Title

A Senior Comprehensive Project
by
name
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I hereby recognize and pledge to fulfill my responsibilities, as defined in the Honor Code, and to maintain the integrity of both myself and the College community as a whole.

Pledge:

name

Acknowledgements

I want to thank everybody for everything. Keep typing to see what happens with a new paragraph.

Start a new paragraph thanking people.

Abstract

This comp is about everything!

Contents

1 Introduction

This purpose of this comp is to show that I've learned something really wonderful.

As a bonus, when I'm done and have passed the oral exam, I will finally graduate!

2 Preliminaries

3 Metric Spaces

In this section we overview some elementary topological concepts about metric spaces. Becoming comfortable with these concepts is an important step towards understanding the framework of Munguìa *et al.* 's paper. We specifically talk about metric spaces, norms, and the completion of metric spaces. Finally, we touch on the concept of Banach spaces, which are particularly important in the paper. Note that many definitions, theorems, etc. come from Gerald Edgar's *Measure, Topology, and Fractal Geometry* and Kaplansky's *Set Theory and Metric Spaces.*

Definition 3.1. A **metric space** is a set S together with a function $d : S \times S \to [0, \infty)$ satisfying the following:

(1) $d(x, y) = 0 \Leftrightarrow x = y$

(2) $d(x, y) \geq 0$ for all $x, y \in X$

(3) $d(x, y) = d(y, x)$

(4) $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle inequality)

The nonnegative real number $d(x, y)$ is called the *distance* between x and y, while the function d itself is known as the *metric* of the set S. A metric space is written as (S, d), but oftentimes the metric is implied and the space is simply referred to as S.

Example 3.2. The set of real numbers $\mathbb{R}$, with $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d(x, y) = |x - y|$$

is a metric space. This is the usual metric used with $\mathbb{R}$.

The complex plane $\mathbb{C}$ has a similar usual metric:

Example 3.3. The complex numbers $\mathbb{C}$ with $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ defined by

$$d(z, w) = |z - w|$$

where $|z|$ is the modulus of z is a metric space.

Generally, algebraic operations are not defined on a metric space, just a distance function. Meanwhile, a vector space (which is not necessarily a metric space) provides the operations of vector addition and scalar multiplication, but without a notion of distance. We can combine a vector space with a *norm*, though, to create a normed vector space — note that all normed vector spaces are also metric spaces.

Definition 3.4. A **normed vector space** $(X, ||\cdot||)$ is a vector space X with a function $||\cdot|| : X \to \mathbb{R}$, called a *norm* on X, such that for all $x, y \in X$ and $k \in \mathbb{R}$:

(1) $||x|| \geq 0$ and $||x|| = 0$ if and only if $x = 0$;

(2) $||kx|| = |k|\,||x||$ (scaling property);

(3) $||x + y|| \leq ||x|| + ||y||$.

These properties are rather intuitive. Property (1) says that a vector has nonnegative length, and the length of x is 0 if and only if x is the 0-vector; property (2) states multiplying a vector by a scalar k multiplies its length by k; finally property (3) is the triangle inequality, which is analogous to property (4) of definition 3.1.

While a norm is defined rather similarly to a metric, the two are not the same. However, we often define a metric on a normed metric space using the norm.

Proposition 3.5. *If* $(X, ||\cdot||)$ *is a normed vector space* X*, then* $d : X \times X \to \mathbb{R}$*, defined by* $d(x, y) = ||x - y||$*, is a metric on* X*.*

The necessary properties for d to be a metric follow immediately from properties (1) and (3) of a norm. If X is a normed vector space, we always use the metric associated with its norm, unless specifically stated otherwise.

A metric defined on a norm has all the properties of a metric discussed earlier, as well as two more — for all $x, y, z \in X$ and $k \in \mathbb{R}$

$$d(x + z, y + z) = d(x + y), \qquad d(kx, ky) = |k|d(x, y).$$

These properties are called *translation invariance* and *homogeneity*, respectively. These properties are not included in a simple metric space because they do not even make sense in that framework — recall that in a space which is only a metric space, we can not add points together or multiply them by scalars.

While there are a variety of norms which can be used on a vector space, the *Euclidean norm* is the most common and most intuitive.

Example 3.6. On $\mathbb{R}^n$, the length of a vector $x = (x_1, x_2, ..., x_n)$ is given by

$$||x := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

This gives the distance from the origin to the point x and is known as the *Euclidean norm.* It should be familiar as it is a result of the Pythagorean theorem.

Example 3.7. On $\mathbb{C}^n$, the most common norm is given by

$$||z|| := \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2} = \sqrt{z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n}$$

Note that while a metric is often derived from a norm, the existence of a metric does not imply a norm — a metric does not even necessarily need to make geometric sense. Take for example what is known as the *discrete metric*:

Example 3.8. On any set X, the discrete metric is defined as

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

4 Lp-Spaces

asdf

5 Chaos

The following is a well known criterion for chaos, known as the Eigenvalue Criterion. [2,3] provide proofs for the Criterion, and [9,10,12,14,15] provide examples using the Criterion.

