BI2113 Ecology and Evolution

Assignment 1

Aumshree P. Shah
20231059

October 2024

Q1) For the population described by the transition matrix given below

$$\begin{pmatrix} 0.4 & 3 \\ 0.32 & 0.7 \end{pmatrix}$$

a) What is the ratio of the total population sizes (i.e. sum of both stage classes in a generation) in two successive generations at equilibrium?
b) At equilibrium what is the fraction of the juveniles in any given generation? This refers to the ratio of juveniles to the total population size.

Solution:

a) Let's start with assuming that J is the fraction of juveniles at equilibrium and similarly A is the fraction of adults at equilibrium. At equilibrium, after any iteration, the fractions of the adults and juveniles will remain the same. (Perron–Frobenius theorem states that every matrix with real positive numbers will have an eigenvalue and a eigenvector)

$$\begin{pmatrix} 0.4 & 3 \\ 0.32 & 0.7 \end{pmatrix} \begin{pmatrix} J \\ A \end{pmatrix} = r \begin{pmatrix} J \\ A \end{pmatrix}$$

Here r is the eigenvalue of the transition matrix which represents the ratio of the population size after each iteraton (since the matix multiplication is the same as multiplication by a scalar at equilibrium) and $\binom{J}{A}$ is the eigenvector of the matrix.
To find the eigenvalue and eigenvector we make use of properties of matrices, in which the eigenvalue of a 2×2 matrix will be the roots of the quadratic equation $A - \lambda I = 0$ where A is our given transition matrix and I is the identity matrix. From this we get:

$$\begin{pmatrix} 0.4 - \lambda & 3 \\ 0.32 & 0.7 - \lambda \end{pmatrix} = 0$$

$$\begin{aligned} \implies \quad & (0.4 - \lambda)(0.7 - \lambda) - 3 \times 0.32 &= 0 \\ \implies \quad & 0.28 - 0.4\lambda - 0.7\lambda + \lambda^2 - 0.96 &= 0 \\ \implies \quad & \lambda^2 - 1.1\lambda - 0.68 &= 0 \end{aligned}$$

$$\implies \lambda = \frac{-(-1.1) \pm \sqrt{1.1^2 + 4 \times 0.68}}{2} \quad \text{or} \quad \lambda \approx 1.541\ldots,\ -0.441\ldots$$

Final Eigenvalue

Since the other eigenvalue is negative (which does not biologically make sense), we choose the greater eigenvalue, which is $r \approx 1.5412$. Hence, the ratio of population sizes $\left(\frac{T_{n+1}}{T_n}\right)$ of any two successive generations at equilibrium is approximately 1.54.

b) To find the ratio of juveniles, we first find the ratio of juveniles to adults to get $\dfrac{J}{A}$. We again make use of the properties of matrices in which the eigenvector of a 2×2 matrix will solve $(A - \lambda I)\vec{v} = 0$ where $\vec{v}$ is the eigenvector. Hence we get:

$$\begin{pmatrix} 0.4 - \lambda & 3 \\ 0.32 & 0.7 - \lambda \end{pmatrix} \begin{pmatrix} J \\ A \end{pmatrix} = 0$$

From this, we get two linear equation as:

$$-(r - 0.4)J + 3A = 0$$
$$0.32J + (r - 0.7)A = 0$$

Where r is the greater eigenvalue found in part *a)*. Solving any of the above equations we get the ratio of adults to juveniles as:

$$\frac{A}{J} = \frac{r - 0.4}{3}$$

But we need to find the ratio of juveniles to the total population, ie we need to find the ratio of $\frac{J}{J+A}$. we see that:

$$\frac{J}{J+A} = \frac{\frac{J}{J}}{\frac{J+A}{J}} = \frac{1}{1 + \frac{A}{J}}$$

Substituting the value of $\frac{A}{J}$ in the above equation we get the fraction of the juveniles in any given generation as: $\dfrac{1}{1 + \frac{r-0.4}{3}}$

Final Eigenvector

Hence we get the final ratio of juveniles to the total population $\left(\frac{J}{J+A}\right)$, by simplifing it to ≈ 0.7244

Q2) Assume a Lotka-Volterra competition scenario with the following parameter values:
Species 1 : $r_1 = 1, K_1 = 200, \alpha_{12} = 0.8$; Species2 : $r_2 = 0.5, K_2 = 300, \alpha_{21} = 2$
Given below are four starting points for the system. For each case, what will be the population sizes of Species 1 and Species 2 at equilibrium? For this your answer should consist of the filled table as well as the four isocline diagrams showing the trajectory of the system (not merely the vectors).

