1 set up

1.1 Lagrangian

A wise choice would be to follow the notation [1611.03827] by Bhupal et al.

$$-\mathcal{L} \supset (Y_\nu)_{ai}\overline{N}_{\mathrm{R}a}\tilde{\phi}^\dagger L_i + \frac{1}{2}\overline{N}_{\mathrm{R}a}M_{ab}N^{\mathrm{c}}_{\mathrm{R}b} + \text{h.c.} \tag{1.1}$$

$\phi = (\phi^+, \phi^0)^{\mathrm{T}}$
$\tilde{\phi} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \phi^- \\ \phi^{0*} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \phi^{0*} \\ -\phi^- \end{smallmatrix}\right)$

$$\begin{aligned} -\mathcal{L} &\supset (Y_\nu)_{ai}\overline{N}_{\mathrm{R}a}(\phi^0\nu_i - \phi^+ l_i) + \frac{1}{2}\overline{N}_{\mathrm{R}a}M_{ab}N^{\mathrm{c}}_{\mathrm{R}b} + \text{h.c.} && (1.2)\\ &= \epsilon_{AB}(Y_\nu)_{ai}\overline{N}_{\mathrm{R}a}L_{iA}\phi_B + \frac{1}{2}\overline{N}_{\mathrm{R}a}M_{ab}N^{\mathrm{c}}_{\mathrm{R}b} + \text{h.c.} && (1.3)\end{aligned}$$

This Lagrangian is written without specifying the basis. For CI parameterization we first define M_a and m_i by the physical mass of N_a and ν_i, where $M_1 \le M_2$ but the basis for ν_i is "as usual".

Neutrino mass matrix is better shown in two-component $N_{\mathrm{R}} = \left(\begin{smallmatrix} 0 \\ n^\dagger \end{smallmatrix}\right)$:

$$\begin{aligned} -\mathcal{L} &\supset \langle\phi_0\rangle(Y_\nu)_{ai}n_a\nu_i + \frac{1}{2}M_{ab}n_a n_b + \text{h.c.} && (1.4)\\ &= \frac{1}{2}\begin{pmatrix}\nu_i & n_a\end{pmatrix}\begin{pmatrix} 0_{ij} & \langle\phi_0\rangle(Y_\nu)_{bi} \\ \langle\phi_0\rangle(Y_\nu)_{aj} & M_{ab}\end{pmatrix}\begin{pmatrix}\nu_j \\ n_b\end{pmatrix} + \text{h.c.} && (1.5)\end{aligned}$$

or we will write down, assuming the notation is understood,

$$M_\nu = \begin{pmatrix} 0 & \langle\phi^0\rangle Y_\nu^{\mathrm{T}} \\ \langle\phi^0\rangle Y_\nu & M \end{pmatrix} \tag{1.6}$$

and perform Autonne–Takagi diagonalization:

$$U_0^{\mathrm{T}} M_\nu U_0 = \mathrm{diag}(m_1, m_2, m_3, M_1, M_2) \tag{1.7}$$

1.2 Casas–Ibarra parameterization

We split this Autonne–Takagi diagonalization procedure to two steps:

$$\begin{aligned} &U_1^{\mathrm{T}} M_\nu U_1 = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right), && (1.8)\\ &\begin{pmatrix} U_2 & 0 \\ 0 & U_3 \end{pmatrix}^{\mathrm{T}} U_1^{\mathrm{T}} M_\nu U_1 \begin{pmatrix} U_2 & 0 \\ 0 & U_3 \end{pmatrix} = \begin{pmatrix} m_{\mathrm{diag}} & 0 \\ 0 & M_{\mathrm{diag}} \end{pmatrix}, && (1.9)\end{aligned}$$

where $m_{\rm diag} = {\rm diag}(m_1, m_2, m_3)$ and $M_{\rm diag} = {\rm diag}(M_1, M_2)$. The result of the first step is well-known in series-expanded form:

