Analysis of NuFIT Best-fit points

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1. Notation

Masses We introduce the following notations on the two heavier neutrino masses[*1] M_I and the three lighter neutrino masses m_i, one of which is zero, as

$$\begin{aligned} \Delta M &:= M_2 - M_1 > 0 & \delta M &:= \Delta M / M_1 & & \\ \Delta M^2 &:= M_2^2 - M_1^2 & \delta M^2 &:= \Delta M^2 / M_1^2 & & \\ \Delta m &:= m_{\text{heavier}} - m_{\text{lighter}} & \rho_m &:= \Delta m / m_{\text{tot}}, & m_{\text{tot}} &:= \sum_i m_i, \end{aligned}$$

$$m_{\text{heavier}} = m_3 \ (m_2), \qquad m_{\text{lighter}} = m_2 \ (m_1) \qquad \text{for NH (IH)}.$$

Numerically, for BFP of NH (IH), $\rho_m \simeq \sqrt{0.5}$ (0.0075) and $m_{\text{tot}} \simeq 5.9$ (9.9) $\times 10^{-11}$ GeV.

Yukawa Yukawa matrix follows the notation of 1611 paper, and the CIP is given by

$$y = \mathrm{i}(v/\sqrt{2})^{-1} \sqrt{M_{\text{diag}}} R \sqrt{m_{\text{diag}}} U^\dagger \tag{1.1}$$

with $v = \sqrt{\langle \phi_0 \rangle} \approx 246\,\text{GeV}$,

$$R = \begin{pmatrix} 0 & +\mathrm{c}_z & \zeta \mathrm{s}_z \\ 0 & -\mathrm{s}_z & \zeta \mathrm{c}_z \end{pmatrix} \quad \text{for NH,} \qquad R = \begin{pmatrix} +\mathrm{c}_z & \zeta \mathrm{s}_z & 0 \\ -\mathrm{s}_z & \zeta \mathrm{c}_z & 0 \end{pmatrix} \quad \text{for IH.} \tag{1.2}$$

Here, $z \in \mathbb{C}$ and $\zeta = \pm 1$:

$$z = w + \mathrm{i}x; \qquad w \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{1.3}$$

and $U = U_{\text{PMNS}}$ follows 1611 paper, i.e., PDG with a phase matrix $\mathrm{diag}(1, \mathrm{e}^{\mathrm{i}\sigma}, 1)$.

Yukawa products We define

$$\begin{aligned} W_1 &= W_{11} = \cosh 2x - \rho_m \cos 2w, & W_{12} &= W_{21}^* = \mathrm{i} \sinh 2x + \rho_m \sin 2w, \\ W_2 &= W_{22} = \cosh 2x + \rho_m \cos 2w, & \mu_I &= \frac{m_{\text{tot}} M_I}{8\pi v^2} \approx 3.9 \ (6.5) \times 10^{-17} M_I/\text{GeV}, \end{aligned}$$

for NH (IH), which leads us to

$$(yy^\dagger)_{IJ} = \frac{m_{\text{tot}} \sqrt{M_I M_J}}{v^2} W_{IJ}, \qquad \Gamma_I = \frac{(yy^\dagger)_{II}}{8\pi} M_I = \mu_I W_I M_I. \tag{1.4}$$

[*1] We can safely neglect the differences between M_I and the (diagonalized) Majorana masses.

Effective neutrino masses Remembering that m_i is given by

$$m_i = \left[U^{\mathrm{T}}\left(-\frac{v^2}{2}y^{\mathrm{T}}M^{-1}y\right)U\right]_{ii} = U_{i\alpha}\left(-\frac{v^2}{2}\sum_I \frac{y_{I\alpha}y_{I\beta}}{M_I}\right)U_{\beta i}, \tag{1.5}$$

we define

$$\widetilde{m_I} := \frac{v^2}{2}\sum_\alpha \frac{|y_{I\alpha}|^2}{M_I} = \frac{v^2}{2}\frac{(yy^\dagger)_{II}}{M_I}, \qquad m_* := 1.66\sqrt{g_*}\frac{8\pi(v^2/2)}{M_{\mathrm{pl}}} \sim 1\times 10^{-12}\,\mathrm{GeV}. \tag{1.6}$$

