第四次作业答案

解答题

1. 计算下列函数图像围成的图形的面积.

(1) $y = 2x^2 + 4x + 3$ 与 $y = -4x - 3$

这两个函数的交点为 $(-3, 9)$, $(-1, 1)$, 面积为 $\int_{-3}^{-1} -4x - 3 - (2x^2 + 4x + 3)\ \mathrm{d}x = \int_{-3}^{-1} (-2x^2 - 8x - 6)\ \mathrm{d}x = (-\frac{2}{3}x^3 - 4x^2 - 6x)|_{-3}^{-1} = (\frac{2}{3} - 4 + 6) - (18 - 36 + 18) = \frac{8}{3}$

(2) $y = x^3$ 与 $y = x$

这两个函数的交点为 $(-1, -1), (0, 0), (1, 1)$, 面积为 $\int_{-1}^{0} x^3 - x\ \mathrm{d}x + \int_{0}^{1} x - x^3\ \mathrm{d}x = \frac{1}{2}$

(3) $y = x^2$ 与 $y = 2^x$. *(你可以将这两个函数的第一个交点的横坐标记为 $a, a \approx -0.77$ 但难以算出具体数值. 你的结果是一个包含 a 的式子)*

这两个函数的交点为 $(a, a^2), (2, 4), (4, 16)$, 面积为 $\int_{a}^{2} 2^x - x^2\ \mathrm{d}x + \int_{2}^{4} x^2 - 2^x\ \mathrm{d}x = (\frac{2^x}{\ln 2} - \frac{x^3}{3})|_a^2 + (\frac{x^3}{3} - \frac{2^x}{\ln 2})|_2^4 = [(\frac{4}{\ln 2} - \frac{8}{3}) - (\frac{2^a}{\ln 2} - \frac{a^3}{3})] + [(\frac{64}{3} - \frac{16}{\ln 2}) - (\frac{8}{3} - \frac{4}{\ln 2})] = \frac{48}{3} - \frac{8 + 2^a}{\ln 2} + \frac{a^3}{3}$

2. 求下列不定积分（以下默认$a > 0$）

(1) $\int \frac{x}{1 + x^4}\ \mathrm{d}x = \int \frac{\mathrm{d}x^2/2}{1 + x^4} \xlongequal{t=x^2} \frac{1}{2} \int \frac{\mathrm{d}t}{1 + t^2} = \frac{1}{2} \arctan t + C = \frac{1}{2} \arctan(x^2) + C$

(2) $\int \frac{1}{\cos x}\ \mathrm{d}x = \int \frac{\cos x\ \mathrm{d}x}{\cos^2 x} = \int \frac{\mathrm{d}(\sin x)}{1 - \sin^2 x} \xlongequal{t=\sin x} \int \frac{\mathrm{d}t}{1 - t^2} \xlongequal{(8)} \frac{1}{2} \ln |\frac{1 + t}{1 - t}| + C = \frac{1}{2} \ln |\frac{1 + \sin x}{1 - \sin x}| + C$
$= \frac{1}{2} \ln |\frac{(1 + \sin x)^2}{(1 - \sin x)(1 + \sin x)}| + C = \frac{1}{2} \ln |\frac{(1 + \sin x)^2}{\cos^2 x}| + C = \ln |\frac{1 + \sin x}{\cos x}| + C$

(3) $\int \sin^3 x \cos^2 x\ \mathrm{d}x = \int \sin^2 x \cos^2 x (\sin x\ \mathrm{d}x) = \int (1 - \cos^2 x) \cos^2 x (-1)\ \mathrm{d}(\cos x) \xlongequal{t=\cos x} -\int (1 - t^2) t^2\ \mathrm{d}t$
$= \int (t^4 - t^2)\ \mathrm{d}t = \frac{t^5}{5} - \frac{t^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$

(4) $\int \sin^2 x \cos^3 x\ \mathrm{d}x = \int \sin^2 x \cos^2 x (\cos x\ \mathrm{d}x) = \int \sin^2 x (1 - \sin^2 x)\ \mathrm{d}(\sin x) \xlongequal{t=\sin x} \int t^2 (1 - t^2)\ \mathrm{d}t$
$= \int (t^2 - t^4)\ \mathrm{d}t = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

