

Team Problems Set

Sample Problems

December 2, 2025

Instructions

This is just a collection of problems for the team round that should give a feel for the problems on the test.
Solutions are left as an exercise to the reader.

Sample Problems

SET 1

Problems 1–5, 5 point each

Problem

Problem 1. Find the largest positive integer n such that $1+2+3+\cdots+n^2$ is divisible by $1+2+3+\cdots+n$.

Problem

Problem 2. Let x , y , and z be positive real numbers such that $(x \cdot y) + z = (x + z) \cdot (y + z)$. What is the maximum possible value of xyz ?

Problem

Problem 3. Find the sum $\frac{2^1}{4^1-1} + \frac{2^2}{4^2-1} + \frac{2^4}{4^4-1} + \frac{2^8}{4^8-1} + \cdots$

Problem

Problem 4. Let a_n denote the number of sequences of 0's and 1's of length n that contain no three consecutive 1's. Find $a_{50} \pmod{1000}$.

Problem

Problem 5. Find the number of ordered triples (a, b, c) of positive integers satisfying $\text{lcm}(a, b) = 72$, $\text{lcm}(b, c) = 600$, and $\text{lcm}(c, a) = 900$.

SET 2

Problems 6–10, 6 points each

Problem

Problem 6. Let $P(x)$ be a polynomial with integer coefficients such that $P(0) = 0$ and $P(P(x)) = x^4 - 4x^3 + 6x^2 - 4x$. Find $P(5)$.

Problem

Problem 7. Find the number of subsets S of $\{1, 2, 3, \dots, 15\}$ such that for any two distinct elements $a, b \in S$, we have $a + b \notin S$.

Problem

Problem 8. Let $\omega = e^{2\pi i/7}$. Compute $\prod_{k=1}^6 (2 - \omega^k - \omega^{-k})$.

Problem

Problem 9. In triangle ABC , $AB = 13$, $BC = 14$, and $CA = 15$. Point P lies inside the triangle such that $\angle APB = \angle BPC = \angle CPA = 120$. Find $AP^2 + BP^2 + CP^2$.

Problem

Problem 10. Find the number of functions $f : \{1, 2, \dots, 10\} \rightarrow \{1, 2, \dots, 10\}$ such that $f(f(f(x))) = x$ for all x and f has no fixed points.

SET 3

Problems 11–15, 7 points each

Problem

Problem 11. Find the number of ways to color the vertices of a regular hexagon with three colors such that no two adjacent vertices have the same color, up to rotational symmetry.

Problem

Problem 12. Let a_1, a_2, \dots, a_n be a sequence defined by $a_1 = 1$, $a_2 = 3$, and $a_n = \frac{a_{n-1}^2 + 2}{a_{n-2}}$ for $n \geq 3$. Find a_{2024} .

Problem

Problem 13. Let S be the set of all 2×2 matrices with entries from $\{0, 1, 2\}$ such that the determinant is congruent to 1 modulo 3. Find $|S|$.

Problem

Problem 14. Find the smallest positive integer n such that n^2 can be expressed as the sum of two perfect cubes in at least two different ways.

Problem

Problem 15. A sequence of real numbers x_1, x_2, \dots, x_{100} satisfies $x_1 = 1$ and $x_{n+1} = x_n + \frac{1}{x_n}$ for $n = 1, 2, \dots, 99$. Find $\lfloor x_{100} \rfloor$.

SET 4

Problems 16–20, 8 points each

Problem

Problem 16. Let G be a graph on 12 vertices where every vertex has degree 5. Find the minimum number of 4-cycles that G must contain.

Problem

Problem 17. Find the coefficient of x^{100} in the expansion of $\frac{1}{(1-x)(1-x^2)(1-x^5)(1-x^{10})}$.

Problem

Problem 18. Let $ABCD$ be a cyclic quadrilateral with $AB = 5$, $BC = 6$, $CD = 7$, and $DA = 8$. Let P be the intersection of diagonals AC and BD . Find $AP \cdot PC$.

Problem

Problem 19. Find the number of sequences $(a_1, a_2, \dots, a_{15})$ where each $a_i \in \{1, 2, 3, 4, 5\}$ and no two consecutive terms are equal, modulo 1000.

Problem

Problem 20. Let $n = 2^{10} \cdot 3^5 \cdot 5^3 \cdot 7^2$. Find the number of divisors d of n such that d^2 also divides n .

SET 5

Problems 21–25, 9 points each

Problem

Problem 21. Let $p(x) = x^4 + x^3 + x^2 + x + 1$. Find the smallest positive integer n such that $p(x)$ divides $x^n - 1$ over the integers.

Problem

Problem 22. Find the number of ways to select 10 distinct positive integers, each at most 100, such that their sum is divisible by 10.

Problem

Problem 23. In tetrahedron $ABCD$, $AB = CD = 10$, $AC = BD = 11$, and $AD = BC = 12$. Find the volume of the tetrahedron.

Problem

Problem 24. Let $S_n = \sum_{k=1}^n k \cdot k!$. Find the largest integer m such that S_{100} is divisible by 10^m .

Problem

Problem 25. Find the number of functions $f : \{1, 2, 3, 4, 5, 6\} \rightarrow \{1, 2, 3, 4, 5, 6\}$ such that $f(f(f(x))) = f(x)$ for all x .

SET 6

Problems 26–30, 10 points each

Problem

Problem 26. Let a_n be the number of regions formed by n hyperplanes in 4-dimensional space in general position. Find a_5 .

Problem

Problem 27. Find the number of non-isomorphic graphs on 6 vertices with exactly 9 edges.

Problem

Problem 28. Let z_1, z_2, \dots, z_{12} be the 12th roots of unity. Compute $\sum_{1 \leq i < j \leq 12} |z_i - z_j|^2$.

Problem

Problem 29. A sequence (a_n) satisfies $a_0 = 1$, $a_1 = 2$, and $a_n = \frac{n}{n-1}a_{n-1} + \frac{1}{n-1}a_{n-2}$ for $n \geq 2$. Find a_{100} .

Problem

Problem 30. Find the number of ways to place 8 rooks on an 8×8 chessboard such that no two rooks attack each other and the rooks occupy exactly 4 of the 8 rows.

SET 7

Problems 31–35, 11 points each

Problem

Problem 31. Let M be a 10×10 matrix with entries from $\{0, 1\}$ such that each row and each column sums to 5. Find the number of such matrices modulo 1000.

Problem

Problem 32. Find the number of ordered 5-tuples $(a_1, a_2, a_3, a_4, a_5)$ of positive integers such that $a_1a_2a_3a_4a_5 = a_1 + a_2 + a_3 + a_4 + a_5$.

Problem

Problem 33. Let $P(x) = x^5 - x^4 - x^3 - x^2 - x - 1$. The five roots of P in the complex plane form a convex pentagon. Find the area of this pentagon.

Problem

Problem 34. A regular octahedron has edge length $\sqrt{2}$. Find the number of ways to color its 6 vertices with 3 colors such that no edge connects two vertices of the same color.

Problem

Problem 35. Let T_n denote the number of spanning trees of the complete graph K_n . Find the largest integer k such that 2^k divides T_{20} .