SGN – Assignment #1

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1 Periodic orbit

Exercise 1

Consider the 3D Earth–Moon Circular Restricted Three-Body Problem with $\mu = 0.012150$.

1) Find the x-coordinate of the Lagrange point L_1 in the rotating, adimensional reference frame with at least 10-digit accuracy.

Solutions to the 3D CRTBP satisfy the symmetry

$$\mathcal{S} : (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \rightarrow (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t).$$

Thus, a trajectory that crosses perpendicularly the $y = 0$ plane twice is a periodic orbit.

2) Given the initial guess $\mathbf{x}_0 = (x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0})$, with

$$\begin{aligned} x_0 &= 1.08892819445324 \\ y_0 &= 0 \\ z_0 &= 0.0591799623455459 \\ v_{x0} &= 0 \\ v_{y0} &= 0.257888699435051 \\ v_{z0} &= 0 \end{aligned}$$

Find the periodic halo orbit that passes through z_0; that is, develop the theoretical framework and implement a differential correction scheme that uses the STM either approximated through finite differences or achieved by integrating the variational equation.

The periodic orbits in the CRTBP exist in families. These can be computed by continuing the orbits along one coordinate, e.g., z_0. This is an iterative process in which one component of the state is varied, while the other components are taken from the solution of the previous iteration.

3) By gradually decreasing z_0 and using numerical continuation, compute the families of halo orbits until $z_0 = 0.034$.

(8 points)

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1) Considering the 3D Earth–Moon Circular Restricted Three-Body Problem with $\mu = 0.012150$, the x-coordinate of the Lagrange points in the rotating, adimensional reference frame with 10-digit accuracy are:

- x_{L1} = 0.8369180073
- x_{L2} = 1.1556799131
- x_{L3} = -1.0050624018

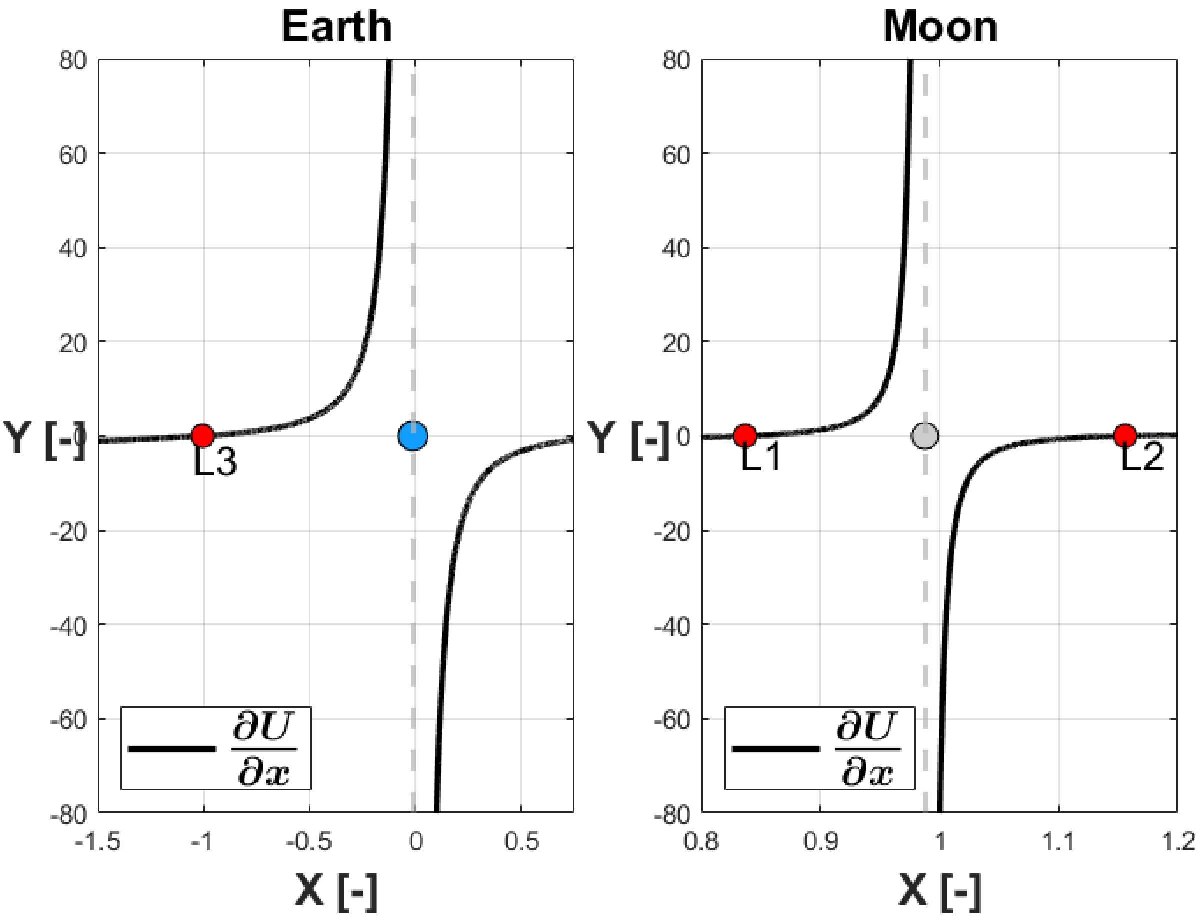


Figure 1: Earth-Moon Lagrange points and potential curve

2+3) STM variational approach derivatives to find the periodic halo orbits:

$$
\begin{aligned}
r_1 &= \sqrt{(x+\mu)^2 + y^2 + z^2} \\
r_2 &= \sqrt{(x+\mu-1)^2 + y^2 + z^2} \\
\omega_{xx} &= 1 - (1-\mu)/r_1^3 + 3(1-\mu)(x+\mu)^2/r_1^5 - \mu/r_2^3 + 3\mu(x-1+\mu)^2/r_2^5 \\
\omega_{yy} &= 1 - (1-\mu)/r_1^3 + 3(1-\mu)y^2/r_1^5 - \mu/r_2^3 + 3\mu y^2/r_2^5 \\
\omega_{zz} &= -(1-\mu)/r_1^3 - \mu/r_2^3 + 3z^2(1-\mu)/r_1^5 + 3\mu z^2/r_2^5 \\
\omega_{xy} &= 3(1-\mu)(x+\mu)y/r_1^5 + 3\mu(x+\mu-1)y/r_2^5 \\
\omega_{zx} &= 3\mu z(\mu+x-1)/r_2^5 + 3z(1-\mu)(\mu+x)/r_1^5 \\
\omega_{yz} &= 3\mu yz/r_2^5 + 3yz(1-\mu)/r_1^5 \\
\mathbf{A} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \omega_{xx} & \omega_{xy} & \omega_{xz} & 0 & 2 & 0 \\ \omega_{yx} & \omega_{yy} & \omega_{yz} & -2 & 0 & 0 \\ \omega_{zx} & \omega_{zy} & \omega_{zz} & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
\tag{1}
$$

Family of 10 CRTBP periodic orbits in Earth-Moon rotating reference frame:

