A Rough Guide to Gravitational Radiation

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June 6, 2019

Outline

Aim is to provide an introduction to Gravitational Waves (GWs) without using General Relativity

- What are GWs ?
- Rough estimates for GW amplitude & Luminosity..
- Why we expect GW background in the nano-Hertz frequency range?
- Bernard F. Schutz, Gravitational waves on the back of an envelope; American Journal of Physics 52, 412 (1984); *https://doi.org/10.1119/1.13627*

GWs: I

General Relativity (GR) defines GWs as ripples in the curvature of space-time that propagate with the speed of light !

It is possible to **COMPUTE** most of the crucial effects of GWs using Newtonian Gravitational Theory, Classical Electrodynamics & some elements of Special Relativity

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- Newtonian gravity involves a scalar potential $\phi_{\rm N}(\boldsymbol{x}, t)$ such that $\nabla \phi_{\rm N} = 4\,\pi\, G\,\rho$, such that

$$\phi_{\rm N}(\boldsymbol{x}, t) = -G \int \frac{\rho(\boldsymbol{y}, t)}{r}\, d^3 y\,, \quad r \equiv |\boldsymbol{x} - \boldsymbol{y}| \tag{1}$$

  A change in $\phi_{\rm N}(\boldsymbol{x}, t)$ due to a change in $\rho(\boldsymbol{y}, t)$ propagate instantaneously

- Special Relativity demands that no information should be able to propagate faster than c: the speed of light

GWs: II

To make $\phi_{\rm N}$ consistent with Special Relativity, we modify it

- $$\phi_{\rm R}(\boldsymbol{x}, t) = -G \int \frac{\rho(\boldsymbol{y}, t - \frac{r}{c})}{r} \, d^3 y \qquad (2)$$

  Dominant effects of GWs can be deducted from such a retarded gravitational potential

  This simple insertion will help us to define GWs in a non-rigorous way

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- $\phi_{\rm R}$ satifies the scalar wave equation $\Box\phi_{\rm R} \equiv \left(\nabla^2 - \frac{\partial^2}{c^2\,\partial^2 t}\right)\,\phi_{\rm R} = 4\,\pi\,G\,\rho$

- Take the spatial gradient of $\phi_{\rm R}$

$$\boldsymbol{\nabla}\phi_{\rm R} = G \int \left(\frac{\rho}{r} - \frac{\partial\rho}{c\,\partial t}\right) \frac{\boldsymbol{x} - \boldsymbol{y}}{r^2} \, d^3 y \tag{3}$$

GWs: III

- If $|\boldsymbol{x}| \gg |\boldsymbol{y}_{\rm mx}|$, we have $r \sim |\boldsymbol{x}|$ & can negelct $1/r$ term in the previous Eq.

$$\boldsymbol{n} \cdot \boldsymbol{\nabla} \phi_{\rm R} \sim \frac{\partial \phi_{\rm R}}{c\, \partial t}\,, \quad \boldsymbol{n} = \boldsymbol{x}/|\boldsymbol{x}| \tag{4}$$

$$\phi_{\rm R}/\lambda \sim 1/c \times \phi_R/T$$

- If we are far away from a GW source, the typical length scale over which $\phi_{\rm R}$ varies is **c** $\times$ the typical time scale over which $\phi_{\rm R}$ changes

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- This is true for a wave traveling at speed c
  This is our Gravitational Wave

- Recall that $\phi_R \sim v^2$. Therefore, the amplitude of GW should be $\sim$

$$h \sim \frac{\Big(time - dependent\ \ part\ \ of\ \ \phi_R \Big)}{c^2} \tag{5}$$

An estimate for the dominate contribution to h: I

Consider a region $|\boldsymbol{x}| \gg |\boldsymbol{y}| b$ & we have far-zone expansion

- $$1/r \equiv |\boldsymbol{x} - \boldsymbol{y}|^{-1} \sim \frac{1}{|\boldsymbol{x}|} + \boldsymbol{y} \cdot \boldsymbol{n}\, |\boldsymbol{x}|^{-2} \tag{6}$$

