# Chaos for Linear Fractal Transformations of Shifts

Lucas Hawk

October 19, 2016

## 1 Theorem 1

The following is a well known criterion for chaos, known as the Eigenvalue Criterion. [2,3] provide proofs for the Criterion, and [9,10,12,14,15] provide examples using the Criterion.

**Theorem 1.** *Let $T : X \to X$ be an operator on a separable complex Banach space $X$. Consider the subspaces*

$$X_0 := \mathrm{Span}\{\mathrm{x} \in \mathrm{X} : \mathrm{T(x)} = \lambda \mathrm{X} \textit{ for some } \lambda \in \mathbb{C}\mathrm{with}|\lambda| < 1\},$$

$$Y_0 := \mathrm{Span}\{\mathrm{x} \in \mathrm{X} : \mathrm{T(x)} = \lambda \mathrm{X} \textit{ for some } \lambda \in \mathbb{C}\mathrm{with}|\lambda| > 1\},$$

$$Z_0 := \mathrm{Span}\{\mathrm{x} \in \mathrm{X} : \mathrm{T(x)} = \mathrm{e}^{\alpha\pi\mathrm{i}}\mathrm{x} \textit{ for some } \alpha \in \mathbb{Q}\}.$$

*If $X_0, Y_0$, and $Z_0$ are all dense in $X$, then $T$ is chaotic.*

Since the set of eigenvalues $\sigma_p(B) = \mathbb{D}$ in our framework, this Criterion says that $\varphi(B)$ is chaotic on $l^p$ if and only if $\varphi(\mathbb{D})$ intersects the unit circle.

## 2 Lemma 2

The following result gives a geometrical description of $\varphi(\mathbb{D})$ when the pole of $\varphi$ lies outside the closed unit disc.

**Lemma 2.** *Let $\varphi$ be a linear fractional transformation(LFT) with $c \neq 0$ and $|d| > |c|$. Then $\varphi(\mathbb{D})$ is the disc $P + r\mathbb{D}$ with center $P$ and radius $r$ given by*

$$P = \frac{b\bar{d} - a\bar{c}}{|d|^2 - |c|^2},\ r = \frac{|bc - ad|}{|d|^2 - |c|^2}.$$

*Proof.* Note that LFTs map circles and lines to circles and lines. Indeed, if $f$ is a LFT and $E$ is a circle or a line in $\mathbb{C}$, the image of $E$, $f(E)$, is mapped to a line if it passes through the pole. If $E$ avoids the pole, $f(E)$ is a circle.

Observe that $\bar{\mathbb{D}} = \{z : |z| \leq 1\}$ (i.e. the closure of $\mathbb{D}$) is clearly a bounded and convex set. Because we imposed that $|d| > |c|$, we have that $|d/c| > 1$ and so the pole at $z = -d/c$ lies outside of $\bar{\mathbb{D}}$. Since LFTs are conformal at every point except at the pole, $\varphi(\bar{\mathbb{D}})$ must be bounded and convex. Furthermore, $\varphi(\bar{\mathbb{D}})$ is a circle whose boundary is $\varphi(\partial\mathbb{D})$, where $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$.

Now, take three distinct points in the unit circle. We choose $z_1 = 1$, $z_2 = -1$, and $z_3 = i$ as they are three very simple points on the unit circle. Since $\varphi$ is linear, it is also one-to-one. Thus, $A = f(z_1)$, $B = f(z_2)$, and $C = f(z_3)$ are three distinct points. Since $z_1$, $z_2$, and $z_3$ are on the unit circle which is equivalent to $\partial\mathbb{D}$, $A$, $B$, and $C$ are in fact three distinct points in the circle $\varphi(\partial\mathbb{D})$. That is, circle circumscribed over $A$, $B$, and $C$ coincides with $\varphi(\partial\mathbb{D})$.

To verify that $\varphi(\partial\mathbb{D})$ indeed has center $P$ and radius $r$, we just need to show that

$$|A - P| = |B - P| = |C - P| = r.$$

For this, we use the equalities $|z|^2 = z\bar{z}$, and $|c+d| = |\bar{c}+\bar{d}|$.