(5) $\int \frac{\sin^2 x}{\cos x}\ \mathrm{d}x = \int \frac{1 - \cos^2 x}{\cos x}\ \mathrm{d}x = \int \frac{1}{\cos x}\ \mathrm{d}x - \int \cos x\ \mathrm{d}x \xlongequal{(2)} \ln |\frac{1 + \sin x}{\cos x}| - \sin x + C$

(6) $\int \sin^2 x \cos^2 x\ \mathrm{d}x = \int (\sin x \cos x)^2\ \mathrm{d}x = \int (\frac{1}{2} \sin 2x)^2\ \mathrm{d}x = \frac{1}{4} \int \sin^2 2x\ \mathrm{d}x = \frac{1}{4} \int \frac{1 - \cos 4x}{2}\ \mathrm{d}x$
$= \frac{1}{8} (\int 1\ \mathrm{d}x - \int \cos 4x\ \mathrm{d}x) = \frac{1}{8} (x - \frac{1}{4} \sin 4x) + C = \frac{x}{8} - \frac{\sin 4x}{32} + C$

(7) $\displaystyle\int \frac{\sin^2 x}{\cos^4 x}\ \mathrm{d}x = \int \tan^2 x(\frac{\mathrm{d}x}{\cos^2 x}) = \int \tan^2 x\ \mathrm{d}(\tan x) \xlongequal{t=\tan x} \int t^2\ \mathrm{d}t = \frac{t^3}{3} + C = \frac{\tan^3 x}{3} + C$

(8) $\displaystyle\int \frac{1}{1-x^2}\ \mathrm{d}x \xlongequal[a=1]{(9)} \frac{1}{2}\ln|\frac{1+x}{1-x}| + C$

(9) $\displaystyle\int \frac{1}{a^2-x^2}\ \mathrm{d}x = \int \frac{1}{2a}(\frac{1}{a-x} + \frac{1}{a+x})\ \mathrm{d}x = \frac{1}{2a}(\int \frac{\mathrm{d}x}{a-x} + \int \frac{\mathrm{d}x}{a+x}) = \frac{1}{2a}(-\ln|a-x| + \ln|a+x|) + C$
$\displaystyle= \frac{1}{2a}\ln|\frac{a+x}{a-x}| + C$

(10) $\displaystyle\int \frac{1}{1+x^2}\ \mathrm{d}x = \arctan x + C$

(11) $\displaystyle\int \frac{1}{a^2+x^2}\ \mathrm{d}x = \frac{1}{a^2}\int \frac{\mathrm{d}x}{1+(\frac{x}{a})^2} \xlongequal{t=\frac{x}{a}} \frac{1}{a^2}\int \frac{a\ \mathrm{d}t}{1+t^2} = \frac{1}{a}\arctan t + C = \frac{1}{a}\arctan\frac{x}{a} + C$

(12) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\ \mathrm{d}x = \arcsin x + C$

(13) $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\ \mathrm{d}x = \frac{1}{a}\int \frac{\mathrm{d}x}{\sqrt{1-(\frac{x}{a})^2}} \xlongequal{t=\frac{x}{a}} \frac{1}{a}\int \frac{a\ \mathrm{d}t}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin\frac{x}{a} + C$

(14) $\displaystyle\int \sqrt{1-x^2}\ \mathrm{d}x \xlongequal[t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]]{x=\sin t} \int \cos t(\cos t\ \mathrm{d}t) = \int \frac{1+\cos 2t}{2}\ \mathrm{d}t = \frac{t}{2} + \frac{\sin 2t}{4} + C = \frac{t}{2} + \frac{\sin t\cos t}{2} + C$
$\displaystyle= \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C$

