1 set up

1.1 Lagrangian

Following the notation [1611.03827] by Bhupal et al. and without specifying the basis,

$$-\mathcal{L} \supset y_{ak}\overline{N}_{\mathrm{R}a}\tilde{\phi}^\dagger L_k + \frac{1}{2}\overline{N}_{\mathrm{R}a}M_{ab}N^{\mathrm{c}}_{\mathrm{R}b} + \text{h.c.} \tag{1.1}$$

$\phi = (\phi^+, \phi^0)^{\mathrm{T}}$
$\tilde{\phi} = \mathrm{i}\sigma_2\phi^* = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}\phi^-\\ \phi^{0*}\end{smallmatrix}\right) = \left(\begin{smallmatrix}\phi^{0*}\\ -\phi^-\end{smallmatrix}\right)$
$\tilde{\phi}^\dagger = (\mathrm{i}\sigma_2\phi^*)^\dagger = \begin{pmatrix}\phi_0 & -\phi^+\end{pmatrix}$

$$-\mathcal{L} \supset y_{ak}\overline{N}_{\mathrm{R}a}(\phi^0\nu_k - \phi^+ l_k) + \frac{1}{2}\overline{N}_{\mathrm{R}a}M_{ab}N^{\mathrm{c}}_{\mathrm{R}b} + \text{h.c.} \tag{1.2}$$

Neutrino mass matrix is better shown in two-component $N_{\mathrm{R}} = \left(\begin{smallmatrix}0\\ n^\dagger\end{smallmatrix}\right)$:

$$-\mathcal{L} \supset \langle\phi_0\rangle(y)_{ai}n_a\nu_i + \frac{1}{2}M_{ab}n_a n_b + \text{h.c.} \tag{1.3}$$

$$= \frac{1}{2}\begin{pmatrix}\nu_i & n_a\end{pmatrix}\begin{pmatrix}0_{ij} & \langle\phi_0\rangle(y)_{bi}\\ \langle\phi_0\rangle(y)_{aj} & M_{ab}\end{pmatrix}\begin{pmatrix}\nu_j\\ n_b\end{pmatrix} + \text{h.c.} \tag{1.4}$$

or we will write down, assuming the notation is understood,

$$M_\nu = \begin{pmatrix}0 & \langle\phi^0\rangle y^{\mathrm{T}}\\ \langle\phi^0\rangle y & M\end{pmatrix} \tag{1.5}$$

and perform Autonne–Takagi diagonalization:

$$U_0^{\mathrm{T}}M_\nu U_0 = \mathrm{diag}(m_1, m_2, m_3, M_1, M_2) \tag{1.6}$$

1.2 Casas–Ibarra parameterization

We split this Autonne–Takagi diagonalization procedure to two steps:

$$U_1^{\mathrm{T}}M_\nu U_1 = \left(\begin{smallmatrix}A & 0\\ 0 & B\end{smallmatrix}\right), \tag{1.7}$$

$$\begin{pmatrix}U_2 & 0\\ 0 & U_3\end{pmatrix}^{\mathrm{T}} U_1^{\mathrm{T}}M_\nu U_1\begin{pmatrix}U_2 & 0\\ 0 & U_3\end{pmatrix} = \begin{pmatrix}m_{\mathrm{diag}} & 0\\ 0 & M_{\mathrm{diag}}\end{pmatrix}, \tag{1.8}$$

where $m_{\mathrm{diag}} = \mathrm{diag}(m_1, m_2, m_3)$ and $M_{\mathrm{diag}} = \mathrm{diag}(M_1, M_2)$. The result of the first step is well-known in series-expanded form:

$$U_1 \simeq \begin{pmatrix}1 - \frac{\langle\phi_0\rangle^2}{2}y^\dagger(MM^*)^{-1}y & \langle\phi_0\rangle y^\dagger M^{*-1}\\ -\langle\phi_0\rangle M^{-1}y & 1 - \frac{\langle\phi_0\rangle^2}{2}M^{-1}yy^\dagger M^{*-1}\end{pmatrix}, \tag{1.9}$$

$$A \simeq -\langle\phi_0\rangle^2 y^{\mathrm{T}}M^{-1}y, \tag{1.10}$$

$$B \simeq M + \frac{\langle\phi_0\rangle^2}{2}\left(yy^\dagger M^{*-1} + M^{*-1}y^*y^{\mathrm{T}}\right) \tag{1.11}$$

The second step is expressed by

$$U_2^{\mathrm{T}} A U_2 = m_{\mathrm{diag}}, \qquad U_3^{\mathrm{T}} B U_3 = M_{\mathrm{diag}}. \tag{1.12}$$