So, we have

$$
\begin{aligned}
|A-P| &= \left|\frac{a+b}{c+d} - \frac{b\bar{d}-a\bar{c}}{|d|^2-|c|^2}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(a+b)(|d|^2-|c|^2)-(b\bar{d}-a\bar{c})(c+d)}{c+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{ad\bar{d}-bc\bar{c}-bc\bar{d}+ad\bar{c}}{c+d}\right|, \text{ by using the first equality} \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(\bar{c}+\bar{d})(bc-ad)}{c+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|\bar{c}+\bar{d}||bc-ad|}{|c+d|}\right) \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|c+d||bc-ad|}{|c+d|}\right), \text{ by using the second equality} \\
&= \frac{bc-ad}{|d|^2-|c|^2} \\
&= r.
\end{aligned}
$$

Showing $|B-P| = r$ is analagous:

$$
\begin{aligned}
|B-P| &= \left|\frac{-a+b}{-c+d} - \frac{b\bar{d}-a\bar{c}}{|d|^2-|c|^2}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(-a+b)(|d|^2-|c|^2)-(b\bar{d}-a\bar{c})(-c+d)}{-c+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{-ad\bar{d}-bc\bar{c}+bc\bar{d}+ad\bar{c}}{-c+d}\right|, \text{ by using the first equality} \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(-\bar{c}+\bar{d})(bc-ad)}{-c+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|-\bar{c}+\bar{d}||bc-ad|}{|-c+d|}\right) \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|-c+d||bc-ad|}{|-c+d|}\right), \text{ by using the second equality} \\
&= \frac{bc-ad}{|d|^2-|c|^2} \\
&= r.
\end{aligned}
$$

Using a third equality, $|ci+d| = |\bar{c}+\bar{d}i|$, we have

$$\begin{aligned}
|C-P| &= \left|\frac{ai+b}{ci+d} - \frac{b\bar{d}-a\bar{c}}{|d|^2-|c|^2}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(ai+b)(|d|^2-|c|^2)-(b\bar{d}-a\bar{c})(ci+d)}{ci+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{ad\bar{d}i - bc\bar{c} - bc\bar{d}i + ad\bar{c}}{ci+d}\right|, \text{ by using the first equality} \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(\bar{c}+\bar{d}i)(ad-bc)}{ci+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|\bar{c}+\bar{d}i||ad-bc|}{|ci+d|}\right) \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|ci+d||ad-bc|}{|ci+d|}\right), \text{ by using the third equality} \\
&= \frac{bc-ad}{|d|^2-|c|^2} \\
&= r.
\end{aligned}$$

Hence the circle circumscribed over the points $A$, $B$, and $C$ indeed has center $P$ and radius $r$. Thus $\varphi(\partial\mathbb{D})$ has center $P$ and radius $r$. Thus $\varphi(\mathbb{D})$ is the disc $P + r\mathbb{D}$. □

# 3 Theorem 3

In [10], DeLaubenfels and Emamirad showed that, for a non-constant polynomial $P(z)$, $P(B)$ (where $B$ is the backwards shift operator) is chaotic on $l^p, 1 \le p \le \infty$ whenever $P(\mathbb{D})$ intersects the unit disc. We provide a generalization of this result which can be applied to Linear Fractional Transformations.

**Theorem 3.** *Let $\varphi$ be a LFT with $c \neq 0$ and $|d| > |c|$. The operator $\varphi(B)$ is chaotic if and only if*

$$\big||d|^2 - |c|^2 - |b\bar{d} - a\bar{c}| < |bc - ad|\big|.$$

*Proof.* We showed in Lemma 2 that $\varphi(\mathbb{D}) = P + r\mathbb{D}$ with center $P$ and radius $r$ given where

$$P = \frac{b\bar{d}-a\bar{c}}{|d|^2-|c|^2},\ r = \frac{|bc-ad|}{|d|^2-|c|^2}.$$

Theorem 1, the Eigenvalue Criterion, showed that $\varphi(B)$ is chaotic on $l^p$ if and only if $\varphi(\mathbb{D})$ intersects the unit circle. So, we have that $\varphi(B)$ is chaotic if and only if the disc $P + r\mathbb{D}$ intersects the unit circle.

In order for the disc to intersect the unit disc, we have two possibilities: the center of the disc $P$ is contained within the unit circle, or $P$ is outside the closed unit disc. If $P$ is inside the unit disc, then $|P| + |r| > 1$; if $P$ is outside the closed unit disc, then we must have $|P| - |r| < 1$.

These conditions lead to

$$-|r| < 1 - |P| < |r|.$$

After substituting in the values of $P$ and $r$, we have

$$-\frac{|bc-ad|}{|d|^2-|c|^2} < 1 - \left|\frac{|b\bar{d}-a\bar{c}|}{|d|^2-|c|^2}\right| < \frac{|bc-ad|}{|d|^2-|c|^2}.$$

