II.2.5 THERMAL DIFFUSIVITY OF BRASS

1. INTRODUCTION:

The thermal diffusivity of a material is defined by the ratio $\kappa/\rho c$ where κ is the thermal conductivity, ρ, the density and c the specific heat of a material. If heat is generated at a point in the material, the speed with which it diffuses out from the point is determined by this ratio. The dimension of this ratio is ((Joule/s.m.K)/ [(kg/m^3) (Joule/kg K)] = m^2/s. It indicates the spread, m^2 per second, of the locally applied heat. For a poor thermal conductor like glass, κ= 1W/mK, ρ = 2500 kg/ m^3 and c = 1 kiloJ/kg K and the thermal diffusivity is 4×10^{-7} m^2/s. For copper the diffusivity has a much higher value. Heat spreads much faster in copper than in glass.

In steady state measurements of heat transport only the conductivity plays a role. The diffusivity has no role to play. But if the heat supplied varies as a function of time then the diffusivity determines how the temperature varies with space and time in the medium. For such a case, the equation satisfied by the temperature T for a one-dimensional problem is obtained as follows.

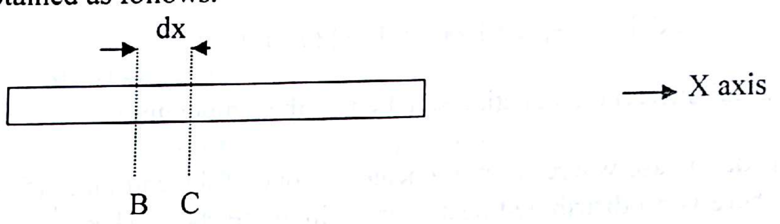

Figure II.2.5.1

Let the cross-section of the rod be A. Heat is propagated along the length of the rod, which is taken along the X direction. Consider two sections of the rod at B and C separated by an infinitesimal distance dx. The temperature at the cross section at B is T. The amount of heat crossing the section at B in the direction of the X-axis is $-\kappa A dT/dx$. The heat passing out through the section at C is $-\kappa A\,[\partial T/\partial x + \partial^2 T/\partial x^2\,dx]$. So the net amount of heat flowing out of the element of length dx in one second is given by

$$\{-\kappa A\,[\partial T/\partial x + \partial^2 T/\partial x^2\,dx]\} - (-\kappa A \partial T/\partial x) = -\kappa A \partial^2 T/\partial x^2\,dx \qquad \text{(II.2.5.1)}$$

The mass of material between the two sections is ρA dx. If the temperature varies with time then the amount of heat used up in one second in heating this element is

$$\rho A\,dx\,c\,\partial T/\partial t \qquad \text{(II.2.5.2)}$$

In addition there may be radiation from the surface to the surroundings. Using the simple Newton's law the heat lost from the surface area of the element between the sections B and C is

$$\varepsilon P\, dx\, (T - T_0) \tag{II.2.5.3}$$

where T_0 is the temperature of the surroundings and P is the perimeter of the cross section of the rod.

The total heat used up by the element in one second is

$$\rho A\, dx\, c\, \partial T/\partial t - \kappa A \partial^2 T/\partial x^2\, dx + \varepsilon P\, dx\, (T - T_0) \tag{II.2.5.4}$$

If an amount of heat λ Adx is supplied for unit time, the law of conservation of energy tells us that the heat supplied must be equal to the heat used up. So

$$\lambda\, Adx = \rho A\, dx\, c\, \partial T/\partial t - \kappa A \partial^2 T/\partial x^2\, dx + \varepsilon P\, dx\, (T - T_0) \tag{II.2.5.5}$$

or

$$-\kappa \partial^2 T/\partial x^2 + \rho\, c\, \partial T/\partial t + \varepsilon\, (P/A)\, (T - T_0) = \lambda \tag{II.2.5.6}$$

This is the partial differential equation satisfied by the temperature.

