1.1.

(1) 有限次；四则

(2) 相近数；远小于

(3) 截断误差；舍入误差

(4) (4)

1.3.

x_1 : 4, $\varepsilon(x_1)=0.5\times10^{-4}$, $\varepsilon_r(x_1)=1.64\times10^{-4}$

x_2 : 2, $\varepsilon(x_2)=0.5\times10^{8}$, $\varepsilon_r(x_2)=0.0098$

x_3 : 3, $\varepsilon(x_3)=0.5$, $\varepsilon_r(x_3)=0.00125$

x_4 : 4, $\varepsilon(x_4)=0.5\times10^{-6}$, $\varepsilon_r(x_4)=1.49\times10^{-4}$

x_5 : 3, $\varepsilon(x_5)=0.5\times10^{-8}$, $\varepsilon_r(x_5)=5.71\times10^{-4}$

1.5.

$S=L^2$, $\Delta S=2L\cdot\Delta L$

$\because \Delta S\leq 1cm^2$

$\therefore 2L\cdot\Delta L\leq 1$

$\therefore \Delta L\leq\frac{1}{2L}=0.005cm$

$\therefore$ 测量的边长误差不超过 0.005cm

1.6.

$\Delta L=0.1\%\times L=0.001\times400=0.4m$

$t'=t\times\frac{L+\Delta L}{L}=60.06s$

$\Delta t=60.06-60=0.06s$

$\varepsilon_t=\frac{\Delta t}{t}=\frac{0.06}{60}=0.1\%$

$\therefore$ 误差 0.06s，相对误差 0.1%

1.11

(1) $y = \frac{1}{1+2x} - \frac{1-x}{1+x} = \frac{2x^2}{(1+2x)(1+x)}$

(2) $y = \sqrt{x+\frac{1}{x}} - \sqrt{x-\frac{1}{x}} = \frac{(x+\frac{1}{x}) - (x-\frac{1}{x})}{\sqrt{x+\frac{1}{x}} + \sqrt{x-\frac{1}{x}}} = \frac{2}{x(\sqrt{x+\frac{1}{x}} + \sqrt{x-\frac{1}{x}})}$

(3) $y = \frac{1-\cos 2x}{x} = \frac{2\sin^2 x}{x}$

(4) $y = \sin(x+\varepsilon) - \sin x = 2\cos(x+\frac{\varepsilon}{2})\sin\frac{\varepsilon}{2}$

2.1.

(4) $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

$\|A\|_1 = \max\{1, 3\} = 3$

$A^T A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$

$\det(\lambda I - A^T A) = \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-5 \end{vmatrix} = \lambda^2 - 6\lambda + 4 = 0$

$\lambda = 3 \pm \sqrt{5}$ $\therefore \lambda_{max} = 3+\sqrt{5}$

$\therefore \|A\|_2 = \sqrt{\lambda_{max}} = 2.287$

$\det(\lambda I - A) = (\lambda-1)(\lambda-2) = 0$

$\lambda_{max} = 2$

$\therefore \rho(A) = 2$

(5) $A = \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}$

同理, $\det(\lambda I - A) = (\lambda - t)(\lambda - \frac{1}{t}) = 0$

$\lambda = t, \frac{1}{t}$

又 $\because t > 1$

$\therefore \rho(A) = \lambda_{max} = t$

$\|A\|_2 = t$

$A^{-1} = \begin{pmatrix} \frac{1}{t} & 0 \\ 0 & t \end{pmatrix}$

$\|A^{-1}\|_2 = t$

$cond(A)_2 = \|A\|_2 \|A^{-1}\|_2 = t \cdot t = t^2$

(b) $A = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$

$det(\lambda I - A) = (\lambda - a)(\lambda - b)(\lambda - c) = 0$

又 $\because c > b > a > 0$

$\therefore \rho(A) = \lambda_{max} = c$

$\|A\|_2 = c$

$A^{-1} = \begin{pmatrix} \frac{1}{a} & & \\ & \frac{1}{b} & \\ & & \frac{1}{c} \end{pmatrix}$

$\|A^{-1}\|_2 = \frac{1}{a}$

$\therefore cond(A)_2 = \|A\|_2 \|A^{-1}\|_2 = c \cdot \frac{1}{a} = \frac{c}{a}$

2.11.

a) $A = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$

$\|A\|_1 = max\{2, 5\} = 5$

$\|A\|_\infty = max\{4, 3\} = 4$

$A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 13 \end{pmatrix}$

$det(\lambda I - A^T A) = \lambda^2 - 15\lambda + 25 = 0$

$\lambda = \frac{15 \pm 5\sqrt{5}}{2}$

$\rho(A) = \sqrt{\lambda_{max}} = 3.618$

$det(\lambda I - A) = \lambda^2 - 3\lambda + 5 = 0$

$\lambda = \frac{3 \pm i\sqrt{11}}{2}$

$\rho(A) = |\lambda_{max}| = \sqrt{5} = 2.236$

2.14.