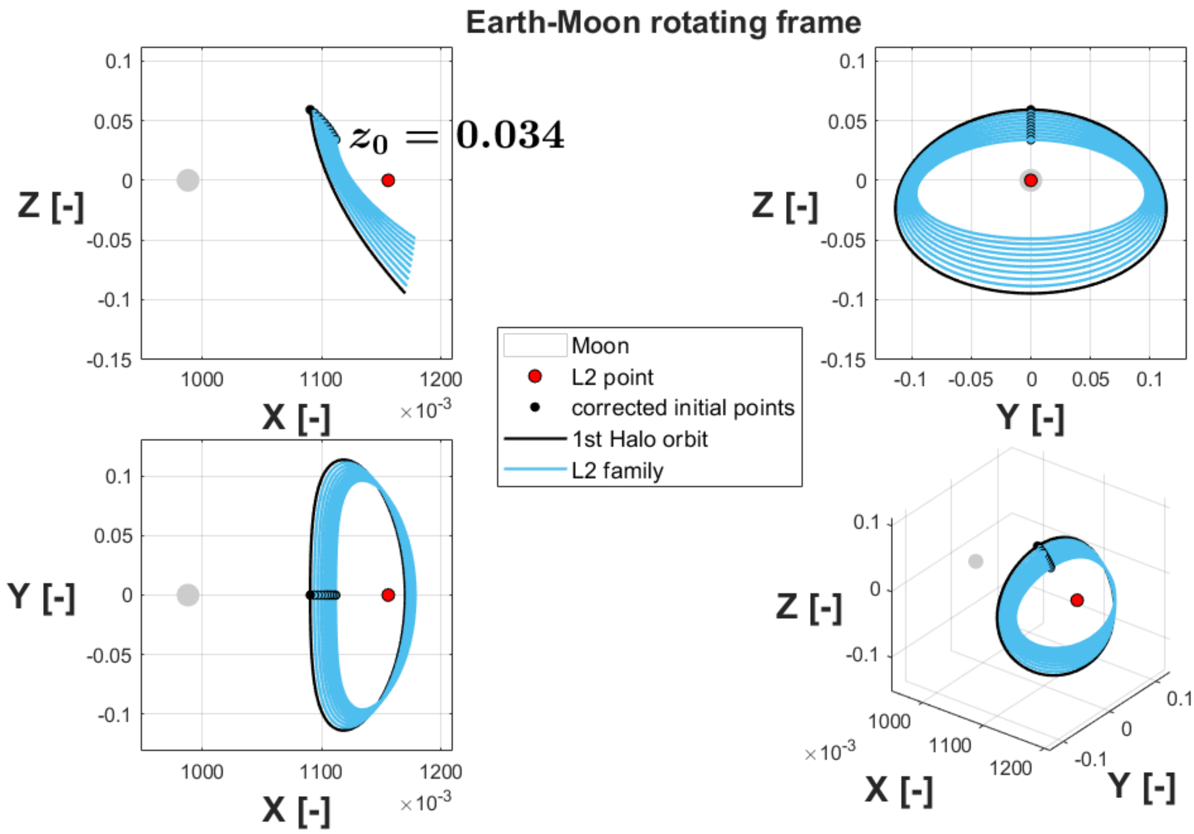


Figure 2: Family of ten halo orbits around L2

2 Impulsive guidance

Exercise 2

The Aphophis close encounter with Earth will occur on April 2029. You shall design a planetary protection guidance solution aimed at reducing the risk of impact with the Earth.

The mission shall be performed with an impactor spacecraft, capable of imparting a $\Delta\mathbf{v} = 0.00005\,\mathbf{v}(t_{\text{imp}})$, where $\mathbf{v}$ is the spacecraft velocity and t_{imp} is the impact time. The spacecraft is equipped with a chemical propulsion system that can perform impulsive manoeuvres up to a total Δv of 5 km/s.

The objective of the mission is to maximize the distance from the Earth at the time of the closest approach. The launch shall be performed between 2024-10-01 (LWO, Launch Window Open) and 2025-02-01 (LWC, Launch Window Close), while the impact with Apophis shall occur between 2028-08-01 and 2029-02-28. An additional Deep-Space Maneuver (DSM) can be performed between LWO+6 and LWC+18 months.

1) Analyse the close encounter conditions reading the SPK kernel and plotting in the time window [2029-01-01; 2029-07-31] the following quantities:

   a) The distance between Apophis and the Sun, the Moon and the Earth respectively.

   b) The evolution of the angle Earth-Apophis-Sun

   c) The ground-track of Apophis for a time-window of 12 hours centered around the time of closest approach (TCA).

2) Formalize an unambiguous statement of the problem specifying the considered optimization variables, objective function, the linear and non-linear equality and inequality constraints, starting from the description provided above. Consider a multiple-shooting problem with $N = 3$ points (or equivalently 2 segments) from t_0 to t_{imp}.

3) Solve the problem with multiple shooting. Propagate the dynamics of the spacecraft considering only the gravitational attraction of the Sun; propagate the post-impact orbit of Apophis using a full n-body integrator. Use an event function to stop the integration at TCA to compute the objective function; read the position of the Earth at t_0 and that of Apophis at t_{imp} from the SPK kernels. Provide the optimization solution, that is, the optimized departure date, DSM execution epoch and the corresponding $\Delta\mathbf{v}$'s, the spacecraft impact epoch, and time and Distance of Closest Approach (DCA) in Earth radii. Suggestion: try different initial conditions.

(11 points)

1) Relevant quantities regarding Earth-Apophis close approach (reference frame ECLIPJ2000 @Sun):

