重庆大学
CHONGQING UNIVERSITY

1.1 (1) 将一个数学问题分解为有限次四则运算

(2) 避免两个相近数作减法计算，避免分母绝对值远小于分子绝对值

(3) 数值分析处理截断误差和舍入误差

(4) 有效数字越多，相对误差越小

1.3 $x_1=0.3040$ 有效4位，误差限 0.5×10^{-4}，相对误差限 1.64×10^{-4} $\frac{误差限}{值}$

$x_2=5.1\times10^9$ 有效2位，误差限 0.5×10^8，相对误差限 0.0098

1.5 正方形边长100cm，面积误差不超过1cm

$S=L^2$ $\Delta S=2L\cdot\Delta L$ $\Delta S\leq 1cm^2$ $\Delta L\leq 0.005cm$

1.6 道长误差 0.1%，400m成绩为60s

误差 0.06s ε_t 0.1%

1.11 改进式子使计算更精确

$$y=\frac{1}{1+2x}-\frac{1-x}{1+x}\ (|x|\ll1)=\frac{2x^2}{(1+2x)(1+x)}$$

$$y=\sqrt{x+\frac{1}{x}}-\sqrt{x-\frac{1}{x}}\ (x>>1)=\frac{(x+\frac{1}{x})-(x-\frac{1}{x})}{\sqrt{x+\frac{1}{x}}+\sqrt{x-\frac{1}{x}}}=\frac{2}{x(\sqrt{x+\frac{1}{x}}+\sqrt{x-\frac{1}{x}})}$$

$$y=\frac{1-\cos 2x}{x}=\frac{2\sin^2 x}{x}\ (x<<1)$$

$$y=\sin(x+\varepsilon)-\sin x=2\cos(x+\frac{\varepsilon}{2})\sin\frac{\varepsilon}{2}$$

2.1 (1) Gauss消去法主元素为零则计算中断，主元素绝对值太小则误差增大

(2) Gauss消去法计算量 $O(\frac{n^3}{3})$ 平方根求解对称正定计算量 $O(\frac{n^3}{6})$

(3) LU分解计量 $O(\frac{n^3}{3})$ 追赶法求解对角占优 $O(5n-3)$

2.2 Gauss消去法

$$A=\begin{pmatrix}1&1&-1\\1&2&-2\\-2&1&1\end{pmatrix}\quad b=\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}1&1&-1&1\\1&2&-2&0\\-2&1&1&1\end{pmatrix}\begin{pmatrix}1&1&-1&1\\0&1&-1&-1\\0&3&-1&3\end{pmatrix}\begin{pmatrix}1&1&-1&1\\0&1&-1&-1\\0&0&2&6\end{pmatrix}\begin{pmatrix}2\\2\\3\end{pmatrix}$$

2.3 列主元消去法

$$A=\begin{pmatrix}-3&2&6\\10&-7&0\\5&-1&5\end{pmatrix}\quad b=\begin{pmatrix}4\\7\\6\end{pmatrix}$$

$$\begin{pmatrix}-3&2&6&4\\10&-7&0&7\\5&-1&5&6\end{pmatrix}\begin{pmatrix}10&-7&0&7\\-3&2&6&4\\5&-1&5&6\end{pmatrix}\begin{pmatrix}10&-7&0&7\\0&-0.1&6&6.1\\0&2.5&5&2.5\end{pmatrix}$$

$$\begin{pmatrix}10&-7&0&7\\0&2.5&5&2.5\\0&-0.1&6&6.1\end{pmatrix}\begin{pmatrix}10&-7&0&7\\0&2.5&5&2.5\\0&0&6.2&6.2\end{pmatrix}\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

2.5 LU分解

$$A=\begin{pmatrix}1&1&-1\\1&2&-2\\-2&1&1\end{pmatrix}\quad b=\begin{pmatrix}1\\0\\1\end{pmatrix}\quad L=\begin{pmatrix}1&&\\1&1&\\-2&3&1\end{pmatrix}\quad U=\begin{pmatrix}1&1&-1\\&1&-1\\&&2\end{pmatrix}$$

$$Ly=b\quad y=\begin{pmatrix}1\\-1\\6\end{pmatrix}\quad Ux=y\quad x=\begin{pmatrix}2\\2\\3\end{pmatrix}$$

$$A=\begin{pmatrix}4&3&2&1\\3&4&3&2\\2&3&4&3\\1&2&3&4\end{pmatrix}\quad b=\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\quad L=\begin{pmatrix}1&&&\\\frac{3}{4}&1&&\\\frac{1}{2}&\frac{6}{7}&1&\\\frac{1}{4}&\frac{5}{7}&&1\end{pmatrix}\quad U=\begin{pmatrix}4&3&2&1\\&\frac{7}{4}&\frac{3}{2}&\frac{5}{4}\\&&\frac{12}{7}&\frac{10}{7}\\&&&\frac{5}{3}\end{pmatrix}$$

$$Ly=b\quad y=\begin{pmatrix}1\\\frac{1}{4}\\-\frac{12}{7}\\0\end{pmatrix}\quad Ux=y\quad x=\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