(15) $\displaystyle\int \sqrt{a^2-x^2}\ \mathrm{d}x = a\int \sqrt{1-(\frac{x}{a})^2}\ \mathrm{d}x \xlongequal{t=\frac{x}{a}} a\int \sqrt{1-t^2}a\ \mathrm{d}t \xlongequal{(14)} a^2(\frac{\arcsin t}{2} + \frac{t\sqrt{1-t^2}}{2}) + C$
$\displaystyle= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{2} + C$

(16) $\displaystyle\int \sqrt{1+x^2}\ \mathrm{d}x \xlongequal[t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)]{x=\tan t} \int \frac{1}{\cos t}(\frac{\mathrm{d}t}{\cos^2 t}) = \int \frac{\cos t\ \mathrm{d}t}{\cos^4 t} = \int \frac{\mathrm{d}(\sin t)}{(1-\sin^2 x)^2} \xlongequal{s=\sin t} \int \frac{\mathrm{d}s}{(1-s^2)^2} = \int \frac{\mathrm{d}s}{(1-s)^2(1+s)^2}$
$\displaystyle= \int \frac{1}{4}(\frac{1}{1-s} + \frac{1}{1+s} + \frac{1}{(1-s)^2} + \frac{1}{(1+s)^2})\ \mathrm{d}s = \frac{1}{4}(-\ln|1-s| + \ln|1+s| + \frac{1}{1-s} - \frac{1}{1+s}) + C$
$\displaystyle= \frac{1}{4}(\ln|\frac{1+s}{1-s}| + \frac{2s}{1-s^2}) + C = \frac{1}{4}(\ln|\frac{1+\sin t}{1-\sin t}| + \frac{2\sin t}{1-\sin^2 t}) + C = \frac{1}{4}(\ln|\frac{1+\frac{x}{\sqrt{x^2+1}}}{1-\frac{x}{\sqrt{x^2+1}}}| + \frac{2\frac{x}{\sqrt{x^2+1}}}{1-\frac{x^2}{x^2+1}}) + C$
$\displaystyle= \frac{1}{4}(\ln|\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}-x}| + 2x\sqrt{x^2+1}) + C = \frac{1}{4}\ln|\frac{(\sqrt{x^2+1}+x)^2}{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}| + \frac{1}{2}x\sqrt{x^2+1} + C$
$\displaystyle= \frac{1}{2}\ln|\sqrt{x^2+1}+x| + \frac{1}{2}x\sqrt{x^2+1} + C$

(17) $\displaystyle\int \sqrt{a^2+x^2}\ \mathrm{d}x = a\int \sqrt{1+(\frac{x}{a})^2}\ \mathrm{d}x \xlongequal{t=\frac{x}{a}} a\int \sqrt{1+t^2}(a\ \mathrm{d}t) \xlongequal{(16)} a^2(\frac{1}{2}\ln|\sqrt{t^2+1}+t| + \frac{1}{2}t\sqrt{t^2+1}) + C$
$\displaystyle= \frac{a^2}{2}\ln|\sqrt{(\frac{x}{a})^2+1} + \frac{x}{a}| + \frac{a^2}{2}\frac{x}{a}\sqrt{(\frac{x}{a})^2+1} + C \xlongequal{C'=C-\frac{a^2}{2}\ln a} \frac{a^2}{2}\ln|\sqrt{x^2+a^2}+x| + \frac{1}{2}x\sqrt{x^2+a^2} + C'$

(18) $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\ \mathrm{d}x \xlongequal[t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)]{x=\tan t} \int \cos t(\frac{\mathrm{d}t}{\cos^2 t}) = \int \frac{1}{\cos t}\ \mathrm{d}t \xlongequal{(2)} \ln|\frac{1+\sin t}{\cos t}| + C = \ln|\frac{1+(x/\sqrt{x^2+1})}{1/\sqrt{x^2+1}}| + C$
$\displaystyle= \ln(x+\sqrt{x^2+1}) + C$

