Title

A Senior Comprehensive Project
by
name
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I hereby recognize and pledge to fulfill my responsibilities, as defined in the Honor Code, and to maintain the integrity of both myself and the College community as a whole.

Pledge:

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name

Acknowledgements

I want to thank everybody for everything. Keep typing to see what happens with a new paragraph.

Start a new paragraph thanking people.

Abstract

This comp is about everything!

Contents

1 Introduction

This purpose of this comp is to show that I've learned something really wonderful.

As a bonus, when I'm done and have passed the oral exam, I will finally graduate!

2 Preliminaries

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3 Chaos

The following is a well known criterion for chaos, known as the Eigenvalue Criterion. [2,3] provide proofs for the Criterion, and [9,10,12,14,15] provide examples using the Criterion.

Theorem 3.1. *Let $T : X \to X$ be an operator on a separable complex Banach space X. Consider the subspaces*

$$X_0 := \mathrm{Span}\{\mathrm{x} \in \mathrm{X} : \mathrm{T(x)} = \lambda\mathrm{X} \textit{ for some } \lambda \in \mathbb{C}\mathrm{with}|\lambda| < 1\},$$

$$Y_0 := \mathrm{Span}\{\mathrm{x} \in \mathrm{X} : \mathrm{T(x)} = \lambda\mathrm{X} \textit{ for some } \lambda \in \mathbb{C}\mathrm{with}|\lambda| > 1\},$$

$$Z_0 := \mathrm{Span}\{\mathrm{x} \in \mathrm{X} : \mathrm{T(x)} = \mathrm{e}^{\alpha\pi\mathrm{i}}\mathrm{x} \textit{ for some } \alpha \in \mathbb{Q}\}.$$

If X_0, Y_0, and Z_0 are all dense in X, then T is chaotic.

Since the set of eigenvalues $\sigma_p(B) = \mathbb{D}$ in our framework, this Criterion says that $\varphi(B)$ is chaotic on l^p if and only if $\varphi(\mathbb{D})$ intersects the unit circle.

Exercise 3.2. sldkfjlskdfjldksjf

Theorem 3.3. *as;ldkfjlasdkjfl*

Proof. alsdkjflaskdjf □

4 Lp-spaces

Exercise 4.1. sldkfjlskdfjldksjf

5 Main results

The following result gives a geometrical description of $\varphi(\mathbb{D})$ when the pole of φ lies outside the closed unit disc.

Lemma 5.1. *Let φ be a linear fractional transformation(LFT) with $c \neq 0$ and $|d| > |c|$. Then $\varphi(\mathbb{D})$ is the disc $P + r\mathbb{D}$ with center P and radius r given by*

$$P = \frac{b\bar{d} - a\bar{c}}{|d|^2 - |c|^2}, \; r = \frac{|bc - ad|}{|d|^2 - |c|^2}.$$

Proof. Note that LFTs map circles and lines to circles and lines. Indeed, if f is a LFT and E is a circle or a line in $\mathbb{C}$, the image of E, $f(E)$, is mapped to a line if it passes through the pole. If E avoids the pole, $f(E)$ is a circle.

Observe that $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$ (i.e. the closure of $\mathbb{D}$) is clearly a bounded and convex set. Because we imposed that $|d| > |c|$, we have that $|d/c| > 1$ and so the pole at $z = -d/c$ lies outside of $\overline{\mathbb{D}}$. Since LFTs are conformal at every point except at the pole, $\varphi(\overline{\mathbb{D}})$ must be bounded and convex. Furthermore, $\varphi(\overline{\mathbb{D}})$ is a circle whose boundary is $\varphi(\partial\mathbb{D})$, where $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$.

Now, take three distinct points in the unit circle. We choose $z_1 = 1$, $z_2 = -1$, and $z_3 = i$ as they are three very simple points on the unit circle. Since φ is linear, it is also one-to-one. Thus, $A = f(z_1)$, $B = f(z_2)$, and $C = f(z_3)$ are three distinct points. Since z_1, z_2, and z_3 are on the unit circle which is equivalent to $\partial\mathbb{D}$, A, B, and C are in fact three distinct points in the circle $\varphi(\partial\mathbb{D})$. That is, circle circumscribed over A, B, and C coincides with $\varphi(\partial\mathbb{D})$.

To verify that $\varphi(\partial\mathbb{D})$ indeed has center P and radius r, we just need to show that

$$|A-P| = |B-P| = |C-P| = r.$$

For this, we use the equalities $|z|^2 = z\bar{z}$, and $|c+d| = |\bar{c}+\bar{d}|$.