Multiplying by $|d|^2 - |c|^2$ gives

$$-|bc-ad| < |d|^2 - |c|^2 - |b\bar{d} - a\bar{c}| < |bc-ad|.$$

So finally, we have

$$\left||d|^2 - |c|^2 - |b\bar{d} - a\bar{c}|\right| < |bc - ad|.$$

So, we have shown that if $\varphi(B)$ is chaotic, then $\left||d|^2 - |c|^2 - |b\bar{d} - a\bar{c}|\right| < |bc - ad|$. The other direction is completely analogous. It requires the exact same algebra, done in reverse, to show that if $\left||d|^2 - |c|^2 - |b\bar{d} - a\bar{c}|\right| < |bc - ad|$, then $P + r\mathbb{D}$ intersects the unit circle, and thus $\varphi(B)$ is chaotic. $\square$

# 4 Definitions

**Definition 1.** A metric space $(X, d)$ is a set $X$ and a function $d$(the distance function) which assigns a real number $d(x, y)$ to every pair $(x, y) \in X$, which satisfies the following properties :

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \Rightarrow x = y$.
3. $d(x, y) = d(y, x)$.
4. $d(x, y) + d(y, z) \geq d(x, z)$. This last property is called the *triangle inequality.*

**Definition 2.** A function $f$ is topologically transitive iff for all nonempty open subsets $U, V$ of $X$, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V$ is nonempty.

**Definition 3.** Let $X$ be a topological space. A set $Q$ is dense in $X$ if for any point $x \in X$ and for any $\epsilon > 0$, there exists a point in $q \in Q$ such that the distance between $x$ and $q$ is less than $\epsilon$. In other words, a set $Q$ is dense in $X$ if every point in $X$ is either in $Q$ or is a limit point in $Q$.

**Definition 4.** A point x is said to be a periodic point of a function $f$ if there exists an integer $n$ such that $f^n(x) = x$. The least positive integer $n$ for which this is true is the period of $x$.

**Definition 5.** Let $(X, d)$ be a metric space. A function $f : X \to X$ is said to be chaotic on $X$ if it satisfies the following three conditions:

1. *$f$ is topologically transitive.*
2. *The set of periodic points in $f$ is dense in $X$.* That is, that every open set in $f$ contains a periodic point.
3. *$f$ has sensitive dependence on initial conditions.* That is, $\exists\delta > 0$ such that for any open set $U$ and for any $x \in U$, there exists a $y \in U$ such that $d(f^{[k]}(x), f^{[k]}(y)) > \delta$ for some k. $\delta$ is called a *sensitivity constant.*

**Definition 6.** A backward shift operator $B$ operates on an element of a sequence to produce the previous element.

e.g. if $X = \{x_1, x_2, \dots\}$, then $B(X) = \{x_2, x_3, \dots\}$.

**Definition 7.** Let $z \in \mathbb{C}$. That is, let $z = x+yi$, where $x$ and $y$ are real numbers. The absolute value or modulus of $z$, denoted $|z|$ is given by

$$|z| = \sqrt{x^2 + y^2}.$$

**Definition 8.** The open unit disc of $\mathbb{C}$, denoted $\mathbb{D}$, is the region in the complex plane defined by

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

**Definition 9.** A mapping $T$ from a vector space $V_1$ to a vector space $V_2$, i.e. $T : V_1 \to V_2$, is a linear transformation iff

$$T(c\vec{u} + c\vec{v}) = cT(\vec{u}) + cT(\vec{v}),$$

for all $\vec{u}, \vec{v} \in V_1$, and all $c \in \mathbb{R}$. The transformation is referred to as an operator if the mapping is from a vector space to itself.

**Definition 10.** Let $U \subset \mathbb{C}$ be open and let $f : U \to \mathbb{C}$. If $f$ is complex differentiable at every point in $U$, $f$ is said to be holomorphic or on $U$.

**Definition 11.** A function $f$ has a pole of order $n$ at $z_0$ if $n$ is the smallest positive integer for which $(z - z_0)^n f(z)$ is holomorphic at $z_0$. A function $f$ has a pole at infinity if $\lim_{z\to\infty} f(z) = \infty$.

**Definition 12.** A Linear Fraction Transformation is a function of the form

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}, ad \neq cb$.

**Definition 13.** A vector space is a set that is closed under finite vector addition and scalar muliplication. A vector space $V$ is complete is every Cauchy sequence of points in $V$ converges to a point in $V$.

**Definition 14.** Let $V$ be a complex vector space. A norm on $V$ is a function

$$|| \cdot || : V \to \mathbb{R}$$

that satisfies the following conditions:

1. $||\vec{v}|| \geq 0, \forall \vec{c} \in V, and ||\vec{v}|| = 0 \Leftrightarrow \vec{v} = 0;$
2. $||\alpha \vec{v}|| = |\alpha| \; ||\vec{v}||, \forall \vec{v} \in V, \alpha \in \mathbb{C};$
3. $||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||, \forall \vec{v}, \vec{w} \in V.$

A vector space equipped with a norm is called a normed vector space.

**Definition 15.** A Banach space is a complete normed vector space.