2.1(4) $A=\begin{pmatrix}1&1\\0&2\end{pmatrix}$ $\|A\|_1=\max(1\ \ 3)=3$

$A^TA=\begin{pmatrix}1&0\\1&2\end{pmatrix}\begin{pmatrix}1&1\\0&2\end{pmatrix}=\begin{pmatrix}1&1\\1&5\end{pmatrix}$ $\det(\lambda I-A^TA)=\begin{vmatrix}\lambda-1&-1\\-1&\lambda-5\end{vmatrix}=\lambda^2-6\lambda+4=0$

$\lambda=3\pm\sqrt{5}$ $-\lambda_{max}=3+\sqrt{5}$ $\|A\|_2=\sqrt{\lambda_{max}}=2.287$

$\det(\lambda I-A)=(\lambda-1)(\lambda-2)=0$ $\lambda_{max}=2$ $\rho(A)=2$

(5) $A=\begin{pmatrix}t&0\\0&\frac{1}{t}\end{pmatrix}$ $t>1$ $\det(\lambda I-A)=(\lambda-t)(\lambda-\frac{1}{t})=0$

$\lambda=t,\ \frac{1}{t}$ $\rho(A)=\lambda_{max}=t$ $\|A\|_2=t$

$A^{-1}=\begin{pmatrix}\frac{1}{t}&0\\0&t\end{pmatrix}$ $\|A^{-1}\|_2=t$ $cond(A)_2=\|A\|_2\|A^{-1}\|_2=t^2$

(6) $A=\begin{pmatrix}a&&\\&b&\\&&c\end{pmatrix}$ $c>b>a>0$ $\det(\lambda I-A)=(\lambda-a)(\lambda-b)(\lambda-c)=0$

$c>b>a>0$ $\therefore \rho(A)=\lambda_{max}=c$

$\|A\|_2=c$ $A^{-1}=\begin{pmatrix}\frac{1}{a}&&\\&\frac{1}{b}&\\&&\frac{1}{c}\end{pmatrix}$ $\|A^{-1}\|_2=\frac{1}{a}$

$cond(A)_2=\|A\|_2\|A^{-1}\|_2=c\cdot\frac{1}{a}=\frac{c}{a}$

2.11 (1) $A=\begin{pmatrix}1&3\\-1&2\end{pmatrix}$ $\|A\|_1=\max\{2,5\}=5$ $\|A\|_\infty=\max\{4,3\}=4$

$A^TA=\begin{pmatrix}2&1\\-1&3\end{pmatrix}$ $\det(\lambda I-A^TA)=\lambda^2-15\lambda+25=0$

$\lambda=\frac{15\pm5\sqrt{5}}{2}$ $\rho(A)=\sqrt{\lambda_{max}}=3.618$

$\det(\lambda I-A)=\lambda^2-3\lambda+5=0$ $\lambda=\frac{3\pm i\sqrt{11}}{2}$ $\rho(A)=|\lambda_{max}|=\sqrt{5}$

2.14 $\underset{A}{\begin{pmatrix}2&1&1\\3&1&2\\1&2&2\end{pmatrix}}\underset{x}{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}}=\underset{b}{\begin{pmatrix}4\\6\\5\end{pmatrix}}$ 扰动 $\|\delta b\|=\frac{1}{2}\times10^{-5}$ $A^{-1}=\begin{pmatrix}\frac{2}{3}&0&-\frac{1}{3}\\\frac{4}{3}&-1&\frac{1}{3}\\-\frac{5}{3}&1&\frac{1}{3}\end{pmatrix}$

$\|A\|_\infty=\max\{4,6,5\}=6$ $\|A^{-1}\|_\infty=\max\{1,\frac{8}{3},3\}=3$

$\|b\|_\infty=\max\{4,6,5\}=6$

$$\frac{1}{\|A\|\cdot\|A^{-1}\|}\frac{\|\delta b\|}{\|b\|}\le\frac{\|\delta x\|}{\|x\|}\le\|A\|\cdot\|A^{-1}\|\cdot\frac{\|\delta b\|}{\|b\|}$$

3.2 $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$

Jacobi迭代 $x_i^{(k+1)}=\frac{1}{a_{ii}}\left(b_i-\sum_{j=1}^{i-1}a_{ij}x_j^{(k)}-\sum_{j=i+1}^{n}a_{ij}x_j^{(k)}\right)$

$X^{(0)}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $X^{(1)}=\begin{pmatrix} 1.5 \\ -2.5 \\ 2 \end{pmatrix}$ $X^{(2)}=\begin{pmatrix} 2.75 \\ -4.25 \\ 3.25 \end{pmatrix}$

Gauss-Seidei迭代 $x_i^{(k+1)}=\frac{1}{a_{ii}}\left(b_i-\sum_{j=1}^{i-1}a_{ij}x_j^{(k+1)}-\sum_{j=i+1}^{n}a_{ij}x_j^{(k)}\right)$

$X^{(0)}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $X^{(1)}=\begin{pmatrix} 1.5 \\ -3.25 \\ 3.625 \end{pmatrix}$ $X^{(2)}=\begin{pmatrix} 3.125 \\ -5.875 \\ 4.9375 \end{pmatrix}$

