# 4月3日. 周四. 第13次课.

## Topic 1: 三角函数有理式的不定积分

[中文课本 (5.4. 三角函数有理式的积分). $P_{215} \sim P_{219}$ ]

(1). 三角函数有理式的定义及通用积分求法:

三角函数有理式是指含$x$的部分只有 $\sin x$ . $\cos x$,

且 $\sin x$ . $\cos x$ 不出现在根号下的式子.

i.e. 由 $\sin x$、$\cos x$ 和实数 只经过+.-.×.÷ 得到的式子。

e.g.1. $\sin x + \cos x$ (√) . $\sin^3 x \cos^5 x$ (√) . $\tan^3 x \cdot \sin x$ . (√).

$\sin\frac{x}{2} + \sin x$ (×) . $\sin x + x$ (×) . $\sqrt{1+\sin x}$ (×)

$\frac{\sin^2 x + \cos^4 x}{\sin^2 x}$ (√). $\sqrt{2}\sin x + \cos x$ .(√). $\frac{1+\sin^2 x}{\cos^3 x + \sin^5 x}$ (√).

用到的三角函数公式:

$\sin 2\theta = 2\sin\theta \cdot \cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$1 = \cos^2\theta + \sin^2\theta$.

所有的三角函数有理式积分, 都可以用下方的换元求解:

取 $t = \tan\frac{x}{2}$.

则 $\sin x = 2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2} = \frac{2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}} = \frac{2\cdot\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1} = \frac{2t}{t^2+1}$.

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{\tan^2\frac{x}{2}+1} = \frac{1-t^2}{t^2+1}$.

$dt = \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}\cdot dx = \frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{2\cdot\cos^2\frac{x}{2}}dx = \frac{\tan^2\frac{x}{2}+1}{2}dx = \frac{t^2+1}{2}dx$

$\therefore dx = \frac{2}{1+t^2}dt$.

经过以上换元, 可以将关于$x$的三角函数有理式不定积分,

转化为关于$t$的有理函数不定积分,

进而可以用上一个Topic的方法求.

e.g.2. 求 $\int \frac{dx}{1+\sin x}$.

$$\xlongequal{t=\tan\frac{x}{2}} \int \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2} \cdot dt = \int \frac{2\cdot dt}{1+t^2+2t}$$

$$= \int \frac{2\cdot dt}{(t+1)^2} \xlongequal{u=t+1} \int \frac{2\cdot du}{u^2} = -\frac{2}{u}+C$$

$$= -\frac{2}{t+1}+C = -\frac{2}{\tan\frac{x}{2}+1}+C$$

e.g.3. 求 $\int \frac{\cot x\cdot dx}{\sin x+\cos x-1}$. $\cot x = \frac{\cos x}{\sin x}$

$$= \int \frac{\cos x \ dx}{\sin x\cdot(\sin x+\cos x-1)} \xlongequal{t=\tan\frac{x}{2}} \int \frac{\frac{1-t^2}{1+t^2}\cdot\frac{2}{1+t^2}\cdot dt}{\frac{2t}{1+t^2}\cdot\left(\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}-1\right)}$$

$$= \int \frac{2\cdot(1-t^2)\cdot dt}{2t\cdot[2t+1-t^2-(1+t^2)]} = \int \frac{(1-t)(1+t)\cdot dt}{t\cdot 2t\cdot(1-t)} = \int \frac{1+t}{2t^2}dt$$

$$= \frac{1}{2}\int\frac{dt}{t^2} + \frac{1}{2}\int\frac{dt}{t} = -\frac{1}{2t}+\frac{\ln t}{2}+C = -\frac{1}{2\tan\frac{x}{2}}+\frac{\ln\tan\frac{x}{2}}{2}+C$$

$\boxed{\text{Try it!}}$ . $\int \frac{dx}{1+\cos x} = \tan\frac{x}{2}+C$ $\int \frac{dx}{\sin x+\cos x+1} = \ln|\tan\frac{x}{2}+1| + C$

## (2). 特殊情形下的换元技巧:

上述换元方式虽然通用,但可能会很麻烦.

e.g.4. $\int \cos^{10}x \cdot \sin x \, dx = \int \cos^{10}x(-d(\cos x)) \xlongequal{t=\cos x} -\int t^{10}dt = -\frac{t^{11}}{11}+C = -\frac{\cos^{11}x}{11}+C$

$\xlongequal{t=\tan\frac{x}{2}} \int \frac{(1-t^2)^{10}}{(1+t^2)^{10}}\cdot\frac{2t}{(1+t^2)^{10}}\cdot\frac{2}{1+t^2}dt$. 虽然理论上可求.但是很难算

以下考虑对特殊的三角函数有理式的换元技巧.