Theorem 5.1. *Let $T : X \to X$ be an operator on a separable complex Banach space X. Consider the subspaces*

$$X_0 := \mathrm{Span}\{\mathrm{x} \in \mathrm{X} : \mathrm{T(x)} = \lambda \mathrm{X} \textit{ for some } \lambda \in \mathbb{C}\mathrm{with}|\lambda| < 1\},$$

$$Y_0 := \mathrm{Span}\{\mathrm{x} \in \mathrm{X} : \mathrm{T(x)} = \lambda \mathrm{X} \textit{ for some } \lambda \in \mathbb{C}\mathrm{with}|\lambda| > 1\},$$

$$Z_0 := \mathrm{Span}\{\mathrm{x} \in \mathrm{X} : \mathrm{T(x)} = \mathrm{e}^{\alpha\pi\mathrm{i}}\mathrm{x} \textit{ for some } \alpha \in \mathbb{Q}\}.$$

If X_0, Y_0, and Z_0 are all dense in X, then T is chaotic.

Since the set of eigenvalues $\sigma_p(B) = \mathbb{D}$ in our framework, this Criterion says that $\varphi(B)$ is chaotic on l^p if and only if $\varphi(\mathbb{D})$ intersects the unit circle.

6 Main results

The following result gives a geometrical description of $\varphi(\mathbb{D})$ when the pole of φ lies outside the closed unit disc.

Lemma 6.1. *Let φ be a linear fractional transformation(LFT) with $c \neq 0$ and $|d| > |c|$. Then $\varphi(\mathbb{D})$ is the disc $P + r\mathbb{D}$ with center P and radius r given by*

$$P = \frac{b\bar{d} - a\bar{c}}{|d|^2 - |c|^2}, \; r = \frac{|bc - ad|}{|d|^2 - |c|^2}.$$

Proof. Note that LFTs map circles and lines to circles and lines. Indeed, if f is a LFT and E is a circle or a line in $\mathbb{C}$, the image of E, $f(E)$, is mapped to a line if it passes through the pole. If E avoids the pole, $f(E)$ is a circle.

Observe that $\bar{\mathbb{D}} = \{z : |z| \leq 1\}$ (i.e. the closure of $\mathbb{D}$) is clearly a bounded and convex set. Because we imposed that $|d| > |c|$, we have that $|d/c| > 1$ and so the pole at $z = -d/c$ lies outside of $\bar{\mathbb{D}}$. Since LFTs are conformal at every point except at the pole, $\varphi(\bar{\mathbb{D}})$ must be bounded and convex. Furthermore, $\varphi(\bar{\mathbb{D}})$ is a circle whose boundary is $\varphi(\partial\mathbb{D})$, where $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$.

Now, take three distinct points in the unit circle. We choose $z_1 = 1$, $z_2 = -1$, and $z_3 = i$ as they are three very simple points on the unit circle. Since φ is linear, it is also one-to-one. Thus, $A = f(z_1)$, $B = f(z_2)$, and $C = f(z_3)$ are three distinct points. Since z_1, z_2, and z_3 are on the unit circle which is equivalent to $\partial\mathbb{D}$, A, B, and C are in fact three distinct points in the circle $\varphi(\partial\mathbb{D})$. That is, circle circumscribed over A, B, and C coincides with $\varphi(\partial\mathbb{D})$.

To verify that $\varphi(\partial\mathbb{D})$ indeed has center P and radius r, we just need to show that

$$|A-P| = |B-P| = |C-P| = r.$$

For this, we use the equalities $|z|^2 = z\bar{z}$, and $|c+d| = |\bar{c}+\bar{d}|$.

So, we have

$$\begin{aligned}
|A-P| &= \left|\frac{a+b}{c+d} - \frac{b\bar{d}-a\bar{c}}{|d|^2-|c|^2}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(a+b)(|d|^2-|c|^2)-(b\bar{d}-a\bar{c})(c+d)}{c+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{ad\bar{d}-bc\bar{c}-bc\bar{d}+ad\bar{c}}{c+d}\right|, \text{ by using the first equality} \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(\bar{c}+\bar{d})(bc-ad)}{c+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|\bar{c}+\bar{d}||bc-ad|}{|c+d|}\right) \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|c+d||bc-ad|}{|c+d|}\right), \text{ by using the second equality} \\
&= \frac{bc-ad}{|d|^2-|c|^2} \\
&= r.
\end{aligned}$$

