

Notes on Error Propagation (PH3244)

Shruti Chakravarty

February 2024

1 Quincke's Method

In this experiment you are measuring the rise/fall of a magnetic liquid in response to the changing magnetic field.

You control the magnetic field by changing the current supplied to the electromagnet. For this, you first use a Gaussmeter to measure the field (H) generated between the two poles of the electromagnet for various values of current supplied (I). Your data table should look like this:

I (A)	H (G)
.	.
.	.
.	.

Then you plot H vs I (aka the Calibration curve); it should be linear, at least up to 2-3 Amps of current. When you perform a linear fit on H vs I, you will obtain an equation:

$$H = k_1 I + H_0 \quad (1)$$

where k_1 and H_0 are the slope and intercept with errors δk_1 and δH_0 , respectively. Note the fit parameters and print them on the graphs attached to your journal.

Next, place the U-tube filled with the salt solution in the center of the gap between the poles of the electromagnet and focus your traveling microscope on the meniscus. Note the height (h_0) when the magnet is off (there might still be some remnant field, depending on when the magnet was turned off last), and start increasing the current in small increments (0.25 or 0.5A). Measure the corresponding rise/fall in the height of the liquid. The change might be small and may not be instantaneous. Be patient and measure the height values carefully. The change will be larger than the least count of the traveling microscope. Restrict the experiment to I values for which H vs I remained linear. Your dataset now will look like this: Now plot $\Delta h = (h - h_0)$ vs H^2 . It should follow a linear trend. Perform another linear fit and get the

I (A)	H (= $k_1 I + H_0$) (G)	H^2 (G^2)	($h - h_0$) (mm)
.	.	.	.
.	.	.	.
.	.	.	.

equation:

$$\Delta h = k_2 H^2 + \Delta h_0 \quad (2)$$

where k_2 and Δh_0 are your slope and intercept with errors δk_2 and $\delta(\Delta h_0)$, respectively.

The mass susceptibility of the solution and the uncertainty in its value is given by:

$$\chi^{sol} = 2g \left(\frac{\Delta h}{H^2} \right) \Rightarrow \chi^{sol} = 2g k_2 \quad (3)$$

and

$$\delta\chi^{sol} = 2g\delta k_2 \quad (4)$$

But the aim is to get to the susceptibility of the salt and not the solution. You get that by subtracting the (diamagnetic) susceptibility of the solvent, i.e. water. The value of χ^{water} is $-7.203 \times 10^{-7} \text{ cm}^3\text{g}^{-1}$ [1] at ambient temperature and pressure as reported in literature. The susceptibility of the solution χ^{sol} can be expressed in terms of χ^{salt} and χ^{water} as:

$$\begin{aligned} \chi^{sol} &= \frac{m_{salt}}{(m_{salt} + m_{water})} \chi^{salt} + \frac{m_{water}}{(m_{salt} + m_{water})} \chi^{water} \\ \Rightarrow \frac{m_{salt}}{(m_{salt} + m_{water})} \chi^{salt} &= \chi^{sol} - \frac{m_{water}}{(m_{salt} + m_{water})} \chi^{water} \\ \Rightarrow \chi^{salt} &= \frac{(m_{salt} + m_{water})}{m_{salt}} \left(\chi^{sol} - \frac{m_{water}}{(m_{salt} + m_{water})} \chi^{water} \right) \\ \Rightarrow \chi^{salt} &= \left(1 + \frac{m_{water}}{m_{salt}} \right) \left(\chi^{sol} - \frac{m_{water}}{(m_{salt} + m_{water})} \chi^{water} \right) \\ \Rightarrow \chi^{salt} &= \chi^{sol} + \frac{m_{water}}{m_{salt}} \chi^{sol} - \frac{m_{water}}{m_{salt}} \chi^{water} \\ \text{Thus, } \chi^{salt} &= \chi^{sol} + \frac{m_{water}}{m_{salt}} (\chi^{sol} - \chi^{water}) \quad (5) \end{aligned}$$

NOTE ABOUT MAGNETIC UNITS:

So, converting between cgs and SI units for electromagnetism can be tricky and sometimes make your brain go wonky. Here, for example, when you calculate χ^{sol} in eq. (3), it has units of $\text{cm}^2\text{s}^{-2}\text{G}^{-2}$. But the literature value for χ^{water} has the units cm^3g^{-1} in the cgs system. How do you go from $\text{cm}^2\text{s}^{-2}\text{G}^{-2}$ to cm^3g^{-1} ?

This happens because, in the cgs system, neither charge nor current are independent physical quantities and are instead defined in terms of mass, length, and time in the cgs-base units. So

$$1 \text{ G} = \frac{g}{Bi \cdot s^2}$$

where $Bi = 0.1 \text{ A}$ is the derived electromagnetic unit (emu) of current [3] called *Biot* or *abAmpere*. In cgs-base units:

$$1 \text{ Biot} = g^{1/2} \text{ cm}^{1/2} \text{ s}^{-1}$$

which gives us

$$1 \text{ G}^2 = \frac{g^2}{s^4 \cdot g \cdot \text{cm} \cdot s^{-2}} = \frac{g}{s^2 \cdot \text{cm}}$$

and therefore the units of χ^{sol} you obtain from your experiment can be rewritten as:

$$\frac{\text{cm}^2}{s^2 \cdot \text{G}^2} = \frac{\text{cm}^2}{s^2 \cdot \frac{g}{s^2 \cdot \text{cm}}} = \frac{\text{cm}^3}{g} \quad !! \quad :)$$