Note that these effective neutrino masses are related to

$$\widetilde{m_I} = \frac{m_{\mathrm{tot}}}{2}W_I. \tag{1.7}$$

2. Neutrino-option condition

The neutrino-option condition is given by, matching at $Q_0 = M_1\,\mathrm{e}^{-3/4}$,

$$\mu^2_{\mathrm{EFT}}(Q_0) = \frac{M_1^2}{16\pi^2}\,\mathrm{Tr}(yy^\dagger). \tag{2.1}$$

Defining

$$f(M_1) := \frac{8\pi^2 v^2\cdot\mu^2_{\mathrm{EFT}}(Q_0)}{M_1^3}\bigg|_{Q=M_1\exp(-3/4)}, \tag{2.2}$$

we can rewrite the neutrino-option condition as

$$f(M_1) = \frac{v^2}{2}\frac{\mathrm{Tr}(yy^\dagger)}{M_1}. \tag{2.3}$$

The right-hand side is parameterized as[*2]

$$\frac{v^2}{2}\frac{\mathrm{Tr}(yy^\dagger)}{M_1} = m_{\mathrm{tot}}\left[\cosh 2x + \frac{\delta M}{2}(\cosh 2x + \rho_m\cos 2w)\right] > m_{\mathrm{tot}}. \tag{2.4}$$

Therefore, the neutrino-option condition gives a constraint

$$f(M_1) > m_{\mathrm{tot}}. \tag{2.5}$$

This is translated to an upper bound on M_1, which is

$$M_1 < 9.4\ (7.9)\times 10^6\,\mathrm{GeV}\quad \text{for NH (IH)} \tag{2.6}$$

Meanwhile, for M_1 below this upper bound, we can always find a solution to the neutrino-option condition. With small δM,

$$\cosh 2x \simeq \frac{f(M_1)}{m_{\mathrm{tot}}}. \tag{2.7}$$

For example, for $M_1 = 4\ (1)\ \times 10^6\,\mathrm{GeV}$, the condition is satisfied with $\cosh 2x \sim 10\ (1000)$.

[*2] Notice that $\cosh 2x = (\widetilde{m_1}+\widetilde{m_2})/m_{\mathrm{tot}}$, where $\widetilde{m_I}$ is an effective neutrino parameter.

3. Leptogenesis

The resulting lepton asymmetry is approximately given by

$$\delta\eta_l \simeq \sum_\alpha \frac{\sum_I \epsilon_{I\alpha}}{D_\alpha}, \tag{3.1}$$

where

$$\epsilon_{I\alpha} = F_{I\alpha} f_{IJ}^{\text{vertex}} + \left(F_{I\alpha} + \frac{M_I}{M_J} F'_{I\alpha}\right)(f_{IJ}^{\text{osc}} + f_{IJ}^{\text{mix}}), \qquad D_\alpha := K_\alpha^{\text{eff}} \min(z_c, z_\alpha). \tag{3.2}$$

For the numerator, we introduced

$$F_{I\alpha} := \left.\frac{\operatorname{Im}\left[y_{I\alpha} y_{J\alpha}^* (yy^\dagger)_{IJ}\right]}{(yy^\dagger)_{II}(yy^\dagger)_{JJ}}\right|_{J=3-I}, \qquad F'_{I\alpha} := \left.\frac{\operatorname{Im}\left[y_{I\alpha} y_{J\alpha}^* (yy^\dagger)_{JI}\right]}{(yy^\dagger)_{II}(yy^\dagger)_{JJ}}\right|_{J=3-I},$$

$$f_{IJ}^{\text{vertex}} := \frac{\Gamma_J}{M_I}\left[1 - \left(1 + \frac{M_J^2}{M_I^2}\right)\ln\left(1 + \frac{M_I^2}{M_J^2}\right)\right], \quad f_{IJ}^{\text{mix}} := \frac{R_{IJ}}{1 + R_{IJ}^2}, \quad f_{IJ}^{\text{osc}} := \frac{R_{IJ}}{1 + \rho_{\text{osc}} R_{IJ}^2},$$

$$R_{IJ} := \frac{M_I \Gamma_J}{M_I^2 - M_J^2}, \qquad \rho_{\text{osc}} = \left(\frac{M_J}{M_I} + \frac{\Gamma_I}{\Gamma_J}\right)^2 \frac{\det\left[\operatorname{Re}(yy^\dagger)\right]}{(yy^\dagger)_{11}(yy^\dagger)_{22}}.$$

while, for the denominator,

$$z_\alpha = 1.25 \log 25 K_\alpha^{\text{eff}}, \qquad z_c = \frac{M_1}{149\,\text{GeV}}, \qquad K_\alpha^{\text{eff}} = \kappa_\alpha \sum_I \frac{\Gamma_I}{H_N} \frac{|y_{I\alpha}|^2}{(yy^\dagger)_{II}}.$$