We also have the expression of the mass eigenstates:

$$\text{lighter}: \quad U_2^\dagger \left[\nu - \langle\phi_0\rangle y^\dagger M^{*-1} n - \frac{\langle\phi_0\rangle^2}{2} y^\dagger (MM^*)^{-1} y\nu\right] \tag{1.13}$$

$$\text{heavier}: \quad U_3^\dagger \left[n + \langle\phi_0\rangle M^{-1} y\nu - \frac{\langle\phi_0\rangle^2}{2} M^{-1} y y^\dagger M^{*-1} n\right] \tag{1.14}$$

$$\tag{1.15}$$

Combining them,

$$\begin{aligned} m_{\mathrm{diag}} &= U_2^{\mathrm{T}} A U_2 && (1.16)\\ &= -\langle\phi_0\rangle^2 U_2^{\mathrm{T}} y^{\mathrm{T}} M^{-1} y U_2 + \mathcal{O}(M\epsilon^4) && (1.17)\\ &= -\langle\phi_0\rangle^2 U_2^{\mathrm{T}} y^{\mathrm{T}} B^{-1} y U_2 + \mathcal{O}(M\epsilon^4) && (1.18)\\ &= -\langle\phi_0\rangle^2 U_2^{\mathrm{T}} y^{\mathrm{T}} U_3 M_{\mathrm{diag}}^{-1} U_3^{\mathrm{T}} y U_2 + \mathcal{O}(M\epsilon^4) && (1.19)\\ &=: R'^{\mathrm{T}} R' + \mathcal{O}(M\epsilon^4) && (1.20) \end{aligned}$$

with $\mathcal{O}(M\epsilon^n) \sim \mathcal{O}(\langle\phi_0\rangle^n / M_1^{n-1})$ and $R' := -\mathrm{i}\langle\phi_0\rangle M_{\mathrm{diag}}^{-1/2} U_3^{\mathrm{T}} y U_2$. Now, the Yukawa coupling is given by $y = \mathrm{i}\langle\phi_0\rangle^{-1} U_3^* M_{\mathrm{diag}}^{1/2} R' U_2^\dagger$.

Here, for two heavy neutrino scenarios, R' can be parameterized as

$$R'_{\mathrm{NH}} = \begin{pmatrix} 0 & +\sqrt{m_2}\cos z & \zeta\sqrt{m_3}\sin z \\ 0 & -\sqrt{m_2}\sin z & \zeta\sqrt{m_3}\cos z \end{pmatrix} = \begin{pmatrix} 0 & +\cos z & \zeta\sin z \\ 0 & -\sin z & \zeta\cos z \end{pmatrix} \sqrt{m_{\mathrm{diag}}}, \tag{1.21}$$

$$R'_{\mathrm{IH}} = \begin{pmatrix} +\sqrt{m_1}\cos z & \zeta\sqrt{m_2}\sin z & 0 \\ -\sqrt{m_1}\sin z & \zeta\sqrt{m_2}\cos z & 0 \end{pmatrix} = \begin{pmatrix} +\cos z & \zeta\sin z & 0 \\ -\sin z & \zeta\cos z & 0 \end{pmatrix} \sqrt{m_{\mathrm{diag}}} \tag{1.22}$$

for normal hierarchy (NH; $m_1 = 0 < m_2 < m_3$) and inverted hierarchy (IH; $m_3 = 0 < m_1 < m_2$) cases. Therefore, the Yukawa couplings are given by

$$y = \mathrm{i}\langle\phi_0\rangle^{-1} U_3^* \sqrt{M_{\mathrm{diag}}} R \sqrt{m_{\mathrm{diag}}} U_2^\dagger, \tag{1.23}$$

which is the Casas–Ibarra parameterization in general basis; R is given by

$$R_{\mathrm{NH}} = \begin{pmatrix} 0 & +\cos z & \zeta\sin z \\ 0 & -\sin z & \zeta\cos z \end{pmatrix}, \qquad R_{\mathrm{IH}} = \begin{pmatrix} +\cos z & \zeta\sin z & 0 \\ -\sin z & \zeta\cos z & 0 \end{pmatrix}. \tag{1.24}$$