Now using the variance formula for calculating the error propagation (assuming no correlation between the sources of errors) we get

$$(\delta\chi^{salt})^2 = \left(\frac{\partial\chi^{salt}}{\partial\chi^{sol}} \delta\chi^{sol} \right)^2 + \left(\frac{\partial\chi^{salt}}{\partial m_{water}} \delta m_{water} \right)^2 + \left(\frac{\partial\chi^{salt}}{\partial m_{salt}} \delta m_{salt} \right)^2 \quad (6)$$

$$\text{Now, } \frac{\partial\chi^{salt}}{\partial\chi^{sol}} = \left(1 + \frac{m_{water}}{m_{salt}} \right);$$

$$\frac{\partial\chi^{salt}}{\partial m_{water}} = \left(\frac{\chi^{sol} - \chi^{water}}{m_{salt}} \right); \text{ and}$$

$$\frac{\partial\chi^{salt}}{\partial m_{salt}} = \frac{m_{water}}{m_{salt}^2} (\chi^{water} - \chi^{sol})$$

Also, $\delta m_{water} = \delta m_{salt} = \delta m$ = least count of the weighing scale. Substituting these into equation (6), and rearranging the terms, we get:

$$\begin{aligned} (\delta\chi^{salt})^2 &= \left(1 + \frac{m_{water}}{m_{salt}} \right)^2 (\delta\chi^{sol})^2 + \left[\left(\frac{\chi^{sol} - \chi^{water}}{m_{salt}} \right)^2 + \frac{m_{water}^2}{m_{salt}^4} (\chi^{water} - \chi^{sol})^2 \right] \delta m^2 \\ \Rightarrow (\delta\chi^{salt})^2 &= \left(1 + \frac{m_{water}}{m_{salt}} \right)^2 (\delta\chi^{sol})^2 + \left(1 + \frac{m_{water}^2}{m_{salt}^2} \right) \left(\frac{\chi^{sol} - \chi^{water}}{m_{salt}} \right)^2 \delta m^2 \end{aligned}$$

Thus,

$$\delta\chi^{salt} = \sqrt{\left(1 + \frac{m_{water}}{m_{salt}} \right)^2 (\delta\chi^{sol})^2 + \left(1 + \frac{m_{water}^2}{m_{salt}^2} \right) \left(\frac{\chi^{sol} - \chi^{water}}{m_{salt}} \right)^2 \delta m^2} \quad (7)$$

So equations (5) and (7) give you $\chi^{salt} \pm \delta\chi^{salt}$ as the final value.

You can now use this value to calculate the molar susceptibility of the salt (χ_m^{salt}), given by:

$$\begin{aligned} \chi_m^{salt} &= \chi^{salt} \times \text{molecular weight of the salt} \quad ; \text{ and} \\ \delta\chi_m^{salt} &= \delta\chi^{salt} \times \text{molecular weight of the salt} \end{aligned}$$

which gives you the final result ($\chi_m^{salt} \pm \delta\chi_m^{salt}$).

2 Ultrasonic Diffraction

In this experiment, you study how light can be diffracted by ultrasonic waves. By observing the diffraction, you can calculate the velocity of the waves. You have a laser, mounted on one end, which produces a light of wavelength 650 nm (red). A piezoelectric transducer crystal connected to an RF oscillator will be the source of the ultrasonic waves. When immersed in the liquid, these waves will travel from one end of the glass tank to the other and get reflected back. This will lead to the formation of a standing wave in the liquid. This standing wave is what acts as a diffraction grating and makes the light diffract. The diffraction pattern will be observed on a screen placed in front of the glass tank. Make sure the direction of propagation of the laser is parallel to the face of the piezoelectric crystal.

You measure the following parameters and make a note of their uncertainties:

- Distance between the crystal and the detector = $L \pm \delta L$
- Distance between the central spot and the n^{th} order spot = $D \pm \delta D$

These are the values that are fixed in this experiment and given to you:

- Wavelength of the laser (λ) = 650nm ($\delta\lambda = 0$)
- Frequency of the crystal (ν) = 3 MHz or 5 MHz ($\delta\nu$ = least count of RF oscillator)

Now you make your observation table, which should look like:

Order n	D (mm)	Diffraction angle $\left(\theta = \tan^{-1}\left(\frac{D}{L}\right)\right)$ (rad)	Wavelength of ultrasonic waves $\left(\Lambda = \frac{n\lambda}{\sin\theta}\right)$ (nm)	Velocity of ultrasonic waves ($v = n\Lambda$) (m/s)
.
.
.