3.1. Numerator

In this subsection, we assume

- the contribution from vertex corrections is negligible,
- $\mu_1, \mu_2 \ll \delta M \ll 1$,

which we should discuss elsewhere (but SI checks the validity). Due to the second assumption, we expand the expressions in terms of μ_I, not of δM.

The numerator is simplified as

$$\sum_I \epsilon_{I\alpha} \simeq \sum_I \epsilon_{I\alpha}^{\cancel{\text{vertex}}} = \left(F_{I\alpha} + \frac{M_I}{M_J} F'_{I\alpha}\right)(f_{IJ}^{\text{osc}} + f_{IJ}^{\text{mix}}), \tag{3.3}$$

and, defining

$$f_{IJ} := f_{IJ}^{\text{mix}} + f_{IJ}^{\text{osc}}, \qquad F_\alpha^\pm := (F_{2\alpha} + F'_{2\alpha}) \pm (F_{1\alpha} + F'_{1\alpha}), \tag{3.4}$$

we evaluate

$$\sum_I \epsilon_{I\alpha}^{\cancel{\text{vertex}}} = \left(F_{I\alpha} + \frac{M_I}{M_J}F'_{I\alpha}\right)(f_{IJ}^{\text{osc}} + f_{IJ}^{\text{mix}}). \tag{3.5}$$

$$= \frac{f_{21}-f_{12}}{2}F_\alpha^- + \left(\frac{M_1-M_2}{M_2}F'_{1\alpha}f_{12} + \frac{M_2-M_1}{M_1}F'_{2\alpha}f_{21}\right), \tag{3.6}$$

where we used $F_\alpha^+ = 0$. Expanding in terms of μ_I (but not in δM),

$$\begin{aligned}\sum_I \epsilon_{I\alpha}^{\cancel{\text{vertex}}} = &\left[\frac{M_1M_2}{M_2^2-M_1^2}(W_1\mu_1 + W_2\mu_2) + \mathcal{O}(\mu_I^3)\right]F_\alpha^- \\ &+ \left(\frac{2M_1\mu_2W_2F'_{1\alpha} + 2M_2\mu_1W_1F'_{2\alpha}}{M_1+M_2} + \mathcal{O}(\mu_I^3)\right).\end{aligned} \tag{3.7}$$

The remaining parts are evaluated as

$$F_\alpha^- = \frac{4\,\mathrm{Re}(yy^\dagger)_{12}\,\mathrm{Im}(y_{1\alpha}^*y_{2\alpha})}{(yy^\dagger)_{11}(yy^\dagger)_{22}} \tag{3.8}$$

$$= \frac{4\,\mathrm{Re}\,W_{12}}{W_1W_2}\,\mathrm{Im}\left[\frac{2}{m_{\text{tot}}}\sum_{ij}R_{2i}\sqrt{m_i}(U^\dagger)_{i\alpha}U_{\alpha j}\sqrt{m_j}(R^\dagger)_{j1}\right] \tag{3.9}$$

$$= \frac{4\,\mathrm{Re}\,W_{12}}{W_1W_2}\left(G_\alpha^{(2)}\zeta\cosh 2x - G_\alpha^{(1)}\sinh 2x\right), \tag{3.10}$$

where

$$G_\alpha^{(1)} = \sum_i \frac{m_i}{m_{\text{tot}}}|U_{\alpha i}|^2, \qquad G_\alpha^{(2)} = \begin{cases}\left(2\sqrt{m_2m_3}/m_{\text{tot}}\right)\mathrm{Im}\,(U_{\alpha2}U_{\alpha3}^*) & \text{(NH)}\\ \left(2\sqrt{m_1m_2}/m_{\text{tot}}\right)\mathrm{Im}\,(U_{\alpha1}U_{\alpha2}^*) & \text{(IH)}\end{cases} \tag{3.11}$$

Note that $G_\alpha^{(1)} \ge |G_\alpha^{(2)}| \ge 0$. Also, for the sub-leading term,

$$F'_{I\alpha} = \frac{2}{m_{\text{tot}}}\sum_{i,j}\frac{\mathrm{Im}\left[W_{JI}\cdot R_{Ii}\sqrt{m_i}(U^\dagger)_{i\alpha}U_{\alpha j}\sqrt{m_j}(R^\dagger)_{jJ}W_{JI}\right]}{W_1W_2}, \tag{3.12}$$

which however is not used hereafter.