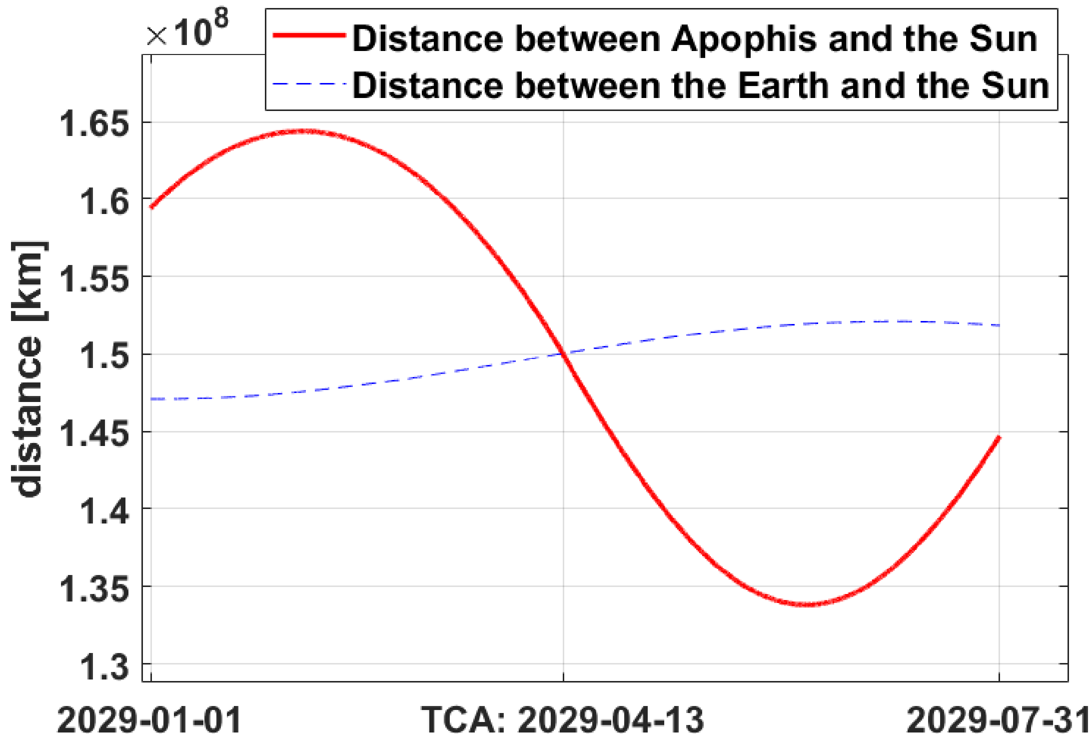


Figure 3: Earth-Sun and Apophis-Sun distances

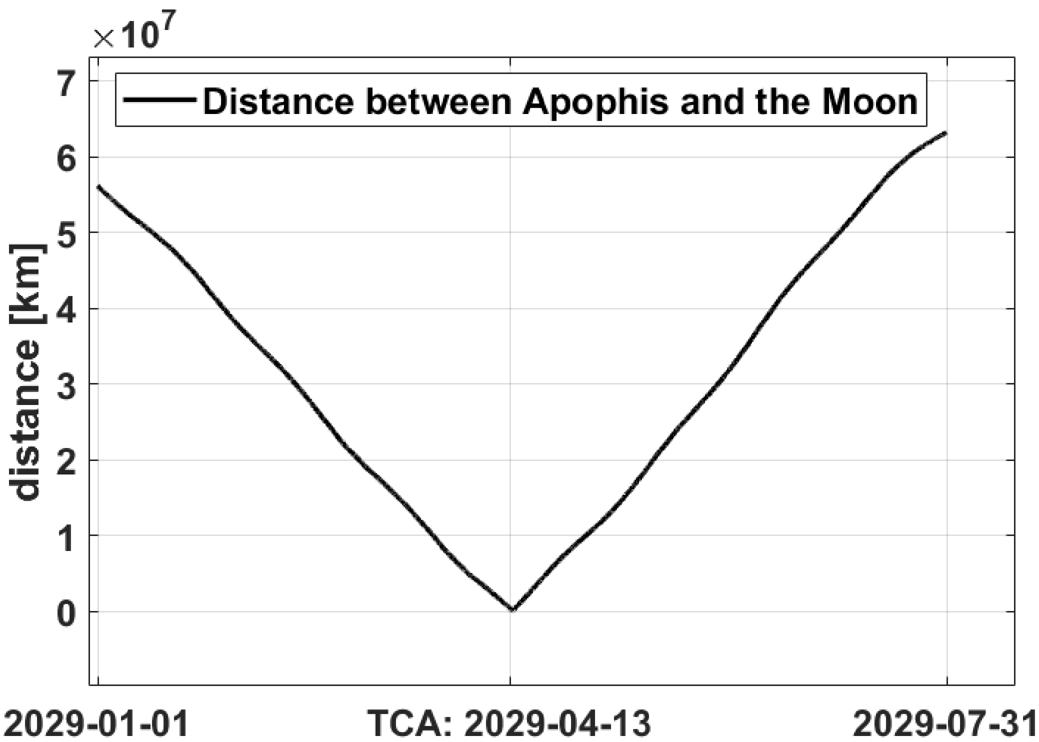


Figure 4: Apophis-Moon distance

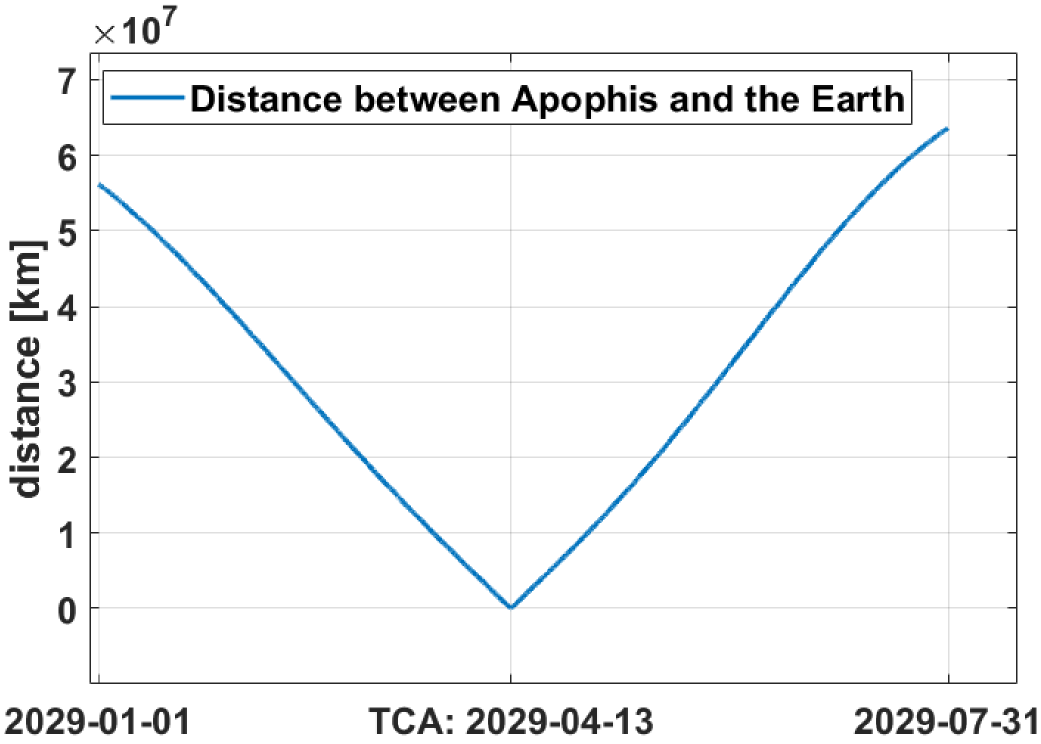


Figure 5: Apophis-Earth distance

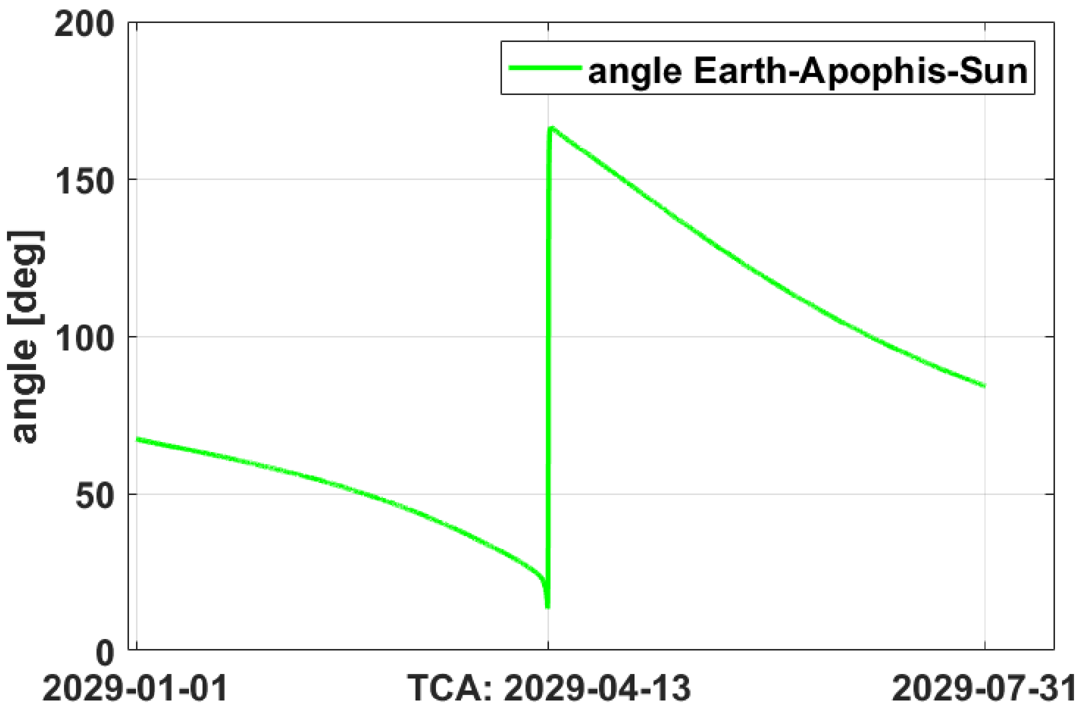
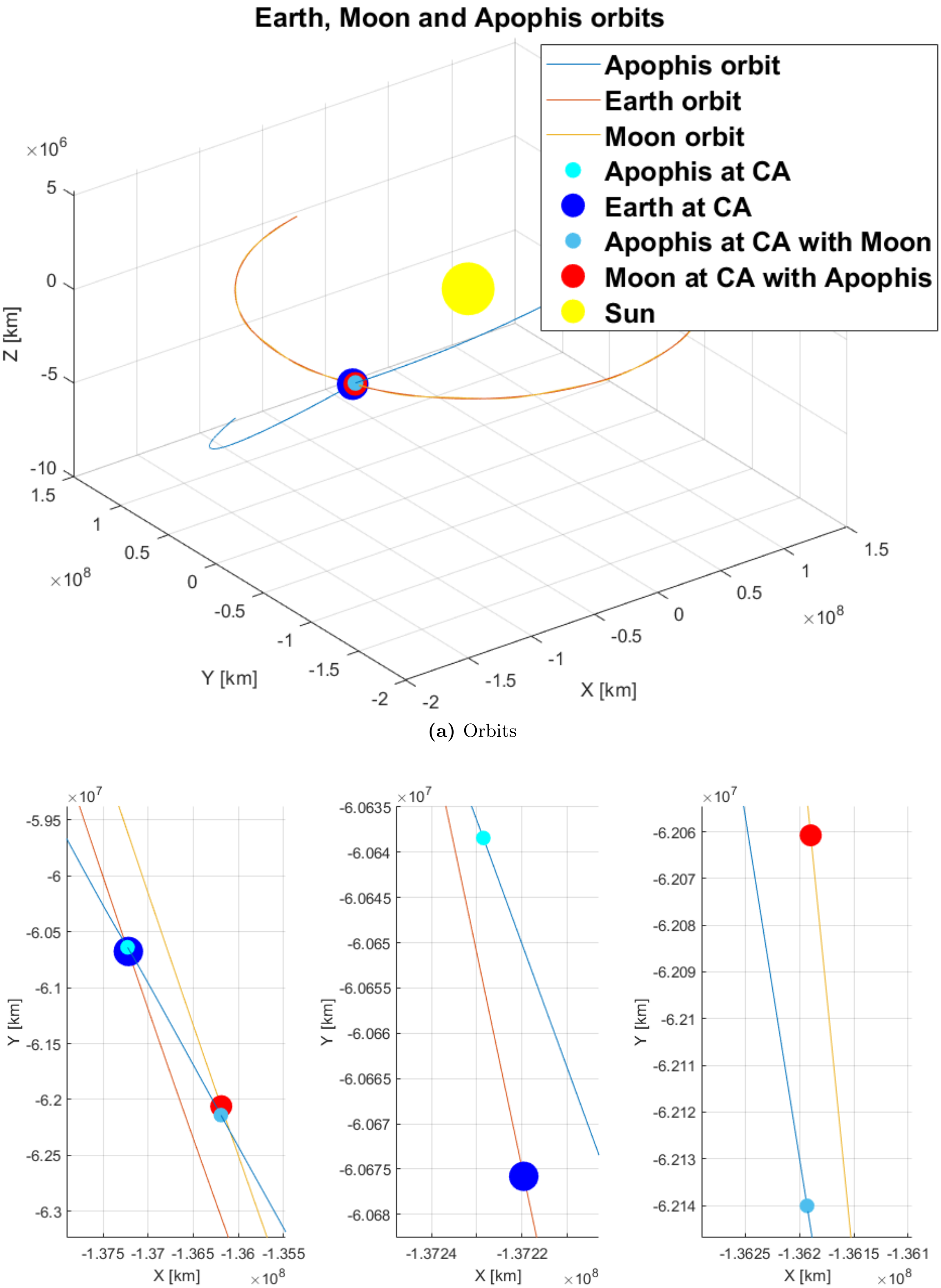


Figure 6: Angle Earth-Apophis-Sun



(a) Orbits

(b) Both CAs

(c) Apophis-Earth CA

(d) Apophis-Moon CA

Figure 7: Earth, Moon and Apophis orbits around the Sun and their close approaches (reference frame ECLIPJ2000 @Sun)

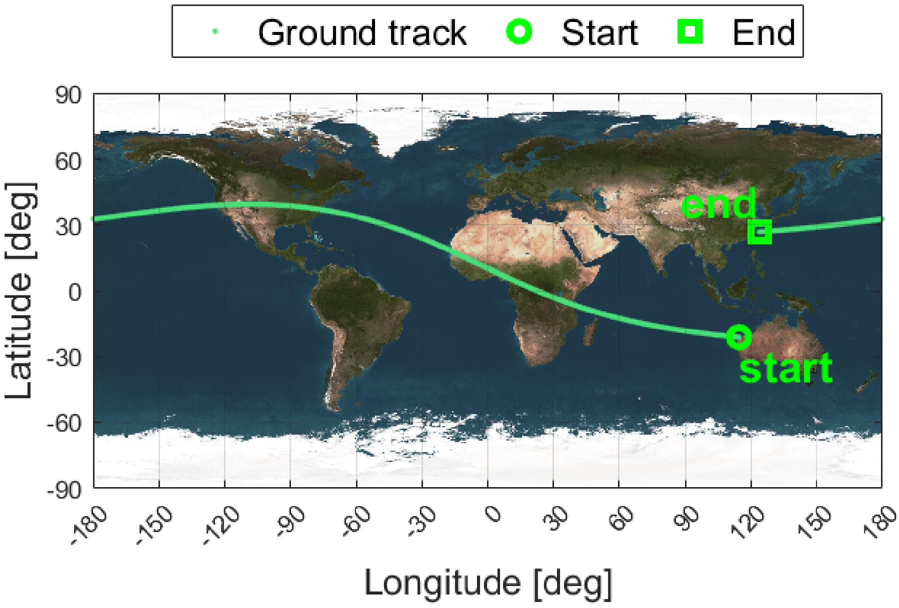


Figure 8: Apophis ground-track for a time-window of 12 hours centered around the time of closest approach (TCA)

Earth-Apophis close approach (Figure 7c) occurs on 2029 APR 13 22:11:24.1 UTC, and the minimum distance between the two bodies is 39448.1349801351 km, that is 6.1848996743 Earth radii. Apophis has a close approach with the Moon (Figure 7d) at different date and distance with respect to the one with Earth: 2029 APR 14 14:28:29.6 UTC, 95973.8054802819 km (15.0473110721 Earth radii). The angle Earth-Apophis-Sun jumps from 13.3744042581 to 166.4496531175 degrees across the close approach; the reason why this happens is the close approach dynamics and it's clear from Figures 3 and 7a, where it's shown that before the close approach the Earth is closer to the Sun, while after the close approach Apophis-Sun distance becomes lower than Earth-Sun distance.

2) The formulation of the NLP problem is:

$$\min_{\underline{\boldsymbol{y}}} f(\underline{\boldsymbol{y}}) \quad s.t. \quad \begin{cases} \boldsymbol{C}_{eq}(\underline{\boldsymbol{y}}) = 0 \\ \boldsymbol{C}(\underline{\boldsymbol{y}}) \leq 0 \\ \underline{\boldsymbol{y}}^{LB} \leq \underline{\boldsymbol{y}} \leq \underline{\boldsymbol{y}}^{UP} \end{cases} \tag{2}$$