So, we have

$$\begin{aligned}
|A-P| &= \left|\frac{a+b}{c+d} - \frac{b\bar{d}-a\bar{c}}{|d|^2-|c|^2}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(a+b)(|d|^2-|c|^2)-(b\bar{d}-a\bar{c})(c+d)}{c+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{ad\bar{d}-bc\bar{c}-bc\bar{d}+ad\bar{c}}{c+d}\right|, \text{ by using the first equality} \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(\bar{c}+\bar{d})(bc-ad)}{c+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|\bar{c}+\bar{d}||bc-ad|}{|c+d|}\right) \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|c+d||bc-ad|}{|c+d|}\right), \text{ by using the second equality} \\
&= \frac{bc-ad}{|d|^2-|c|^2} \\
&= r.
\end{aligned}$$

Showing $|B-P|=r$ is analagous:

$$\begin{aligned}
|B-P| &= \left|\frac{-a+b}{-c+d} - \frac{b\bar{d}-a\bar{c}}{|d|^2-|c|^2}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(-a+b)(|d|^2-|c|^2)-(b\bar{d}-a\bar{c})(-c+d)}{-c+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{-ad\bar{d}-bc\bar{c}+bc\bar{d}+ad\bar{c}}{-c+d}\right|, \text{ by using the first equality} \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(-\bar{c}+\bar{d})(bc-ad)}{-c+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|-\bar{c}+\bar{d}||bc-ad|}{|-c+d|}\right) \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|-c+d||bc-ad|}{|-c+d|}\right), \text{ by using the second equality} \\
&= \frac{bc-ad}{|d|^2-|c|^2} \\
&= r.
\end{aligned}$$

Using a third equality, $|ci+d| = |\bar{c}+\bar{d}i|$, we have

$$\begin{aligned}
|C-P| &= \left|\frac{ai+b}{ci+d} - \frac{b\bar{d}-a\bar{c}}{|d|^2-|c|^2}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(ai+b)(|d|^2-|c|^2)-(b\bar{d}-a\bar{c})(ci+d)}{ci+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{ad\bar{d}i-bc\bar{c}-bc\bar{d}i+ad\bar{c}}{ci+d}\right|, \text{ by using the first equality} \\
&= \frac{1}{|d|^2-|c|^2}\left|\frac{(\bar{c}+\bar{d}i)(ad-bc)}{ci+d}\right| \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|\bar{c}+\bar{d}i||ad-bc|}{|ci+d|}\right) \\
&= \frac{1}{|d|^2-|c|^2}\left(\frac{|ci+d||ad-bc|}{|ci+d|}\right), \text{ by using the third equality} \\
&= \frac{bc-ad}{|d|^2-|c|^2} \\
&= r.
\end{aligned}$$

Hence the circle circumscribed over the points A, B, and C indeed has center P and radius r. Thus $\varphi(\partial\mathbb{D})$ has center P and radius r. Thus $\varphi(\mathbb{D})$ is the disc $P + r\mathbb{D}$. □

In [10], DeLaubenfels and Emamirad showed that, for a non-constant polynomial $P(z)$, $P(B)$ (where B is the backwards shift operator) is chaotic on $l^p, 1 \leq p \leq \infty$ whenever $P(\mathbb{D})$ intersects the unit disc. We provide a generalization of this result which can be applied to Linear Fractional Transformations.

Theorem 5.2. *Let φ be a LFT with $c \neq 0$ and $|d| > |c|$. The operator $\varphi(B)$*

is chaotic if and only if

$$\big||d|^2 - |c|^2 - |b\bar{d} - a\bar{c}| < |bc - ad|\big|.$$

Proof. We showed in Lemma 2 that $\varphi(\mathbb{D}) = P + r\mathbb{D}$ with center P and radius r given where

$$P = \frac{b\bar{d} - a\bar{c}}{|d|^2 - |c|^2},\ r = \frac{|bc - ad|}{|d|^2 - |c|^2}.$$

Theorem 1, the Eigenvalue Criterion, showed that $\varphi(B)$ is chaotic on l^p if and only if $\varphi(\mathbb{D})$ intersects the unit circle. So, we have that $\varphi(B)$ is chaotic if and only if the disc $P + r\mathbb{D}$ intersects the unit circle.

In order for the disc to intersect the unit disc, we have two possibilities: the center of the disc P is contained within the unit circle, or P is outside the closed unit disc. If P is inside the unit disc, then $|P| + |r| > 1$; if P is outside the closed unit disc, then we must have $|P| - |r| < 1$.

These conditions lead to

$$-|r| < 1 - |P| < |r|.$$

After substituting in the values of P and r, we have

$$-\frac{|bc - ad|}{|d|^2 - |c|^2} < 1 - \left|\frac{|b\bar{d} - a\bar{c}|}{|d|^2 - |c|^2}\right| < \frac{|bc - ad|}{|d|^2 - |c|^2}.$$

Multiplying by $|d|^2 - |c|^2$ gives

$$-|bc - ad| < |d|^2 - |c|^2 - |b\bar{d} - a\bar{c} < |bc - ad|.$$

So finally, we have

$$\big||d|^2 - |c|^2 - |b\bar{d} - a\bar{c}|\big| < |bc - ad|.$$

So, we have shown that if $\varphi(B)$ is chaotic, then $\big||d|^2 - |c|^2 - |b\bar{d} - a\bar{c}|\big| < |bc - ad|$. The other direction is completely analogous. It requires the exact same algebra, done in reverse, to show that if $\big||d|^2 - |c|^2 - |b\bar{d} - a\bar{c}|\big| < |bc - ad|$, then $P + r\mathbb{D}$ intersects the unit circle, and thus $\varphi(B)$ is chaotic. $\square$

References

[1] Burton, David M., *Elementary Number Theory*, Second Ed., W.C. Brown Publishers, 1989.

[2] Duren, P., Khavinson, D., Shapiro, H.S., and Sundberg, C., Contractive zero-divisors in Bergman spaces, *Pacific J. Math*, **157** (1993), 37-56.

[3] Fraleigh, John B., *A First Course in Abstract Algebra*, Fifth Ed., Addison-Wesley, 1994.