Let us consider a case where we put a heater on one of the end faces (face at x = 0) of the rod and there is no distributed heat source along the rod. Then $\lambda = 0$ all along the rod. Then the equation satisfied by the temperature T is

$$\kappa \partial^2 T/\partial x^2 - \rho\, c\, \partial T/\partial t - \varepsilon\, (P/A)\, (T - T_0) = 0 \qquad (0<x<L) \tag{II.2.5.7}$$

$$\partial^2 T/\partial x^2 - (\rho c/\kappa)\, \partial T/\partial t - \varepsilon\, (P/A\kappa)\, (T - T_0) = 0 \tag{II.2.5.8}$$

Let us assume that over the end face the heat flowing per unit time Q varies as $\exp(-\iota\omega t)$ i.e.

$$Q = Q_0 \exp(-\iota\omega t) \tag{II.2.5.9}$$

Then the temperature also will vary as

$$T(x, t) = \theta(x) \exp(-\iota\omega t) + T_0 \tag{II.2.5.10}$$

Then

$$d^2\theta/dx^2 + \iota(\omega\rho c/\kappa)\, \theta - \varepsilon\, (P/A\kappa)\, \theta = 0 \tag{II.2.5.11}$$

Writing $\varepsilon\,(P/A\kappa) = a$ and $(\omega\rho c/\kappa) = b$ equation (11) can be written as

$$d^2\theta/dx^2 - (a - \iota b)\theta = 0 \qquad \text{(II.2.5.12)}$$

This equation has to be solved subject to certain boundary conditions. One condition will be determined by

$$-\kappa A\,(\partial T/\partial x)_{x=0} = Q \qquad \text{(II.2.5.13)}$$

The other boundary condition can be imposed by keeping the end of the rod at L at a fixed temperature T_0.

Writing $a - ib = \eta^2$ and putting $u = \eta x$ (II.2.5.14)

$$d^2\theta/du^2 - \theta = 0 \qquad \text{(II.2.5.15)}$$

The solution of this equation is

$$\theta\,(x) = C\exp(\eta x) + B\exp(-\eta x) \qquad \text{(II.2.5.16)}$$

The temperature T (x, t) varies as

$$T\,(x, t) = [C\exp(\eta x) + B\exp(-\eta x)]\exp(-\iota\omega t) + T_0 \qquad \text{(II.2.5.17)}$$

$$\eta = (\alpha - \iota\beta) = (a - \iota b)^{1/2} = (a^2+b^2)^{1/4}\exp(-\iota\phi/2) \qquad \text{(II.2.5.17a)}$$

where $\tan\phi = b/a$ (II.2.5.17b)

$$T\,(x, t) = [C\exp(\alpha x)\exp(-\iota\beta x) + B\exp(-\alpha x)\exp(+\iota\beta x)]\exp(-\iota\omega t) \qquad \text{(II.2.5.18)}$$

$$\alpha = (a^2+b^2)^{1/4}\cos(\phi/2) \qquad \text{(II.2.5.19 a)}$$
$$\beta = (a^2+b^2)^{1/4}\sin(\phi/2) \qquad \text{(II.2.5.19 b)}$$

α and β have the dimension of inverse of length. If $\alpha L>5$, we may take $T = T_0$ at L and put $C = 0$ in the term $C\exp(\alpha+\iota\beta)\,x$ in (II.2.5.18). We assume this to be true and put $C = 0$.

Then the temperature distribution is given by

$$T(x, t) = B \exp(-(\alpha-\iota\beta)x) \exp(-\iota\omega t) + T_0 \qquad (II.2.5.20)$$

The temperature gradient at $x = 0$ is then given by

$$dT/dx = - [(\alpha-\iota\beta)] B \exp(-\iota\omega t) \qquad (II.2.5.21)$$

The boundary condition at $x = 0$ is

$$-\kappa A \, |dT/dx| = Q_0 \exp(-\iota\omega t) \qquad (II.2.5.22)$$

This gives

$$B = [Q_0 / (\kappa A\} (\alpha^2+\beta^2)^{1/2}] \qquad (II.2.5.23)$$
$$= [Q_0 / (\kappa A\} (a^2+\beta^2)^{1/4}]$$