(19) $\displaystyle\int \frac{1}{\sqrt{a^2+x^2}}\ \mathrm{d}x = \frac{1}{a}\int \frac{\mathrm{d}x}{\sqrt{1+(\frac{x}{a})^2}} \xlongequal{t=\frac{x}{a}} \frac{1}{a}\int \frac{a\ \mathrm{d}t}{\sqrt{1+t^2}} \xlongequal{(18)} \ln(t+\sqrt{t^2+1}) + C$
$\displaystyle= \ln(\frac{x}{a} + \sqrt{(\frac{x}{a})^2+1}) + C \xlongequal{C'=C-\ln a} \ln(x+\sqrt{x^2+a^2}) + C'$

(20) $\int \frac{\sqrt{4-x^2}}{x}\ dx \xlongequal[t\in[-\frac{\pi}{2},\frac{\pi}{2}]]{x=2\sin t} \int \frac{2\cos t}{2\sin t}(2\cos t\ dt) = 2\int \frac{\cos^2 t}{\sin t}\ dt = 2\int \frac{1-\sin^2 t}{\sin t}\ dt = 2(\int \frac{dt}{\sin t} - \int \sin t\ dt)$
$= 2(\ln|\frac{1-\cos t}{\sin t}| + \cos t) + C = 2(\ln|\frac{1-\sqrt{1-(\frac{x}{2})^2}}{x/2}| + \sqrt{1-(\frac{x}{2})^2}) + C = 2\ln|\frac{2-\sqrt{4-x^2}}{x}| + \sqrt{4-x^2} + C$

(21) $\int \sqrt{1+2x}\ dx = \int \sqrt{1+2x}\frac{d(1+2x)}{2} \xlongequal{t=1+2x} \int \sqrt{t}\frac{dt}{2} = \frac{t^{3/2}}{3} + C = \frac{(1+2x)\sqrt{1+2x}}{3} + C$

(22) $\int x\sqrt{2x^2+7}\ dx = \int \sqrt{2x^2+7}\frac{d(2x^2+7)}{4} \xlongequal{t=2x^2+7} \frac{1}{4}\int \sqrt{t}\ dt = \frac{1}{4}\frac{2}{3}t^{\frac{3}{2}} + C = \frac{1}{6}(2x^2+7)^{\frac{3}{2}} + C$

(23) $\int (2x^{\frac{3}{2}}+1)^{\frac{2}{3}}\sqrt{x}\ dx = \int (2x^{\frac{3}{2}}+1)^{\frac{2}{3}}\frac{d(2x^{\frac{3}{2}}+1)}{3} \xlongequal{t=2x^{\frac{3}{2}}+1} \int \frac{1}{3}t^{\frac{2}{3}}\ dt = \frac{1}{5}t^{\frac{5}{3}} + C = \frac{1}{5}(2x^{\frac{3}{2}}+1)^{\frac{5}{3}} + C$

(24) $\int \frac{2x-1}{\sqrt{1-x^2}}\ dx = \int \frac{2x\ dx}{\sqrt{1-x^2}} - \int \frac{dx}{\sqrt{1-x^2}} = \int \frac{-d(1-x^2)}{\sqrt{1-x^2}} - \arcsin x \xlongequal{t=1-x^2} -\int \frac{dt}{\sqrt{t}} - \arcsin x$
$= -2\sqrt{t} - \arcsin x + C = -2\sqrt{1-x^2} - \arcsin x + C$

(25) $\int \frac{1}{(a^2-x^2)^{\frac{3}{2}}}\ dx \xlongequal[t\in[-\frac{\pi}{2},\frac{\pi}{2}]]{x=a\sin t} \int \frac{a\cos t\ dt}{(a^2-a^2\sin^2 t)^{\frac{3}{2}}} = \int \frac{a\cos t\ dt}{a^3\cos^3 t} = \frac{1}{a^2}\int \frac{dt}{\cos^2 t} = \frac{1}{a^2}\tan t + C$
$= \frac{1}{a^2}\frac{x/a}{\sqrt{1-(\frac{x}{a})^2}} + C = \frac{1}{a^2}\frac{x}{\sqrt{a^2-x^2}} + C$

(26) $\int \frac{x^3}{\sqrt{1-x^8}}\ dx = \int \frac{d(x^4)}{4\sqrt{1-x^8}} \xlongequal{t=x^4} \int \frac{dt}{4\sqrt{1-t^2}} = \frac{\arcsin t}{4} + C = \frac{\arcsin(x^4)}{4} + C$