Now, the average wavelength (Λ) and velocity (v) can be simply calculated from the table. Let us look at how the error propagates in this experiment. Firstly we will have uncertainty in the value of θ calculated, i.e. $\delta\theta$. To estimate this, and make our lives simple, let $y = \tan\theta = \frac{D}{L}$. This implies that:

$$\delta y = y \sqrt{\left(\frac{\delta D}{D}\right)^2 + \left(\frac{\delta L}{L}\right)^2} \quad ; \text{ and}$$

$$\delta y = \sec^2\theta \delta\theta$$

We can eliminate δy by writing:

$$\sec^2\theta \delta\theta = \tan\theta \sqrt{\left(\frac{\delta D}{D}\right)^2 + \left(\frac{\delta L}{L}\right)^2}$$

$$\Rightarrow \quad \delta\theta = \sin\theta \cos\theta \sqrt{\left(\frac{\delta D}{D}\right)^2 + \left(\frac{\delta L}{L}\right)^2}$$

Considering the right-angled triangle made by L and D , we get:

$$\sin\theta = \frac{D}{\sqrt{D^2 + L^2}} \quad \text{and} \quad \cos\theta = \frac{L}{\sqrt{D^2 + L^2}}$$

This means we can write:

$$\delta\theta = \frac{DL}{(D^2 + L^2)} \sqrt{\left(\frac{\delta D}{D}\right)^2 + \left(\frac{\delta L}{L}\right)^2} \quad (8)$$

This uncertainty in θ travels and becomes $\delta\Lambda$ which can be estimated as:

$$(\delta\Lambda)^2 = \left(\frac{\partial\Lambda}{\partial\theta} \delta\theta \right)^2 + \left(\frac{\partial\Lambda}{\partial\lambda} \delta\lambda \right)^2$$

Since, $\frac{\partial\Lambda}{\partial\theta} = -\Lambda \cot\theta$ and $\delta\lambda = 0$

we get:

$$\delta\Lambda = \Lambda \frac{\delta\theta}{\tan\theta} \quad (9)$$

And similarly,

$$\delta\nu = \nu \sqrt{\left(\frac{\delta\nu}{\nu} \right)^2 + \left(\frac{\delta\Lambda}{\Lambda} \right)^2} \quad (10)$$

And just like that you are done with another experiment! :)

3 Curie-Weiss Law

In this experiment, you will be observing the thermal variation of the capacitance of a ceramic and polymer capacitor and verify that the former follows the Curie-Weiss Law. You set a fixed sine voltage (e.g. $V_{in} = 1V$, $f = 1kHz$) as the input and measure the voltage across the capacitors and the resistors as the temperature (of the furnace) increases. These voltages allow you to calculate the impedance of the capacitor (Z) in terms of the $1k\Omega$ resistor connected to it, which in turn gives you the capacitance through the relation $Z = \frac{1}{2\pi fC}$, where f is the frequency of the input voltage and C the desired capacitance.

Your dataset should look something like this:

	Ceramic Capacitor				Polymer Capacitor			
Temperature (°C)	V_C (V)	V_R (V)	$Z_{ceramic}$ $\left(= \frac{V_C}{V_R} \cdot R \right) (\Omega)$	C (μF)	V_C (V)	V_R (V)	$Z_{polymer}$ (Ω)	C (μF)
.
.
.

Uncertainties in these observations include:

- Least count of the temperature controller $\rightarrow \delta T$
- Least count of the DMM used for voltage measurement $\rightarrow \delta V (= \delta V_C = \delta V_R)$
- Least count of the signal generator $\rightarrow \delta f$

Since here we have:

$$Z = \frac{V_C}{V_R} R = \frac{1}{2\pi fC}$$

$$\Rightarrow C = \frac{V_R}{2\pi f V_C R} \quad (11)$$

And it follows from this expression that:

$$\delta C = C \sqrt{\left(\frac{1}{V_C^2} + \frac{1}{V_R^2} \right) \delta V^2 + \left(\frac{\delta f}{f} \right)^2} \quad (\because \delta R = 0) \quad (12)$$

So now that you have $C \pm \delta C$, you plot $\left(\frac{1}{C} \right)$ vs T . If it looks linear, perform a linear fit (if it doesn't, I'm sorry, but you might have to repeat the experiment (;_:_)). The fit will return an equation of the form:

$$\frac{1}{C} = k_3 T + \frac{1}{C_0} \quad (13)$$

Since the Curie Weiss Law expects the ceramic's dielectric constant to be inversely proportional to temperature, i.e. $\epsilon = \left(\frac{C}{T - \Theta} \right)$, the x-intercept of this linear plot allows you to calculate the Curie Temperature (Θ) which is a measure of the interaction strength between the electric dipoles within the dielectric material.

So,

$$\Theta = \frac{-1}{k_3 C_0} \quad \text{and} \quad \delta \Theta = T \sqrt{\left(\frac{\delta k_3}{k_3} \right)^2 + \left(\frac{\delta \left(\frac{1}{C_0} \right)}{\left(\frac{1}{C_0} \right)} \right)^2}$$

And now you also have $(\Theta \pm \delta \Theta)$ to report.

