Expressions for the Limber approximation

David Alonso

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1 Full-sky expressions

The angular power spectrum between two contributions is:

$$C_\ell^{ij} = 4\pi \int_0^\infty \frac{dk}{k}\,\mathcal{P}_\Phi(k)\Delta_\ell^i(k)\Delta_\ell^j(k). \tag{1}$$

The expressions for density, RSD, magnification, lensing convergence and CMB lensing are:

$$\Delta_\ell^D(k) = \int dz p_z(z) b(z) T_\delta(k,z) j_\ell(k\chi(z)) \tag{2}$$

$$\Delta_\ell^{RSD}(k) = \int dz \frac{(1+z)p_z(z)}{H(z)} T_\theta(k,z) j_\ell''(k\chi(z)) \tag{3}$$

$$\Delta_\ell^M(k) = \ell(\ell+1)\int \frac{dz}{H(z)}\frac{W^M(z)}{\chi(z)} T_{\phi+\psi}(k,z) j_\ell(k\chi(z)), \tag{4}$$

$$\Delta_\ell^L(k) = -\frac{1}{2}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\int \frac{dz}{H(z)}\frac{W^L(z)}{\chi(z)} T_{\phi+\psi}(k,z) j_\ell(k\chi(z)), \tag{5}$$

$$\Delta_\ell^{IA}(k) = \sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\int dz\, p_z(z)\, b_{\rm IA}(z)\, T_\delta(k,z)\,\frac{j_\ell(k\chi(z))}{(k\chi(z))^2}, \tag{6}$$

$$\Delta_\ell^C(k) = -\frac{\ell(\ell+1)}{2}\int_0^{\chi_*} d\chi \frac{\chi_*-\chi}{\chi\chi_*} T_{\phi+\psi}(k,z) j_\ell(k\chi), \tag{7}$$

$$\Delta_\ell^{ISW}(k) = 2\int_0^{\chi_*} d\chi\, a(\chi)\, T_{\dot\phi} \tag{8}$$

where

$$W^M(z) \equiv \int_z^\infty dz' p_z(z') \frac{2-5s(z')}{2}\frac{\chi(z')-\chi(z)}{\chi(z')} \tag{9}$$

$$W^L(z) \equiv \int_z^\infty dz' p_z(z') \frac{\chi(z')-\chi(z)}{\chi(z')} \tag{10}$$

Writing $\mathcal{P}_\Phi(k) = k^3 P(k)/(2\pi^2 T_\delta^2(k, z=0))$ we can rewrite:

$$C_\ell^{ij} = \frac{2}{\pi}\int_0^\infty dk\, k^2\, P_k \tilde\Delta_\ell^i(k)\tilde\Delta_\ell^j(k) \tag{11}$$

where P_k is the matter power spectrum at $z = 0$ and:

$$\tilde{\Delta}^D_\ell(k) = \int d\chi H(\chi)\, p_z(\chi)\, b(\chi)\, D(\chi)\, j_\ell(k\chi) \tag{12}$$

$$\tilde{\Delta}^{RSD}_\ell(k) = \int d\chi H(\chi)\, p_z(\chi)\, f(\chi)\, D(\chi) \frac{\left[(k\chi)^2 - \ell(\ell-1)\right] j_\ell(k\chi) - 2(k\chi)\, j_{\ell+1}(k\chi)}{(k\chi)^2} \tag{13}$$

$$\tilde{\Delta}^M_\ell(k) = -\frac{3H_0^2\Omega_M \ell(\ell+1)}{k^2} \int d\chi\, W^M(\chi)\, \frac{D(\chi)}{\chi\, a(\chi)} j_\ell(k\chi), \tag{14}$$

$$\tilde{\Delta}^L_\ell(k) = \frac{3H_0^2\Omega_M}{2k^2} \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \int d\chi\, W^L(\chi) \frac{D(\chi)}{\chi\, a(\chi)} j_\ell(k\chi), \tag{15}$$

$$\tilde{\Delta}^{IA}_\ell(k) = \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \int d\chi\, H(\chi)\, p_z(\chi)\, b_{\rm IA}(\chi)\, D(\chi)\, \frac{j_\ell(k\chi)}{(k\chi)^2}, \tag{16}$$

$$\tilde{\Delta}^C_\ell(k) = \frac{3H_0^2\Omega_M \ell(\ell+1)}{2k^2} \int d\chi\, \Theta(\chi; 0, \chi_*) \frac{\chi_* - \chi}{\chi\chi_*} \frac{D(\chi)}{a(\chi)} j_\ell(k\chi), \tag{17}$$

$$\tilde{\Delta}^{ISW}_\ell(k) = \frac{3H_0^2\Omega_M}{k^2} \int d\chi\, \Theta(\chi; 0, \chi_*) H(\chi) D(\chi) [1 - f(\chi)] j_\ell(k\chi) \tag{18}$$

where $\Theta(\chi; \chi_1, \chi_2)$ is 1 if $\chi_1 < \chi < \chi_2$ and 0 otherwise.

