1. set up

1.1. Lagrangian

Following the notation [1611.03827] by Bhupal et al. and without specifying the basis,

$$-\mathcal{L} \supset y_{ak}\overline{N}_{\mathrm{R}a}\tilde{\phi}^\dagger L_k + \frac{1}{2}\overline{N}_{\mathrm{R}a}M_{ab}N^{\mathrm{c}}_{\mathrm{R}b} + \text{h.c.} \tag{1.1}$$

$\phi = (\phi^+, \phi^0)^{\mathrm{T}}$
$\tilde{\phi} = \mathrm{i}\sigma_2\phi^* = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \phi^- \\ \phi^{0*} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \phi^{0*} \\ -\phi^- \end{smallmatrix}\right)$
$\tilde{\phi}^\dagger = (\mathrm{i}\sigma_2\phi^*)^\dagger = \begin{pmatrix}\phi_0 & -\phi^+\end{pmatrix}$

$$-\mathcal{L} \supset y_{ak}\overline{N}_{\mathrm{R}a}(\phi^0\nu_k - \phi^+ l_k) + \frac{1}{2}\overline{N}_{\mathrm{R}a}M_{ab}N^{\mathrm{c}}_{\mathrm{R}b} + \text{h.c.} \tag{1.2}$$

Neutrino mass matrix is better shown in two-component $N_{\mathrm{R}} = \left(\begin{smallmatrix} 0 \\ n^\dagger \end{smallmatrix}\right)$:

$$-\mathcal{L} \supset \langle\phi_0\rangle(y)_{ai}n_a\nu_i + \frac{1}{2}M_{ab}n_an_b + \text{h.c.} \tag{1.3}$$

$$= \frac{1}{2}\begin{pmatrix}\nu_i & n_a\end{pmatrix}\begin{pmatrix} 0_{ij} & \langle\phi_0\rangle(y)_{bi} \\ \langle\phi_0\rangle(y)_{aj} & M_{ab}\end{pmatrix}\begin{pmatrix}\nu_j \\ n_b\end{pmatrix} + \text{h.c.} \tag{1.4}$$

or we will write down, assuming the notation is understood,

$$M_\nu = \begin{pmatrix} 0 & \langle\phi^0\rangle y^{\mathrm{T}} \\ \langle\phi^0\rangle y & M \end{pmatrix} \tag{1.5}$$

and perform Autonne–Takagi diagonalization:

$$U_0^{\mathrm{T}}M_\nu U_0 = \operatorname{diag}(m_1, m_2, m_3, M_1, M_2) \tag{1.6}$$

1.2. Casas–Ibarra parameterization

We split this Autonne–Takagi diagonalization procedure to two steps:

$$U_1^{\mathrm{T}}M_\nu U_1 = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right), \tag{1.7}$$

$$\begin{pmatrix} U_2 & 0 \\ 0 & U_3 \end{pmatrix}^{\mathrm{T}} U_1^{\mathrm{T}}M_\nu U_1\begin{pmatrix} U_2 & 0 \\ 0 & U_3 \end{pmatrix} = \begin{pmatrix} m_{\mathrm{diag}} & 0 \\ 0 & M_{\mathrm{diag}} \end{pmatrix}, \tag{1.8}$$

where $m_{\mathrm{diag}} = \operatorname{diag}(m_1, m_2, m_3)$ and $M_{\mathrm{diag}} = \operatorname{diag}(M_1, M_2)$. The result of the first step is well-known in series-expanded form:

$$U_1 \simeq \begin{pmatrix} 1 - \frac{\langle\phi_0\rangle^2}{2}y^\dagger(MM^*)^{-1}y & \langle\phi_0\rangle y^\dagger M^{*-1} \\ -\langle\phi_0\rangle M^{-1}y & 1 - \frac{\langle\phi_0\rangle^2}{2}M^{-1}yy^\dagger M^{*-1} \end{pmatrix}, \tag{1.9}$$

$$A \simeq -\langle\phi_0\rangle^2 y^{\mathrm{T}}M^{-1}y, \tag{1.10}$$

$$B \simeq M + \frac{\langle\phi_0\rangle^2}{2}\left(yy^\dagger M^{*-1} + M^{*-1}y^*y^{\mathrm{T}}\right) \tag{1.11}$$

The second step is expressed by

$$U_2^{\mathrm{T}} A U_2 = m_{\mathrm{diag}}, \qquad U_3^{\mathrm{T}} B U_3 = M_{\mathrm{diag}}. \tag{1.12}$$