(27) $\int \sin x\sin 2x\ dx = \int 2\sin^2 x\cos x\ dx = 2\int \sin^2 x\ d(\sin x) \xlongequal{t=\sin x} 2\int t^2\ dt = \frac{2}{3}t^3 + C = \frac{2}{3}\sin^3 x + C$

证明题

1. *已知 $F(t)$ 是 $f(t)$ 的一个原函数, 则 $F(g(x))$ 是 $f(g(x))g'(x)$ 的一个原函数.*

   证明: 因为 $F(t)$ 是 $f(t)$ 的一个原函数, 所以 $F'(t) = f(t)$.

   所以 $(F(g(x)))' = F'(g(x))g'(x) = f(g(x))g'(x)$. 所以$F(g(x))$ 是 $f(g(x))g'(x)$ 的一个原函数.

2. *在3月10日课上的例题中,我们通过两种不同的换元方式得到了 $\int \frac{dx}{\sin x}$ 的不同结果:*

$$\int \frac{dx}{\sin x} = -\frac{1}{2}\ln|\frac{1+\cos x}{1-\cos x}| + C$$
$$\int \frac{dx}{\sin x} = \ln|\tan\frac{x}{2}| + C$$

   请验证: $-\frac{1}{2}\ln|\frac{1+\cos x}{1-\cos x}|$ 和 $\ln|\tan\frac{x}{2}|$ 的定义域和周期相同, 且它们的差值是一个常数.

   [Hint: 你可以把它们的差值当成新的函数,利用求导来证明这个新函数在一个周期内为常数.]

   记 $f(x) = -\frac{1}{2}\ln|\frac{1+\cos x}{1-\cos x}|$, $g(x) = \ln|\tan\frac{x}{2}|$, $F(x) = f(x) - g(x)$. 我们希望证明 $F(x)$在定义域上为常数.

   则 $f(x)$ 的定义域为 $\{x \in R|\cos x \neq \pm 1\} = \{x \in R|x \neq k\pi, k \in Z\}$

   $g(x)$ 的定义域为 $\{x \in R|\frac{x}{2} \neq k\pi + \frac{\pi}{2}(k \in Z), \tan\frac{x}{2} \neq 0\} = \{x \in R|\frac{x}{2} \neq k\frac{\pi}{2}, k \in Z\} = \{x \in R|x \neq k\pi, k \in Z\}$.

   因此$f(x)$ 与 $g(x)$ 定义域相同, 均为 $\{x \in R|x \neq k\pi, k \in Z\}$, 这也是 $F(x)$的定义域.

   验证可知,$f(x+2\pi) = f(x), f(x+\pi) = -f(x); g(x+2\pi) = g(x), g(x+\pi) = -g(x)$.

   因此$f(x)$ 与 $g(x)$ 的周期相同, 均为 2π, 这也是 $F(x)$ 的周期. 我们要验证 $F(x)$ 是一个常数, 只要验证 $F(x)$ 在一个周期(比如 $(0,\pi)\cup(\pi,2\pi)$)上是一个常数即可.

$$\begin{aligned}
F(x) &= f(x) - g(x) \\
&= -\frac{1}{2}\ln|\frac{1+\cos x}{1-\cos x}| - \ln|\tan\frac{x}{2}| \\
F'(x) &= -\frac{1}{2}\frac{1-\cos x}{1+\cos x}\frac{(-\sin x)(1-\cos x)-(1+\cos x)\sin x}{(1-\cos x)^2} - \frac{1}{\tan\frac{x}{2}}\frac{1}{\cos^2\frac{x}{2}}\frac{1}{2} \\
&= -\frac{1}{2}\frac{1-\cos x}{1+\cos x}\frac{-2\sin x}{(1-\cos x)^2} - \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} \\
&= \frac{\sin x}{1-\cos^2 x} - \frac{1}{\sin x} \\
&= \frac{\sin x}{\sin^2 x} - \frac{1}{\sin x} \\
&= \frac{1}{\sin x} - \frac{1}{\sin x} \\
&= 0
\end{aligned}$$

$F'(x)=0$ 告诉我们, $F(x)$ 在连续的区间上是常数.i.e. F(x) 在 $(0,\pi)$ 上是一个常数, 在$(\pi,2\pi)$ 上也是一个常数, 但是这两个常数不一定是同一个. 我们需要进行检验.

