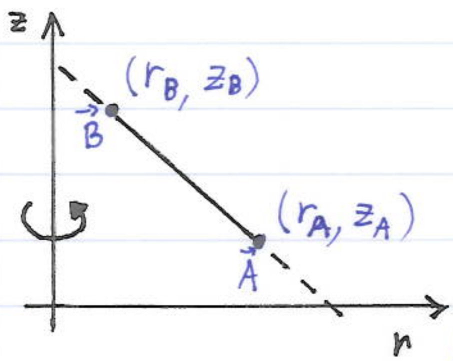
Cone :: intercept

Find distance t to intersection $\vec{w}$ of a straight line with a conical surface.

$$\text{Line: } \vec{w} = \vec{p} + \vec{u}\, t \qquad (1)$$

2-D RZ meshes containing straight-line segments yield Faces shaped as conical ribbons upon rotation into 3-D about the z-axis.



Direction vector: $(r_B - r_A \,;\, z_B - z_A)$

Normal vector: $(z_B - z_A \,;\, r_A - r_B)$

Line through points $\vec{A}, \vec{B}$:

$$(r - r_A)(z_B - z_A) = (z - z_A)(r_B - r_A)$$

becomes a conical ribbon with $r = \sqrt{x^2 + y^2}$.

Define: $\Delta z \equiv z_B - z_A \quad (2) \quad ; \quad \Delta r \equiv r_B - r_A \quad (3)$

$$(r - r_A)\,\Delta z = (z - z_A)\,\Delta r \qquad (4)$$

Special case disk: $\Delta z = 0 \,;\, \Delta r \neq 0 \implies z = z_A = z_B$

Special case cylinder: $\Delta r = 0 \,;\, \Delta z \neq 0 \implies r = r_A = r_B$

Line from Eq.(1): $\vec{w} \equiv (x, y, z) \,;\, \vec{p} \equiv (p_x, p_y, p_z) \,;\, \vec{u} \equiv (u_x, u_y, u_z)$

$$\implies x = p_x + u_x t \,;\, y = p_y + u_y t \,;\, z = p_z + u_z t \qquad (5)$$

In order to find the intersection of Line (1) with Cone (4), assume that $\Delta z \neq 0$ at first. In the end verify that the obtained solution also covers the $\Delta z = 0$ special case.

$$(4) \implies r = r_A + (z - z_A)\frac{\Delta r}{\Delta z}$$

/ 2

$$r^2 = r_A^2 + 2r_A(z - z_A)\frac{\Delta r}{\Delta z} + (z - z_A)^2 \frac{\Delta r^2}{\Delta z^2} \ ; \ r^2 = x^2 + y^2$$

Substituting from (5) to obtain the equation for $t = ?$

$$p_x^2 + 2p_x u_x t + u_x^2 t^2 + p_y^2 + 2p_y u_y t + u_y^2 t^2 =$$
$$= r_A^2 + 2r_A(p_z + u_z t - z_A)\frac{\Delta r}{\Delta z} + (p_z + u_z t - z_A)^2 \frac{\Delta r^2}{\Delta z^2}$$

Define: $r_p^2 \equiv p_x^2 + p_y^2$ (6) ; $z_D \equiv p_z - z_A$ (7)

$$r_p^2 + 2(p_x u_x + p_y u_y)t + (u_x^2 + u_y^2)t^2 =$$
$$= r_A^2 + 2r_A z_D \frac{\Delta r}{\Delta z} + 2r_A u_z \frac{\Delta r}{\Delta z} t + z_D^2 \frac{\Delta r^2}{\Delta z^2} + 2z_D \frac{\Delta r^2}{\Delta z^2} u_z t + u_z^2 \frac{\Delta r^2}{\Delta z^2} t^2$$

Multiply by $\Delta z^2 \neq 0$:

$$r_p^2 \Delta z^2 + 2\Delta z^2 (p_x u_x + p_y u_y)t + \Delta z^2 (u_x^2 + u_y^2)t^2 =$$
$$= r_A^2 \Delta z^2 + 2r_A z_D \Delta z \Delta r + 2r_A u_z \Delta z \Delta r\, t + z_D^2 \Delta r^2 +$$
$$+ 2z_D \Delta r^2 u_z t + u_z^2 \Delta r^2 t^2$$

