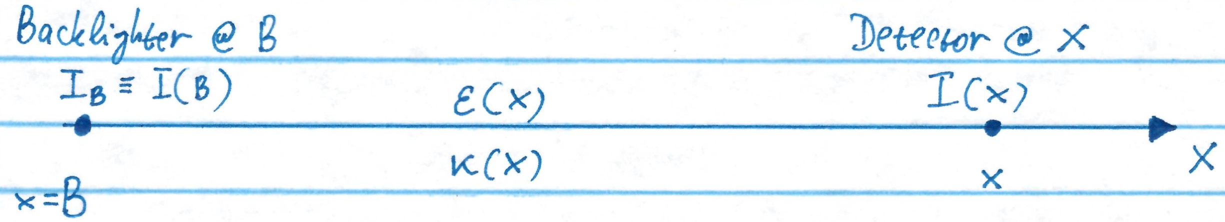
RADIATION TRANSPORT EQUATION

pmh_2021_0929

SOLUTION IN 1-D, STEADY-STATE



$I(x) = ?$, specific intensity in $W/(cm^2 \cdot sr \cdot eV)$

$\varepsilon(x)$, emissivity in $W/(cm^3 \cdot sr \cdot eV)$, known

$\kappa(x)$, opacity in cm^{-1}, known

Radiation transport equation: $I'(x) = \varepsilon(x) - \kappa(x)\, I(x)$

with the boundary condition: $I(B) = I_B$ (backlighter)

where $' \equiv \frac{d}{dx}$ known

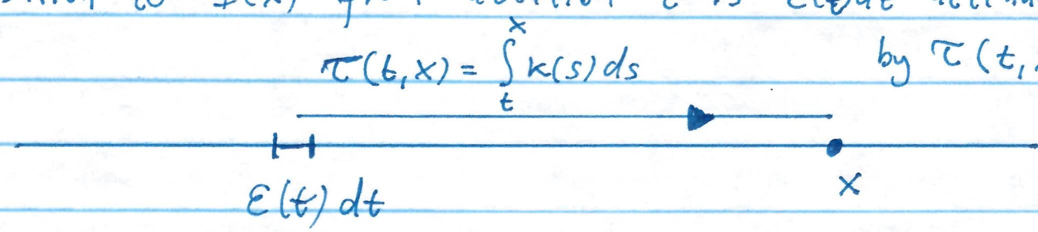
Optical depth: $\tau(t,x) \equiv \int_t^x \kappa(s)\, ds$

then $\tau(x,x) = 0$ and $\tau'(t,x) \equiv \frac{d\tau}{dx} = \kappa(x)$

Solution:

$$I(x) = I_B\, e^{-\tau(B,x)} + \int_B^x \varepsilon(t)\, e^{-\tau(t,x)}\, dt$$

Contribution to $I(x)$ from location t is $\varepsilon(t)dt$ attenuated by $\tau(t,x)$.



Solution verification:

Backlighter boundary condition:

$$I(B) = I_B \underbrace{e^{-\tau(B,B)}}_{1} + \underbrace{\int_B^B \varepsilon(t)\, e^{-\tau(t,x)}\, dt}_{\emptyset} = I_B \checkmark$$

Note: Leibniz integral rule:

The derivative $F'(x)$ of function $F(x)$ defined as

$$F(x) \equiv \int_{l(x)}^{u(x)} f(x,t)\, dt, \text{ is } F'(x) = u'(x)\, f(x, u(x)) - l'(x) f(x, l(x)) + \int_{l(x)}^{u(x)} \frac{\partial f}{\partial x}(x,t)\, dt.$$

LHS

$$I'(x) = I_B\, e^{-\tau(B,x)} (-\kappa(x)) + 1 \cdot \varepsilon(x) \cdot e^{-\tau(x,x)} - 0 + \int_B^x \varepsilon(t)\, e^{-\tau(t,x)} (-\kappa(x))\, dt$$

$$I'(x) = \varepsilon(x) - \kappa(x) \underbrace{\left[I_B\, e^{-\tau(B,x)} + \int_B^x \varepsilon(t)\, e^{-\tau(t,x)}\, dt \right]}_{I(x)}$$

$$I'(x) = \varepsilon(x) - \kappa(x)\, I(x) \quad \checkmark$$

Uniform slab of thickness Δx:

added on 2022_0708

$B = 0$; $x = \Delta x$

$\mathcal{E}(t) \equiv \mathcal{E} = \text{const}$; $\kappa(s) \equiv \kappa = \text{const}$

no backlighter: $I_B = 0$

$$I(\Delta x) = \int_0^{\Delta x} \mathcal{E}\, e^{-\int_t^{\Delta x} \kappa\, ds}\, dt = \mathcal{E} \int_0^{\Delta x} e^{-\kappa \int_t^{\Delta x} ds} =$$

$$= \mathcal{E} \int_0^{\Delta x} e^{-\kappa(\Delta x - t)}\, dt = \mathcal{E}\, e^{-\kappa \Delta x} \int_0^{\Delta x} e^{\kappa t}\, dt =$$

$$= \mathcal{E}\, e^{-\kappa \Delta x} \left. \frac{e^{\kappa t}}{\kappa} \right|_0^{\Delta x} = \frac{\mathcal{E}}{\kappa} e^{-\kappa \Delta x} \left(e^{\kappa \Delta x} - 1 \right)$$

$$I(\Delta x) = \frac{\mathcal{E}}{\kappa} \left(1 - e^{-\kappa \Delta x} \right)$$

With backlighter $I_B \neq 0$: and $\Delta x \to L$
we have Eq. (2) from CPC 207, 415 (2016).

$$I(L) = I_B\, e^{-\kappa L} + \frac{\mathcal{E}}{\kappa} \left(1 - e^{-\kappa L} \right)$$