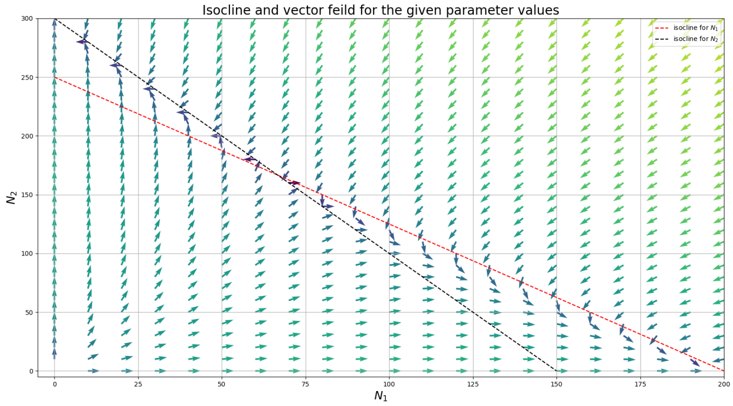
	Starting Populatoin Sizes		Equilibrium population sizes	
S. No.	Species 1	Species 2	Species 1	Species 2
1	70	175		
2	70	75		
3	30	120		
4	30	100		

Table 1

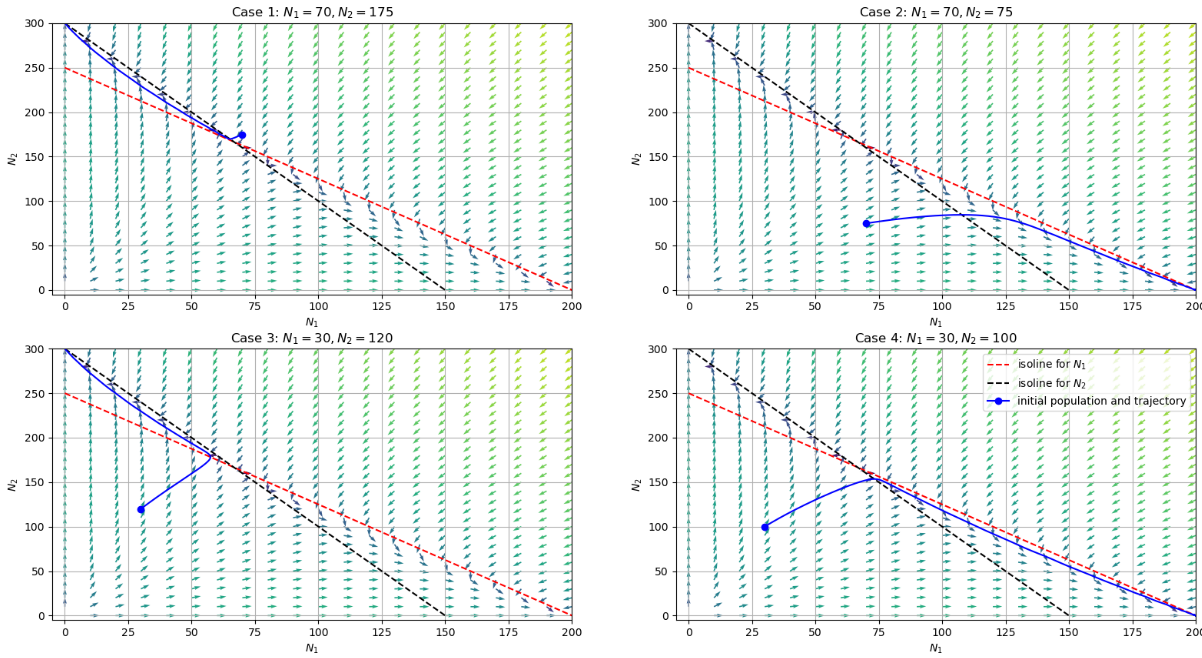
Solution

We first plot the isoline diagram with its vector feild using python libraries where N_1 is the population of species 1 and N_2 is the population of species 2:

Here we make the length of the vector equal to get a proper visulization and represent the strength of those vectors by colours which have a linear change with logarithmic increase in magnitute.