4 Thermal Diffusivity

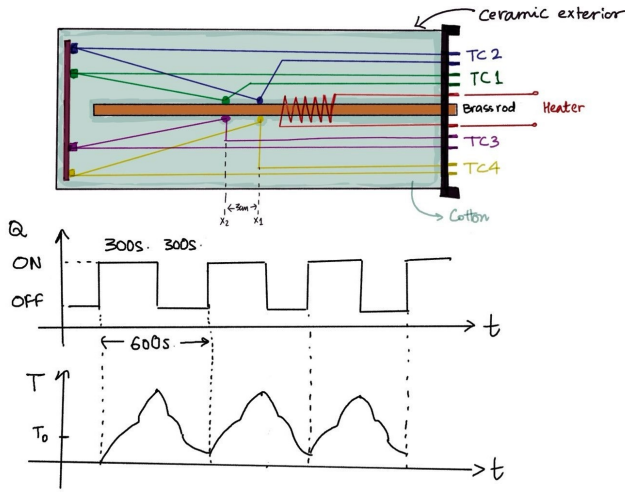


Figure 1: Top: Schematic of the setup; Bottom: Periodicity of the supplied heat pulse at $x = x_1$, and the expected variation in temperature response at $x = x_2$

In this experiment, you observe how the temperature of a thermally conducting material responds to a time-varying heat pulse, and how, by measuring these oscillations, you can calculate the thermal diffusivity (D), which is an intrinsic quantity of the material and is defined as the ratio of the thermal conductivity (κ) and the product of the density (ρ) and specific heat (C) of the material.

$$D = \frac{\kappa}{\rho C} \quad (14)$$

The material provided in this experiment is a brass rod, around which a nichrome wire is wrapped around the center (Fig. 1). Cu/CuNi (T-type) thermocouples are used to read the temperature of the brass rod at 2 fixed locations. There are 4 thermocouples with 2 junctions each (TC1, TC2, TC3, TC4). One junction is placed on the brass rod, and the other on a copper block (which is isolated from the heater). TC1 and TC2 are on the same side and are placed 3cm apart. Since TC2 is closer to the heater, it will read a higher temperature than TC1. The heat pulse incident at x_1 will diffuse through the brass rod and reach x_2 and by measuring the temperature oscillations in this region, we will calculate the diffusivity of the material.

The procedure follows a specific protocol of switching the heater ON and OFF at 5 min intervals, which means that the period of the heat pulse is 600s ($= t_{ON} + t_{OFF}$). Figure 2 shows the protocol that has to be followed for data collection.

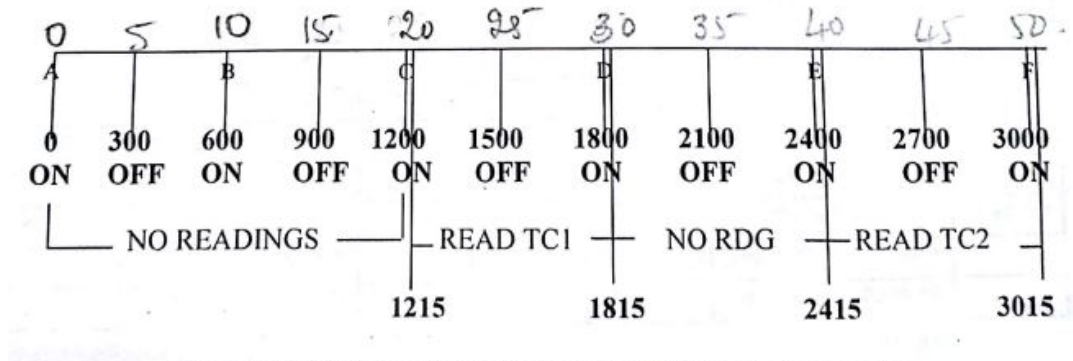


Figure 2: Schematic showing the protocol for data collection. Numbers denote time in seconds. ON & OFF refer to switching the heater on and off.

You begin taking measurements 15s after the third cycle starts, with the selector switch at I_1 , i.e., you measure the voltage across TC1 at 1215s and then continue to do the same every 30s. The wait time of 1200s (2 cycles) allows the rod to thermalize and the thermocouple reading to start varying periodically with time. After measuring the voltage across TC1 for 1 cycle (1215-1815s), selector switch is turned to I_2 and the voltage across TC2 is measured from 2415-3015s (1 cycle). Note that for 600s in between (1815-2415s), no data is taken. This basically means that you track the temporal variation of the temperature at x_1 for 1 cycle (using TC1) and that on x_2 for another (using TC2) with a gap of 1 cycle between them. This is how your dataset will look like:

Now, we need to see how the errors propagate in this experiment. The direct observations in this experiment are of time and voltage, which will have an uncertainty equal to the least count of the stopwatch/timer (δt) and voltmeter (δV), respectively. Both V_{TC1} and V_{TC2} will have the same uncertainty, i.e. $\delta V_{TC1} = \delta V_{TC2} = \delta V$. Also $\delta t = \delta \tau$.

Period $\tau = 600$ s					Distance between the thermocouple junctions $(x_2 - x_1) = 3$ cm				
t (s)	$\theta = \frac{2\pi t}{\tau}$ (rad)	V_{TC1} (mV)	$V_{TC1}\cos\theta$ (mV)	$V_{TC1}\sin\theta$ (mV)	t (s)	$\theta = \frac{2\pi t}{\tau}$ (rad)	V_{TC2} (mV)	$V_{TC1}\cos\theta$ (mV)	$V_{TC1}\sin\theta$ (mV)
1215		.			2415		.		
1245		.			2445		.		
1275		.			2475		.		
1305		.			2505		.		
.		.			.		.		
.		.			.		.		
.		.			.		.		
1785		.			2985		.		
1815		.			3015		.		