So the temperature distribution is given by

$$T(x, t) = [Q_0/ (\kappa A (a^2+b^2)^{1/4}] \exp(-\alpha x) \exp -\iota(\omega t - \beta x) \qquad (II.2.5.24)$$

The temperature is not in phase with the heating. The amplitude of the temperature oscillation is a function of x and varies as

$$Amp(T(x)) = [Q_0/ (\kappa A (a^2+b^2)^{1/4})]\exp(-\alpha x) \qquad (II.2.5.26)$$

The ratio of amplitudes at x_1 and x_2 is

$$Amp(x_2)/Amp(x_1) = \exp[-\alpha (x_2-x_1)] \qquad (II.2.5.27)$$

The difference in phase between the temperature oscillations at two points x_1 and x_2 is

$$\phi(x_1) - \phi(x_2) = \beta (x_1-x_2) \qquad (II.2.5.28)$$

One can find α and β and hence the thermal diffusivity of the material by measuring the ratio of amplitudes at two points separated by a known distance and by measuring the phase difference of the temperature wave between two points separated by a known distance.

However producing a sinusoidal heating is difficult. We can produce a periodic heating at the end $x = 0$ by supplying a known current to a heater in a periodic fashion as shown below.

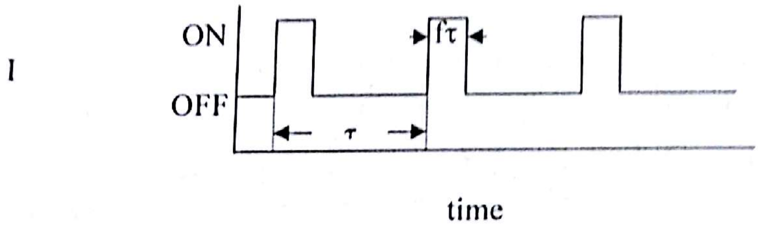

Figure II.2.5.2 Shows periodic heating at $x = 0$.

The current is switched on for times $n\tau \le t < (n+f)\tau$; n is integral and $0<f<1$
The current is switched off for times $(n+f)\tau<t<(n+1)\tau$

(i.e.) $$I = I_0 \text{ for } n\tau \le t < (n+f)\tau; \text{ and } 0 \text{ for } (n+f)\tau<t<(n+1)\tau \quad \text{(II.2.5.29)}$$

τ is the period of current variation. So the heat input Q varies as

$$Q = H = I^2R \qquad \text{for } n\tau\le t < (n+f)\tau;$$

And

$$= 0 \qquad \text{for } (n+f)\tau<t<(n+1)\tau \quad \text{(II.2.5.30)}$$

Such a periodic function Q(t) can be expanded in a Fourier series

$$Q(t) = H_0 + \Sigma_{n=1}^{\infty} [H_n \exp(\iota n\omega_0 t) + H_{-n} \exp(-\iota n\omega_0 t)] \quad \text{(II.2.5.31)}$$

$$H_n = H^*_{-n}; \qquad \omega_0 = 2\pi/\tau$$

The coefficients H_0 and H_n are obtained from the equations

$$H_0 = (1/\tau)\int_0^{f\tau} H\, dt = Hf = I_0^2Rf \quad \text{(II.2.5.32)}$$

$$H_{-n} = (1/\tau)\int_0^{f\tau} H \exp(-\iota n\omega_0 t)\, dt = -\iota H (1/2\pi n)[1-\exp(-\iota 2\pi nf)] \quad \text{(II.2.5.33)}$$

The corresponding temperature distribution is given by

$$T(x,t) = T_0 + (Hf/\kappa A)(L-x) + \Sigma_{n=1}^{\infty} [B_n \exp(-(1+\iota)\sqrt{n}\, \eta x/\sqrt{2})\exp(-\iota n\omega_0 t) + B_{-n} \exp(-(1-\iota)\sqrt{n}\, \eta x/\sqrt{2})\exp(\iota n\omega_0 t)] \quad \text{(II.2.5.34)}$$

with $\quad B_{-n} = B^*_n$

and $\quad B_n = [\, H_{-n} / (\, \kappa A\sqrt{n}\, \eta)]\, \exp(-\iota\pi/4)$ (II.2.5.35)

Thus the temperature will show some periodic variation as shown in figure.

