Chaos for Linear Fractal Transformations of Shifts

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1 Lemma 2

Lemma 1. *Let φ be a linear fractional transformation as in (2) and $|d| > |c|$. Then $\varphi(\mathbb{D})$ is the disc $P + r\mathbb{D}$ with center P and radius r given by*

$$P = \frac{b\bar{d} - a\bar{c}}{|d|^2 - |c|^2}, \; r = \frac{bc - ad}{|d|^2 - |c|^2}.$$

asdf

2 Definitions

Definition 1. A metric space (X, d) is a set X and a function d(the distance function) which assigns a real number $d(x, y)$ to every pair $(x, y) \in X$, which satisfies the following properties :

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \Rightarrow x = y$.
3. $d(x, y) = d(y, x)$.
4. $d(x, y) + d(y, z) \geq d(x, z)$. This last property is called the *triangle inequality*.

Definition 2. A function f is topologically transitive iff for all nonempty open subsets U, V of X, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V$ is nonempty.

Definition 3. Let X be a topological space. A set Q is dense in X if for any point $x \in X$ and for any $\epsilon > 0$, there exists a point in $q \in Q$ such that the distance between x and q is less than ϵ. In other words, a set Q is dense in X if every point in X is either in Q or is a limit point in Q.

Definition 4. A point x is said to be a periodic point of a function f if there exists an integer n such that $f^n(x) = x$. The least positive integer n for which this is true is the period of x.

Definition 5. Let (X, d) be a metric space. A function $f : X \to X$ is said to be chaotic on X if it satisfies the following three conditions:

1. *f is topologically transitive.*
2. *The set of periodic points in f is dense in X.* That is, that every open set in f contains a periodic point.
3. *f has sensitive dependence on initial conditions.* That is, $\exists \delta > 0$ such that for any open set U and for any $x \in U$, there exists a $y \in U$ such that $d(f^{[k]}(x), f^{[k]}(y)) > \delta$ for some k. δ is called a *sensitivity constant.*

Definition 6. A backward shift operator B operates on an element of a sequence to produce the previous element.

e.g. if $X = \{x_1, x_2, \dots\}$, then $B(X) = \{x_2, x_3, \dots\}$.

Definition 7. Let $z \in \mathbb{C}$. That is, let $z = x + iy$, where x and y are real numbers. The absolute value or modulus of z, denoted $|z|$ is given by

$$|z| = \sqrt{x^2 + y^2}.$$

Definition 8. The open unit disc of $\mathbb{C}$, denoted $\mathbb{D}$, is the region in the complex plane defined by

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Definition 9.