$$U_1 \simeq \begin{pmatrix} 1 - \frac{\langle\phi_0\rangle^2}{2} Y_\nu^\dagger (MM^*)^{-1} Y_\nu & \langle\phi_0\rangle Y_\nu^\dagger M^{*-1} \\ -\langle\phi_0\rangle M^{-1} Y_\nu & 1 - \frac{\langle\phi_0\rangle^2}{2} M^{-1} Y_\nu Y_\nu^\dagger M^{*-1} \end{pmatrix}, \tag{1.10}$$

$$A \simeq -\langle\phi_0\rangle^2 Y_\nu^{\rm T} M^{-1} Y_\nu, \tag{1.11}$$

$$B \simeq M + \frac{\langle\phi_0\rangle^2}{2} \left(Y_\nu Y_\nu^\dagger M^{*-1} + M^{*-1} Y_\nu^* Y_\nu^{\rm T} \right) \tag{1.12}$$

The second step is expressed by

$$U_2^{\rm T} A U_2 = m_{\rm diag}, \qquad U_3^{\rm T} B U_3 = M_{\rm diag}. \tag{1.13}$$

We also have the expression of the mass eigenstates:

$$\text{lighter}: \quad U_2^\dagger \left[\nu - \langle\phi_0\rangle Y_\nu^\dagger M^{*-1} n - \frac{\langle\phi_0\rangle^2}{2} Y_\nu^\dagger (MM^*)^{-1} Y_\nu \nu \right] \tag{1.14}$$

$$\text{heavier}: \quad U_3^\dagger \left[n + \langle\phi_0\rangle M^{-1} Y_\nu \nu - \frac{\langle\phi_0\rangle^2}{2} M^{-1} Y_\nu Y_\nu^\dagger M^{*-1} n \right] \tag{1.15}$$

$$\tag{1.16}$$

Combining them,

$$\begin{aligned} m_{\rm diag} &= U_2^{\rm T} A U_2 && (1.17) \\ &= -\langle\phi_0\rangle^2 U_2^{\rm T} Y_\nu^{\rm T} M^{-1} Y_\nu U_2 + \mathcal{O}(\epsilon^4) && (1.18) \\ &= -\langle\phi_0\rangle^2 U_2^{\rm T} Y_\nu^{\rm T} B^{-1} Y_\nu U_2 + \mathcal{O}(\epsilon^4) && (1.19) \\ &= -\langle\phi_0\rangle^2 U_2^{\rm T} Y_\nu^{\rm T} U_3 M_{\rm diag}^{-1} U_3^{\rm T} Y_\nu U_2 + \mathcal{O}(\epsilon^4) && (1.20) \end{aligned}$$

with $\mathcal{O}(M\epsilon^n) \sim \mathcal{O}(\langle\phi_0\rangle^n / M_1^{n-1})$. Now $R := -i\langle\phi_0\rangle M_{\rm diag}^{-1/2} U_3^{\rm T} Y_\nu U_2 m_{\rm diag}^{-1/2}$ satisfies $R^{\rm T} R = 1$, which is Casas–Ibarra parameterization in general basis,

$$Y_\nu = i\langle\phi_0\rangle^{-1} U_3^* \sqrt{M_{\rm diag}} R \sqrt{m_{\rm diag}} U_2^\dagger. \tag{1.21}$$

We have not yet defined the lepton basis. We can assume that we have been using, from the beginning, the charged lepton mass basis for L. Then, we identify the PMNS matrix (U_{li} in Eq. (14.1) of PDG2018, where i for mass and l for gauge indices) as

$$U_{\rm PMNS} \simeq U_2. \tag{1.22}$$

Similarly, the basis for $N_{\rm R}$ is such that $U_3 \simeq 1$, which corresponds to

$$M = B + \mathcal{O}(\epsilon^2) = U_3^* M_{\rm diag} U_3^\dagger + \mathcal{O}(\epsilon^2) = M_{\rm diag} + \mathcal{O}(\epsilon^2), \tag{1.23}$$

i.e., the basis in which $M_{ab} \simeq \mathrm{diag}(M_1, M_2)$ with $0 < M_1 \leq M_2$. These basis choice gives the well-known Casas–Ibarra parameterization,[1]

$$Y_\nu = i\langle\phi_0\rangle^{-1}\sqrt{M_{\mathrm{diag}}}R\sqrt{m_{\mathrm{diag}}}U^\dagger_{\mathrm{PMNS}}, \tag{1.24}$$

where

$$\begin{aligned}U_{\mathrm{PMNS}} = &\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & +c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ +s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \\ &\times \mathrm{diag}(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2}).\end{aligned} \tag{1.25}$$