3.4 判定收敛性

(1) $A=\begin{pmatrix} 5 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix}$ $\begin{cases} 5 \geq 2+1 \\ 3 \geq 1+2 \\ 2 \geq 1+1 \end{cases}$ 对角占优 不可约 $\begin{cases} \text{Jacobi收敛} \\ \text{GaussSeidei收敛} \end{cases}$

(2) $A=\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ $B_J=-\begin{pmatrix} 0 & 2 \\ \frac{3}{2} & 0 \end{pmatrix}$ $\rho(B_J)=\sqrt{3}>1$ Jacobi不收敛

$B_S=\begin{pmatrix} 0 & -2 \\ 0 & 3 \end{pmatrix}$ $\rho(B_S)=3>1$ GaussSeidei不收敛

3.6 $\begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}$ (1) 若A正定，则 $\begin{cases} \begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix}=1-a^2>0 \quad a\in(-1,1) \\ \begin{vmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{vmatrix}=1-3a^2+2a^3>0 \quad a\in(-\frac{1}{2},1)\cup(1,+\infty) \end{cases}$

(2) 若Jacobi收敛 $B_J=D^{-1}(L+U)=\begin{pmatrix} 0 & -a & -a \\ -a & 0 & -a \\ -a & -a & 0 \end{pmatrix}$

$|\lambda I-B_J|=\lambda^3-3a^2\lambda+2a^3=0$

$\lambda=-a$ 或 $-2a$

$\rho(B_J)<1$ $|-2a|<1$ $a\in(-\frac{1}{2},\frac{1}{2})$

4.2 (1) 乘幂法求矩阵

$$A=\begin{pmatrix}6&2&1\\2&3&1\\1&1&1\end{pmatrix}\quad v_0=\begin{pmatrix}1\\1\\1\end{pmatrix}\quad u_0=\frac{v_0}{\max(v_0)}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$v_1=Au_0=\begin{pmatrix}9\\6\\3\end{pmatrix}\quad u_1=\frac{v_1}{\max(v_1)}=\begin{pmatrix}1\\\frac{2}{3}\\\frac{1}{3}\end{pmatrix}\quad \lambda=9$$

$$v_2=Au_1\begin{pmatrix}\frac{23}{3}\\\frac{13}{3}\\2\end{pmatrix}\quad u_2=\frac{v_2}{\max(v_2)}=\begin{pmatrix}1\\\frac{13}{23}\\\frac{6}{23}\end{pmatrix}\quad \lambda=\frac{23}{3}$$

$$\lambda=\frac{23}{3}\quad x=\begin{pmatrix}1\\\frac{13}{23}\\\frac{6}{23}\end{pmatrix}$$

4.7 $A=\begin{pmatrix}1&2&0\\2&-1&1\\0&1&3\end{pmatrix}$ Jacobi求特征值特征向量

$\max\limits_{i\neq j}|a_{ij}|=2$ 取 $p=1$ $q=2$ $a=\cot 2\theta_K=\dfrac{a_{qq}-a_{pp}}{2a_{pq}}=\dfrac{-1-1}{2\times 2}=-\dfrac{1}{2}<0$

令 $t=\tan\theta_K$ $t^2+2at-1=0$ $a<0$

$t=-\dfrac{1}{|a|+\sqrt{1+a^2}}=\dfrac{1-\sqrt{5}}{2}$ $\cos\theta_K=\dfrac{1}{\sqrt{1+t^2}}=0.8507$ $\sin\theta_K=t\cos\theta_K=0.5257$

$$S_1=\begin{pmatrix}0.8507&-0.5257&0\\0.5257&0.8507&0\\0&0&1\end{pmatrix}\quad T_1=S_1^TAS_1=\begin{pmatrix}2.2362&0&0.5257\\0&-2.2362&0.8507\\0.5257&0.8507&3\end{pmatrix}$$

$\max\limits_{i\neq j}|a_{ij}|=0.8507$，取 $p=2$ $q=3$ $a=2\cot\theta_K=\dfrac{t_{qq}-t_{pp}}{2t_{pq}}=3.0776>0$

$t=\dfrac{1}{3.0776+\sqrt{1+3.0776^2}}=0.1584$ $\cos\theta_K=\dfrac{1}{\sqrt{1+t^2}}=0.9877$

$\sin\theta_K=t\cos\theta_K=0.1565$ $S_2=\begin{pmatrix}1&0&0\\0&0.9877&0.1565\\0&-0.1565&0.9877\end{pmatrix}$

$$T_2=S_2'T_1S_2=\begin{pmatrix}\overset{\lambda_1}{2.2362}&-0.0820&0.5193\\-0.0820&\overset{\lambda_2}{-2.3710}&0\\0.5193&0&\overset{\lambda_3}{3.1349}\end{pmatrix}$$

$$Q=S_1S_2=\begin{array}{c}\begin{pmatrix}0.8507&-0.5192&-0.0823\\0.5257&0.8402&0.1331\\0&-0.1565&0.9877\end{pmatrix}\\\begin{matrix}u_1&\quad u_2&\quad u_3\end{matrix}\end{array}$$