Form this we see that the given competition scenario forms a feild with one unstable equilibrium point. Hence at the equilibrium the population of any one of the species will go extint and the other population will be at its maximum capacity. Now we plot the trajectory of the given four different types of starting population sizes: Here we see that in two of the cases, case-2 and case-3, where the



starting population sizes is [70, 75] and [30, 120] respectively, the species N_1 goes to extinct whereas in the case-2 and case-4, the N_1 species two goes to extinct.
Here we assume that the population follows a continious change which doesn't happen in real population but since the population number are large it doesn't affect our result that much, aslo the population will ossilate at the equilibrium. From these results, we fill the table as follows:

Final population values

	Starting Population Sizes		Equilibrium population sizes	
S. No.	Species 1	Species 2	Species 1	Species 2
1	70	175	0	300
2	70	75	200	0
3	30	120	0	300
4	30	100	200	0

Q3 a) Suppose we have a hypothetical insect population with a cohort lifetable that looks as shown in **??** . Over time, a population of this insect will be increasing or decreasing in numbers?

Age (in days)	**No. surviving**	**Avg. No. of eggs per capita**
Egg	250	0
1	239	0
2	210	0
3	156	0.551
4	129	0.412
5	109	0.251
6	76	0.99
7	43	0.95
8	32	0.71
9	19	0.49
10	9	0.135
11	0	0

Table 2: Survival and Egg Production Data

Solution:

a) For this, We first find the fraction of surviving people by dividing the number of surviving people (l_x) by 250, the number of eggs per capita (m_x) is given, using this we calculate $l_x m_x$ or the average number of offsprings by that particular age. Then we find the net per capita reproductive rate ($\sum l_x m_x$), using the cohort lifetable as follows:

Age (days)	**No. surviving**	**Fraction surviving (l_x)**	**No. eggs per capita (m_x)**	$l_x m_x$
Egg	250	1.000	0	0
1	239	0.956	0	0
2	210	0.840	0	0
3	156	0.624	0.551	0.344
4	129	0.516	0.412	0.213
5	109	0.436	0.251	0.109
6	76	0.304	0.990	0.301
7	43	0.172	0.950	0.164
8	32	0.128	0.710	0.091
9	19	0.076	0.490	0.037
10	9	0.036	0.135	0.005
11	0	0.000	0	0
				$\sum l_x m_x = 1.264$

Table 3: Complete Survival and Egg Production Data

Conclusion

$\sum l_x m_x = 1.264$ means each approximately individual produces 1.26 offspring in his lifetime. Because the net per capita rate for the given cohort lifetable is greater than 1, the population over time will increase.

Q3b) Suppose there was a mutation in the insect population mentioned in Part a due to which this insect now lives two days longer with an enhanced egg-output during late life. However, the insect now starts reproducing one day later. The modified life-table is given to the right. From the perspective of long-term dynamics, is this a beneficial mutation or a harmful one?

Age (in days)	**No. surviving**	**Avg. No. of eggs per capita**
Egg	250	0
1	239	0
2	210	0
3	156	0
4	129	0.412
5	109	0.251
6	76	0.99
7	43	0.95
8	32	0.71
9	19	0.49
10	9	0.27
11	9	0.27
12	9	0.27
13	0	0

Table 4: Survival and Egg Production Data

Age (days)	**No. surviving**	**Fraction surviving (l_x)**	**No. eggs per capita (m_x)**	$l_x m_x$
Egg	250	1.000	0	0
1	239	0.956	0	0
2	210	0.840	0	0
3	156	0.624	0	0
4	129	0.516	0.412	0.213
5	109	0.436	0.251	0.109
6	76	0.304	0.990	0.301
7	43	0.172	0.950	0.164
8	32	0.128	0.710	0.091
9	19	0.076	0.490	0.037
10	9	0.036	0.270	0.010
11	9	0.036	0.270	0.010
12	9	0.036	0.270	0.010
13	0	0.000	0	0
				$\sum l_x m_x = 0.945$

Table 5: Survival and Egg Production Data

$\sum l_x m_x = 0.945$ means each approximately individual produces only 0.945 offspring in his lifetime. Because the net per capita rate for the given cohort lifetable in part b) is less than 1, the population over time will decrease.

Conclusion

Hence from the perspective of long term dynamic the given mutation is a harmful one for the insect. As it not only decreases the rate of growth, but it actually makes it negative meaning after a finite amount of time the population will go extinct. The cause of the mutation giving lesser number of eggs per capita is due to the fact that the mutation skips the egg production by 1 day during which, without mutation the insect produces the most number of eggs compared to any age class, and in the latter age groups the population is alrealy low which doesn't contribute much to the total eggs produced.