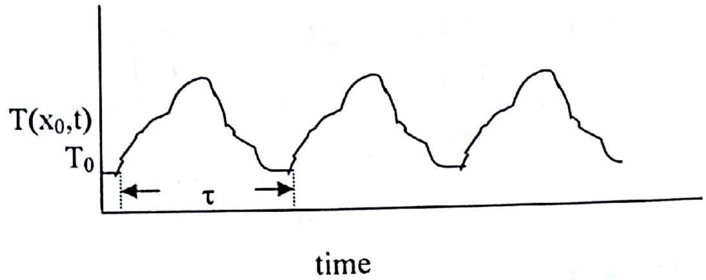

Figure II.2.5.3 shows the temperature variation with time at a point x_0 along the length of the rod.

We measure the temperature-time graph at two points. We Fourier transform the graphs to get the Fourier component at the frequency ω_0. This is done as follows. The complex amplitude B_1 of the Fourier component varying as $\exp(-\iota\omega_0 t)$ is found from the integral

$$B_1 \exp(-(1+\iota)\eta x_1/\sqrt{2}) = (1/\tau)\int_0^{\tau} T(x_1,t)\exp(\iota\omega_0 t)\, dt \qquad \text{(II.2.5.36)}$$

The RP of the integral on the right is

$$\text{RP of } I(x_1) = (1/\tau)\int_0^{\tau} T(x_1,t)\cos(\omega_0 t)\, dt \qquad \text{(II.2.5.37)}$$

And the imaginary part is

$$\text{IP of } I(x_1) = (1/\tau)\int_0^{\tau} T(x_1,t)\sin(\omega_0 t)\, dt \qquad \text{(II.2.5.38)}$$

We divide the time interval from 0 to τ into 20 equal parts, 0, $\tau/20, 2\tau/20, \ldots m\tau/20, \ldots \tau$. Take the temperature $T(x_1, m\tau/20)$ and multiply it by $\cos(2\pi m/20)$ and $\sin(2\pi m/20)$. We do this for each value of m. The real part of the integral is then given by

$$\text{RP of } I(x_1) = (1/\tau)(\tau/20)\Big[(1/2)\, T(x_1,0) + \sum_{m=1}^{19} T(x_1, m\tau/20)\cos(2\pi m/20) + (1/2)\, T(x_1,\tau)\Big] \qquad \text{(II.2.5.39)}$$

and the imaginary part of the integral is

$$\text{IP of } I(x_1) = (1/\tau)(\tau/20)\,[\,0 + \sum_{m=1}^{19} T(x_1, m\tau/20)\sin(2\pi m/20) + 0\,] \qquad \text{(II.2.5.40)}$$

This is the Simpson's rule for integration.

Once the RP and IP of $I(x_1)$ is found, the amplitude of the temperature varying at frequency ω_0 at x_1 is

$$\text{Amp of } \theta(x_1,\omega_0) = [\,\text{RP}^2 \text{ of } I(x_1) + \text{IP}^2 \text{ of } I(x_1)]^{1/2} \qquad \text{(II.2.5.41)}$$

And the phase $\phi(x_1)$ is

$$\text{Phase of } \theta(x_1,\omega_0) = \phi(x_1) = \tan^{-1}[\text{IP of } I(x_1)/\text{RP of } I(x_1)] \qquad \text{(II.2.5.42)}$$

We repeat this calculation for the temperature at the point x_2 to get Amplitude and phase of $\theta(x_2,\omega_0)$.