① $t=\sin x$ 型换元 $\to dt=\cos x\cdot dx$

如果 $\frac{f(x)}{\cos x}$ 中 $\cos x$ 都以偶数次方出现 (e.g. $\cos^2 x$, $\cos^4 x$).

则可作换元 $t=\sin x$，利用 $\cos^2 x=1-\sin^2 x$ 代入. 转化为关于 $t$ 的有理函数.

e.g. 5. $\int \frac{\cos^3 x}{1+\sin^2 x}dx$ （$\overset{t=\tan\frac{x}{2}}{=}\int \frac{\left(\frac{1-t^2}{1+t^2}\right)^3}{1+\left(\frac{2t}{1+t^2}\right)^2}\cdot\frac{2}{1+t^2}dt$. 很难算）.

$$=\int\frac{\cos^2 x}{1+\sin^2 x}\cdot(\cos x\cdot dx)=\int\frac{1-\sin^2 x}{1+\sin^2 x}\cdot d(\sin x)\overset{t=\sin x}{=\!=}\int\frac{1-t^2}{1+t^2}dt=\int\frac{2-(1+t^2)}{1+t^2}dt$$

$$=2\cdot\int\frac{dt}{1+t^2}-\int 1\cdot dt=2\arctan t-t+C=2\cdot\arctan(\sin x)-\sin x+C.$$

② $t=\cos x$ 型换元 $\to dt=-\sin x\cdot dx$

如果 $\frac{f(x)}{\sin x}$ 中 $\sin x$ 都以偶数次方出现 (e.g. $\sin^2 x$, $\sin^4 x$).

则可作换元 $t=\cos x$，利用 $\sin^2 x=1-\cos^2 x$ 代入. 转化为关于 $t$ 的有理函数.

e.g. 6. $\int\frac{\sin^3 x}{1+\cos^2 x}dx$

$$=\int\frac{\sin^2 x}{1+\cos^2 x}\cdot(\sin x\cdot dx)=\int\frac{1-\cos^2 x}{1+\cos^2 x}\cdot(-d(\cos x))\overset{t=\cos x}{=\!=}\int\frac{t^2-1}{t^2+1}\cdot dt$$

$$=\int\left(1-\frac{2}{t^2+1}\right)\cdot dt=\int 1\cdot dt-2\cdot\int\frac{dt}{t^2+1}=t-2\arctan t+C=\cos x-2\arctan(\cos x)+C$$

③ $t=\tan x$ 型换元

如果 $f(x)$ 能写成只含 $\tan x$ 的式子，可作换元 $t=\tan x$.

进而 $dt=\frac{1}{\cos^2 x}dx=(1+t^2)\cdot dx$, $dx=\frac{1}{1+t^2}dt$.

$\left(\frac{1}{\cos^2 x}=1+\tan^2 x\right)$

转化为关于 $t$ 的有理函数.

特例: $f(x)$ 的分子. 分母中每一项中, $\sin x$ 与 $\cos x$ 的次数相加都是偶数.

$$\sin^k x\cdot\cos^{2n-k}x=\left(\frac{\sin x}{\cos x}\right)^k\cdot(\cos^2 x)^n=\tan^k x\cdot\left(\frac{1}{1+\tan^2 x}\right)^n.$$

e.g.7. $\int \tan^4 x\, dx$.

$$\xlongequal{t=\tan x} \int t^4 \cdot \frac{dt}{1+t^2} = \int (t^2-1) + \frac{1}{1+t^2}\, dt$$

$$\begin{array}{r} t^2-1 \\ t^2+1 \overline{\big)\, t^4 \qquad} \\ \underline{t^4+t^2} \\ -t^2 \\ \underline{-t^2-1} \\ 1 \end{array}$$

$$= \frac{t^3}{3} - t + \arctan t + C = \frac{\tan^3 x}{3} - \tan x + x + C.$$

e.g. 8. $\int \tan^3 x\, dx$

way 1: $t = \tan x$. way 2: $t = \frac{1}{\cos x}$.