3.2. Denominator

Now we are to evaluate

$$K_\alpha^{\text{eff}} = \kappa_\alpha\sum_I\frac{\Gamma_I}{H_N}\frac{|y_{I\alpha}|^2}{(yy^\dagger)_{II}} = \frac{\kappa_\alpha}{8\pi H_N}\sum_I\frac{2M_I^2}{v^2}\sum_{i,j}\left[R_{Ii}\sqrt{m_i}(U^\dagger)_{i\alpha}U_{\alpha j}\sqrt{m_j}(R^\dagger)_{jI}\right], \tag{3.13}$$

where we can assume $M_1 \simeq M_2$ since μ_I does not appear in this expression. Hence,

$$K_\alpha^{\text{eff}} \approx \frac{\kappa_\alpha}{8\pi H_N}\frac{2M_1^2}{v^2}\sum_{I,i,j}\left[R_{Ii}\sqrt{m_i}(U^\dagger)_{i\alpha}U_{\alpha j}\sqrt{m_j}(R^\dagger)_{jI}\right] \tag{3.14}$$

$$= \frac{\kappa_\alpha}{8\pi H_N}\frac{2M_1^2}{v^2}\sum_{i,j}\left[U_{\alpha j}\sqrt{m_j}(R^\dagger R)_{ji}\sqrt{m_i}(U^\dagger)_{i\alpha}\right] \tag{3.15}$$

$$= \frac{\kappa_\alpha m_{\text{tot}}}{m_*}\left(G_\alpha^{(1)}\cosh 2x - G_\alpha^{(2)}\zeta\sinh 2x\right). \tag{3.16}$$

♣Here some $\zeta(3)$ is missing; readers should amend it.

3.3. Result

We now have a simple analytic expression for $\delta\eta_l$ under the assumptions

- vertex contribution is negligible,
- $\mu_1, \mu_2 \ll \delta M \ll 1$,
- the term in the second line of Eq. 3.7 is negligible:

with $z_* = \min(z_c, z_\alpha)$, ♣$\zeta(3)$ **should be included**

$$\delta\eta_l \simeq -\sum_\alpha \frac{1}{\kappa_\alpha z_*} \frac{M_1 M_2}{M_2^2 - M_1^2} \frac{m_*}{m_{\mathrm{tot}}} \frac{4(W_1\mu_1 + W_2\mu_2)\,\mathrm{Re}\,W_{12}}{W_1 W_2} \frac{G_\alpha^{(1)} \tanh 2x - \zeta G_\alpha^{(2)}}{G_\alpha^{(1)} - \zeta G_\alpha^{(2)} \tanh 2x} \tag{3.17}$$

$$= -\sum_\alpha \frac{1}{\kappa_\alpha z_*} \frac{M_2}{M_2 - M_1} \frac{m_* M_1}{8\pi v^2} \frac{4\rho_m \cosh 2x \sin 2w}{\cosh^2 2x - \rho_m^2 \cos^2 2w} \left(1 + \frac{\rho_m \cos 2w}{2\cosh 2x} \delta' M\right) G_\alpha, \tag{3.18}$$

where $\delta' M := 2\Delta M/(M_1 + M_2) \simeq \delta M$. Here, the PMNS-matrix dependence is contained in

$$G_\alpha := \frac{G_\alpha^{(1)} \tanh 2x - \zeta G_\alpha^{(2)}}{G_\alpha^{(1)} - \zeta G_\alpha^{(2)} \tanh 2x}. \tag{3.19}$$