Let's start propagating these errors. First, δt will travel to $\delta\theta$ as follows:

$$\delta\theta = \sqrt{\left(\frac{\partial\theta}{\partial t}\delta t\right)^2 + \left(\frac{\partial\theta}{\partial\tau}\delta\tau\right)^2}$$

$$\Rightarrow \delta\theta = \theta\sqrt{\left(\frac{1}{t^2} + \frac{1}{\tau^2}\right)(\delta t)^2}$$

Both δV and $\delta\theta$ will contribute to the uncertainty in $V_{TC(1,2)}\cos\theta$ and $V_{TC(1,2)}\sin\theta$. For clarity, let $V_r = V_{TC}\cos\theta$ and $V_i = V_{TC}\sin\theta$. Then, the uncertainty in the values stored in columns 4 and 5 is:

$$\delta V_r = \sqrt{\left(\frac{\partial V_r}{\partial V_{TC}}\delta V_{TC}\right)^2 + \left(\frac{\partial V_r}{\partial\theta}\delta\theta\right)^2}$$

$$\Rightarrow \delta V_r = \sqrt{(\cos\theta\delta V)^2 + (-\sin\theta.V_{TC}\delta\theta)^2}$$

Similarly,

$$\delta V_i = \sqrt{(\sin\theta\delta V)^2 + (\cos\theta.V_{TC}\delta\theta)^2}$$

Our data set gives us the temperature-time graph at two points on the brass rod. The Fourier transform of these graphs gives the Fourier component at frequency $\omega = 2\pi/\tau = \theta/t$. The complex amplitude of the Fourier component varying as $e^{-i\omega t}$ is calculated from the integral:

$$I = \frac{1}{\tau} \int_0^\tau T(x, t)e^{i\omega t} dt \quad (15)$$

whose real and imaginary parts are:

$$I_r = \frac{1}{\tau} \int_0^\tau T(x, t)\cos\theta dt \quad \text{and} \quad I_i = \frac{1}{\tau} \int_0^\tau T(x, t)\sin\theta dt$$

These integrals can be calculated using Simpson's rule for integration, by dividing the time interval from 0 to τ in 20 equal parts: 0, $\tau/20$, $2\tau/20$, ..., $m\tau/20$, ..., τ (hence the data-taking interval of 30 s); and calculating $T(x, t)\cos\theta$ and $T(x, t)\sin\theta$ at each $t = m\tau/20$.

Using this rule, a definite integral $\int_a^b f(x)dx$, approximated by dividing it into m parts of equal width $\Delta x = \left(\frac{b-a}{m}\right)$ is written as [4] :

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 4f(x_{n-1}) + f(x_n) \right]$$

In our case, $x = t$, $f(x) = T(x, t)\cos\omega t$ or $T(x, t)\sin\omega t$ and $\Delta x = \frac{\tau}{20}$.

Putting these values, we get I_r and I_i :

$$\begin{aligned} I_r = & \left(\frac{1}{\tau}\right) \left(\frac{\tau}{20}\right) \left(\frac{1}{3}\right) \left[T(x_1, 0) + T(x_1, \tau) + \right. \\ & 4 \left\{ T(x_1, 30)\cos\left(\frac{\pi}{10}\right) + T(x_1, 90)\cos\left(\frac{3\pi}{10}\right) + \dots + T(x_1, 570)\cos\left(\frac{19\pi}{10}\right) \right\} + \\ & \left. 2 \left\{ T(x_1, 60)\cos\left(\frac{2\pi}{10}\right) + T(x_1, 120)\cos\left(\frac{4\pi}{10}\right) + \dots + T(x_1, 540)\cos\left(\frac{18\pi}{10}\right) \right\} \right] \quad (16) \end{aligned}$$

$$\begin{aligned} I_i = & \left(\frac{1}{\tau}\right) \left(\frac{\tau}{20}\right) \left(\frac{1}{3}\right) \left[0 + 0 + \right. \\ & 4 \left\{ T(x_1, 30)\sin\left(\frac{\pi}{10}\right) + T(x_1, 90)\sin\left(\frac{3\pi}{10}\right) + \dots + T(x_1, 570)\sin\left(\frac{19\pi}{10}\right) \right\} + \\ & \left. 2 \left\{ T(x_1, 60)\sin\left(\frac{2\pi}{10}\right) + T(x_1, 120)\sin\left(\frac{4\pi}{10}\right) + \dots + T(x_1, 540)\sin\left(\frac{18\pi}{10}\right) \right\} \right] \quad (17) \end{aligned}$$

This is exactly what you are doing when you calculate the quantities SUM_r and SUM_i given by:

$$SUM_r = V_{0r} + 4(V_{1r} + V_{3r} + \dots V_{19r}) + 2(V_{2r} + V_{4r} + \dots + V_{18r}) + V_{20r} \quad \text{and}$$

$$SUM_i = 2[2(V_{1i} + V_{3i} + \dots V_{19i}) + (V_{2i} + V_{4i} + \dots + V_{18i})]$$

which are basically the quantities inside the square braces in eqs. (16) and (17).