where $\underline{\boldsymbol{y}}$ is the vector of variables, $\boldsymbol{C}_{eq}(\underline{\boldsymbol{y}})$ contains the equality constraints, $\boldsymbol{C}(\underline{\boldsymbol{y}})$ contains the inequality ones, $\underline{\boldsymbol{y}}^{LB}$ and $\underline{\boldsymbol{y}}^{UP}$ are lower and upper boundaries. The considered optimization variables are:

$$\underline{\boldsymbol{y}}(t_1, t_2, t_3, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \tag{3}$$

where t_1 is the departure epoch, t_2 is the deep space maneuver epoch, t_3 is the impact epoch, while $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are the respective states in position and velocities (reference frame ECLIPJ2000 @Sun), for a total of 21 variables.
The objective function is the n-body propagation of the state $\boldsymbol{x}_3$ after the impact up to the Apophis-Earth close approach, with the function to minimize being minus the distance between the two celestial bodies at CA.
To solve the problem equality constraints are imposed on positions and velocities: $\boldsymbol{x}_1$ must coincide with Earth position at t_1, $\boldsymbol{x}_2$ has to be equal to the last state of first propagated segment and $\boldsymbol{x}_3$ must coincide with Apophis state at t_3. In addition, we impose a constraint on the velocity the spacecraft imparts to the asteroid at impact, that is equal to $0.00005\underline{\boldsymbol{v}}(t_{imp})$.
Inequality constraints are represented by the 5 km/s limit in the maximum Δv the propulsion

system of the spacecraft can provide.
Lower and upper boundaries are the start and the end of launch, deep-space maneuver and impact time windows, that are:

- launch window: 2024-10-01 to 2025-02-01;
- deep-space maneuver window: 2025-04-01 to 2026-08-01;
- impact window: 2028-08-01 to 2029-02-28.