Showing $|B-P|=r$ is analagous:

$$\begin{aligned}
|B-P| &= \left|\frac{-a+b}{-c+d} - \frac{b\bar{d}-a\bar{c}}{|d|^2-|c|^2}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(-a+b)(|d|^2-|c|^2)-(b\bar{d}-a\bar{c})(-c+d)}{-c+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{-ad\bar{d}-bc\bar{c}+bc\bar{d}+ad\bar{c}}{-c+d}\right|, \text{ by using the first equality} \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(-\bar{c}+\bar{d})(bc-ad)}{-c+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|-\bar{c}+\bar{d}||bc-ad|}{|-c+d|}\right) \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|-c+d||bc-ad|}{|-c+d|}\right), \text{ by using the second equality} \\
&= \frac{bc-ad}{|d|^2-|c|^2} \\
&= r.
\end{aligned}$$

Using a third equality, $|ci+d| = |\bar{c}+\bar{d}i|$, we have

$$\begin{aligned}
|C-P| &= \left|\frac{ai+b}{ci+d} - \frac{b\bar{d}-a\bar{c}}{|d|^2-|c|^2}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(ai+b)(|d|^2-|c|^2)-(b\bar{d}-a\bar{c})(ci+d)}{ci+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{ad\bar{d}i - bc\bar{c} - bc\bar{d}i + ad\bar{c}}{ci+d}\right|, \text{ by using the first equality} \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(\bar{c}+\bar{d}i)(ad-bc)}{ci+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|\bar{c}+\bar{d}i||ad-bc|}{|ci+d|}\right) \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|ci+d||ad-bc|}{|ci+d|}\right), \text{ by using the third equality} \\
&= \frac{bc-ad}{|d|^2-|c|^2} \\
&= r.
\end{aligned}$$

Hence the circle circumscribed over the points A, B, and C indeed has center P and radius r. Thus $\varphi(\partial\mathbb{D})$ has center P and radius r. Thus $\varphi(\mathbb{D})$ is the disc $P + r\mathbb{D}$. □

In [10], DeLaubenfels and Emamirad showed that, for a non-constant polynomial $P(z)$, $P(B)$ (where B is the backwards shift operator) is chaotic on $l^p, 1 \leq p \leq \infty$ whenever $P(\mathbb{D})$ intersects the unit disc. We provide a generalization of this result which can be applied to Linear Fractional Transformations.

Theorem 6.2. *Let φ be a LFT with $c \neq 0$ and $|d| > |c|$. The operator $\varphi(B)$*

is chaotic if and only if

$$\big||d|^2 - |c|^2 - |b\bar{d} - a\bar{c}| < |bc - ad|\big|.$$

Proof. We showed in Lemma 2 that $\varphi(\mathbb{D}) = P + r\mathbb{D}$ with center P and radius r given where

$$P = \frac{b\bar{d} - a\bar{c}}{|d|^2 - |c|^2},\ r = \frac{|bc - ad|}{|d|^2 - |c|^2}.$$

Theorem 1, the Eigenvalue Criterion, showed that $\varphi(B)$ is chaotic on l^p if and only if $\varphi(\mathbb{D})$ intersects the unit circle. So, we have that $\varphi(B)$ is chaotic if and only if the disc $P + r\mathbb{D}$ intersects the unit circle.

In order for the disc to intersect the unit disc, we have two possibilities: the center of the disc P is contained within the unit circle, or P is outside the closed unit disc. If P is inside the unit disc, then $|P| + |r| > 1$; if P is outside the closed unit disc, then we must have $|P| - |r| < 1$.

These conditions lead to

$$-|r| < 1 - |P| < |r|.$$

After substituting in the values of P and r, we have

$$-\frac{|bc - ad|}{|d|^2 - |c|^2} < 1 - \left|\frac{|b\bar{d} - a\bar{c}|}{|d|^2 - |c|^2}\right| < \frac{|bc - ad|}{|d|^2 - |c|^2}.$$

Multiplying by $|d|^2 - |c|^2$ gives

$$-|bc - ad| < |d|^2 - |c|^2 - |b\bar{d} - a\bar{c} < |bc - ad|.$$

So finally, we have

$$\big||d|^2 - |c|^2 - |b\bar{d} - a\bar{c}|\big| < |bc - ad|.$$

So, we have shown that if $\varphi(B)$ is chaotic, then $\big||d|^2 - |c|^2 - |b\bar{d} - a\bar{c}|\big| < |bc - ad|$. The other direction is completely analogous. It requires the exact same algebra, done in reverse, to show that if $\big||d|^2 - |c|^2 - |b\bar{d} - a\bar{c}|\big| < |bc - ad|$, then $P + r\mathbb{D}$ intersects the unit circle, and thus $\varphi(B)$ is chaotic. $\square$

References

[1] Burton, David M., *Elementary Number Theory*, Second Ed., W.C. Brown Publishers, 1989.

[2] Duren, P., Khavinson, D., Shapiro, H.S., and Sundberg, C., Contractive zero-divisors in Bergman spaces, *Pacific J. Math*, **157** (1993), 37-56.

[3] Fraleigh, John B., *A First Course in Abstract Algebra*, Fifth Ed., Addison-Wesley, 1994.