$F(\frac{\pi}{2}) = -\frac{1}{2}\ln|\frac{1+\cos\frac{\pi}{2}}{1-\cos\frac{\pi}{2}}| - \ln|\tan\frac{\pi}{4}| = -\frac{1}{2}\ln|\frac{1}{1}| - \ln|1| = 0;$

$F(\frac{3\pi}{2}) = -\frac{1}{2}\ln|\frac{1+\cos\frac{3\pi}{2}}{1-\cos\frac{3\pi}{2}}| - \ln|\tan\frac{3\pi}{4}| = -\frac{1}{2}\ln|\frac{1}{1}| - \ln|-1| = 0$

因此, $F(x)$ 在$(0,\pi)\cup(\pi,2\pi)$)上恒为0. 由周期性可知, $F(x)$ 在定义域上恒为0.

因此, $f(x)=g(x)$ 恒成立, 即 $-\frac{1}{2}\ln|\frac{1+\cos x}{1-\cos x}| = \ln|\tan\frac{x}{2}|$ 恒成立.

3. *已知 $x=g(t)$ 是一个可逆函数, 且 $F(t)$ 是 $f(g(t))g'(t)$的一个原函数, 则 $F(g^{-1}(x))$ 是 $f(x)$的原函数.*

   证明: 因为 $F(t)$ 是 $f(g(t))g'(t)$的一个原函数, 所以 $F'(t)=f(g(t))g'(t)$.

   所以 $(F(g^{-1}(x)))' = F'(g^{-1}(x))(g^{-1}(x))' = f(g(g^{-1}(x)))g'(g^{-1}(x))\frac{1}{g'(g^{-1}(x))} = f(x)$

   所以$F(g^{-1}(x))$ 是 $f(x)$的原函数.

4. *双曲函数是与三角函数具有相似性质的一类函数,也可以用于第二消元法中. 它们的定义如下:*

$$\text{双曲正弦}\textit{(hyperbolic sine):}\quad \sinh x = \frac{e^x - e^{-x}}{2}$$
$$\text{双曲余弦}\textit{(hyperbolic cosine):}\quad \cosh x = \frac{e^x + e^{-x}}{2}$$
$$\text{双曲正切}\textit{(hyperbolic tangent):}\quad \tanh x = \frac{\sinh x}{\cosh x}$$

   这三个函数的反函数分别记为 $arsinhx$, $arcoshx$, $artanhx$.

   请根据以上定义证明下方结论:

   (1) $\cosh^2 x - \sinh^2 x = 1$ *(这一点和三角函数的*$\cos^2 x + \sin^2 x = 1$ *非常像)*

   (2) $arsinhx = \ln(x+\sqrt{x^2+1}), (arsinhx)' = \frac{1}{\sqrt{x^2+1}}.$

   (3) $arcoshx = \ln(x+\sqrt{x^2-1}), (arsinhx)' = \frac{1}{\sqrt{x^2-1}}.$

   (4) $artanhx = \frac{1}{2}\ln\frac{1+x}{1-x}, (artanhx)' = \frac{1}{1-x^2}.$

   [如果你对双曲函数感兴趣,可以参考英文课本 Section 3.11 Hyperbolic Functions (P259-P264) 或中文课本附录 A.3.3 双曲函数及常用公式 (P451-P457).]

(1) $\cosh^2 x - \sinh^2 x = (\frac{e^x+e^{-x}}{2})^2 - (\frac{e^x-e^{-x}}{2})^2 = \frac{e^{2x}+2+e^{-2x}}{4} - \frac{e^{2x}-2+e^{-2x}}{4} = \frac{4}{4} = 1$

(2) $\mathrm{arsinh}x$ 是 $y = \frac{e^x-e^{-x}}{2}$ 的反函数.