3) The multiple shooting problem solution varies depending on the chosen set of initial conditions. Many different solutions have been found using the following type of initial conditions and varying the parameters within the boundaries given by the text of the problem: $\underline{\boldsymbol{x}}_{\mathbf{0}}$ comprises 21 parameters: launch time, deep space maneuver time, impact time, Earth state at launch time + launch Δv, state at deep space maneuver time + deep space maneuver Δv, Apophis state at impact time (position and velocity vectors, that is 6 parameters per state). Using the following example set of initial conditions:

$$\underline{\boldsymbol{x}}_{\mathbf{0}} = \begin{bmatrix} 781572608.989753\ [s] \\ 827784307.513450\ [s] \\ 906878954.791016\ [s] \\ 143890498.720087\ [km] \\ 36375361.9959268\ [km] \\ 25719.7664184030\ [km] \\ -7.14430136920452\ [km/s] \\ 29.6805521756719\ [km/s] \\ 1.61098964897752\ [km/s] \\ -144237294.855872\ [km] \\ 73225319.8450688\ [km] \\ 5590789.68918057\ [km] \\ -12.4989167052946\ [km/s] \\ -22.9118929883880\ [km/s] \\ -0.551506863769696\ [km/s] \\ 113070286.591064\ [km] \\ 30679579.4769331\ [km] \\ 1019606.72977353\ [km] \\ -5.35923358001503\ [km/s] \\ 35.6865075729688\ [km/s] \\ -2.03150293249896\ [km/s] \end{bmatrix} \tag{4}$$

this solution is obtained:

Launch	`2024-10-07-11:16:45.130 UTC`		
DSM	`2026-03-27-13:35:36.917 UTC`		
Impact	`2028-09-26-18:45:39.713 UTC`		
TCA	`2029-04-13-23:14:04.791 UTC`		
$\Delta \mathbf{v}_L$ [km/s]	0.2562056344	0.9142493135	1.5705427270
$\Delta \mathbf{v}_{DSM}$ [km/s]	-1.5938053946	2.6101722385	-0.1522070819
Norm of total $\Delta \mathbf{v}$ [km/s]	4.8973255564		
DCA without impact [Re]	6.1848996743		
DCA after impact [Re]	24.3892771860		

Table 1: Guidance solution for the impactor mission

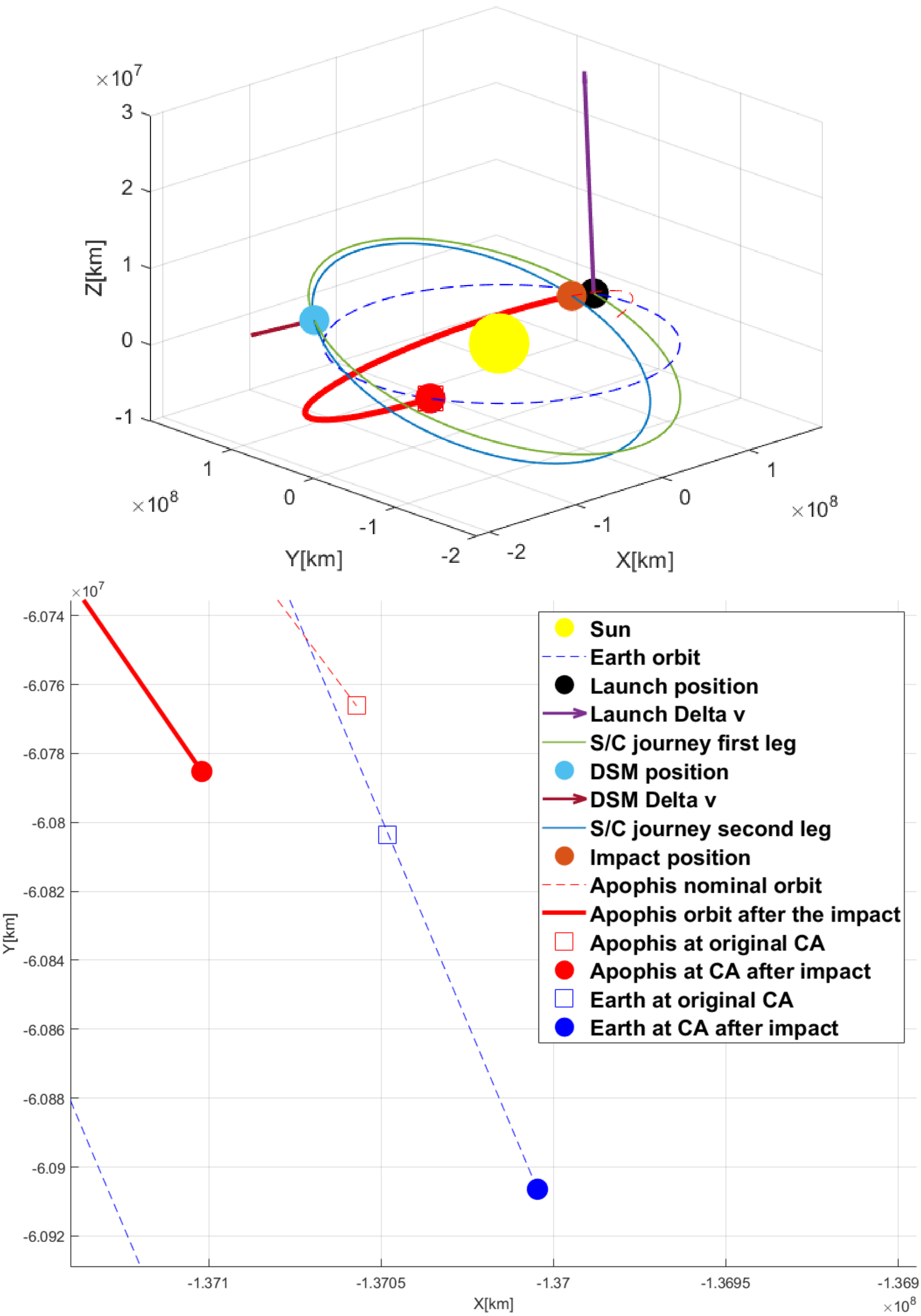


Figure 9: Spacecraft journey from launch from Earth to impact with Apophis, and Earth and Apophis orbits after the impact, with a close-up on old (squares) and new (circles) close approaches (reference frame ECLIPJ2000 @Sun)

3 Continuous guidance

Exercise 3

A low-thrust option is being considered for an Earth-Venus transfer*. Provide a *time-optimal* solution under the following assumptions: the spacecraft moves in the heliocentric two-body problem, Venus instantaneous acceleration is determined only by the Sun gravitational attraction, the departure date is fixed, and the spacecraft initial and final states are coincident with those of the Earth and Venus, respectively.

1) Using the PMP, write down the spacecraft equations of motion, the costate dynamics, and the zero-finding problem for the unknowns $\{\boldsymbol{\lambda}_0, t_f\}$ with the appropriate transversality condition.