Define: $f \equiv \Delta z^2 (p_x u_x + p_y u_y)$ (8) ; $g \equiv \Delta z \cdot r_A \cdot \Delta r$ (9)

$$r_p^2 \Delta z^2 + 2ft + \Delta z^2 (u_x^2 + u_y^2)t^2 = r_A^2 \Delta z^2 + 2g z_D + 2g u_z t + z_D^2 \Delta r^2 +$$
$$+ 2z_D \Delta r^2 u_z t + u_z^2 \Delta r^2 t^2$$

$$\left[\Delta z^2 (u_x^2 + u_y^2) - (u_z \Delta r)^2\right] t^2 + 2\left[f - g u_z - z_D u_z \Delta r^2\right] t +$$
$$+ \Delta z^2 (r_p^2 - r_A^2) - 2g z_D - z_D^2 \Delta r^2 = 0$$

$$\left[\Delta z^2(\mu_x^2+\mu_y^2) - (\mu_z \Delta r)^2\right] t^2 + 2\left[f - \mu_z\left(g + z_D \Delta r^2\right)\right] t \;+$$

$$+\; \Delta z^2\left(r_P^2 - r_A^2\right) - z_D\left(2g + z_D \Delta r^2\right) = 0 \qquad (10)$$

Define: $h \equiv g + z_D \Delta r^2 \quad (11) \;\Rightarrow\; a t^2 + b t + c = 0 \quad (12)$

$$\text{with: } \left.\begin{aligned} a &\equiv \Delta z^2(\mu_x^2+\mu_y^2) - (\mu_z \Delta r)^2 \\ b &\equiv 2\left[f - \mu_z h\right] \\ c &\equiv \Delta z^2\left(r_P^2 - r_A^2\right) - z_D(g+h) \end{aligned}\right\} \downarrow \text{ solve for } t = ?$$

Special case disk: $\Delta z = 0 \Rightarrow f = 0$ and $g = 0$; $\Delta r \neq 0$

$$(10) \Rightarrow -(\mu_z \cancel{\Delta r})^2 t^2 - 2\mu_z z_d \cancel{\Delta r^2}\, t - z_D^2 \cancel{\Delta r^2} = 0$$

$$\mu_z^2 t^2 + 2\mu_z z_d t + z_D^2 = 0$$

$$(z_D + \mu_z t)^2 = 0$$

$$p_z - z_A + \mu_z t = 0 \iff z_A = p_z + \mu_z t,$$

which agrees with Eq. (5) when demanding $z = z_A = z_B$.
Therefore, Eqs. (10), (12) can be used also if $\Delta z = 0$, even though their derivation assumed $\Delta z \neq 0$.

Special case cylinder: $\Delta r = 0 \Rightarrow g = 0$ and $f = \Delta z^2(p_x\mu_x + p_y\mu_y)$; $\Delta z \neq 0$

$$(10) \Rightarrow \cancel{\Delta z^2}(\mu_x^2+\mu_y^2)t^2 + 2\cancel{\Delta z^2}(p_x\mu_x + p_y\mu_y)t + \cancel{\Delta z^2}(p_x^2+p_y^2-r_A^2) = 0$$

which also follows from (5) while requiring $x^2 + y^2 = r_A^2 = r_B^2$.

$$(p_x + \mu_x t)^2 + (p_y + \mu_y t)^2 = r_A^2$$

$$(\mu_x^2+\mu_y^2)t^2 + 2(p_x\mu_x + p_y\mu_y)t + (p_x^2+p_y^2-r_A^2) = 0$$

Define: $s \equiv p_x \mu_x + p_y \mu_y$

$$D \equiv s^2 - (\mu_x^2 + \mu_y^2)(p_x^2 + p_y^2 - r_A^2)$$

then for a cylinder: $t = \dfrac{-s \pm \sqrt{D}}{\mu_x^2 + \mu_y^2} \Rightarrow (\mu_x^2 + \mu_y^2)^2 t^2 = (-s \pm \sqrt{D})^2$

Verify $x^2 + y^2 = r_A^2$:

$$(p_x + \mu_x t)^2 + (p_y + \mu_y t)^2 = p_x^2 + p_y^2 + 2st + (\mu_x^2 + \mu_y^2) t^2 =$$