From the ratio of the amplitude of $\theta(x_1,\omega_0)$ to the amplitude $\theta(x_2,\omega_0)$, using the equation (II.2.5.27), one obtains α

$$\alpha = \ln(\,|\theta(x_1, \omega_0)|/\,|\theta(x_2,\omega_0)|)/\,(x_2-x_1) \qquad \text{(II.2.5.43)}$$

One can also obtain β from the measurement of the phase difference using equation (II.2.5.28).

$$\beta = [\phi(x_2) - \phi(x_1)]/\,(x_2-x_1) \qquad \text{(II.2.5.44)}$$

From this knowing τ one obtains the diffusivity $\kappa/\rho c$.

$$\begin{aligned}\alpha\beta &= (a^2+b^2)^{1/2}\cos(\phi/2)\sin(\phi/2)\\ &= (1/2)\,(a^2+b^2)^{1/2}\sin\phi = b/2 = (\omega\rho c/2\kappa)\end{aligned} \qquad \text{(II.2.5.45)}$$

So the diffusivity

$$D = \kappa/\rho c = \omega/(2\alpha\beta) \qquad \text{(II.2.5.46)}$$

The diffusivity can be calculated from a measurement of α and β.

2. APPARATUS REQUIRED

A DC power supply giving 15 V maximum and 2 A maximum, the thermal diffusivity box, DC differential amplifier, and a DMM reading in DC 200 mV range.

3. EXPERIMENTAL SET-UP

The schematic of the experimental set up is shown in Figure II.2.5.4. This experiment is designed to be performed on a brass rod. Take a brass rod (1) of 30 cm length and diameter about 5 mm (3/16"). A small heater (3) of about 10 Ohms is wound on the centre of the brass rod. Two copper-constantan thermocouples (4) and (5) are attached to the brass rod with superglue. The distance between the junctions is 3 cm. Symmetrically to the right of the heater two more junctions (4') and (5') are fixed with superglue to the brass rod. These junctions are not shown in the figure. The cold ends of the two couples (4) and (5) are embedded in a copper block so that they are at the same constant temperature. Similarly the cold ends of the thermocouple (4') and (5') are embedded in a copper block.

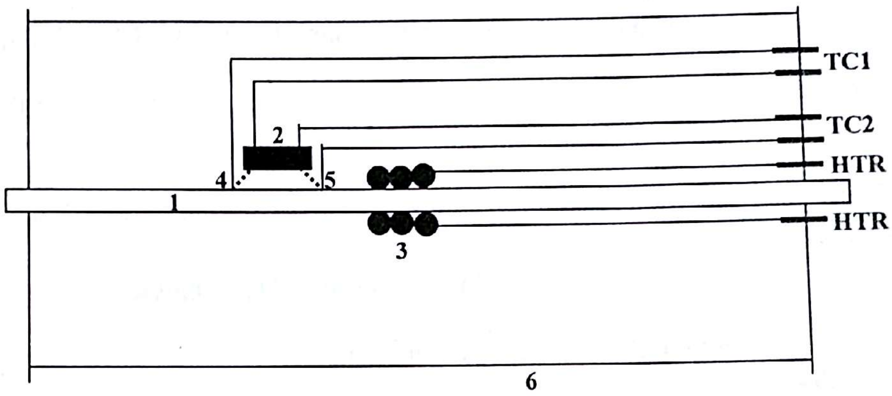

Figure:II.2.5.4 1 is brass tube; 2:Copper block in which cold junctions of the thermocouples 4 and 5 are embedded. Dotted line – constantan wire; continuous line Copper wire
3 Wire-wound heater
6 Box with lids at the ends The box is stuffed with cotton wool to prevent convection. The right lid carries terminals for leads. Two more thermocouples 4' and 5' are also provided to the right of the heater (not shown In Figure.)

The thermocouple leads are brought out to four pairs of banana terminals on the lid of the outer tube (6). Two pairs marked TC1 and TC2 correspond to the thermocouples (4) and (5). Those marked TC3 and TC4 correspond to the pair of thermocouples (5') and (4'). The heater leads are brought out to two big banana terminals. The rod is surrounded by a tightly wound layer of cotton. On this a layer of aluminium foil is wound. The box (6) is filled loosely with cotton to prevent convection currents.

The heater is connected to a power supply. A voltage of 5 Vis set on the power supply. The current through the heater coil will be approximately 0.5 amp. The current through the coil and the voltage across the coil gives the heat supplied. A switch in the source is toggled on for 5 minutes and off for five minutes regularly. This interrupts the current periodically. (An automatic electronic switch with adjustable period can also be used). One pair of thermocouples, either TC1 and TC2 or TC3 and TC4 is connected to the input terminals I1 and I2 on the DC differential amplifier. TC1 is farther away from the heater than TC2. So TC2 will indicate a higher temperature than TC1. The thermocouple voltages are amplified by a DC amplifier of amplification 100 and read on a milli-voltmeter.