  This leads to (if we neglect $\mathcal{O}(|\boldsymbol{x}|^{-2})$ terms)

  $$\phi_{\rm R} = -\frac{G}{|\boldsymbol{x}|} \int \rho(\boldsymbol{y}, t_r)\, d^3y\,, \quad t_r = t - \frac{r}{c} \tag{7}$$

- Let $t_0 = t - \frac{|\boldsymbol{x}|}{c}$ & this leads to $t_r \sim t_0 - \boldsymbol{y} \cdot \boldsymbol{n}/c$
- Expand $\rho(t_r)$ about t_0

$$\phi_R = \frac{-G}{|\boldsymbol{x}|} \left\{ \int \left[\rho(t_0) - \frac{\dot{\rho}}{c}\, \boldsymbol{n} \cdot \boldsymbol{y} + \frac{\ddot{\rho}}{2\,c^2}\, (\boldsymbol{n} \cdot \boldsymbol{y})^2 + \ldots \right] d^3y \right\} \tag{8}$$

An estimate for the dominate contribution to h: II

- First term

$$\int \rho(t_0) d^3 y \equiv M \tag{9}$$

- Let $\boldsymbol{v} = \frac{d\boldsymbol{y}}{dt}$; the second term involves

$$n_i \int \dot{\rho} y_i \, d^3 y = n_i \int \rho \, v_i \, d^3 y = \boldsymbol{n} \cdot \boldsymbol{P} \tag{10}$$

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- Third term contains

$$\int \ddot{\rho} \, y_i \, y_j \, d^3 \, y = \ddot{I}_{ij} \tag{11}$$

where $I_{ij}(t) \equiv \int \rho(t) \, y_i \, y_i \, d^3 y$ is the quadrupole moment of gravitating source & $\ddot{I}_{ij} = \int \rho \, v_i \, v_j \, d^3 y$

- Retarded potential becomes

$$\phi_R \sim -\frac{G\,M}{|\boldsymbol{x}|} + \frac{G\,\boldsymbol{n} \cdot \boldsymbol{P}}{c\,|\boldsymbol{x}|} - \frac{G}{2\,c^2} \, \frac{\ddot{I}_{ij} \, n_i \, n_j}{|\boldsymbol{x}|} \,, \tag{12}$$

An estimate for the dominate contribution to h: III

- This leads to

$$h \sim \frac{G}{2\,c^4}\,\frac{\ddot{I}_{ij}\,n_i\,n_j}{|\boldsymbol{x}|} \tag{13}$$

- Note that h depends only on the components of I_{ij} along $\boldsymbol{n}$, the direction of propagation of the wave & this is due to the fact that we are dealing with **scalar waves**

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- In GR, GWs are *ripples in the curvature of space-time* & space-time & its disturbances are described by tensors
- $I_{ij} \rightarrow$ transverse components of trace-free tensor $\mathcal{I}_{ij} = I_{ij} - \frac{\delta_{ij}}{3}\,I_{kk}$
- In GR, we have
$$h_{ij}^{TT} = \frac{2\,G}{c^4\,r}\,\ddot{\mathcal{I}}_{ij}(t - r/c) \tag{14}$$
This implies that spherically symmetric motion **WILL NOT** produce GWs. Any spherically symmetric tensor $\propto \delta_{ij}$ & hence $\mathcal{I}_{ij}$ vanishes

An estimate for GW luminosity :I

- In classical ED, the dominant order multipole radiation fron a charge distribution is the dipole radiation. The vector potential A_i in the wave-zone

$$A_j = \frac{1}{c\, r}\, \dot{d}_j(t_r) \tag{15}$$

The $1/r$ EM fields $\boldsymbol{E}$ & $\boldsymbol{B}$ depend only on the components of $\dot{\boldsymbol{d}}$ transverse to $\boldsymbol{n}$; $d_j^{\mathrm{T}} \equiv P_{jk}\, d_k\,,\ \ P_{jk} = \delta_{jk} - n_j\, n_k$

- The Larmor formula provides the expression for EM luminosity

$$\mathcal{L}_{\mathrm{EM}} = \frac{2}{3\, c^3}\, \ddot{d}_j\, \ddot{d}_j\,,\ \ d_j = e\, y_i \tag{16}$$