$A=\begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$ $b=\begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix}$

$A^{-1}=\begin{pmatrix} \frac{2}{3} & 0 & -\frac{1}{3} \\ \frac{4}{3} & -1 & \frac{1}{3} \\ -\frac{5}{3} & 1 & \frac{1}{3} \end{pmatrix}$

$\therefore \|A\|_\infty = \max\{4,6,5\}=6$ $\|A^{-1}\|_\infty = \max\{1,\frac{8}{3},3\}=3$ $\|b\|_\infty=\max\{4,6,5\}=6$

$$\frac{1}{\|A\|\cdot\|A^{-1}\|}\frac{\|\delta_b\|}{\|b\|}\le\frac{\|\delta_x\|}{\|x\|}\le\|A\|\cdot\|A^{-1}\|\frac{\|\delta_b\|}{\|b\|}$$

$$\therefore \frac{1}{6\times3}\times\frac{0.5\times10^{-5}}{6}\le\frac{\|\delta_x\|}{\|x\|}\le 6\times3\times\frac{0.5\times10^{-5}}{6}$$

即 $\frac{\|\delta_x\|}{\|x\|}\in[4.6\times10^{-8},\ 1.5\times10^{-5}]$

3.2.

(1) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$

Jacobi 迭代: $x_i^{(k+1)}=\frac{1}{a_{ii}}(b_i-\sum_{j=1}^{i-1}a_{ij}x_j^{(k)}-\sum_{j=i+1}^{n}a_{ij}x_j^{(k)})$

取 $X^{(0)}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\therefore X^{(1)}=\begin{pmatrix} 1.5 \\ -2.5 \\ 2 \end{pmatrix}$

$X^{(2)}=\begin{pmatrix} 2.75 \\ -4.25 \\ 3.25 \end{pmatrix}$

Gauss-Seidel 迭代: $x_i^{(k+1)}=\frac{1}{a_{ii}}(b_i-\sum_{j=1}^{i-1}a_{ij}x_j^{(k+1)}-\sum_{j=i+1}^{n}a_{ij}x_j^{(k)})$

取 $X^{(0)}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\therefore X^{(1)} = \begin{pmatrix} 1.5 \\ -3.25 \\ 3.625 \end{pmatrix}$

$X^{(2)} = \begin{pmatrix} 3.125 \\ -5.875 \\ 4.9375 \end{pmatrix}$

2.1

(1) 计算中断；误差增大

(2) $O\left(\frac{n^3}{3}\right)$；$O\left(\frac{n^3}{6}\right)$

(3) $O\left(\frac{n^3}{3}\right)$；$O(5n-3)$

2.2.

(1) $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -2 & 0 \\ -2 & 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 6 \end{pmatrix}$$

$\therefore X = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

2.3

(1) $A = \begin{pmatrix} -3 & 2 & 6 \\ 10 & -7 & 0 \\ 5 & -1 & 5 \end{pmatrix}$ $b = \begin{pmatrix} 4 \\ 7 \\ 6 \end{pmatrix}$

$$\begin{pmatrix} -3 & 2 & 6 & 4 \\ 10 & -7 & 0 & 7 \\ 5 & -1 & 5 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 10 & -7 & 0 & 7 \\ -3 & 2 & 6 & 4 \\ 5 & -1 & 5 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 10 & -7 & 0 & 7 \\ 0 & -0.1 & 6 & 6.1 \\ 0 & 2.5 & 5 & 2.5 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 10 & -7 & 0 & 7 \\ 0 & 2.5 & 5 & 2.5 \\ 0 & -0.1 & 6 & 6.1 \end{pmatrix} \longrightarrow \begin{pmatrix} 10 & -7 & 0 & 7 \\ 0 & 2.5 & 5 & 2.5 \\ 0 & 0 & 6.2 & 6.2 \end{pmatrix}$$

$\therefore X = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

2.5 LU分解

(1) $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$L = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ -2 & 3 & 1 \end{pmatrix}$ $U = \begin{pmatrix} 1 & 1 & -1 \\ & 1 & -1 \\ & & 2 \end{pmatrix}$

$Ly = b \Rightarrow y = \begin{pmatrix} 1 \\ -1 \\ 6 \end{pmatrix}$

$Ux = y \Rightarrow x = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

(2) $A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$

$L = \begin{pmatrix} 1 & & & \\ \frac{3}{4} & 1 & & \\ \frac{1}{2} & \frac{6}{7} & 1 & \\ \frac{1}{4} & \frac{5}{7} & & 1 \end{pmatrix}$

$U = \begin{pmatrix} 4 & 3 & 2 & 1 \\ & \frac{7}{4} & \frac{3}{2} & \frac{5}{4} \\ & & \frac{12}{7} & \frac{10}{7} \\ & & & \frac{5}{3} \end{pmatrix}$

$Ly = b \Rightarrow y = \begin{pmatrix} 1 \\ \frac{1}{4} \\ -\frac{12}{7} \\ 0 \end{pmatrix}$

$$Ux = y \Rightarrow \quad x = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

3.4.