We also have the expression of the mass eigenstates:

$$\text{lighter}: \quad U_2^{\dagger}\left[\nu - \langle\phi_0\rangle y^{\dagger} M^{*-1} n - \frac{\langle\phi_0\rangle^2}{2} y^{\dagger}(MM^*)^{-1} y\nu\right] \tag{1.13}$$

$$\text{heavier}: \quad U_3^{\dagger}\left[n + \langle\phi_0\rangle M^{-1} y\nu - \frac{\langle\phi_0\rangle^2}{2} M^{-1} y y^{\dagger} M^{*-1} n\right] \tag{1.14}$$

$$\tag{1.15}$$

Combining them,

$$\begin{aligned}
m_{\mathrm{diag}} &= U_2^{\mathrm{T}} A U_2 && (1.16)\\
&= -\langle\phi_0\rangle^2 U_2^{\mathrm{T}} y^{\mathrm{T}} M^{-1} y U_2 + \mathcal{O}(M\epsilon^4) && (1.17)\\
&= -\langle\phi_0\rangle^2 U_2^{\mathrm{T}} y^{\mathrm{T}} B^{-1} y U_2 + \mathcal{O}(M\epsilon^4) && (1.18)\\
&= -\langle\phi_0\rangle^2 U_2^{\mathrm{T}} y^{\mathrm{T}} U_3 M_{\mathrm{diag}}^{-1} U_3^{\mathrm{T}} y U_2 + \mathcal{O}(M\epsilon^4) && (1.19)\\
&=: R'^{\mathrm{T}} R' + \mathcal{O}(M\epsilon^4) && (1.20)
\end{aligned}$$

with $\mathcal{O}(M\epsilon^n) \sim \mathcal{O}(\langle\phi_0\rangle^n / M_1^{n-1})$ and $R' := -\mathrm{i}\langle\phi_0\rangle M_{\mathrm{diag}}^{-1/2} U_3^{\mathrm{T}} y U_2$. Now, the Yukawa coupling is given by $y = \mathrm{i}\langle\phi_0\rangle^{-1} U_3^* M_{\mathrm{diag}}^{1/2} R' U_2^{\dagger}$.

Here, for two heavy neutrino scenarios, R' can be parameterized as

$$R'_{\mathrm{NH}} = \begin{pmatrix} 0 & +\sqrt{m_2}\cos z & \zeta\sqrt{m_3}\sin z \\ 0 & -\sqrt{m_2}\sin z & \zeta\sqrt{m_3}\cos z \end{pmatrix} = \begin{pmatrix} 0 & +\cos z & \zeta\sin z \\ 0 & -\sin z & \zeta\cos z \end{pmatrix}\sqrt{m_{\mathrm{diag}}}, \tag{1.21}$$

$$R'_{\mathrm{IH}} = \begin{pmatrix} +\sqrt{m_1}\cos z & \zeta\sqrt{m_2}\sin z & 0 \\ -\sqrt{m_1}\sin z & \zeta\sqrt{m_2}\cos z & 0 \end{pmatrix} = \begin{pmatrix} +\cos z & \zeta\sin z & 0 \\ -\sin z & \zeta\cos z & 0 \end{pmatrix}\sqrt{m_{\mathrm{diag}}} \tag{1.22}$$

for normal hierarchy (NH; $m_1 = 0 < m_2 < m_3$) and inverted hierarchy (IH; $m_3 = 0 < m_1 < m_2$) cases. Therefore, the Yukawa couplings are given by

$$y = \mathrm{i}\langle\phi_0\rangle^{-1} U_3^* \sqrt{M_{\mathrm{diag}}} R \sqrt{m_{\mathrm{diag}}} U_2^{\dagger}, \tag{1.23}$$

which is the Casas–Ibarra parameterization in general basis; R is given by

$$R_{\mathrm{NH}} = \begin{pmatrix} 0 & +\cos z & \zeta\sin z \\ 0 & -\sin z & \zeta\cos z \end{pmatrix}, \qquad R_{\mathrm{IH}} = \begin{pmatrix} +\cos z & \zeta\sin z & 0 \\ -\sin z & \zeta\cos z & 0 \end{pmatrix}. \tag{1.24}$$

