Primordial Black Holes (in Dark Matter clothing)

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1 Length Scales

Distribution of orbits

From Ref. [1] (Sec. 2), the probability distribution for the semi-major axis a and the eccentricity e of the binary PBH orbits is given by:

$$\mathrm{d}P(a,e) = \frac{3}{4} f^{3/2} \bar{x}^{-3/2} a^{1/2} e (1-e^2)^{-3/2} \,\mathrm{d}a \,\mathrm{d}e \,. \tag{1}$$

Here, f is the fraction of PBHs in DM and $\bar{x}$ is the mean physical separation of PBHs at matter-radiation equality $z = z_\mathrm{eq}$:

$$\overline{x} = \frac{1}{(1+z_\mathrm{eq}) f^{1/3}} \left(\frac{8\pi G}{3H_0^2}\frac{M_\mathrm{BH}}{\Omega_{DM}}\right)^{1/3} \,. \tag{2}$$

Note that there could be some extra factors of $4\pi/3$ in here, depending on your precise definition of $\bar{x}$ (see e.g. Eq. 2 in Ref. [2]).

Range of parameter values

The maximum eccentricity is given by (Ref. [1], Eq. 6):

$$e_\mathrm{max} = \sqrt{1 - f^{3/2}\left(\frac{a}{\bar{x}}\right)^{3/2}} \,. \tag{3}$$

The distance of closest approach of the PBHs - *peri-BH*, r_peri - is given by:

$$r_\mathrm{peri} = a(1-e) \,. \tag{4}$$

Then periBH lies in the range:

$$r_\mathrm{peri} \in a[1 - e_\mathrm{max}, 1] \,. \tag{5}$$

We note also that the requirement for forming a binary is that the physical separation of the BH is

$$x < f^{1/3}\bar{x} \,, \tag{6}$$

which means that the semi-major axis is limited to:

$$a \leq \alpha f^{1/3} \bar{x} \,, \tag{7}$$

where α is a numerical factor of $\mathcal{O}(1)$, set to 1 in Ref. [1].

Distribution of periBH

We now change variables from eccentricity to periBH:

$$\mathrm{d}P(a, r_{\rm peri}) = \frac{3}{4} f^{3/2} \bar{x}^{-3/2} a^{-1/2} \left(1 - \frac{r_{\rm peri}}{a}\right) \left(\frac{2r_{\rm peri}}{a} - \frac{r_{\rm peri}^2}{a^2}\right)^{-3/2} \mathrm{d}a\, \mathrm{d}r_{\rm peri}\,. \tag{8}$$

Or, in terms of the dimensionless periBH $\mathfrak{r} = r_{\rm peri}/a$:

$$\mathrm{d}P(a, \mathfrak{r}) = \frac{3}{4} f^{3/2} \bar{x}^{-3/2} a^{1/2} \left(1 - \mathfrak{r}\right) \left(2\mathfrak{r} - \mathfrak{r}^2\right)^{-3/2} \mathrm{d}a\, \mathrm{d}\mathfrak{r}\,. \tag{9}$$

Note here, that the PDF is valid over the range:

$$a \in [0, f^{1/3}\bar{x}]\,, \tag{10}$$

$$\mathfrak{r} \in [1 - e_{\rm max}(a), 1]\,. \tag{11}$$

We can rearrange the limits:

$$a \in [0, \frac{\bar{x}}{f} \left(1 - (1-\mathfrak{r})^2\right)^{2/3}]\,, \tag{12}$$

$$\mathfrak{r} \in [0, 1]\,, \tag{13}$$

and then integrate over a:

$$\mathrm{d}P(\mathfrak{r}) = \frac{1}{2} \frac{(1-\mathfrak{r})}{\sqrt{1 - (1-\mathfrak{r})^2}} \mathrm{d}\mathfrak{r}\,. \tag{14}$$

As a sanity check, we can integrate over all $\mathfrak{r} \in [0, 1]$ and we obtain $1/2$. This matches the original normalisation of the PDFs from Ref. [1] (my guess is that they normalised to $1/2$ because one PBH is one-half of a PBH binary).

See the jupyter notebook or plots folder for some visualisations.

References

[1] M. Sasaki, T. Suyama, T. Tanaka and S. Yokoyama, *Primordial Black Hole Scenario for the Gravitational-Wave Event GW150914*, *Phys. Rev. Lett.* **117** (2016) 061101, [1603.08338].

[2] Y. Ali-Haïmoud, E. D. Kovetz and M. Kamionkowski, *The merger rate of primordial-black-hole binaries*, 1709.06576.