Q4) The government is planning to reintroduce 10 Namibian cheetahs to Madhya Pradesh's Gandhi Sagar Wildlife Sanctuary in January 2025. At the same time, they plan to introduce 30 Sangai deers from Manipur into the sanctuary as prey. No further introduction of either prey or the predator is planned. The government has decided to celebrate the introduction after a few years through a major celebration in the park inviting many world leaders. Park authorities have determined that the optimal viewing densities for the dignitaries would be 15 cheetahs and 18 sangais in the park. They have also determined that the cheetah-sangai system can be modelled as a Lotka-Volterra preypredator system

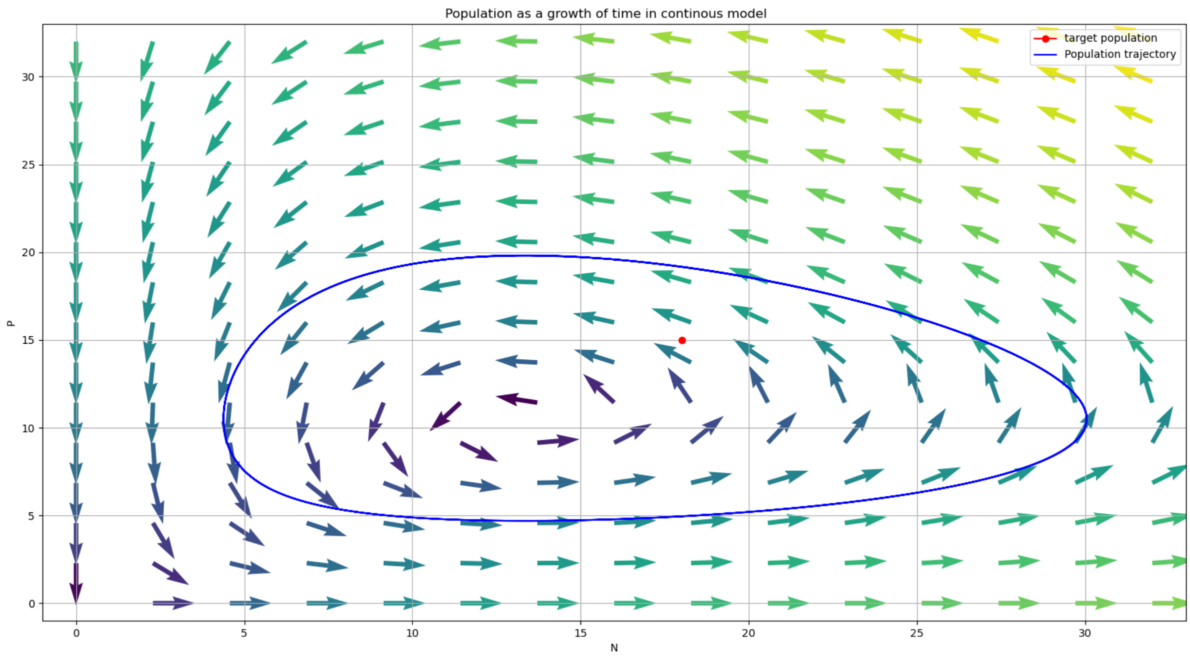
$$\frac{\mathrm{d}N}{\mathrm{d}t} = r_1 N - CNP$$
$$\frac{\mathrm{d}P}{\mathrm{d}t} = -d_2 P + gCNP$$

where N and P denote the prey and the predator population respectively, and the other constants have their usual meaning as per the Lotka-Volterra prey-predator model. As per data from the scientists, the yearly rates are $r_1 = 1.05, C = 0.1, d_2 = 0.6, g = 0.45$.
Based on this information, in which year will the celebratory event happen for the first time? Assume that all the assumptions of the Lotka-Volterra continuous prey-predator model are true for this system, and one needs to have both the cheetah's and the sangai's numbers to be simultaneously as per specifications.