4. (a few) Discussion

4.1. Hierarchy

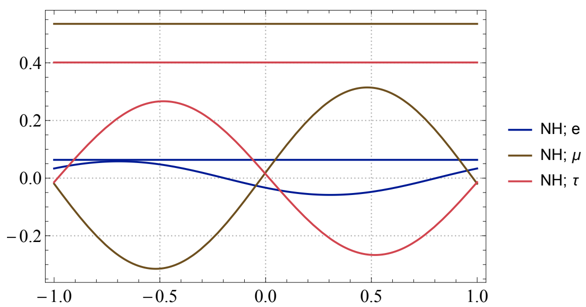
Let us evaluate the factors $G_\alpha^{(a)}$ at the NuFIT 4.0 best-fit points. For normal hierarchy,

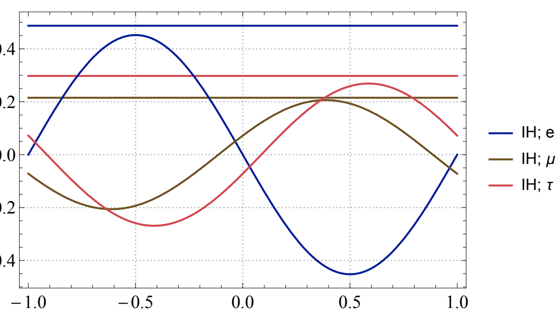
$$G_\alpha^{(1)} = \begin{pmatrix} 0.0634 \\ 0.535 \\ 0.401 \end{pmatrix}, \quad G_\alpha^{(2)} = \begin{pmatrix} -0.0334 & -0.0477 \\ +0.0194 & +0.314 \\ +0.0140 & -0.266 \end{pmatrix} \begin{pmatrix} \cos\sigma \\ \sin\sigma \end{pmatrix}, \quad \begin{pmatrix} \rho_m \\ m_{\mathrm{tot}} \end{pmatrix} = \begin{pmatrix} 0.708 \\ 5.89 \times 10^{-2}\,\mathrm{eV} \end{pmatrix}$$

and for inverted hierarchy,

$$G_\alpha^{(1)} = \begin{pmatrix} 0.487 \\ 0.215 \\ 0.298 \end{pmatrix}, \quad G_\alpha^{(2)} = \begin{pmatrix} 0 & -0.452 \\ +0.0720 & +0.193 \\ -0.0720 & +0.259 \end{pmatrix} \begin{pmatrix} \cos\sigma \\ \sin\sigma \end{pmatrix}, \quad \begin{pmatrix} \rho_m \\ m_{\mathrm{tot}} \end{pmatrix} = \begin{pmatrix} 0.00746 \\ 9.95 \times 10^{-2}\,\mathrm{eV} \end{pmatrix}.$$

Note that the leptogenesis works better in normal hierarchy due to the larger ρ_m,





4.2. Strict lower bound

Let us derive an analytic upper bound $\overline{\delta\eta_l}$, where the absolute value of analytic expression (3.18) is always smaller than it. Since $\sin 2w/(\cosh^2 2x - \rho_m^2 \cos^2 2w)$ is maximal for $w = \pi/4$,

$$\overline{\delta\eta_l} = \frac{1}{z_*}\frac{M_2}{M_2 - M_1}\frac{m_* M_1}{8\pi v^2}\frac{4\rho_m}{\cosh 2x}\left(1 + \frac{\rho_m \cos 2w}{2\cosh 2x}\delta' M\right)\sum_\alpha \frac{G_\alpha}{\kappa_\alpha}, \tag{4.1}$$

$$= \frac{1}{z_*}\frac{M_2}{M_2 - M_1}\frac{2.81 M_1 \rho_m}{10^{18}\,\mathrm{GeV}\cdot\cosh 2x}\left(1 + \frac{\rho_m \cos 2w}{2\cosh 2x}\delta' M\right)\sum_\alpha \frac{G_\alpha}{\kappa_\alpha}. \tag{4.2}$$

Thus, assuming $M_1 \gg 10^{10}\,\mathrm{GeV}$, we require $\delta M \ll 1$:

$$\overline{\delta\eta_l} \simeq \frac{1}{z_*\delta M}\frac{2.81 M_1 \rho_m}{10^{18}\,\mathrm{GeV}}\sum_\alpha \frac{G_\alpha}{\kappa_\alpha \cosh 2x}. \tag{4.3}$$

We can numerically maximize $\sum_\alpha G_\alpha/\cosh 2x$ for x; around the best-fit point, its maximum value is ~ 1.5 for both hierarchies. Therefore, to get $\delta\eta_l \sim 3\times 10^{-8}$, we need

$$\frac{M_1}{\delta M} \sim 2\times 10^{10}\,\mathrm{GeV} \quad (2\times 10^{12}\,\mathrm{GeV}) \tag{4.4}$$

for NH (IH) if G_α has no special cancellation.