1.3 Higgs potential

Following [1611.03827],

$$V = -\mu^2(\phi^\dagger\phi) + \lambda(\phi^\dagger\phi)^2, \tag{1.26}$$

which gives $m_h^2 = 2\mu^2 = 2\lambda v^2$ with $\langle\phi\rangle = v/\sqrt{2}$.

We calculate the threshold correction to μ^2 and λ by matching the effective potential. The difference of the EFT, in which the heavy neutrinos are integrated out, and the full theory is given by **[TODO: Here M_a is physical mass]**

$$\Delta V(\phi; Q) = V_{\mathrm{full}}(\phi; Q) - V_{\mathrm{EFT}}(\phi; Q) \tag{1.27}$$

$$= \sum_{a=1,2} \frac{-2}{64\pi^2} M_a(\phi)^4 \left(\frac{M_a(\phi)^2}{Q^2} - \frac{3}{2}\right), \tag{1.28}$$

where Q is the matching scale and M_a is the physical masses of the heavier neutrinos. Expanding ΔV in terms of ϕ,

$$\Delta V = (\mathrm{const.}) - \Delta\mu^2|\phi|^2 + \Delta\lambda|\phi|^4 + \mathcal{O}\left(|\phi|^6\right), \tag{1.29}$$

where **[TODO: Now M_a is the diagonal majorana masses...]**

$$\Delta\mu^2 = -\sum_{a=1,2} \frac{H_a}{8\pi^2} M_a^2 \left(1 - \log\frac{M_a^2}{Q^2}\right), \tag{1.30}$$

$$\Delta\lambda = -\frac{1}{16\pi^2}\Big[f_1 \,\mathrm{Tr}(YY^\dagger Y^* Y^{\mathrm{T}}) + f_2\,\mathrm{Tr}(YY^\dagger YY^\dagger) + f_3 H_1^2 + f_4 H_2^2\Big]; \tag{1.31}$$

[1] If we took the basis in which $M_{ab} \simeq \mathrm{diag}(-M_1, -M_2)$, then $U_3 = \mathrm{diag}(-i, -i)$ and we can remove i, but it is less plausible and we put i in the parameterization.

the coefficients are $H_1 = (YY^\dagger)_{11}$, $H_2 = (YY^\dagger)_{22}$, and

$$f_1 = \frac{2M_1M_2}{M_2^2 - M_1^2}\log\frac{M_2}{M_1}, \qquad f_2 = \frac{M_2^2\log(M_2^2/Q^2) - M_1^2\log(M_1^2/Q^2)}{M_2^2 - M_1^2} - 1,$$
$$f_3 = 2 - \frac{2M_2\log(M_2/M_1)}{M_2 - M_1}, \quad f_4 = 2 - \frac{2M_1\log(M_2/M_1)}{M_2 - M_1};$$

for $M_2 \simeq M_1$, they approach to $f_1 = 1$, $f_2 = \log(M_1^2/Q^2)$, and $f_3 = f_4 = 0$, which gives

$$\Delta\mu^2 \simeq -\frac{M_1^2}{8\pi^2}\,\mathrm{Tr}(YY^\dagger)\left(1 - \log\frac{M_1^2}{Q^2}\right), \tag{1.32}$$

$$\Delta\lambda \simeq -\frac{1}{16\pi^2}\Big[\mathrm{Tr}(YY^\dagger Y^* Y^{\mathrm{T}}) + \mathrm{Tr}(YY^\dagger YY^\dagger)\log\frac{M_1^2}{Q^2}\Big]. \tag{1.33}$$