The uncertainty in SUM is:

$$\delta(SUM) = \sqrt{\left(\frac{\partial(SUM)}{\partial V_0}\delta V_0\right)^2 + \left(\frac{\partial(SUM)}{\partial V_1}\delta V_1\right)^2 + \left(\frac{\partial(SUM)}{\partial V_2}\delta V_2\right)^2 + \dots + \left(\frac{\partial(SUM)}{\partial V_{19}}\delta V_{19}\right)^2 + \left(\frac{\partial(SUM)}{\partial V_{20}}\delta V_{20}\right)^2}$$

Since δV_{xm} and δV_{ym} has already been calculated $\forall m$; $\delta(SUM)$ can be calculated from the expression:

$$\delta(SUM) = \sqrt{(1 \cdot \delta V_0)^2 + (4 \cdot \delta V_1)^2 + (2 \cdot \delta V_2)^2 + \dots + (4 \cdot \delta V_{19})^2 + (1 \cdot \delta V_{20})^2} \quad (18)$$

Thus, from eqs. (16) and (17) we can now get the real and imaginary parts of the integral given by eq.(15) at $x = x_1$:

$$I_{r1} = \frac{(SUM_r)_1}{60} \quad I_{i1} = \frac{(SUM_i)_1}{60}$$

and at $x = x_2$:

$$I_{r2} = \frac{(SUM_r)_2}{60} \quad I_{i2} = \frac{(SUM_i)_2}{60}$$

and the amplitude ($Amp(x_1)$) and phase ($\phi(x_1)$) of the temperature oscillation at x_1 , having frequency $\omega = \theta/\tau$ at $x = x_1$ is:

$$Amp(x_1) = \sqrt{I_{r1}^2 + I_{i1}^2} \quad \phi(x_1) = \tan^{-1} \left(\frac{I_{i1}}{I_{r1}} \right) \quad (19)$$

Similarly at $x = x_2$:

$$Amp(x_2) = \sqrt{I_{r2}^2 + I_{i2}^2} \quad \phi(x_2) = \tan^{-1} \left(\frac{I_{i2}}{I_{r2}} \right) \quad (20)$$

The uncertainties propagate as:

$$\begin{aligned} \delta(I_{r1,2}) &= \frac{\delta(SUM_r)_{1,2}}{60} & \delta(I_{i1,2}) &= \frac{\delta(SUM_i)_{1,2}}{60} \\ \delta Amp(x_{1,2}) &= \frac{1}{Amp(x_{1,2})} \sqrt{(I_{r1,2} \delta I_{r1,2})^2 + (I_{i1,2} \delta I_{i1,2})^2} \end{aligned} \quad (21)$$

and

$$\delta \phi(x_{1,2}) = \frac{\sin(2\phi(x_{1,2}))}{2} \sqrt{\left(\frac{\delta I_{r1,2}}{I_{r1,2}} \right)^2 + \left(\frac{\delta I_{i1,2}}{I_{i1,2}} \right)^2} \quad (22)$$

The ratio of the two amplitudes gives α :

$$\alpha = - \frac{\ln \left(\frac{Amp(x_2)}{Amp(x_1)} \right)}{(x_2 - x_1)} \quad (23)$$

and the difference between the phase at the two locations gives β :

$$\beta = \frac{\phi(x_1) - \phi(x_2)}{(x_1 - x_2)} \quad (24)$$

which have the uncertainties:

$$\begin{aligned} \delta \alpha &= \frac{1}{(x_1 - x_2)} \sqrt{\left[\ln \left(\frac{Amp(x_2)}{Amp(x_1)} \right) \frac{1}{(x_1 - x_2)} \right]^2 \cdot (\delta x_1^2 + \delta x_2^2) + \left(\frac{\delta Amp(x_1)}{Amp(x_1)} \right)^2 + \left(\frac{\delta Amp(x_2)}{Amp(x_2)} \right)^2} \\ \delta \beta &= \frac{1}{(x_1 - x_2)} \sqrt{(\delta \phi(x_1))^2 + (\delta \phi(x_2))^2 + \left(\frac{\phi(x_1) - \phi(x_2)}{x_1 - x_2} \right)^2 (\delta x_1^2 + \delta x_2^2)} \end{aligned}$$

But since we do not actually measure the positions $x_{1,2}$ and are simply given the distance between the junctions $x_2 - x_1$ to be 3cm, in our case, $\delta x_1 = \delta x_2 = 0$. This means:

$$\delta \alpha = \frac{1}{(x_1 - x_2)} \sqrt{\left(\frac{\delta Amp(x_1)}{Amp(x_1)} \right)^2 + \left(\frac{\delta Amp(x_2)}{Amp(x_2)} \right)^2} \quad (25)$$

and

$$\delta \beta = \frac{1}{(x_1 - x_2)} \sqrt{(\delta \phi(x_1))^2 + (\delta \phi(x_2))^2} \quad (26)$$

So eqs. (23), (24), (25) and (26) together give you $\alpha \pm \delta \alpha$ and $\beta \pm \delta \beta$.

Finally, we are in a position to calculate the diffusivity D defined in eq.(14), which is the aim of this experiment. D relates to α and β as:

$$D = \frac{\omega}{2\alpha\beta} = \frac{\pi}{\tau\alpha\beta} \quad (27)$$

and has an error

$$\delta D = D \sqrt{\left(\frac{\delta\tau}{\tau}\right)^2 + \left(\frac{\delta\alpha}{\alpha}\right)^2 + \left(\frac{\delta\beta}{\beta}\right)^2} \quad (28)$$

which gives you the final result $D \pm \delta D$!