4.3. With neutrino option

In Section 2, we saw that the neutrino-option condition is satisfied even for smaller M_1 with huge $\cosh 2x$:

$$\cosh 2x \simeq \frac{f(M_1)}{m_{\mathrm{tot}}} = \frac{8\pi^2 v^2 \mu^2_{\mathrm{EFT}}(Q_0)}{M_1^3 m_{\mathrm{tot}}} \approx \frac{4.8\times 10^{10}\,\mathrm{GeV}^4}{M_1^3 m_{\mathrm{tot}}} \tag{4.5}$$

This equality is precise for smaller δM, which we need to have an adequate $\delta\eta_l$:

$$\overline{\delta\eta_l} \simeq \frac{3\times 10^{-8}}{z_*}\left(\frac{M_1/(\delta M)^{1/4}}{5.9\times 10^7\,\mathrm{GeV} \quad (1.6\times 10^8\,\mathrm{GeV})}\right)^4 \sum_\alpha \frac{G_\alpha}{\kappa_\alpha}\bigg|_{\cosh 2x \text{ for n.o.}}. \tag{4.6}$$

Or, using the upper-bound value of M_1,

$$\overline{\delta\eta_l} \simeq \frac{3\times 10^{-8}}{z_*}\left(\frac{M_1}{9.4\times 10^6\,\mathrm{GeV}}\right)^4 \frac{6.3\times 10^{-4}}{\delta M}\sum_\alpha \frac{G_\alpha}{\kappa_\alpha}\bigg|_{\cosh 2x \text{ for n.o.}}. \tag{4.7}$$

for NH and

$$\overline{\delta\eta_l} \simeq \frac{3\times 10^{-8}}{z_*}\left(\frac{M_1}{7.9\times 10^6\,\mathrm{GeV}}\right)^4 \frac{5.6\times 10^{-6}}{\delta M}\sum_\alpha \frac{G_\alpha}{\kappa_\alpha}\bigg|_{\cosh 2x \text{ for n.o.}}. \tag{4.8}$$

for IH; therefore, for IH, we need a special cancellation in δM or G_α.[*3]

[*3] One may also notice that $G_\alpha \to 1$ for $\cosh 2x \gg 1$.

A. Leptogenesis with smaller mass splitting

In Section 3.1 we assume $\mu_I \ll \delta M \ll 1$ and expand the formulae in terms of m_{tot}, or take a limit of $R_{IJ} \ll 1$:

$$R_{IJ} \simeq \frac{\mu_J W_J}{2\delta M} \ll 1. \tag{A.1}$$

For $\delta M \ll \mu_I$, we instead should expand the formulae with $R_{IJ} \gg 1$. Then,

$$\sum_I \epsilon_{I\alpha}^{\cancel{\text{vertex}}} \approx \frac{f_{21} - f_{12}}{2} F_\alpha^-, \tag{A.2}$$

where

$$\frac{f_{21} - f_{12}}{2} \approx \frac{\delta M}{\mu_1}\left(\frac{W_1 W_2}{2\cosh 2x(\cosh^2 2x - \rho_m^2)} + \frac{2\cosh 2x}{W_1 W_2}\right). \tag{A.3}$$

Denominator is calculated as before, and the asymmetry is given by ♣$\zeta(3)$ **should be included**

$$\delta\eta_l \approx -\sum_\alpha \frac{8\pi v^2 m_* \delta M}{z_* \kappa_\alpha m_{\text{tot}}^2 M_1}\left(\frac{W_1 W_2}{2\cosh 2x(\cosh^2 2x - \rho_m^2)} + \frac{2\cosh 2x}{W_1 W_2}\right)\frac{4\,\mathrm{Re}\,W_{12}}{W_1 W_2} G_\alpha \tag{A.4}$$