2) Adimensionalize the problem using as reference length $LU = 1$ AU† and reference mass $MU = m_0$, imposing that $\mu = 1$. Report all the adimensionalized parameters.

3) Solve the problem considering the following data:

   - Launch date: `2023-05-28-14:13:09.000 UTC`
   - Spacecraft mass: $m_0 = 1000$ kg
   - Electric propulsion properties: $T_{\max} = 800$ mN, $I_{sp} = 3120$ s

   To obtain an initial guess for the costate, generate random numbers such that $\lambda_{0,i} \in [-20; +20]$, while $t_f < 2\pi$. Report the obtained solution in terms of $\{\boldsymbol{\lambda}_0, t_f\}$ and the error with respect to the target. Assess your results exploiting the properties of the Hamiltonian in problems that are not time-dependent and time-optimal solutions.

4) Solve the problem for a lower thrust level $T_{\max} = [500]$ mN. Tip: exploit numerical continuation.

(11 points)

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*Read the necessary gravitational constants and planets positions from SPICE. Use the kernels provided on WeBeep for this assignment.

†Read the value from SPICE

1) Two Point Boundary Value problem:

$$\min_{\underline{u}\in U}\int_{t_0}^{t_f} l(\underline{\boldsymbol{x}},\underline{\boldsymbol{u}},\underline{\boldsymbol{t}})\,dt\ s.t.\begin{cases}\dot{\underline{\boldsymbol{x}}}=f(\underline{\boldsymbol{x}},\underline{\boldsymbol{u}},\underline{\boldsymbol{t}})\\ \underline{\boldsymbol{x}}(t_0)=\underline{\boldsymbol{x}}_0\\ x_i(t_f)=x_{f,i},\ i=1,...,I<n\end{cases}$$

$$\text{system dynamics: }\begin{cases}\dot{\underline{\boldsymbol{r}}}=\underline{v}\\ \dot{\underline{\boldsymbol{v}}}=-\frac{\mu}{r^3}\underline{\boldsymbol{r}}+u^*\left(m,\lambda_v,\lambda_m\right)\frac{T_{\max}}{m}\hat{\underline{\boldsymbol{\alpha}}}^*\text{ with optimal }\hat{\underline{\boldsymbol{\alpha}}}^*=-\frac{\boldsymbol{\lambda_v}}{\lambda_v}\\ \dot{m}=-u^*\left(m,\lambda_v,\lambda_m\right)\frac{T_{\max}}{I_{sp}g_0}\end{cases}$$

$$\text{costates dynamics: }\begin{cases}\dot{\underline{\boldsymbol{\lambda}}}_{\boldsymbol{r}}=-3\frac{\mu}{r^5}\left(\underline{\boldsymbol{r}}\cdot\underline{\boldsymbol{\lambda}}_v\right)\underline{\boldsymbol{r}}+\frac{\mu}{r^3}\underline{\boldsymbol{\lambda}}_v\\ \dot{\underline{\boldsymbol{\lambda}}}_{\boldsymbol{v}}=-\underline{\boldsymbol{\lambda}}_r\\ \dot{\lambda}_m=-u^*\left(m,\lambda_v,\lambda_m\right)\frac{\lambda_v T_{\max}}{m^2}\end{cases}$$

$$\text{initial conditions: }\begin{cases}\underline{\boldsymbol{r}}\left(t_0\right)=\underline{\boldsymbol{r}}_0\\ \underline{\boldsymbol{v}}\left(t_0\right)=\underline{\boldsymbol{v}}_0\\ m\left(t_0\right)=m_0\end{cases}$$

$$\text{final conditions }F(\mathbf{x})=0:\begin{cases}\underline{\boldsymbol{r}}\left(t_f\right)-\underline{\boldsymbol{r}}_f=0\\ \underline{\boldsymbol{v}}\left(t_f\right)-\underline{\boldsymbol{v}}_f=0\\ \lambda_m\left(t_f\right)=0\text{ transversality condition}\end{cases}$$

$$\text{unknowns }\mathbf{x}:\begin{cases}\underline{\boldsymbol{\lambda}}_{\mathbf{0}}=\begin{bmatrix}\underline{\boldsymbol{\lambda}}_{\boldsymbol{r}}\left(t_0\right)\\ \underline{\boldsymbol{\lambda}}_{\boldsymbol{v}}\left(t_0\right)\\ \lambda_m\left(t_0\right)\end{bmatrix}\\ t_f\end{cases}\tag{5}$$

$$\text{objective function }l(\underline{x},\underline{u},\underline{t})\text{ for a time-optimal solution: }J:=\int_{t_0}^{t_f}1\,dt$$

$$\text{set of admissible controls }U:=\{(u,\hat{\underline{\boldsymbol{\alpha}}})\,s.t.\ u\in[0,1],\ ||\hat{\underline{\boldsymbol{\alpha}}}||=1\}$$

$$\text{problem statement: }\min_{(u,\hat{\underline{\boldsymbol{\alpha}}})\in U}J=\int_{t_0}^{t_f}1\,dt\ s.t.\begin{cases}\dot{\underline{\boldsymbol{r}}}=\underline{\boldsymbol{v}}\\ \dot{\underline{\boldsymbol{v}}}=-\frac{\mu}{r^3}\underline{\boldsymbol{r}}-u^*\left(m,\lambda_v,\lambda_m\right)\frac{T_{\max}}{m}\frac{\boldsymbol{\lambda_v}}{\lambda_v}\\ \dot{m}=-u^*\left(\lambda_v,\lambda_m\right)\frac{T_{\max}}{I_{sp}g_0}\\ \underline{\boldsymbol{r}}\left(t_0\right)=\underline{\boldsymbol{r}}_0\\ \underline{\boldsymbol{v}}\left(t_0\right)=\underline{\boldsymbol{v}}_0\\ m\left(t_0\right)=m_0\\ \underline{\boldsymbol{r}}\left(t_f\right)-\underline{\boldsymbol{r}}_f=0\\ \underline{\boldsymbol{v}}\left(t_f\right)-\underline{\boldsymbol{v}}_f=0\\ \lambda_m\left(t_f\right)=0\end{cases}$$

2) Using as reference length LU = 1 AU and reference mass MU = m_0, imposing that $\mu =$ 1, first we retrieve the adimensionalized reference time

$$TU = \sqrt{\frac{LU^3}{\mu_{Sun}}} = 5022642.8865662800 \tag{6}$$

and then we compute the other adimensionalized parameters:

$\mathbf{r}_0$	-0.4009314856	-0.9306291891	0.0000487948
$\mathbf{v}_0$	0.9023300052	-0.3991722255	0.0000420730
m_0	1		
I_{sp}	0.0006211869		
T_{max}	0.1349053511		
g_0	1653.7119512003		
GM	1		

Table 2: Adimensionalized quantities ($T_{\text{max}} = 800$ mN)

3+4) Solution with $T_{\text{max}} = 800$ mN:

$\boldsymbol{\lambda}_{0,r}$	1.8451393123	-51.2011105752	0.4009042646
$\boldsymbol{\lambda}_{0,v}$	3.0569000243	-6.1331349492	0.8758324004
$\lambda_{0,m}$	0.0064900331		
t_f	`2023-10-17-03:10:58.043 UTC`		
TOF [days]	141.4758885100		

Table 3: Time-optimal Earth-Venus transfer solution ($T_{\text{max}} = 800$ mN)

$\lvert\lvert\mathbf{r}_f(t_f) - \mathbf{r}_V(t_f)\rvert\rvert$	[km]	$1.7116124249 * 10^{-3}$
$\lvert\lvert\mathbf{v}_f(t_f) - \mathbf{v}_V(t_f)\rvert\rvert$	[m/s]	$6.0720043994 * 10^{-7}$

Table 4: Final state error with respect to Venus' center ($T_{\text{max}} = 800$ mN)

