SOLID-ANGLE INTEGRATION WITH A RAY BUNDLE

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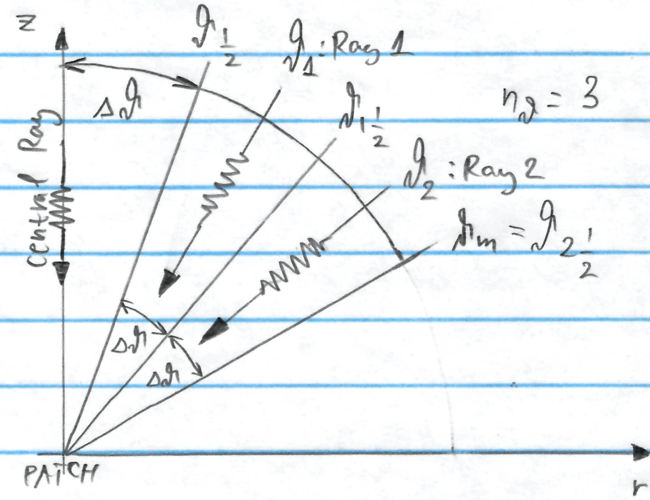
Solid angle under spherical cap spanned by polar angle ϑ:

$$\Omega = 2\pi(1-\cos\vartheta) = 4\pi \sin^2\frac{\vartheta}{2}$$

On input choose $\vartheta_m > 0$ and $n_\vartheta > 0$ (number of intervals in ϑ)

$$\Delta\vartheta = \frac{\vartheta_m}{n_\vartheta}$$



Central Ray will contribute weighted by solid angle element

$$4\pi \sin^2\frac{\Delta\vartheta}{2} \equiv \Delta\Omega_0$$

All other Rays are placed at: $\vartheta_i = i\cdot\Delta\vartheta + \dfrac{\Delta\vartheta}{2}\,;\quad i = 1, 2, \ldots, n_\vartheta - 1$

Rays at ϑ_i are spread uniformly over 2π in φ so that the corresponding solid-angle element is approximately $\Delta\Omega_0$.

Solid angle of i-th ribbon:

$$\Omega_i = 2\pi\left[\left(1-\cos\vartheta_{i+\frac{1}{2}}\right) - \left(1-\cos\vartheta_{i-\frac{1}{2}}\right)\right]$$

$$= 2\pi\left(\cos\vartheta_{i-\frac{1}{2}} - \cos\vartheta_{i+\frac{1}{2}}\right)$$

Using: $\cos\alpha - \cos\beta = 2\sin\dfrac{\beta+\alpha}{2}\,\sin\dfrac{\beta-\alpha}{2}$

$$\Omega_i = 4\pi \sin\frac{\vartheta_{i+\frac{1}{2}} + \vartheta_{i-\frac{1}{2}}}{2} \sin\frac{\vartheta_{i+\frac{1}{2}} - \vartheta_{i-\frac{1}{2}}}{2}$$

$$= 4\pi \sin\frac{\Delta\vartheta + 2i\,\Delta\vartheta}{2} \sin\frac{\Delta\vartheta}{2}$$

$$= 4\pi \sin\left[\left(i+\frac{1}{2}\right)\Delta\vartheta\right] \sin\frac{\Delta\vartheta}{2}$$

The number of Rays over 2π in φ at ϑ_i:

$$n_\varphi(i) \approx \frac{\Omega_i}{\Delta\Omega_0} = \frac{4\pi \sin\left[\left(i+\frac{1}{2}\right)\Delta\vartheta\right] \sin(\Delta\vartheta/2)}{4\pi \sin^2(\Delta\vartheta/2)}$$

$$n_\varphi(i) \approx \frac{\sin\left[\left(i+\frac{1}{2}\right)\Delta\vartheta\right]}{\sin(\Delta\vartheta/2)} \qquad (\blacksquare)$$

Special case: $i = 1$ and $\Delta\vartheta \to 0$

$$n_\varphi(1) \approx \frac{\sin\frac{3}{2}\Delta\vartheta}{\sin\frac{\Delta\vartheta}{2}} \approx \frac{\frac{3}{2}\Delta\vartheta}{\frac{\Delta\vartheta}{2}} = 3$$

Design choice: take ($\blacksquare$) result, round it to the nearest integer, put a floor $=4$ under it

$\Rightarrow n_\varphi(1) = 4$ and $n_\varphi(i>1) \geq 4$.

Azimuthal spacing of Rays at ϑ_i: $\Delta\varphi_i = \dfrac{2\pi}{n_\varphi(i)}$

Solid angle element: $\Delta\Omega_i = \dfrac{\Omega_i}{n_\varphi(i)}$