For the first two cycles of heating no thermocouple readings are taken. In this time the heat will spread to the end of the rod and the temperature at the thermocouple junctions will start varying periodically with time.

We start counting the time from the instant the heater is switched on after two cycles. Call this instant zero. Then we start noting the DMM reading with the selector switch on the differential amplifier at I1. I1 is connected to TC1. **It is enough to note V_+ (i.e. positive reading) on DMM.** ***It is not necessary to note V_- by using the reversing switch. The offset, if any, will remain roughly constant. As we will find the amplitude and phase of the thermocouple signals at the period of heating by Fourier analysis, the nearly constant offset will make no contribution.*** Starting with 15 seconds after the heater is switched on (at time 1215 s) the amplified emf V_+ is noted every 30 seconds. ***One must not forget to switch off the heater at 1500 s and switch it on again at 1800s.*** **Thermocouple reading must be taken from 1215 s after switching on the heater to 1815 seconds. This will cover one period.**

After taking the reading at 1815 s, the selector switch is turned to I_2. This will now read TC2. **No readings are taken till the heater is switched on again at 2400s. .** Then readings of V_+ for the second thermocouple are taken for a full cycle.

Figure II.2.5.3 illustrates schematically the sequence in which readings are taken.

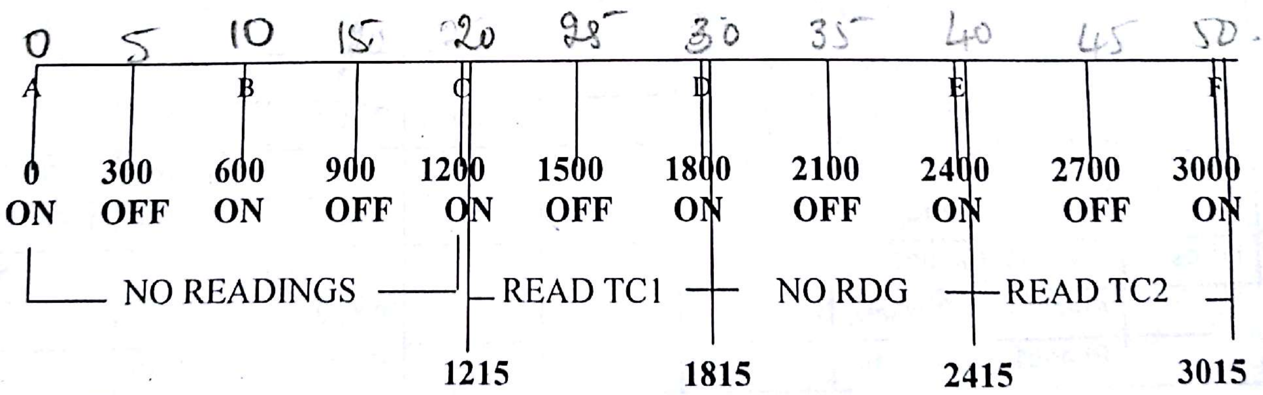

Figure II.2.5.5 Schematic showing how readings are to be taken
Numbers denote time in seconds. ON and OFF
Refer to switching heater on and off.

A sample set of readings is shown in Table II.2.5.1.