$$= -\sum_\alpha \frac{8\pi v^2 m_* \delta M}{z_* \kappa_\alpha m_{\text{tot}}^2 M_1}\left(\frac{W_1 W_2}{2\cosh 2x(\cosh^2 2x - \rho_m^2)} + \frac{2\cosh 2x}{W_1 W_2}\right)\frac{4\sin 2w}{W_1 W_2} G_\alpha, \tag{A.5}$$

where $W_1 W_2 = \cosh^2 2x - \rho_m^2 \sin^2 2w$. The asymmetry is maximized with $w = \pi/4$;

$$\overline{\delta\eta_l} = \sum_\alpha \frac{16\pi v^2 m_* \delta M}{z_* \kappa_\alpha m_{\text{tot}}^2 M_1} \frac{5\cosh^2 2x - \rho_m^2}{\left(\cosh^2 2x - \rho_m^2\right)^2 \cosh 2x} G_\alpha. \tag{A.6}$$

B. Appendix for kappa

We evaluate the expressions in our draft,

$$\kappa_\alpha = 2\sum_{I,J} \frac{\mathrm{Re}\big[y_{I\alpha}y^*_{J\alpha}\big]\mathrm{Re}\left[\left(yy^\dagger\right)_{IJ}\right] - \mathrm{Im}\big[y_{I\alpha}y^*_{J\alpha}\big]\mathrm{Im}\big[y_{I\alpha}y^*_{J\alpha}\big]}{(y^\dagger y)_{\alpha\alpha}\left[(yy^\dagger)_{II} + (yy^\dagger)_{JJ}\right]} \left(1 - 2\mathrm{i}\,\frac{M_I - M_J}{\Gamma_I + \Gamma_J}\right)^{-1}. \tag{B.1}$$

We get

$$\begin{aligned}\kappa_\alpha &= 2\sum_I \frac{|y_{I\alpha}|^2\,\mathrm{Re}(yy^\dagger)_{II}}{(y^\dagger y)_{\alpha\alpha}\cdot 2(yy^\dagger)_{II}} \\ &\quad + 2\frac{\mathrm{Re}[y_{1\alpha}y^*_{2\alpha}]\,\mathrm{Re}[(yy^\dagger)_{12}] - [\mathrm{Im}(y_{1\alpha}y^*_{2\alpha})]^2}{(y^\dagger y)_{\alpha\alpha}\left[(yy^\dagger)_{11} + (yy^\dagger)_{22}\right]}\left(1 - 2\mathrm{i}\frac{M_1 - M_2}{\Gamma_1 + \Gamma_2}\right)^{-1} \\ &\quad + 2\frac{\mathrm{Re}[y_{2\alpha}y^*_{1\alpha}]\,\mathrm{Re}[(yy^\dagger)_{21}] - [\mathrm{Im}(y_{2\alpha}y^*_{1\alpha})]^2}{(y^\dagger y)_{\alpha\alpha}\left[(yy^\dagger)_{11} + (yy^\dagger)_{22}\right]}\left(1 - 2\mathrm{i}\frac{M_2 - M_1}{\Gamma_1 + \Gamma_2}\right)^{-1} \end{aligned} \tag{B.2}$$

$$\begin{aligned} &= 1 + 2\frac{\mathrm{Re}[y_{1\alpha}y^*_{2\alpha}]\,\mathrm{Re}[(yy^\dagger)_{12}] - [\mathrm{Im}(y_{1\alpha}y^*_{2\alpha})]^2}{(y^\dagger y)_{\alpha\alpha}\left[(yy^\dagger)_{11} + (yy^\dagger)_{22}\right]} \\ &\qquad \times \left[\left(1 - 2\mathrm{i}\frac{M_1 - M_2}{\Gamma_1 + \Gamma_2}\right)^{-1} + \left(1 - 2\mathrm{i}\frac{M_2 - M_1}{\Gamma_1 + \Gamma_2}\right)^{-1}\right] \end{aligned} \tag{B.3}$$

$$= 1 + 2\frac{\mathrm{Re}[y_{1\alpha}y^*_{2\alpha}]\,\mathrm{Re}[(yy^\dagger)_{12}] - [\mathrm{Im}(y_{1\alpha}y^*_{2\alpha})]^2}{(y^\dagger y)_{\alpha\alpha}\left[(yy^\dagger)_{11} + (yy^\dagger)_{22}\right]}\frac{2(\Gamma_1 + \Gamma_2)^2}{(\Gamma_1 + \Gamma_2)^2 + 4(\Delta M)^2}. \tag{B.4}$$

Now we should explicitly evaluate the Yukawa products.