- For gravitating systems, linear & angular momenta provide electric & magnetic *type* dipole moments & they are conserved $\boldsymbol{\mu} = \frac{1}{c} \sum_a \boldsymbol{y}^a \times \boldsymbol{d}^a = \frac{1}{c} \sum_a \boldsymbol{y}^a \times m_a \boldsymbol{v}^a = \frac{1}{c} \sum_a \boldsymbol{L}^a$

An estimate for GW luminosity :II

$\mathcal{L}_{\rm GW} \propto \overset{(3)}{I_{ij}} \, \overset{(3)}{I_{ij}}$ & dimensional consideration require us to have G/c^5

- Explicit calculations in GR provides

$$\mathcal{L}_{GW} = \frac{G}{5\,c^5}\,\mathcal{I}_{ij}^{(3)}\,\mathcal{I}_{ij}^{(3)} \tag{17}$$

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- ${}^{(3)}I_{ij} \sim M\,R^2/T^3 \sim M\,V^3/R$

$$\mathcal{L}_{GW} \sim \frac{G}{c^5}\,(M/R)^2\;V^6 \sim L_0\;(r_{Sch}/R)^2\;(V/c)^6\;, \tag{18}$$

where $L_0 = \frac{c^5}{G} \sim 3.6 \times 10^{52}\,\mathrm{J}/s$ & $r_{\rm Sch} = G\,M/c^2$

- $\mathcal{L}_{GW}$ is maximal if $R \sim r_{\rm Sch}$ & $V \sim c$

  Compact objects, having time-dependent quadrupole moment, moving with velocities $\sim c$ are copious sources of GWs

Stochastic GW background from massive BH binaries

How GWs affect TOAs: I

- Nano-Hz– μHz GWs are expected to be extra-galactic
- A travel time delay between the source-Earth path and the source-pulsar path is expected
- GWs that present in timing residuals at time t depends on the 'difference' $\Delta h_{+,\times} = h^{p}_{+,\times} - h^{E}_{+,\times}$
- If we consider "Earth term," $h^{E}_{+,\times}(t)$ at t, we need to consider 'pulsar term," $h^{p}_{+,\times}(t - d/c)$ for a pulsar of distance d
- Passing GWs induces a shift in the pulsar's (constant intrinsic) spin frequency, ν_0

How GWs affect TOAs: II

- If a GW propagates along the positive z direction

$$z(t) = \frac{\nu_0 - \nu(t)}{\nu_0} = \frac{\alpha^2 - \beta^2}{2(1+\gamma)} \Delta h_+(t) + \frac{\alpha\,\beta}{1+\gamma} \Delta h_\times(t) \; .$$

  α, β, and γ provide a pulsar's direction cosines w.r.t the x, y, and z axes (Detweiler 1979)

- Observable timing residuals, induced by the GWs, are given by the integral over time of the above Equation

- $R(t) = \int_0^t \frac{\nu_0 - \nu(t')}{\nu_0} \delta t'$

- It's derivation is straightforward

Stochastic GWB from massive BH binaries ?: I

Cosmological population of MBH binaries is expected to provide a diffusive GW background for PTAs !!
We need to compute # of sources in a frequency interval $\Delta f = 1/T_{\rm obs}$

- $$\Delta N = \left(\frac{dN}{df}\right)\Delta f \;= \frac{dN}{dt}\left(\frac{df}{dt}\right)^{-1}\Delta f \tag{19}$$

- $$\Delta N \propto \frac{dN}{dt}\left({\mathcal{M}_c}^{-5/3}\, f_{\rm GW}^{-11/3}\right)\Delta f \tag{20}$$

- There are some 10^{11} galaxies in our Universe and each galaxy is likely to experience one merger with another galaxy in Hubble time (10^{10} year)

- $$\rightarrow \tfrac{dN}{dt} \sim 10 \text{ mergers/year}$$

Stochastic GWB from massive BH binaries ?: I

- $\rightarrow$ a rough estimate for the number of binary BH sources in a frequency interval $\Delta f = 1/T_{\rm obs}$
-
$$\Delta N \sim 3 \times 10^{12} \left(\frac{\mathcal{M}}{10^9\, M_\odot}\right)^{-5/3} \left(\frac{f_{\rm GW}}{10^{-8}\, Hz}\right)^{-11/3} \left(\frac{T_{\rm obs}}{10\, yr}\right)^{-1} \times \left(\frac{dN/dt}{10 \rm merger/yr}\right)$$
- This is clearly $>> 1$