We have not yet defined the lepton basis. We can assume that we have been using, from the beginning, the charged lepton mass basis for L. Then, we identify the PMNS matrix (U_{li} in Eq. (14.1) of PDG2018, where i for mass and l for gauge indices) as

$$U_{\mathrm{PMNS}} \simeq U_2. \tag{1.25}$$

Similarly, the basis for N_{R} is such that $U_3 \simeq 1$, which corresponds to

$$M = B + \mathcal{O}(\epsilon^2) = U_3^* M_{\mathrm{diag}} U_3^\dagger + \mathcal{O}(\epsilon^2) = M_{\mathrm{diag}} + \mathcal{O}(\epsilon^2), \tag{1.26}$$

i.e., the basis in which $M_{ab} \simeq \mathrm{diag}(M_1, M_2)$ with $0 < M_1 \le M_2$. These basis choice gives the well-known Casas–Ibarra parameterization,[*1]

$$y = \mathrm{i}\langle\phi_0\rangle^{-1}\sqrt{M_{\mathrm{diag}}}R\sqrt{m_{\mathrm{diag}}}U_{\mathrm{PMNS}}^\dagger, \tag{1.27}$$

where

$$\begin{aligned} U_{\mathrm{PMNS}} = &\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & +c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ +s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \\ &\times \mathrm{diag}(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2}). \end{aligned} \tag{1.28}$$

1.3. Higgs potential

Here we calculate the threshold correction to the Higgs potential from the consistency of the one-loop effective potential V_{eff}. The theory with and without the right-handed neutrinos, which we call the "full" theory and the EFT, should have the same effective potential at some matching scale Q_0. Hence, comparing the effective potential, we can derive the corrections for the tree-level Higgs potential.

Following [1611.03827], we denote the tree-level Higgs potential by

$$V = -\mu^2|\phi|^2 + \lambda|\phi|^4, \tag{1.29}$$

which gives $m_h^2 = 2\mu^2 = 2\lambda v^2$ with $\langle\phi\rangle = v/\sqrt{2}$. The effective potential is given by, at the one-loop level,

$$V_{\mathrm{eff}}(Q) = V(Q) + V^{(1)}(Q), \tag{1.30}$$

and the difference between the two theories are

$$\Delta V := V^{(1)}_{\mathrm{full}}(\phi; Q) - V^{(1)}_{\mathrm{EFT}}(\phi; Q) = \sum_{a=1,2} \frac{-2}{64\pi^2} M_a(\phi)^4 \left(\log\frac{M_a(\phi)^2}{Q^2} - \frac{3}{2}\right). \tag{1.31}$$

Therefore, at the matching scale Q_0, the tree-level potential should satisfy

$$V_{\mathrm{full}}(Q_0) - V_{\mathrm{EFT}}(Q_0) = -\Delta V, \tag{1.32}$$

which gives the threshold corrections.

The difference ΔV is expanded in terms of ϕ as

$$\Delta V = (\mathrm{const.}) - \Delta\mu^2|\phi|^2 + \Delta\lambda|\phi|^4 + \mathcal{O}\left(|\phi|^6\right), \tag{1.33}$$

[*1] If we took the basis in which $M_{ab} \simeq \mathrm{diag}(-M_1, -M_2)$, then $U_3 = \mathrm{diag}(-i, -i)$ and we can remove i, but it is less plausible and we put i in the parameterization.

where **[TODO: Now M_a is the diagonal majorana masses...]**

$$\Delta\mu^2 = -\sum_{a=1,2}\frac{H_a}{8\pi^2}M_a^2\left(1-\log\frac{M_a^2}{Q^2}\right), \tag{1.34}$$

$$\Delta\lambda = -\frac{1}{16\pi^2}\Bigl[f_1\operatorname{Tr}(YY^\dagger Y^* Y^{\mathrm{T}}) + f_2\operatorname{Tr}(YY^\dagger YY^\dagger) + f_3 H_1^2 + f_4 H_2^2\Bigr]; \tag{1.35}$$

the coefficients are $H_1 = (YY^\dagger)_{11}$, $H_2 = (YY^\dagger)_{22}$, and

$$f_1 = \frac{2M_1M_2}{M_2^2-M_1^2}\log\frac{M_2}{M_1}, \qquad f_2 = \frac{M_2^2\log(M_2^2/Q^2) - M_1^2\log(M_1^2/Q^2)}{M_2^2-M_1^2} - 1,$$
$$f_3 = 2 - \frac{2M_2\log(M_2/M_1)}{M_2-M_1}, \qquad f_4 = 2 - \frac{2M_1\log(M_2/M_1)}{M_2-M_1};$$

for $M_2 \simeq M_1$, they approach to $f_1 = 1$, $f_2 = \log(M_1^2/Q^2)$, and $f_3 = f_4 = 0$, which gives

$$\Delta\mu^2 \simeq -\frac{M_1^2}{8\pi^2}\operatorname{Tr}(YY^\dagger)\left(1-\log\frac{M_1^2}{Q^2}\right), \tag{1.36}$$

$$\Delta\lambda \simeq -\frac{1}{16\pi^2}\Bigl[\operatorname{Tr}(YY^\dagger Y^* Y^{\mathrm{T}}) + \operatorname{Tr}(YY^\dagger YY^\dagger)\log\frac{M_1^2}{Q^2}\Bigr]. \tag{1.37}$$