Table II.2.5.1
Power supply voltage 5 V

Period T	600	S						
Distance between thermocouple junctions				3	Cm			
time s	TC1 mV	TC1Cos	TC1Sin		Time s	TC2 mV	TC2Cos	TC2sin
1215	51.9	51.26098	8.119224		2415	58.3	57.58218	9.120438
1245	54.3	48.38126	24.65246		2445	73.6	65.57754	33.41476
1275	60.8	42.99094	42.99325		2475	86.8	61.37522	61.37852
1305	67.6	30.68749	60.23319		2505	98.3	44.62398	87.58762
1335	74.1	11.58826	73.18827		2535	107.4	16.79594	106.0785
1365	79.7	-12.4725	78.71802		2565	114.9	-17.981	113.4843
1395	85	-38.5945	75.73286		2595	121.6	-55.2128	108.3425
1425	89.3	-63.1497	63.13955		2625	127.4	-90.0927	90.07815
1455	93.1	-82.9566	42.25895		2655	132	-117.618	59.91601
1485	96.9	-95.7085	15.14874		2685	136.4	-134.723	21.32392
1515	100.3	-99.0634	-15.7015		2715	136.5	-134.817	-21.3685
1545	98.1	-87.4022	-44.5473		2745	121.4	-108.161	-55.1278
1575	91.8	-64.9037	-64.9211		2775	107.8	-76.2159	-76.2363
1605	84.9	-38.5328	-75.652		2805	97.3	-44.1607	-86.7013
1635	78.1	-12.2055	-77.1404		2835	88.2	-13.784	-87.1163
1665	73	11.43171	-72.0993		2865	81.1	12.70016	-80.0994
1695	68	30.88209	-60.583		2895	74.7	33.92488	-66.5522
1725	63.3	44.76827	-44.7515		2925	69.2	48.94098	-48.9226
1755	59	52.5747	-26.775		2955	64.3	57.29752	-29.1802
1785	55.5	54.81852	-8.67064		2985	60.1	59.36204	-9.38929
1815	52.1	51.45677	8.161559		3015	60.6	59.85183	9.4931
		-215.507	-6.63604			Sum	-333.451	130.2172
I1Cos	-10.77	I1sin	-0.33		I2cos	16.67	I2sin	6.51
	Amp1	10.78047				Amp2	17.89877	
	Phase1	3.172483				Phase2	2.769398	

Alpha 0.169

Beta 0.134

Diffusivity = 0.23 cm^2/s

The first column gives the time (measured from the time heater is switched on before readings are taken. Second column gives the readings of TC1 in mV from 1215 to 1815. In the third column are given values of (TC1 cos ($2\pi t/T$) where time t is from the first column and T is 600 s, the period of heating. Similarly the fourth column is TC1 Sin($2\pi t/T$). **The row marked sum gives the sum of all rows in the column 3 (or column 4) minus half the sum of the readings in the first and last rows.** The column value next to I1cos is the [(sum in column 3) x 30/600] where 30 seconds is the interval at which temperatures are measured. Similarly the value next to I1sin is the [(sum in column 4) x30/600]. Amp1 is $[(I1\cos)^2+(I1\sin)^2]^{1/2}$. Since both $I_1\cos$ and $I_1\sin$ are both negative, the angle is in the third quadrant. So phase 1 is Atan (I1sin/I1cos) + π.

A similar analysis of the readings TC2 gives Amp 2 and phase 2. The value of α is obtained from II.2.5.43 and the value of β from (II.2.5.44). The value of diffusivity is calculated from the values of α and β using (II.2.5.45). If the heater current is not switched off and on exactly with a period of 600 s and if the readings are not taken at exact intervals of30s,phase measurement will have a large error.

The aim of the experiment is (i) to illustrate Fourier analysis and (ii) to measure thermal diffusivity. Fourier analysis occurs whenever a function varies periodically either in space or in time or both. For example in a crystal the electron charge density varies periodically in space. X ray diffraction spots occur from different Fourier components of this charge density. Therefore it is necessary to understand how Fourier analysis is done.

Measurement of thermal diffusivity is very important for materials of poor thermal conductivity. Here the specimen will be thin and the frequency of periodic heating will be high. A whole branch of research called Photo-acoustic spectroscopy uses such periodic heating methods.

QUESTIONS:

1. I have an AC current, which is perfectly sinusoidal with a frequency f. If this signal is Fourier analyzed, at what frequencies will the Fourier components be non-zero?

2. White light is a mixture of colours. When it is passed through a prism the colours are separated. Does the prism Fourier analyze the white light?

3. Will heat diffuse faster in a poor conductor than in copper?

4. Indicate what uses photo-acoustic spectroscopy can be put to.