首先, $y = \frac{e^x-e^{-x}}{2}$ 是奇函数, 定义域为 $(-\infty,+\infty)$, 值域为 $(-\infty,+\infty)$, 在定义域上单调递增, 因此在整个定义域上都有反函数 $\mathrm{arsinh}x$, 且 $\mathrm{arsinh}x$ 也是奇函数, 它的定义域与值域也都是$(-\infty,+\infty)$.

$y = \frac{e^x-e^{-x}}{2} \Leftrightarrow e^{2x} - 2ye^x - 1 = 0$. 所以 e^x是方程 $t^2-2yt-1$的大于 0 的解, 由此可知 $e^x = \frac{2y+\sqrt{4y^2+4}}{2} = y+\sqrt{y^2+1}$. 所以$x = \ln(y+\sqrt{y^2+1})$.

所以 $\mathrm{arsinh}x = \ln(x+\sqrt{x^2+1}), x \in (-\infty,+\infty)$.

对它求导可知, $(\mathrm{arsinh}x)' = \frac{1}{x+\sqrt{x^2+1}}(1+\frac{1}{2\sqrt{x^2+1}}2x) = \frac{1}{\sqrt{x^2+1}}$

(3) $\mathrm{arcosh}x$ 是 $y = \frac{e^x+e^{-x}}{2}$ 的反函数.

首先, $\frac{e^x+e^{-x}}{2}$ 是偶函数, 定义域为 $(-\infty,+\infty)$, 值域为 $(1,+\infty)$. 在 $(0,+\infty)$ 上单调递增, 在$(-\infty,0)$ 上单调递减. 因此 $\frac{e^x+e^{-x}}{2}$ 并非在整个定义域上都有反函数. 我们说的 $\mathrm{arcosh}x$ 指的是 $y = \frac{e^x+e^{-x}}{2}$ 在 $[0,+\infty)$上的反函数. $\mathrm{arcosh}x$ 的定义域是 $[1,+\infty)$, 值域是$[0,+\infty)$.

$y = \frac{e^x+e^{-x}}{2} \Leftrightarrow e^{2x} - 2ye^x + 1 = 0$. 所以 e^x是方程 $t^2-2yt+1$的大于 0 的解, 由此可知 $e^x = \frac{2y+\sqrt{4y^2-4}}{2} = y+\sqrt{y^2-1}$.所以$x = \ln(y+\sqrt{y^2-1})$.

所以 $\mathrm{arcosh}x = \ln(x+\sqrt{x^2-1}), x \in [1,+\infty)$.

对它求导可知, $(\mathrm{arcosh}x)' = \frac{1}{x+\sqrt{x^2-1}}(1+\frac{1}{2\sqrt{x^2-1}}2x) = \frac{1}{\sqrt{x^2-1}}$

(4) $\mathrm{artanh}x$ 是 $y = \frac{e^x-e^{-x}}{e^x+e^{-x}}$的反函数.

首先, $y = \frac{e^x-e^{-x}}{e^x+e^{-x}}$ 是奇函数, 定义域为 $(-\infty,+\infty)$, 值域为 $(-1,1)$, 在定义域上单调递增, 因此在整个定义域上都有反函数 $\mathrm{artanh}x$, 且 $\mathrm{arsinh}x$ 也是奇函数, 它的定义域是 $(-1,1)$, 值域是$(-\infty,+\infty)$.

$y = \frac{e^x-e^{-x}}{e^x+e^{-x}} \Leftrightarrow e^{2x}(1-y) = 1+y \Leftrightarrow x = \frac{1}{2}\ln\frac{1+y}{1-y}$.

所以 $\mathrm{artanh}x = \frac{1}{2}\ln\frac{1+x}{1-x}, x \in (-1,1)$.

对它求导可知, $(\mathrm{artanh}x)' = \frac{1}{2}\frac{1-x}{1+x}\frac{1-x+1+x}{(1-x)^2} = \frac{1}{1-x^2}$