第二次作业答案

解答题

1. 请写出下列函数在 $x=0$ 处的带 Peano 余项的泰勒公式，要求展开到 x^5

   (1) $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \mathrm{o}\big(x^5\big)$

   (2) $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \mathrm{o}\big(x^5\big)$

   (3) $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \mathrm{o}\big(x^5\big)$

   (4) $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + +\frac{1}{120}x^5 + \mathrm{o}\big(x^5\big)$

   (5) $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \mathrm{o}\big(x^5\big)$

   (6) $(1+x)^\alpha = 1 + \alpha x + \frac{1}{2}\alpha(\alpha-1)x^2 + \frac{1}{6}\alpha(\alpha-1)(\alpha-2)x^3 + \frac{1}{24}\alpha(\alpha-1)(\alpha-2)(\alpha-3)x^4 + \frac{1}{120}\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)x^5 + \mathrm{o}\big(x^5\big)$

   (7) $\cos x \ln(1+x) = x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{3}{40}x^5 + \mathrm{o}\big(x^5\big)$

   (8) $e^x \cdot \sqrt{1+x} = 1 + \frac{3}{2}x + \frac{7}{8}x^2 + \frac{17}{48}x^3 + \frac{11}{128}x^4 + \frac{107}{3840}x^5 + \mathrm{o}\big(x^5\big)$

   (9) $\frac{e^x + e^{-x}}{2} = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \mathrm{o}\big(x^5\big)$

   (10) $\frac{1}{2}\ln(\frac{1-x}{1+x}) = -x - \frac{1}{3}x^3 - \frac{1}{5}x^5 + \mathrm{o}\big(x^5\big)$

   (11) $\sin^2 x = x^2 - \frac{1}{3}x^4 + \mathrm{o}\big(x^5\big)$

   (12) $\cos x^3 = 1 + \mathrm{o}\big(x^5\big)$

2. 用泰勒公式计算下列极限

   (1) $-\frac{1}{2}$

   (2) $\frac{1}{2}$

   (3) $-\frac{1}{16}$

   (4) $\frac{1}{3}$

   (5) $\frac{1}{2}$

3. 用泰勒公式求下列高阶导数

   (1) $f^{(5)}(0) = -8$

(2) $f^{(3)}(0) = -3\sin 1$

过程提示：当 $x \to 0$ 时，$y = e^x \to 1$，因此要将 $\sin(y)$ 在 $y=1$ 处展开为

$$\sin y = \sin 1 + \cos 1 \cdot (y-1) + \frac{-\sin 1}{2}(y-1)^2 + \frac{-\cos 1}{6}(y-1)^3 + o((y-1)^3)$$

而 $y - 1 = e^x - 1 = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$

则 $y - 1 \sim x$，则 $o((y-1)^3) = o(x^3)$

将 $y - 1 = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$ 代入上式化简，写成 x 在 0 处的泰勒展开，只保留不超过 x^3 的项，计算可得 x^3 的系数为 $-\frac{\sin 1}{2}$，则 $f^{(3)}(0) = 3! \cdot -\frac{\sin 1}{2} = -3\sin 1$

证明题

1. (1) 令 $f(x) = e^x$。当 $x > 0$ 时，存在 $c \in (0, x)$，使得

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}f^{(3)}(c) \cdot x^3$$

由于 $f^{(3)}(x) = e^x > 0$，则 $f^{(3)}(c) > 0$。又 $x > 0$，则

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}f^{(3)}(c) \cdot x^3 > 1 + x + \frac{1}{2}x^2$$

当 $x < 0$ 时，道理同上，因为 $x < 0$，则

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}f^{(3)}(c) \cdot x^3 < 1 + x + \frac{1}{2}x^2$$

(2) 令 $f(x) = \sin x$。当 $x \in (0, \frac{\pi}{2})$ 时，存在 $c \in (0, x)$，使得

$$\sin x = x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(c)}{6}x^3$$

由于 $f''(x) = -\sin x, f^{(3)}(x) = -\cos x$，则 $f''(0) = 0$，且由于 $c \in (0, \frac{\pi}{2})$，有 $\frac{f^{(3)}(c)}{6} < 0$。则

$$\sin x = x + \frac{f^{(3)}(c)}{6}x^3 < x$$

同理，存在 $d \in (0, x)$，使得

$$\sin x = x - \frac{1}{6}x^3 + \frac{f^{(5)}(d)}{5!}x^5$$

又 $f^{(5)}(d) = \cos d > 0$，则

$$\sin x = x - \frac{1}{6}x^3 + \frac{f^{(5)}(d)}{5!}x^5 > x - \frac{1}{6}x^3$$