Solution with $T_{\text{max}} = 500$ mN:

$\boldsymbol{\lambda}_{0,r}$	-1.1729673837	-51.0783249879	-0.9889087148
$\boldsymbol{\lambda}_{0,v}$	4.6985334877	-6.2649934001	0.6615674107
$\lambda_{0,m}$	0.0056795354		
t_f	`2024-01-01-04:01:55.651 UTC`		
TOF [days]	217.4328633020		

Table 5: Time-optimal Earth-Venus transfer solution ($T_{\text{max}} = 500$ mN)

$\lvert\lvert\mathbf{r}_f(t_f) - \mathbf{r}_V(t_f)\rvert\rvert$	[km]	$2.5659643633 * 10^{-4}$
$\lvert\lvert\mathbf{v}_f(t_f) - \mathbf{v}_V(t_f)\rvert\rvert$	[m/s]	$8.3857650248 * 10^{-8}$

Table 6: Final state error with respect to Venus' center ($T_{\text{max}} = 500$ mN)

Validation:

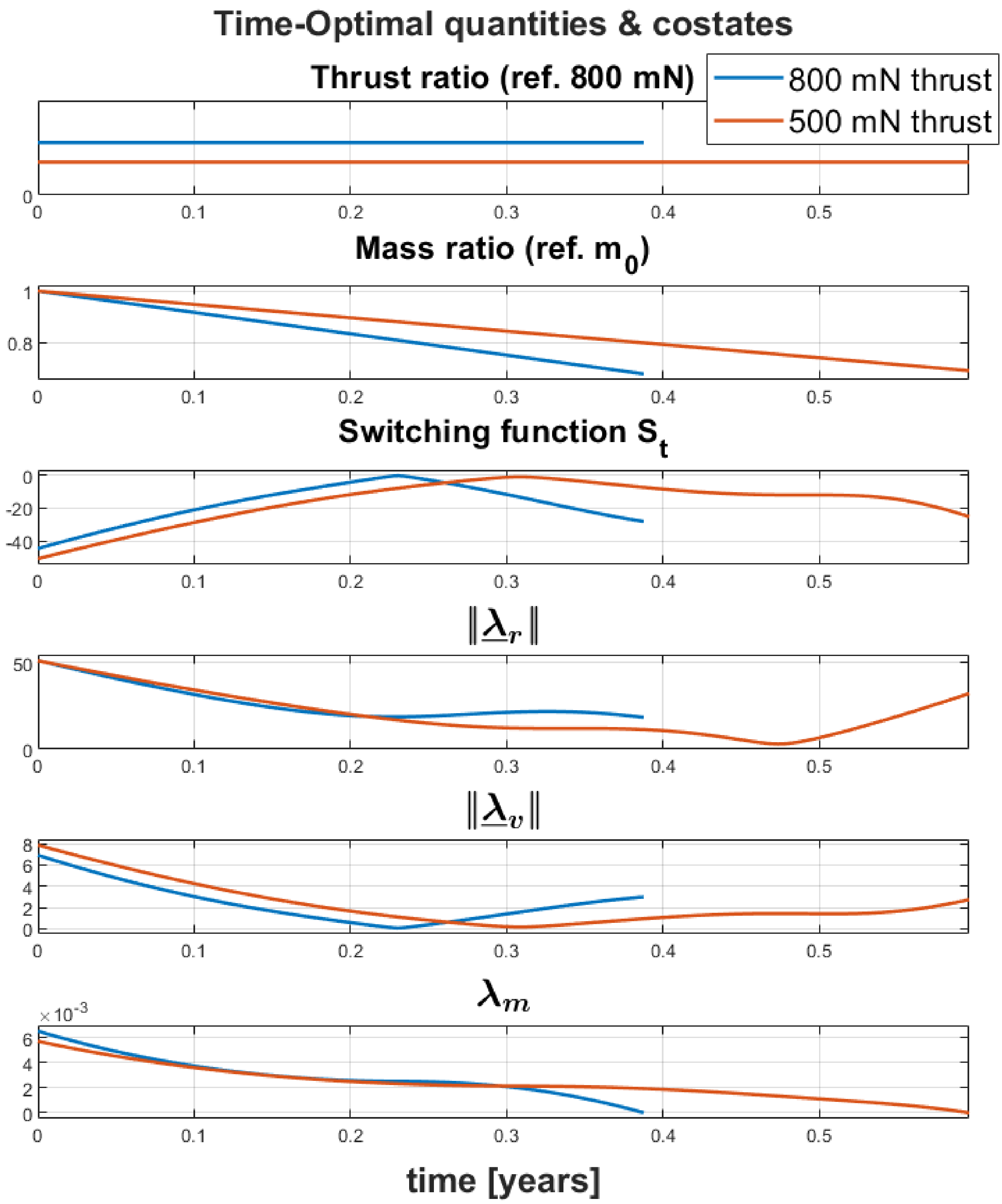
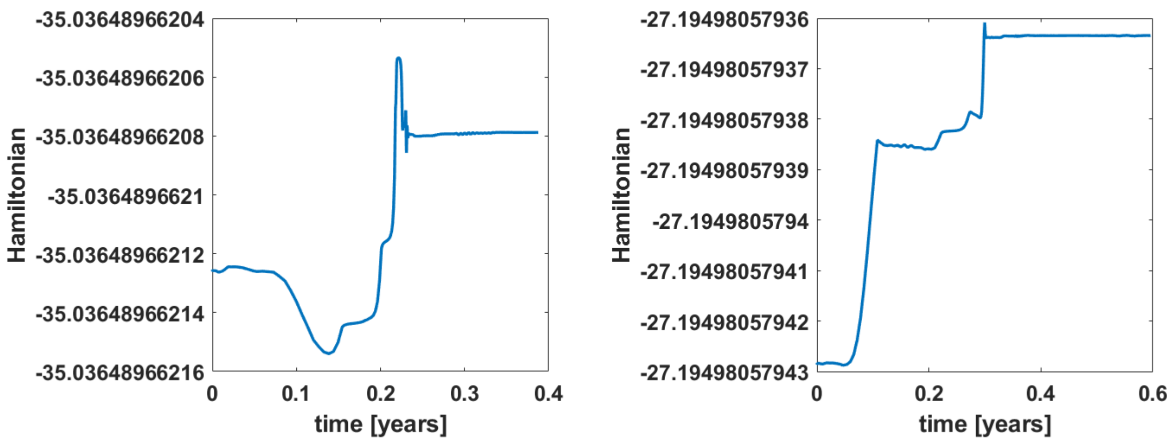


Figure 10: Thrust ratio, Mass ratio, Switching function and scalar costates

As expected for time-optimal solutions, the switching function is negative throughout the entire maneuver; before t_f λ_m is higher than zero and it's always descending monotone till arrival epoch, where it coincides with 0. Moreover, as shown in Figure 11, in problems that are not time-dependent the Hamiltonian is constant along an optimal solution.



(a) 800 mN **(b)** 500 mN

Figure 11: Hamiltonian almost perfectly constant

Thrust angles have been plotted directly along the trajectory, so it's possible to perceive immediately the physical meaning of the costates, and how the solution changes for a different thrust.