Solution:
We determine the year of the celebratory event be doing the following:

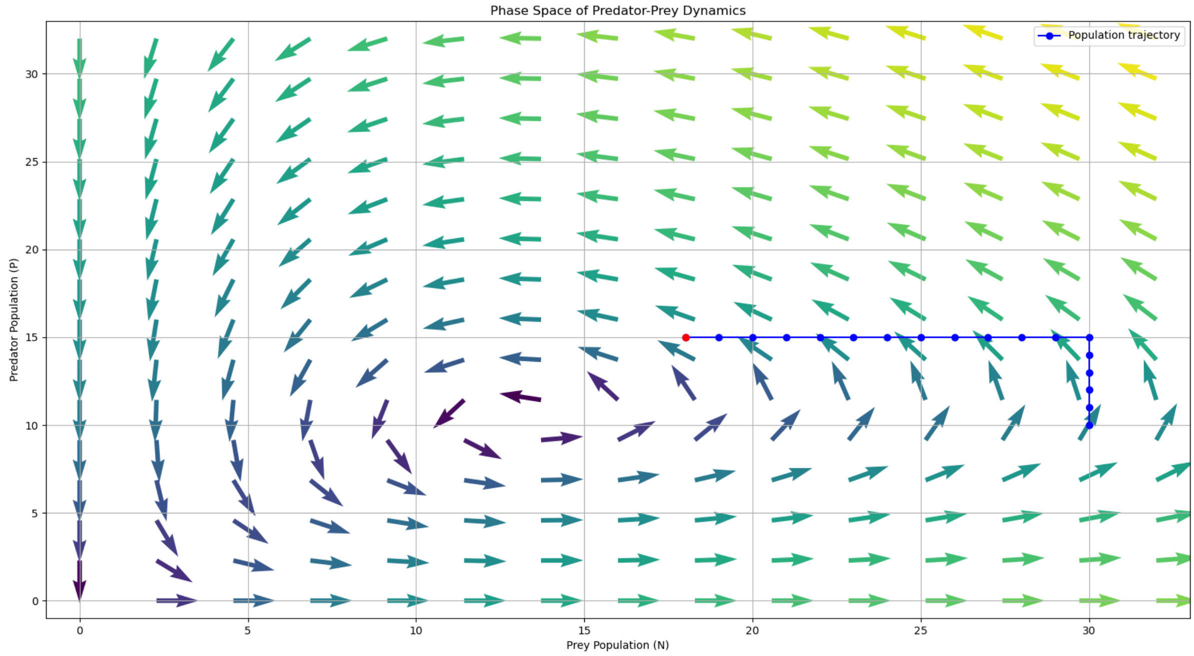
(a) First we plot the vector map of the Prey-Preditor dynamics.
(b) Then we run the system for with the initial population of [30, 10].
(c) We then see the time evolution of the system and determine the time period using it.

Plotting the vector feild with time evoultion we get:



We run the population over the timescale of 30 years, we observe that the trajectory of the population ossilates constantaly. This implies that the population never reaches the required number and hence the celebratory event never occours.

But here we have assumed that the population which is given by the dynamics is continious, which is **not the case in real life**. If we assume that the population at any given point is given by (a, b) then the population only has 4 ways in which it can change which are: $(a+1, b), (a, b+1), (a-1, b), (a, b-1)$. After the change in population it will have a different change in vector which will drive it to a different point.
Now how do we determine in which of the 4 directions it goes to? Well a simple way to determine that will be to see which direction the vector feild favours the most. So we start at $(30, 10)$ and see the time evolution of the system by iterating over steps rather than time.
Plotting the evolution of the system using the discrete step method mentioned above we get:



Here we see that the population does infact reach the desired value after **seventeen** iterations. But we arrive at a problem here. We need to determine the time taken to reach the required point but each iteration will have differnet time, if the vector aligns to x or y axis and has high magnitute, then the population will increase rapidly. Whereas if the population has a small magnitute and is aligned halfway between the axis then the time taken for that iteration would be larger. For this we complicate the code used which is given in remarks which makes use of the method mentioned above (taking $\frac{1}{\cos\theta|\vec{r}|}$ as the time taken where $\vec{r}$ is the vector at the given point) to get the total time as: 2.895 years. This happens with some probablity as we have assumed that even on a small prefrence in the angle the population will move in that direction.

Final Conclusion

If we assume the continious trajectory growth of the system, then we get a ossilating cycle. But real systems doesn't show continious growth.
If instead we try to plot it using the discreate populations and some other assumptions we do infact reach the celebratory event in 2.895 years. This means if we assume the population to be discreate, the **celebratory event on earliest may occour around November 2027**.

REMARKS

Code for question 2 and question 4 is uploaded HERE. or you can paste `https://github.com/LAUGHINGCATMEME/Biology-Assignment`.

In question 4, if the population is at (10, 10) making a angle of 44^o, in the model it moves 1 unit to left but in reality it has almost the same probablity of moving up. Hence the actual predicted value has some finite probablity attached to it.