4.1 Calculations on example dataset from manual

As an example, calculations performed on the sample dataset given in the manual are shown below. I assume δt (or $\delta\tau$) = 1s and $\delta V = 0.1$ mV (these are the only parameters that you actually measure in the experiment; all other parameters are inferred from these).

Least counts: $\delta t = \delta \tau = 1s$, $\delta V = 0.1mV$																															
m	at x = x ₁										at x = x ₂																				
	Data collected										Data collected																				
	timestamp	V _{TC1}	t	θ	δθ	V _{r1} = V _{TC1} cosθ	δV _{r1}	V _{i1} = V _{TC1} sinθ	δV _{i1}	timestamp	V _{TC2}	t	θ	δθ	V _{r2} = V _{TC2} cosθ	δV _{r2}	V _{i2} = V _{TC2} sinθ	δV _{i2}													
	s	mV	s	rad	rad	mV	mV	mV	mV	s	mV	s	rad	rad	mV	mV	mV	mV													
0	1215	51.9	0	0.000	0.000	51.900	0.000	0.000	0.000	2415	58.3	0	0.000	0.000	58.300	0.000	0.000	0.000													
1	1245	54.3	30	0.314	0.001	51.642	0.010	16.780	0.030	2445	73.6	30	0.314	0.001	69.998	0.012	22.744	0.038													
2	1275	60.8	60	0.628	0.001	49.188	0.041	35.737	0.057	2475	86.8	60	0.628	0.001	70.223	0.055	51.020	0.076													
3	1305	67.6	90	0.942	0.002	39.734	0.095	54.690	0.069	2505	98.3	90	0.942	0.002	57.779	0.128	79.526	0.093													
4	1335	74.1	120	1.257	0.002	22.898	0.162	70.473	0.053	2535	107.4	120	1.257	0.002	33.188	0.220	102.143	0.071													
5	1365	79.7	150	1.571	0.003	0.000	0.228	79.700	0.000	2565	114.9	150	1.571	0.003	0.000	0.309	114.900	0.000													
6	1395	85	180	1.885	0.003	-26.266	0.276	80.840	0.090	2595	121.6	180	1.885	0.003	-37.576	0.373	115.648	0.121													
7	1425	89.3	210	2.199	0.004	-52.489	0.287	72.245	0.209	2625	127.4	210	2.199	0.004	-74.884	0.388	103.069	0.282													
8	1455	93.1	240	2.513	0.005	-75.319	0.248	54.723	0.341	2655	132	240	2.513	0.005	-106.790	0.333	77.588	0.459													
9	1485	96.9	270	2.827	0.005	-92.157	0.152	29.944	0.468	2685	136.4	270	2.827	0.005	-129.724	0.204	42.150	0.626													
10	1515	100.3	300	3.142	0.006	-100.300	0.000	0.000	0.565	2715	136.5	300	3.142	0.006	-136.500	0.000	0.000	0.732													
11	1545	98.1	330	3.456	0.006	-93.299	0.187	-30.315	0.576	2745	121.4	330	3.456	0.006	-115.458	0.221	-37.515	0.681													
12	1575	91.8	360	3.770	0.007	-74.268	0.363	-53.959	0.499	2775	107.8	360	3.770	0.007	-87.212	0.407	-63.363	0.561													
13	1605	84.9	390	4.084	0.007	-49.903	0.499	-68.686	0.363	2805	97.3	390	4.084	0.007	-57.192	0.548	-78.717	0.398													
14	1635	78.1	420	4.398	0.008	-24.134	0.580	-74.278	0.188	2835	88.8	420	4.398	0.008	-27.441	0.633	-84.454	0.206													
15	1665	73	450	4.712	0.008	0.000	0.609	-73.000	0.000	2865	81.1	450	4.712	0.008	0.000	0.651	-81.100	0.000													
16	1695	68	480	5.027	0.009	21.013	0.575	-64.672	0.187	2895	74.7	480	5.027	0.009	23.084	0.608	-71.044	0.197													
17	1725	63.3	510	5.341	0.009	37.207	0.483	-51.211	0.351	2925	69.2	510	5.341	0.009	40.675	0.509	-55.984	0.370													
18	1755	59	540	5.655	0.010	47.732	0.345	-34.679	0.475	2955	64.3	540	5.655	0.010	52.020	0.363	-37.795	0.500													
19	1785	55.5	570	5.969	0.010	52.784	0.180	-17.150	0.554	2985	60.1	570	5.969	0.010	57.158	0.188	-18.572	0.580													
20	1815	52.1	600	6.283	0.011	52.100	0.000	0.000	0.575	3015	60.6	600	6.283	0.011	60.600	0.000	0.000	0.647													
SUMr				δSUMr				SUMi				δSUMi				SUMr				δSUMr				SUMi				δSUMi			
-640.837				4.662				80.359				4.786				-921.700				5.235				541.491				5.596			
lr				δlr				li				δli				lr				δlr				li				δli			
-10.681				0.078				1.339				0.080				-15.362				0.087				9.025				0.093			
Amp(x1)				δAmp(x1)				φ(x1)				δφ(x1)				Amp(x2)				δAmp(x2)				φ(x2)				δφ(x2)			
10.764				0.078				-0.125				-0.007				17.817				0.089				-0.531				-0.005			
α										δα																					
-0.168										0.003																					
β										δβ																					
-0.135										0.003																					
D										δD																					
0.230										0.007																					

