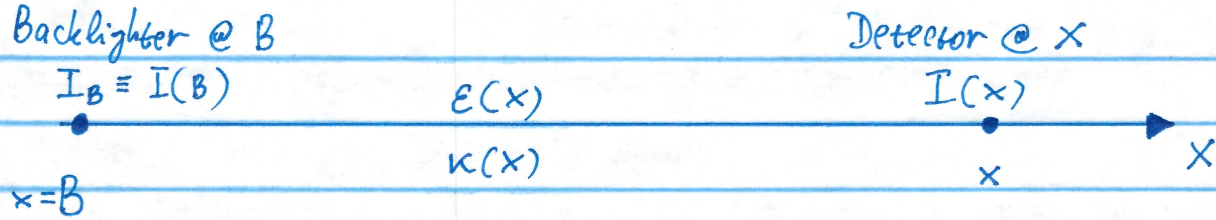
RADIATION TRANSPORT EQUATION

pmh_2021_0929

SOLUTION IN 1-D, STEADY-STATE



$I(x) = ?$, specific intensity in $W/(cm^2 \cdot sr \cdot eV)$

$\varepsilon(x)$, emissivity in $W/(cm^3 \cdot sr \cdot eV)$, known

$\kappa(x)$, opacity in cm^{-1}, known

Radiation transport equation: $I'(x) = \varepsilon(x) - \kappa(x)\, I(x)$

with the boundary condition: $I(B) = I_B$ (backlighter) ← known

where $' \equiv \frac{d}{dx}$

Optical depth: $\tau(t,x) \equiv \int_t^x \kappa(s)\, ds$

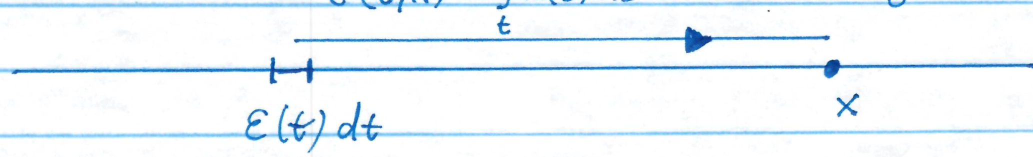
then $\tau(x,x) = 0$ and $\tau'(t,x) \equiv \frac{d\tau}{dx} = \kappa(x)$

Solution:

$$I(x) = I_B\, e^{-\tau(B,x)} + \int_B^x \varepsilon(t)\, e^{-\tau(t,x)}\, dt$$

Contribution to $I(x)$ from location t is $\varepsilon(t)dt$ attenuated by $\tau(t,x)$.

$\tau(t,x) = \int_t^x \kappa(s)\, ds$



Solution verification:

Backlighter boundary condition:

$$I(B) = I_B \underbrace{e^{-\tau(B,B)}}_{1} + \underbrace{\int_B^B \varepsilon(t)\, e^{-\tau(t,x)}\, dt}_{\emptyset} = I_B \quad \checkmark$$

Note: Leibniz integral rule:

The derivative $F'(x)$ of function $F(x)$ defined as

$$F(x) \equiv \int_{l(x)}^{u(x)} f(x,t)\, dt, \text{ is } F'(x) = u'(x) f(x, u(x)) - l'(x) f(x, l(x)) + \int_{l(x)}^{u(x)} \frac{\partial f}{\partial x}(x,t)\, dt .$$

LHS

$$I'(x) = I_B e^{-\tau(B,x)} (-\kappa(x)) + 1 \cdot \varepsilon(x) \cdot e^{-\tau(x,x)} - 0 + \int_B^x \varepsilon(t)\, e^{-\tau(t,x)} (-\kappa(x))\, dt$$

$$I'(x) = \varepsilon(x) - \kappa(x) \underbrace{\left[I_B e^{-\tau(B,x)} + \int_B^x \varepsilon(t) e^{-\tau(t,x)}\, dt \right]}_{I(x)}$$

$$I'(x) = \varepsilon(x) - \kappa(x)\, I(x) \quad \checkmark$$