(1) $A = \begin{pmatrix} 5 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix}$

$5 > 2+1$

$3 \geq 1+2$

$2 \geq 1+1$

$\therefore$ A对角占优

又$\because$ A不可约

$\therefore$ Jacobi迭代和Gauss-Seidel迭代均收敛

(2) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

$B_J = -\begin{pmatrix} 0 & 2 \\ \frac{3}{2} & 0 \end{pmatrix}$

$\rho(B_J) = \sqrt{3} > 1$

$\therefore$ Jacobi不收敛

$B_S = \begin{pmatrix} 0 & -2 \\ 0 & 3 \end{pmatrix}$

$\rho(B_S) = 3 > 1$

$\therefore$ Gauss-Seidel不收敛

3.6.

(1) $\begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix} = 1 - a^2 > 0$

$\therefore a \in (-1, 1)$

$$\begin{vmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{vmatrix} = 1 - 3a^2 + 2a^3 > 0$$

$\therefore a \in (-\frac{1}{2}, 1) \cup (1, +\infty)$

综上, $a \in (-\frac{1}{2}, 1)$

(2) $B_J = D^{-1}(L+U) = \begin{pmatrix} 0 & -a & -a \\ -a & 0 & -a \\ -a & -a & 0 \end{pmatrix}$

$|\lambda I - B_J| = \lambda^3 - 3a^2\lambda + 2a^3 = 0$

$\lambda = -a$ 或 $-2a$

又 $\because \rho(B_J) < 1$

$\therefore |-2a| < 1$

$\therefore a \in (-\frac{1}{2}, \frac{1}{2})$

4.2.

(1) $A = \begin{pmatrix} 6 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$v_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad u_0 = \frac{v_0}{\max(v_0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$v_1 = Au_0 = \begin{pmatrix} 9 \\ 6 \\ 3 \end{pmatrix}$ $\quad u_1 = \frac{v_1}{\max(v_1)} = \begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ $\quad \lambda = 9$

$v_2 = Au_1 = \begin{pmatrix} \frac{23}{3} \\ \frac{13}{3} \\ 2 \end{pmatrix}$ $\quad u_2 = \frac{v_2}{\max(v_2)} = \begin{pmatrix} 1 \\ \frac{13}{23} \\ \frac{6}{23} \end{pmatrix}$ $\quad \lambda = \frac{23}{3}$

$\therefore \lambda = \frac{23}{3}$, $\quad x = \begin{pmatrix} 1 \\ \frac{13}{23} \\ \frac{6}{23} \end{pmatrix}$

4.7.

$$A=\begin{pmatrix}1&2&0\\2&-1&1\\0&1&3\end{pmatrix}$$

$\max\limits_{i\neq j}|a_{ij}|=2$，故取 $p=1,\ q=2$

$$a=\cot 2\theta_k=\frac{a_{qq}-a_{pp}}{2a_{pq}}=\frac{-1-1}{2\times2}=-\frac{1}{2}<0$$

令 $t=\tan\theta_k$. $t^2+2at-1=0$

$\because a<0$

$$\therefore t=-\frac{1}{|a|+\sqrt{1+a^2}}=\frac{1-\sqrt{5}}{2}$$

$$\therefore \cos\theta_k=\frac{1}{\sqrt{1+t^2}}=0.8507 \qquad \sin\theta_k=t\cos\theta_k=0.5257$$

$$\therefore S_1=\begin{pmatrix}0.8507&-0.5257&0\\0.5257&0.8507&0\\0&0&1\end{pmatrix}$$

$$\therefore T_1=S_1^T A S_1=\begin{pmatrix}2.2362&0&0.5257\\0&-2.2362&0.8507\\0.5257&0.8507&3\end{pmatrix}$$

$\max\limits_{i\neq j}|t_{ij}|=0.8507$，故取 $p=2,\ q=3$

$$\therefore a=2\cot\theta_k=\frac{t_{qq}-t_{pp}}{2t_{pq}}=\frac{3+2.2362}{2\times0.8507}=3.0776>0$$

$$\text{同理得 } t=\frac{1}{3.0776+\sqrt{1+3.0776^2}}=0.1584$$

$$\therefore \cos\theta_k=\frac{1}{\sqrt{1+t^2}}=0.9877 \qquad \sin\theta_k=t\cos\theta_k=0.1565$$

$$\therefore S_2=\begin{pmatrix}1&0&0\\0&0.9877&0.1565\\0&-0.1565&0.9877\end{pmatrix}$$

$$\therefore T_2=S_2^T T_1 S_2=\begin{pmatrix}2.2362&-0.0820&0.5193\\-0.0820&-2.3710&0\\0.5193&0&3.1349\end{pmatrix}$$

$$Q = S_1 S_2 = \begin{pmatrix} 0.8507 & -0.5192 & -0.0823 \\ 0.5257 & 0.8402 & 0.1331 \\ 0 & -0.1565 & 0.9877 \end{pmatrix}$$

$\therefore \lambda_1 = 2.2362 \quad \lambda_2 = -2.3710 \quad \lambda_3 = 3.1349$

$$u_1 = \begin{pmatrix} 0.8507 \\ 0.5257 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} -0.5192 \\ 0.8402 \\ -0.1565 \end{pmatrix} \quad u_3 = \begin{pmatrix} -0.0823 \\ 0.1331 \\ 0.9877 \end{pmatrix}$$