  This ensures a diffuse GW background in the PTA GW frequency window from merging massive BHs in the universe

- It is NOT very difficult to show that its characteristic strain spectrum is given by $h_c(f) = A_{1yr} \times (f/{\rm yr}^{-1})^{-2/3}$

$h_c(f)$ derivation : I

This derivation for $h_c(f)$ is essentially due to Phinney

- Let $\mathcal{E}_{gw}$ be the total present day energy density in GWs
- Let $\Omega_{gw}(f)$ be the RATIO between the present-day energy density per logarithmic frequency interval, in GWs AND the rest-mass energy density $\rho_c c^2$ that would be required to close the universe $\rho_c = 3H_0^2/(8\pi G)$
- $\mathcal{E}_{gw} \equiv \int_0^\infty \rho_c c^2 \Omega_{gw}(f)\, df/f$........(1)

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- $\mathcal{E}_{gw} \equiv \int_0^\infty \rho_c c^2 \Omega_{gw}(f)\, df/f$........(1)
- Let $N(z)dz$ be the number of events in unit comoving volume between redshifts z and $z + dz$
- $\mathcal{E}_{gw} \equiv \int_0^\infty N(z) \frac{1}{1+z}\, dE_{gw}\, dz$
  $\mathcal{E}_{gw}$ must be equal to sum of the energy densities radiated at each redshift, divided by $(1+z)$
- $\mathcal{E}_{gw} \equiv \int_0^\infty \int_0^\infty N(z) \frac{1}{1+z} \frac{dE_{gw}}{df_r} f_r \frac{df_r}{f_r}\, dz$ $\qquad f = f_r/(1+z)$

$h_c(f)$ derivation : II

- $\mathcal{E}_{gw} = \int_0^\infty \int_0^\infty N(z) \frac{1}{1+z} f_r \frac{dE_{gw}}{df_r} \, dz \, \frac{df}{f}$

- We have already argued that
$\mathcal{E}_{gw} \equiv \int_0^\infty \rho_c c^2 \Omega_{gw}(f) \, df/f \, ...(1)$

- Equating the above two equations $\rightarrow$

$$\rho_c c^2 \Omega_{gw}(f) = \int_0^\infty N(z) \frac{1}{1+z} \left(f_r \frac{dE_{gw}}{df_r} \right) \Bigg|_{f_r = f(1+z)} dz \, .$$

- The energy density in GWs per log frequency interval is equal to the comoving number density of event remnants, times the (redshifted) energy each event produced per log frequency interval.

$h_c(f)$ derivation : III

- It is fairly easy to compute $\frac{dE_{gw}}{df_r}$ associated with a massive BH binary inspiraling along circular orbits
- $$\frac{dE_{gw}}{df_r} = \frac{\pi}{3}\frac{1}{G}\frac{(G\mathcal{M})^{5/3}}{(\pi f_r)^{1/3}} \quad \text{for } f_{\min} < f_r < f_{\max} \,, \tag{21}$$
- This leads to $\Omega_{gw}(f) = \frac{8\pi^{5/3}}{9}\frac{1}{c^2 H_0^2}(G\mathcal{M})^{5/3} f^{2/3} N_0 \langle (1+z)^{-1/3} \rangle$
  $N_0 = \int_0^\infty N(z)\, dz$ is the present-day comoving number density of merged remnants
- It is convenient to introduce h_c: the characteristic amplitude of the gravitational wave spectrum over a logarithmic frequency interval as $\mathcal{E}_{gw} \equiv \int_0^\infty \rho_c c^2 \Omega_{gw}(f)\, df/f \equiv \int_0^\infty \frac{\pi}{4}\frac{c^2}{G} f^2 h_c^2(f) \frac{df}{f}$,
- For massive BH binaries,
  $h_c^2(f) = \frac{4}{3\pi^{1/3}}\frac{1}{c^2}\frac{(G\mathcal{M})^{5/3}}{f^{4/3}} N_0 \langle (1+z)^{-1/3} \rangle$
- $h_c \propto f^{-2/3}$