way 1: $\xlongequal{t=\tan x} \int t^3 \cdot \frac{dt}{1+t^2} = \int t - \frac{t}{1+t^2}\, dt$

$$\begin{array}{r} t \\ t^2+1 \overline{\big)\, t^3 \qquad} \\ \underline{t^3+t} \\ -t \end{array}$$

$$= \int t \cdot dt - \int \frac{t \cdot dt}{1+t^2} = \frac{t^2}{2} - \int \frac{\frac{1}{2} d(t^2)}{1+t^2} = \frac{t^2}{2} - \frac{1}{2} \ln(1+t^2) + C$$

$$= \frac{1}{2} \tan^2 x - \frac{1}{2} \ln(1+\tan^2 x) + C$$

way 2: $= \int \tan^2 x \cdot \cos x \left( \frac{\tan x}{\cos x} \cdot dx \right) = \int \left( \frac{1}{\cos^2 x} - 1 \right) \cdot \cos x \cdot d\left( \frac{1}{\cos x} \right)$

$$\xlongequal{t=\frac{1}{\cos x}} \int (t^2-1) \frac{1}{t}\, dt = \int t - \frac{1}{t}\, dt = \frac{t^2}{2} - \ln|t| + C$$

$$= \frac{1}{2} \cdot \frac{1}{\cos^2 x} - \ln\left| \frac{1}{\cos x} \right| + C$$

way 2. 简单一些.

三角函数的不定积分可能有许多种不同换元方法.

难度也会不同.

e.g.9. $\int \frac{dx}{\tan x + 1} \xlongequal{t=\tan x} \int \frac{1}{t+1} \frac{dt}{t^2+1}$

$$\frac{1}{(t+1)(t^2+1)} = \frac{a}{t+1} + \frac{bt+c}{t^2+1}$$

$$1 = a\cdot(t^2+1) + (bt+c)\cdot(t+1).$$

$$0\cdot t^2 + 0\cdot t + 1 = (a+b)\cdot t^2 + (b+c)\cdot t + (a+c).$$

$$\therefore \begin{cases} 0 = a+b \\ 0 = b+c \\ 1 = a+c \end{cases} \Rightarrow \begin{cases} a = \frac{1}{2} \\ b = -\frac{1}{2} \\ c = \frac{1}{2}. \end{cases}$$

$$\therefore \sim = \int \frac{\frac{1}{2}}{t+1} + \frac{\frac{t}{2}+\frac{1}{2}}{t^2+1}\,dt = \frac{1}{2}\cdot\int\frac{dt}{t+1} + \frac{1}{2}\int\frac{t\,dt}{t^2+1} + \frac{1}{2}\cdot\int\frac{dt}{t^2+1}$$

$$= \ln|t+1| + \frac{1}{4}\cdot\ln(1+t^2) + \frac{1}{2}\arctan t + C$$

$$= \ln|\tan x+1| + \frac{1}{4}\cdot\ln(1+\tan^2 x) + \frac{1}{2}x + C.$$

e.g. 10. $\displaystyle\int \frac{dx}{\sin^2 x + 2\sin x\cos x + 2\cos^2 x}$

令 $t=\tan x$. $\displaystyle \sin^2 x = \frac{\sin^2 x}{\sin^2 x+\cos^2 x} = \frac{\tan^2 x}{\tan^2 x+1} = \frac{t^2}{t^2+1}$

$$2\sin x\cdot\cos x = \frac{2\sin x\cdot\cos x}{\sin^2 x+\cos^2 x} = \frac{2\cdot\tan x}{\tan^2 x+1} = \frac{2t}{t^2+1}$$

$$2\cdot\cos^2 x = \frac{2\cos^2 x}{\sin^2 x+\cos^2 x} = \frac{2}{\tan^2 x+1} = \frac{2}{t^2+1}$$

$$dx = \frac{dt}{1+t^2} \qquad \therefore \sim = \int \frac{1}{\frac{t^2}{t^2+1} + \frac{2t}{t^2+1} + \frac{2}{t^2+1}}\,\frac{dt}{1+t^2}$$

$$= \int\frac{dt}{t^2+2t+2} = \int\frac{dt}{(t+1)^2+1} = \arctan(t+1) + C$$

$$= \arctan(\tan x+1) + C.$$

(*) (不需要掌握)

e.g. 11. $\int \frac{\sin x \cdot \cos x}{\sin^2 x + \cos^4 x} dx$.