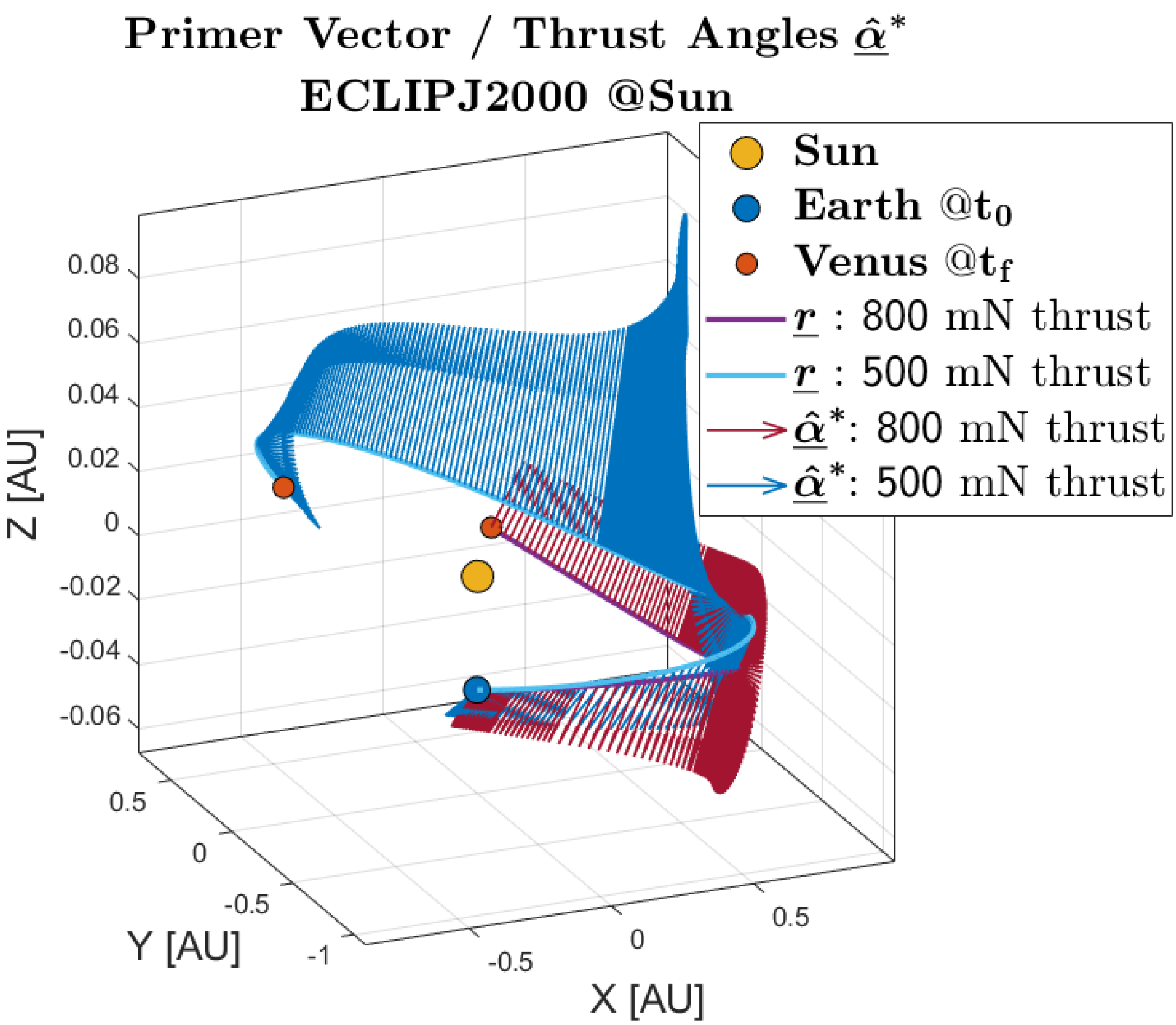


Figure 12: Primer vector / thrust angles

The $\underline{\boldsymbol{\lambda}_v}$ costate, whose direction is $-\underline{\hat{\alpha}}^*$:

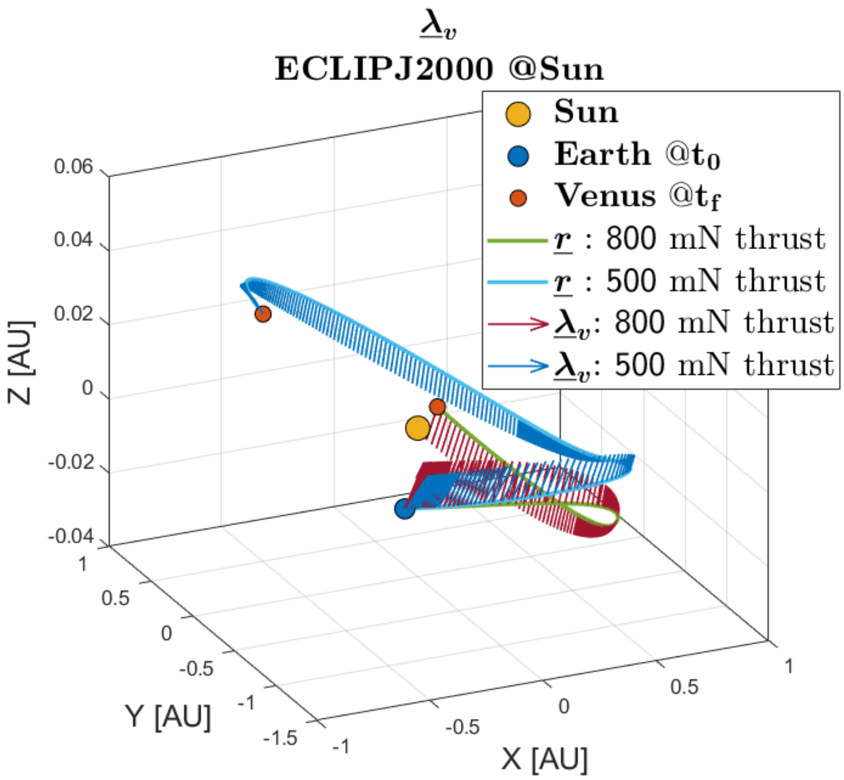


Figure 13: Costate $\underline{\boldsymbol{\lambda}_v}$

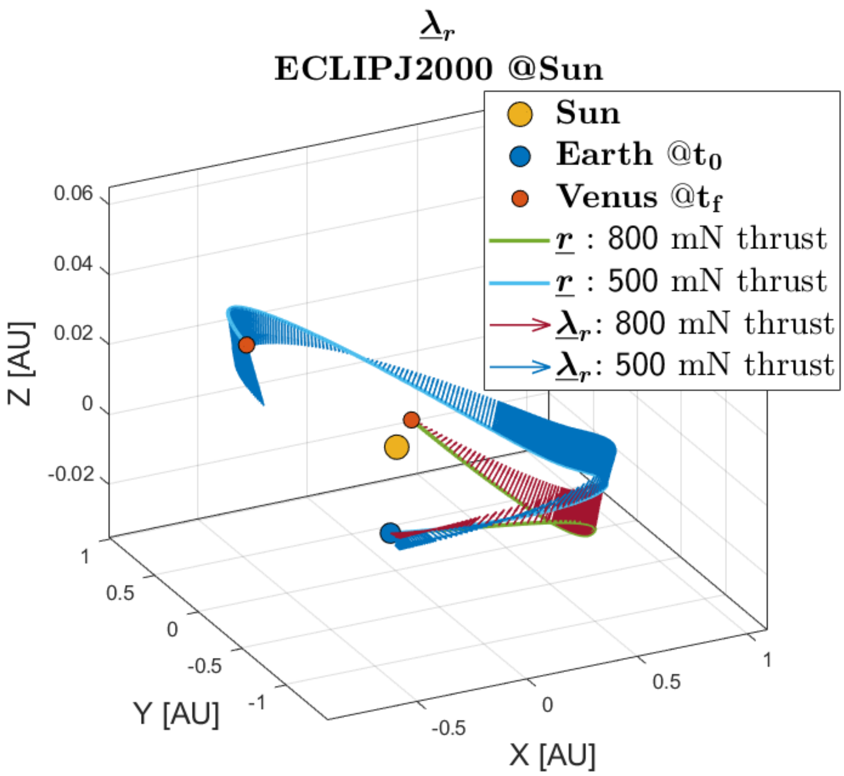


Figure 14: Costate $\underline{\boldsymbol{\lambda}_r}$