Let us define

$$g_{\alpha i} := \frac{m_i}{m_{\mathrm{tot}}}\left|U_{\alpha i}\right|^2, \qquad h_\alpha := \begin{cases}\left(2\sqrt{m_2 m_3}/m_{\mathrm{tot}}\right)\left(U_{\alpha 2}U^*_{\alpha 3}\right) & \text{(NH)} \\ \left(2\sqrt{m_1 m_2}/m_{\mathrm{tot}}\right)\left(U_{\alpha 1}U^*_{\alpha 2}\right) & \text{(IH)}\end{cases} \tag{B.5}$$

which leads

$$G^{(1)}_\alpha = \sum_i g_{\alpha i}, \qquad G^{(2)}_\alpha = \mathrm{Im}\,h_\alpha. \tag{B.6}$$

Then we evaluate the Yukawa product as

$$\begin{aligned} y_{I\alpha}y^*_{J\alpha} &= \frac{1}{v^2/2}\sqrt{M_I M_J}(R\sqrt{m_{\mathrm{diag}}}U^\dagger)_{I\alpha}(U\sqrt{m_{\mathrm{diag}}}R^\dagger)_{\alpha J} && \text{(B.7)} \\ &= \frac{1}{v^2/2}\sum_{i,j} U_{\alpha i}U^*_{\alpha j}(\sqrt{m_{\mathrm{diag}}}R^\dagger)_{iJ}\sqrt{M_I M_J}(R\sqrt{m_{\mathrm{diag}}})_{Ij} && \text{(B.8)} \\ &= \frac{m_{\mathrm{tot}}\sqrt{M_I M_J}}{v^2/2}\left[\sum_i R_{Ii}g_{\alpha i}(R^\dagger)_{iJ} + \frac{1}{2}\left(h_\alpha R^*_{J2}R_{I3} + h^*_\alpha R^*_{J3}R_{I2}\right)\right]. && \text{(B.9)} \end{aligned}$$

For inverted hierarchy, we just replace the indices as usual. Then we get

$$\mathrm{Re}\,(y_{1\alpha}y^*_{2\alpha}) = \frac{m_{\mathrm{tot}}\sqrt{M_1 M_2}}{v^2}\Big[(g_{\alpha 3} - g_{\alpha 2})\sin 2w + (\mathrm{Re}\,h_\alpha)\zeta\cos 2w\Big], \tag{B.10}$$

$$\mathrm{Im}\,(y_{1\alpha}y^*_{2\alpha}) = \frac{m_{\mathrm{tot}}\sqrt{M_1 M_2}}{v^2}\Big[(g_{\alpha 3} + g_{\alpha 2})\sinh 2x - (\mathrm{Im}\,h_\alpha)\zeta\cosh 2x\Big], \tag{B.11}$$

which leads

$$\frac{\mathrm{Re}[y_{1\alpha}y_{2\alpha}^*]\,\mathrm{Re}[(yy^\dagger)_{12}] - [\mathrm{Im}(y_{1\alpha}y_{2\alpha}^*)]^2}{(y^\dagger y)_{\alpha\alpha}\,[(yy^\dagger)_{11} + (yy^\dagger)_{22}]} \tag{B.12}$$

$$= \frac{[(g_{\alpha 3} - g_{\alpha 2})\sin 2w + (\mathrm{Re}\,h_\alpha)\zeta\cos 2w]\,\mathrm{Re}\,W_{12} - \left(G_\alpha^{(1)}\sinh 2x - \zeta G_\alpha^{(2)}\cosh 2x\right)^2}{\frac{v^2}{m_{\mathrm{tot}}M_1M_2}(y^\dagger y)_{\alpha\alpha}(M_1W_1 + M_2W_2)}, \tag{B.13}$$

and, for $M_1 \simeq M_2$,

$$(y^\dagger y)_{\alpha\alpha} \simeq \frac{2M_1 m_{\mathrm{tot}}}{v^2}\left(G_\alpha^{(1)}\cosh 2x - \zeta G_\alpha^{(2)}\sinh 2x\right). \tag{B.14}$$