Figure 3: Temperature-time graphs at positions x_1 and x_2

The final value of diffusivity is found to be $D = (0.23 \pm 0.01) \text{ cm}^2\text{s}^{-1}$

The diffusivity of Brass reported in literature is $D = 0.341 \text{ cm}^2\text{s}^{-1}$ [2, 5].

$$\Rightarrow \quad \% \text{ error} = \left| \frac{0.23 - 0.341}{0.341} \right| \times 100 = 32.5 \%$$

Variation of V_{TC1} and V_{TC2} with time at x_1 and x_2

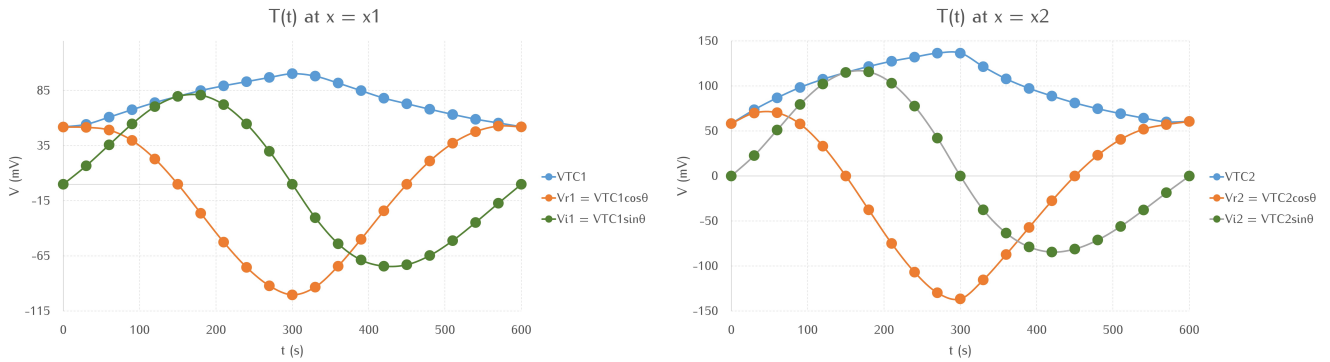


Figure 4: Temperature-time graphs at positions x_1 and x_2

Variation of δV_r and δV_i with time at x_1 and x_2

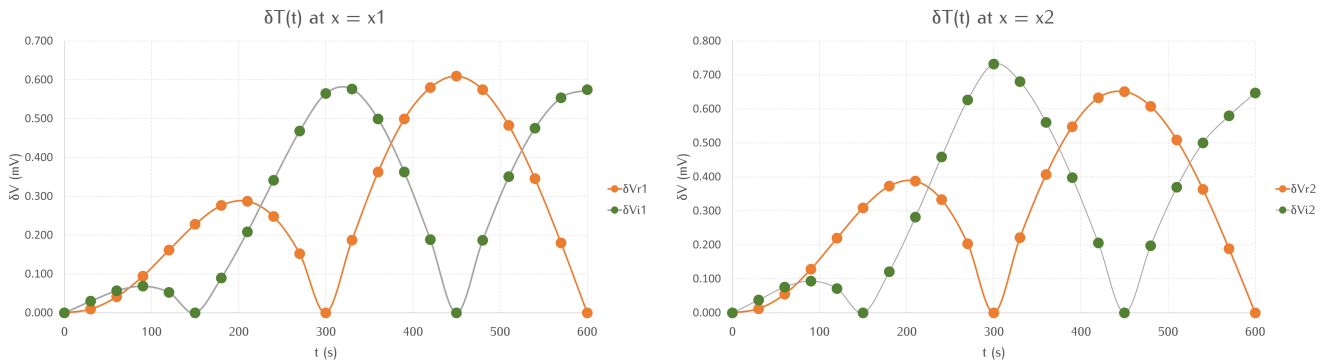


Figure 5: Time dependence of uncertainties in V_r and V_i at x_1 and x_2

That's it! You've done it! :) Make sure you take care of the dimensions and units of the values you report and think about why you get the values and what factors affect their accuracy or lack thereof. Write your thoughts in the "Discussion" section of your journal. All the best! :)

References

- [1] G. P. Arrighini, M. Maestro, and R. Moccia. Magnetic Properties of Polyatomic Molecules. I. Magnetic Susceptibility of H_2O , NH_3 , CH_4 , H_2O_2 . *The Journal of Chemical Physics*, 49(2):882–889, 09 2003.
- [2] Randall F. Barron and Barron. Brian R. *Appendix B: Properties of Materials at 300 K*, pages 462–463. John Wiley & Sons, Ltd, 2011.
- [3] J. Gyllenbok. *Encyclopaedia of Historical Metrology, Weights, and Measures: Volume 1*. Science Networks. Historical Studies. Springer International Publishing, 2018.
- [4] E. Süli and D.F. Mayers. *An Introduction to Numerical Analysis*. An Introduction to Numerical Analysis. Cambridge University Press, 2003.
- [5] Mohammad Reza Talaghat and Seyyed Mohammad Jokar. Determination of heat transfer parameters by use of finite integral transform and experimental data for regular geometric shapes. *Heat and Mass Transfer*, 53:3529–3544, 2017.