2 Limber approximation

The Limber approximation is

$$j_\ell(x) \simeq \sqrt{\frac{\pi}{2\ell+1}}\, \delta\left(\ell + \frac{1}{2} - x\right). \tag{19}$$

Thus for each k and ℓ we can define a radial distance $\chi_\ell \equiv (\ell + 1/2)/k$.

The expressions above can be rewritten as:

$$\tilde{\Delta}^D_\ell(k) = \frac{1}{k}\sqrt{\frac{\pi}{2\ell+1}}\, p_z(\chi_\ell)\, b(\chi_\ell)\, D(\chi_\ell)\, H(\chi_\ell) \tag{20}$$

$$\tilde{\Delta}^{RSD}_\ell(k) = \frac{1}{k}\sqrt{\frac{\pi}{2\ell+1}} \left[\frac{1+8\ell}{(2\ell+1)^2}\, p_z(\chi_\ell)\, f(\chi_\ell)\, D(\chi_\ell)\, H(\chi_\ell) - \right. \tag{21}$$

$$\left. \frac{4}{2\ell+3}\sqrt{\frac{2\ell+1}{2\ell+3}} p_z(\chi_{\ell+1})\, f(\chi_{\ell+1})\, D(\chi_{\ell+1})\, H(\chi_{\ell+1}) \right] \tag{22}$$

$$\tilde{\Delta}^M_\ell(k) = \frac{1}{k}\sqrt{\frac{\pi}{2\ell+1}} \left(-\frac{3\Omega_{M,0}H_0^2 \ell(\ell+1)}{k^2} \frac{D(\chi_\ell)}{a(\chi_\ell)\chi_\ell} W^M(\chi_\ell)\right) \tag{23}$$

$$\tilde{\Delta}^L_\ell(k) = \frac{1}{k}\sqrt{\frac{\pi}{2\ell+1}} \frac{3\Omega_{M,0}H_0^2}{2k^2} \sqrt{\frac{(\ell+2)!}{(\ell-2)}} \frac{D(\chi_\ell)}{a(\chi_\ell)\chi_\ell} W^L(\chi_\ell) \tag{24}$$

$$\tilde{\Delta}^{IA}_\ell(k) = \frac{1}{k}\sqrt{\frac{\pi}{2\ell+1}} \frac{\sqrt{(\ell+2)(\ell+1)\ell(\ell-1)}}{(\ell+1/2)^2}\, p_z(\chi_\ell)\, b_{\rm IA}(\chi_\ell)\, D(\chi_\ell)\, H(\chi_\ell) \tag{25}$$

$$\tilde{\Delta}^C_\ell(k) = \frac{1}{k}\sqrt{\frac{\pi}{2\ell+1}} \frac{3\Omega_{M,0}H_0^2 \ell(\ell+1)}{2k^2} \frac{D(\chi_\ell)}{a(\chi_\ell)\chi_\ell} \frac{\chi_* - \chi_\ell}{\chi_*} \Theta(\chi_\ell; 0, \chi_*) \tag{26}$$

$$\tilde{\Delta}^{ISW}_\ell(k) = \frac{1}{k}\sqrt{\frac{\pi}{2\ell+1}} \frac{3\Omega_{M,0}H_0^2}{k^2} H(\chi_\ell) D(\chi_\ell)\, [1 - f(\chi_\ell)] \tag{27}$$

In the limit $\ell \gg 1/2$ these simplify to

$$\tilde{\Delta}^i \equiv \frac{1}{k}\sqrt{\frac{\pi}{2\ell+1}} \lambda^i \tag{28}$$

where

$$
\begin{aligned}
\lambda_\ell^{D}(k) &= p_z(\chi_\ell) b(\chi_\ell) D(\chi_\ell) H(\chi_\ell) && (29)\\
\lambda_\ell^{RSD}(k) &= 0 && (30)\\
\lambda_\ell^{M}(k) &= -3\Omega_{M,0} H_0^2 \frac{\chi_\ell D(\chi_\ell)}{a(\chi_\ell)} W^M(\chi_\ell) && (31)\\
\lambda_\ell^{L}(k) &= \frac{3}{2}\Omega_{M,0} H_0^2 \frac{\chi_\ell D(\chi_\ell)}{a(\chi_\ell)} W^L(\chi_\ell) && (32)\\
\lambda_\ell^{IA}(k) &= p_z(\chi_\ell) b_{IA}(\chi_\ell) D(\chi_\ell) H(\chi_\ell) && (33)\\
\lambda_\ell^{C}(k) &= \frac{3}{2}\Omega_{M,0} H_0^2 \frac{\chi_\ell D(\chi_\ell)}{a(\chi_\ell)} \frac{\chi_* - \chi_\ell}{\chi_*} \Theta(\chi_\ell; 0, \chi_*) && (34)\\
\lambda_\ell^{ISW} &= \frac{3\Omega_{M,0} H_0^2}{k^2} H(\chi_\ell) D(\chi_\ell)[1 - f(\chi_\ell)] && (35)
\end{aligned}
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