The neutrino-option scenario requires $\mu^2_{\text{full}} = 0$, i.e.,

$$\mu^2_{\text{full}}(Q_0) = \mu^2_{\text{EFT}}(Q_0) - \Delta\mu^2(Q_0) = 0. \tag{1.38}$$

Therefore, the condition for the neutrino-option scenario is summarized as

$$\mu^2_{\text{EFT}}(Q_0) = -\frac{M_1^2}{8\pi^2}\operatorname{Tr}(YY^\dagger)\left(1-\log\frac{M_1^2}{Q_0^2}\right) \tag{1.39}$$

at the one-loop level, or using $Q_0 = M_1\,\mathrm{e}^{-3/4}$,

$$\mu^2_{\text{EFT}}(Q_0) = +\frac{M_1^2}{16\pi^2}\operatorname{Tr}(YY^\dagger). \tag{1.40}$$

A. Comparison

Here we compare our results with literature; we use magenta color for the variables/notations in other papers, while black letters are in our notation

$$V = -\mu^2(\phi^\dagger\phi) + \lambda(\phi^\dagger\phi)^2, \qquad m_h^2 = 2\mu^2 = 2\lambda v^2, \qquad v := \sqrt{2}\langle\phi\rangle,$$

or $v \sim 246\,\text{GeV}$, $\mu^2 \sim (88\,\text{GeV})^2 \sim 7700\,\text{GeV}^2$, and $\lambda \sim 0.13$.

Brivio–Trott **??** uses

$$V(H^\dagger H) = -\frac{m_0^2+\Delta m^2}{2}\left(H^\dagger H\right) + (\lambda_0+\Delta\lambda)\left(H^\dagger H\right)^2 + \cdots \tag{A.1}$$

and derives

$$\Delta m^2 = -\frac{|\omega_p|^2 M_p^2}{4\pi^2}\left(1+\log\frac{\mu^2}{M_p^2}\right) = \frac{1}{8\pi^2}\left[M_1^2|\omega_1|^2 + M_2^2|\omega_2|^2\right], \tag{A.2}$$

which is consistent with our results as $\Delta\mu^2 = \Delta m^2/2$.[*2]

Casas et al. ?? uses effective potential approach:

$$V = V_{\rm SM} + \Delta V_\nu, \qquad V_{\rm tree} = -\frac{1}{2}m^2\phi^2 + \frac{1}{8}\lambda\phi^2 + \Omega, \tag{A.3}$$

where $v = \langle\phi\rangle = 246\,\mathrm{GeV} = v = \sqrt{2}\langle\phi\rangle$, so (fixing a typo) $V_{\rm tree} = -m^2\phi^2 + (\lambda/2)\phi^4$. Then,

$$\Delta V_\nu = -\frac{1}{32\pi^2}\left[m_{\nu_1}^4\log\frac{m_{\nu_1}^2}{\mu^2} + m_{\nu_2}^4\log\frac{m_{\nu_2}^2}{\mu^2}\theta(\mu - M)\right] \tag{A.4}$$

and

$$\Delta_{\rm th}V = -\frac{2}{64\pi^2}m_{\nu_2}^4\log\frac{m_{\nu_2}^2}{\mu^2} = \Delta V_{\mu>M} - \Delta V_{\mu<M}, \tag{A.5}$$

where one should note that $\mu^2 = Q^2\,\mathrm{e}^{3/2}$. Their results, which is evaluated at $\mu = M$,

$$\Delta_{\rm th}m^2 = \frac{1}{16\pi^2}Y_\nu^2 M^2, \qquad \Delta_{\rm th}\lambda = -\frac{5}{16\pi^2}Y_\nu^4, \tag{A.6}$$

are consistent with our results; note that $\Delta_{\rm th}m^2 = \Delta\mu^2$ and $\Delta_{\rm th}\lambda/2 = \Delta\lambda$.

[*2] Their $\Delta\lambda$, which SI thinks incorrect, is totally inconsistent with our results.