We have not yet defined the lepton basis. We can assume that we have been using, from the beginning, the charged lepton mass basis for L. Then, we identify the PMNS matrix (U_{li} in Eq. (14.1) of PDG2018, where i for mass and l for gauge indices) as

$$U_{\mathrm{PMNS}} \simeq U_2. \tag{1.25}$$

Similarly, the basis for N_R is such that $U_3 \simeq 1$, which corresponds to

$$M = B + \mathcal{O}(\epsilon^2) = U_3^* M_\mathrm{diag} U_3^\dagger + \mathcal{O}(\epsilon^2) = M_\mathrm{diag} + \mathcal{O}(\epsilon^2), \tag{1.26}$$

i.e., the basis in which $M_{ab} \simeq \mathrm{diag}(M_1, M_2)$ with $0 < M_1 \le M_2$. These basis choice gives the well-known Casas–Ibarra parameterization,[*1]

$$y = \mathrm{i} \langle \phi_0 \rangle^{-1} \sqrt{M_\mathrm{diag}} R \sqrt{m_\mathrm{diag}} U_\mathrm{PMNS}^\dagger, \tag{1.27}$$

where

$$\begin{aligned} U_\mathrm{PMNS} = &\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & +c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ +s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \\ &\times \mathrm{diag}(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2}). \end{aligned} \tag{1.28}$$

1.3 Higgs potential

Following [1611.03827],

$$V = -\mu^2 (\phi^\dagger \phi) + \lambda (\phi^\dagger \phi)^2, \tag{1.29}$$

which gives $m_h^2 = 2\mu^2 = 2\lambda v^2$ with $\langle \phi \rangle = v/\sqrt{2}$.

We calculate the threshold correction to μ^2 and λ by matching the effective potential. The difference of the EFT, in which the heavy neutrinos are integrated out, and the full theory is given by **[TODO: Here M_a is physical mass]**

$$\begin{aligned} \Delta V(\phi; Q) &= V_\mathrm{full}(\phi; Q) - V_\mathrm{EFT}(\phi; Q) && (1.30) \\ &= \sum_{a=1,2} \frac{-2}{64\pi^2} M_a(\phi)^4 \left(\frac{M_a(\phi)^2}{Q^2} - \frac{3}{2} \right), && (1.31) \end{aligned}$$

where Q is the matching scale and M_a is the physical masses of the heavier neutrinos. Expanding ΔV in terms of ϕ,

$$\Delta V = (\mathrm{const.}) - \Delta\mu^2 |\phi|^2 + \Delta\lambda |\phi|^4 + \mathcal{O}\left(|\phi|^6 \right), \tag{1.32}$$

where **[TODO: Now M_a is the diagonal majorana masses...]**

$$\begin{aligned} \Delta\mu^2 &= -\sum_{a=1,2} \frac{H_a}{8\pi^2} M_a^2 \left(1 - \log \frac{M_a^2}{Q^2} \right), && (1.33) \\ \Delta\lambda &= -\frac{1}{16\pi^2} \Big[f_1 \operatorname{Tr}(YY^\dagger Y^* Y^\mathrm{T}) + f_2 \operatorname{Tr}(YY^\dagger YY^\dagger) + f_3 H_1^2 + f_4 H_2^2 \Big]; && (1.34) \end{aligned}$$

[*1] If we took the basis in which $M_{ab} \simeq \mathrm{diag}(-M_1, -M_2)$, then $U_3 = \mathrm{diag}(-i, -i)$ and we can remove i, but it is less plausible and we put i in the parameterization.

the coefficients are $H_1 = (YY^\dagger)_{11}$, $H_2 = (YY^\dagger)_{22}$, and

$$f_1 = \frac{2M_1 M_2}{M_2^2 - M_1^2} \log \frac{M_2}{M_1}, \qquad f_2 = \frac{M_2^2 \log(M_2^2/Q^2) - M_1^2 \log(M_1^2/Q^2)}{M_2^2 - M_1^2} - 1,$$
$$f_3 = 2 - \frac{2M_2 \log(M_2/M_1)}{M_2 - M_1}, \qquad f_4 = 2 - \frac{2M_1 \log(M_2/M_1)}{M_2 - M_1};$$

for $M_2 \simeq M_1$, they approach to $f_1 = 1$, $f_2 = \log(M_1^2/Q^2)$, and $f_3 = f_4 = 0$, which gives

$$\Delta\mu^2 \simeq -\frac{M_1^2}{8\pi^2} \operatorname{Tr}(YY^\dagger) \left(1 - \log \frac{M_1^2}{Q^2}\right), \tag{1.35}$$

$$\Delta\lambda \simeq -\frac{1}{16\pi^2} \left[\operatorname{Tr}(YY^\dagger Y^* Y^{\mathrm{T}}) + \operatorname{Tr}(YY^\dagger YY^\dagger) \log \frac{M_1^2}{Q^2}\right]. \tag{1.36}$$