \frac{\frac{1}{6}x^3 + (\frac{1}{24} - \frac{1}{12})x^4 + o(x^4)}{(-\frac{1}{6}+\frac{1}{2})x^3 + o(x^4)}$$

$$= \lim_{x \to 0} \frac{\frac{1}{6}x^3 - \frac{1}{24}x^4 + o(x^4)}{\frac{1}{3}x^3 + o(x^4)}$$

$\underline{o(1)} \to 0$

$$= \lim_{x \to 0} \frac{\frac{1}{6} - \frac{1}{24}x + o(x)}{\frac{1}{3} + o(x)}$$

$0 \quad 0$

$0$

$\frac{o(x)}{x} \to 0$

$$= \frac{\frac{1}{6}}{\frac{1}{3}}$$

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(3) $\lim_{x \to 0} \frac{1 - x^2 - (1 - x^2 + \frac{1}{2}x^4)}{x - (2x)^3}$

$$= \frac{-\frac{1}{2}x^4}{8x^4} = -\frac{1}{16}$$

$$e^{-x^2} = 1 + (-x^2) + \frac{1}{2}(-x^2)^2 + o(x^4)$$

(4). $\lim\limits_{x \to 0} \dfrac{\tan x - x}{(x \cdot \tan x)\sin x}$

$= \lim\limits_{x \to 0} \dfrac{\tan x - x}{x^3}$

$= \lim\limits_{x \to 0} \dfrac{\dfrac{\sin x}{\cos x} - x}{x^3}$

$= \lim\limits_{x \to 0} \dfrac{\sin x - x\cos x}{\underbrace{\cos x}_{1} \cdot x^3}$

$= \lim\limits_{x \to 0} \dfrac{\sin x - x\cos x}{x^3}$

$= \lim\limits_{x \to 0} \dfrac{\left(x - \frac{1}{6}x^3 + o(x^3)\right) - x\left(1 - \frac{1}{2}x^2 + o(x^3)\right)}{x^3}$

$= \lim\limits_{x \to 0} \dfrac{x - \frac{1}{6}x^3 - x + \frac{1}{2}x^3 + o(x^3)}{x^3}$.

$$= \frac{1}{3}.$$

(5) $$\lim_{x \to 0} \frac{e^x - x - 1}{x(e^x - 1)}$$

$$= \lim_{x \to 0} \frac{1 + x + \frac{1}{2}x^2 + o(x^2) - x - 1}{x(\cancel{1} + x + \frac{x^2}{2} + o(x^2) \cancel{-1})}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2}x^2 + o(x^2)}{x^2 + \boxed{\frac{1}{2}x^3 + x \cdot o(x^2)}}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2}x^2 + o(x^2)}{x^2 + o(x^2)}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2} + o(1)}{1 + o(1)}$$

$$= \frac{1}{2}$$

1. (1) ①

$x>0$ $f(x)=e^x$ $t\in(0,x)$.

$$e^x = 1+x+\frac{1}{2}x^2+\boxed{\frac{f^{(3)}(t)}{3!}x^3}$$

$$e^x = 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3)$$

$o(x^2)$

只需证 $f^{(3)}(t)\geq 0$

$f^{(3)}(x)=e^x$ $f^{(3)}(t)=e^t>0$