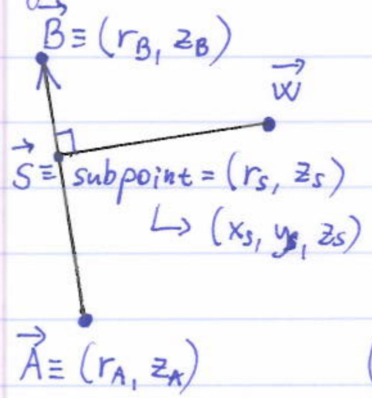
$$= p_x^2 + p_y^2 + 2s \frac{-s \pm \sqrt{D}}{\mu_x^2 + \mu_y^2} + \frac{1}{\mu_x^2 + \mu_y^2}\left[s^2 \mp 2s\sqrt{D} + D\right]$$

$$= p_x^2 + p_y^2 + \frac{1}{\cancel{\mu_x^2 + \mu_y^2}}\Big[\underbrace{-2s^2 \cancel{\pm 2s\sqrt{D}} + s^2}_{\cancel{-s^2}} \cancel{\mp 2s\sqrt{D}} + \underbrace{D}_{\cancel{s^2} - \cancel{(\mu_x^2 + \mu_y^2)}(p_x^2 + p_y^2 - r_A^2)}\Big]$$

$$= p_x^2 + p_y^2 - (p_x^2 + p_y^2 - r_A^2) = \underline{\underline{r_A^2}} \quad \checkmark$$

$\boxed{\text{Cone::subpoint, Cone::distance}}$

Drop the perpendicular from $\vec{w}$ onto *this's line in the $\hat{z}\vec{w}$ plane to get the subpoint; return the SIGNED distance $\vec{w}$ and the subpoint given the orientation of the Cone's defining segment $\overrightarrow{AB}$.



$\vec{w} = (x_w, y_w, z_w) \to (\underbrace{r_w}_{\sqrt{x_w^2 + y_w^2}}, z_w)$

$\vec{S} = \vec{A} + (\vec{B} - \vec{A})t$, find t so that

$(\vec{w} - \vec{S}) \perp (\vec{B} - \vec{A})$

2-D

$(\vec{S} - \vec{w}) \perp (\vec{B} - \vec{A})$

$(\vec{B}-\vec{A}) = (\Delta r, \Delta z)$; see (2), (3)

$(\vec{S}-\vec{w}) = (r_A + t\Delta r - r_w,\ z_A + t\Delta z - z_w)$

$(\vec{S}-\vec{w})\cdot(\vec{B}-\vec{A}) = (r_A - r_w)\Delta r + t\Delta r^2 + (z_A - z_w)\Delta z + t\Delta z^2 \overset{!}{=} 0$

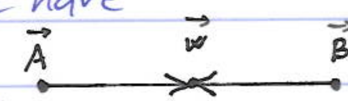
$$t = \frac{(r_w - r_A)\Delta r + (z_w - z_A)\Delta z}{\Delta r^2 + \Delta z^2} \qquad (13)$$

$|\text{distance}| = |\vec{S}-\vec{w}|$, sign chosen according to pmh_2014_1125, p. 2 (Vector3d :: get_turn) with $\vec{B}-\vec{A} \to \vec{w}$ and $\vec{w}-\vec{A} \to \vec{v}$ therein.

To go from 2-D to 3-D calculate angle $\varphi = \text{atan2}(y_w, x_w)$, then $x_s = r_s \cos\varphi$, $y_s = r_s \sin\varphi$.
Note: $\vec{S}$ is undefined when $x_w = y_w = 0$, but the distance remains valid.

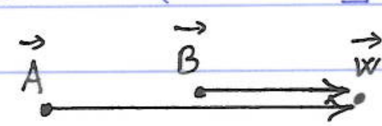
$\boxed{\text{Cone :: contains, Vector3d :: is\_between}}$

Conical ribbon defined in 2-D RZ plane "contains" point $\vec{w}$ if $\vec{w}$ "is_between" points $\vec{A}, \vec{B}$ that define the ribbon.
Assuming that points $\vec{A}, \vec{B}, \vec{w}$ are colinear, and identifying $\vec{A} \equiv$ tail, $\vec{B} \equiv$ head, we have



is_between = true for $\vec{w}-\vec{A}$, $\vec{w}-\vec{B}$ antiparallel
[scalar product $(\vec{w}-\vec{A})\cdot(\vec{w}-\vec{B}) < 0$]

is_between = true for $\vec{w}=\vec{A}$ or $\vec{w}=\vec{B}$ [$(\vec{w}-\vec{A})\cdot(\vec{w}-\vec{B}) = 0$]



is_between = false for $\vec{w}-\vec{A}$, $\vec{w}-\vec{B}$ parallel
[scalar product $(\vec{w}-\vec{A})\cdot(\vec{w}-\vec{B}) > 0$]