way 1. $t = \tan x$; way 2. $t = \sin x$; way 3. $t = \cos x$.

①. way 1: $(t = \tan x)$

$$\sin x \cdot \cos x = \frac{\sin x \cdot \cos x}{\sin^2 x + \cos^2 x} = \frac{\tan x}{\tan^2 x + 1} = \frac{t}{t^2+1}.$$

$$\sin^2 x = \frac{\sin^2 x}{\sin^2 x + \cos^2 x} = \frac{\tan^2 x}{\tan^2 x + 1} = \frac{t^2}{t^2+1}.$$

$$\cos^4 x = \frac{\cos^4 x}{1^2} = \frac{\cos^4 x}{(\sin^2 x + \cos^2 x)^2} = \frac{1}{(\tan^2 x + 1)^2} = \frac{1}{(t^2+1)^2}$$

$$\therefore \sim \overset{t=\tan x}{=} \int \frac{\frac{t}{t^2+1}}{\frac{t^2}{t^2+1} + \frac{1}{(t^2+1)^2}} \cdot \frac{dt}{t^2+1} = \int \frac{t}{t^2 + \frac{1}{t^2+1}} \cdot \frac{dt}{t^2+1}$$

$$= \int \frac{t \cdot dt}{t^4 + t^2 + 1} = \int \frac{\frac{1}{2} \cdot d(t^2)}{t^4 + t^2 + 1} \overset{u=t^2}{=\!=} \frac{1}{2} \int \frac{du}{u^2 + u + 1}$$

$$= \frac{1}{2} \cdot \int \frac{du}{(u+\frac{1}{2})^2 + \frac{3}{4}} = \frac{1}{2} \cdot \int \frac{1}{\frac{3}{4}} \cdot \frac{du}{[\frac{2}{\sqrt{3}}(u+\frac{1}{2})]^2 + 1}$$

$$\overset{s=\frac{2}{\sqrt{3}}(u+\frac{1}{2})}{=\!=} \frac{1}{2} \int \frac{1}{\frac{3}{4}} \cdot \frac{\frac{\sqrt{3}}{2} ds}{s^2+1} = \frac{\sqrt{3}}{3} \cdot \arctan s + C.$$

$$= \frac{\sqrt{3}}{3} \arctan\left(\frac{2}{\sqrt{3}}(\tan^2 x + \tfrac{1}{2})\right) + C.$$

② . ($t = \sin x$)

$$\int \frac{\sin x \cdot \cos x}{\sin^2 x + \cos^4 x} \cdot dx = \int \frac{\sin x \cdot d(\sin x)}{\sin^2 x + (1-\sin^2 x)^2} \xlongequal{t=\sin x} \int \frac{t \cdot dt}{t^2 + (1-t^2)^2}$$

$$\xlongequal{u=t^2} \int \frac{\frac{1}{2} \cdot du}{u + (1-u)^2} = \frac{1}{2} \cdot \int \frac{du}{u^2 - u + 1}$$ . 和①差不多难.

③ . ($t = \cos x$) .

$$\int \frac{\sin x \cdot \cos x}{\sin^2 x + \cos^4 x} dx = \int \frac{\cos x \cdot (-d(\cos x))}{1 - \cos^2 x + \cos^4 x} \xlongequal{t=\cos x} -\int \frac{t \cdot dt}{1 - t^2 + t^4}$$

$$\xlongequal{u=t^2} -\frac{1}{2} \int \frac{du}{u^2 - u + 1}.$$ 和②差不多难

$\boxed{\text{Try it!}}$ $\int \tan^8 x \, dx = ?$ $\int \frac{1}{1+\sin^2 x} dx = ?$