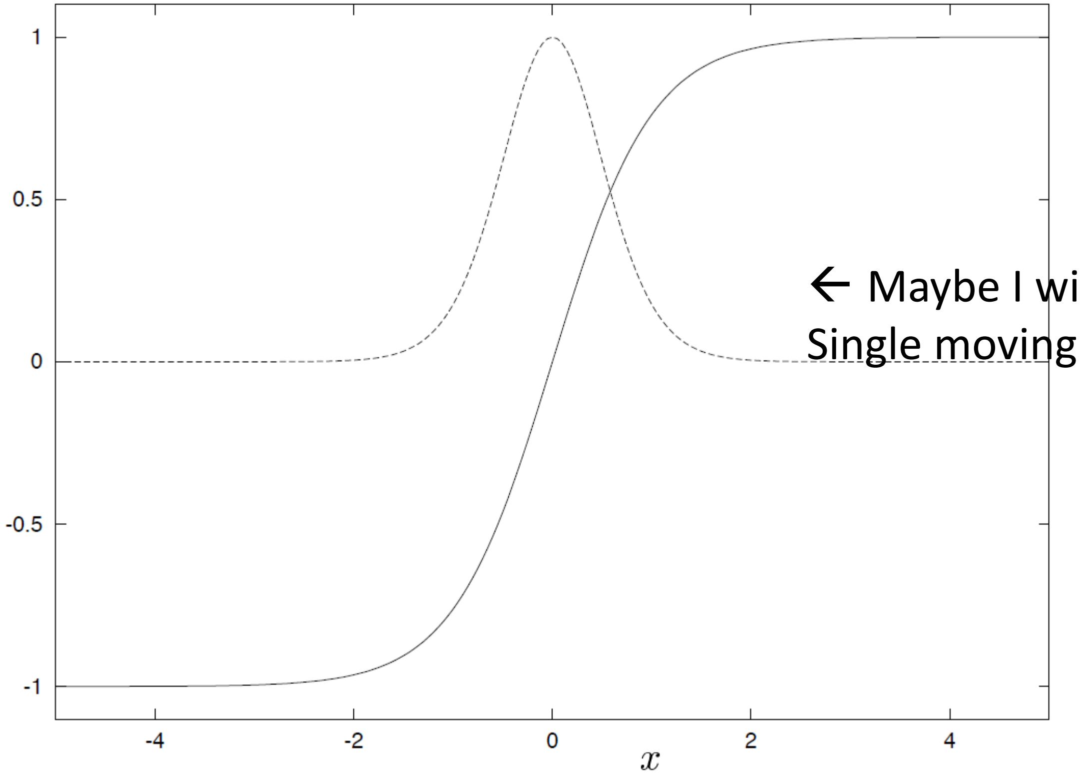



$\phi^4$ *kink (solid curve) and its energy density (dashed curve)*[1]

← Maybe I will also put an animation of a Single moving kink here

# Kinks

## Topological solutions in classical field theory

# Contents

# Topology and vacuum structure

Consider: $$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)$$

→ $$\partial_\mu\partial^\mu\phi + \frac{dU}{d\phi} = 0, \qquad V = \int_{-\infty}^{\infty}\left(\frac{1}{2}\phi'^2 + U(\phi)\right)dx, \qquad T = \frac{1}{2}\int_{-\infty}^{\infty}\dot{\phi}^2\,dx$$

Denote the vacuum manifold as:

$$\mathcal{V} = \{\phi_0, such\ that\ \phi_0' = \dot{\phi}_0 = 0, and\ U(\phi_0) = U_{min}\}$$

# Topology and vacuum structure

The existence of topological solitons depends on there being multiple vacua $\equiv$ non-trivial $\pi_0(\mathcal{V})$

Remember:

- Field configuration defines a map $\phi^\infty : S_\infty^{d-1} \to \mathcal{V}$
- Homotopy class $\pi_{d-1}(\mathcal{V})$ defines topological character of the field configuration $\phi_\infty$

For $d = 1$: $S_\infty^{d-1}$ has two points, $\pm\infty \in \mathbb{R}$, which are mapped into $\mathcal{V}$ by $\phi^\infty$, where the components of $\mathcal{V}$ are classified by $\pi_{d-1}(\mathcal{V})$, the set of topologically distinct vacua

A finite energy field configuration is then classified by $(\phi_-, \phi_+) \in \pi_0(\mathcal{V}) \times \pi_0(\mathcal{V})$ where $\phi_\pm = \lim_{x \to \pm\infty} \phi(x)$

# Topology and vacuum structure

Take now the set $(\phi_-, \phi_+) \in \pi_0(\mathcal{V}) \times \pi_0(\mathcal{V})$:

If $\phi_- = \phi_+$, the field can be continuously transformed (with a finite amount of energy needed) into a constant vacuum solution with zero energy $\phi(x) = \phi_+$

If $\phi_- \neq \phi_+$, the field cannot be continuously transformed (with only a finite amount of energy needed) into constant vacuum solution

→ We get a stable solution

What makes this solution a kink?

# Topology and vacuum structure

To characterize what makes a soliton a kink we want to look at the energy and express it, if possible, in terms of our topological data $(\phi_-, \phi_+) \in \pi_0(\mathcal{V}) \times \pi_0(\mathcal{V})$

Consider

$$\left(\frac{1}{\sqrt{2}}\phi'^2 \pm \sqrt{U(\phi)}\right)^2 \geq 0.$$

Integrating over space and expanding the braket yields

$$\int_{-\infty}^{\infty}\left(\frac{1}{2}\phi' + U(\phi)\right)dx \geq \pm\int_{-\infty}^{\infty}\sqrt{2U(\phi)}\phi' dx$$

# Topology and vacuum structure

Hence for the static case we get

$$E \geq \left| \int_{-\infty}^{\infty} \sqrt{2U(\phi)}\phi' dx \right| = \left| \int_{\phi_-}^{\phi_+} \sqrt{2U(\phi)} dx \right| \ ,$$

which also holds for time dependent fields (since $T \geq 0$).

Since $U(\phi) \geq 0$ we may write $U(\phi) = \frac{1}{2}\left(\frac{dW}{d\phi}\right)^2$ so that:

$$E \geq |W(\phi_+) - W(\phi_-)|.$$

This expression, where the energy is bound from below solely by the topological data, is known as **Bogomolny bound.**

→ Equality if:

$$\phi' = \pm\sqrt{2U(\phi)}.$$

The solutions with the + sign are called **kinks**, those with the minus sign **antikinks** (defining equation)

# $\phi^4$- kinks and kink-antikink solutions

Consider the theory with

$$U(\phi) = \mu + \nu\phi^2 + \lambda\phi^4.$$

Requiring $\pi_0(\mathcal{V}) = \mathbb{Z}_2$ and $U_{min} = 0$ we get

$$U(\phi) = \lambda(m^2 - \phi^2)^2 \text{ with degenerated minima } \phi = m \text{ and } \phi = -m$$

so overall:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \lambda(m^2 - \phi^2)^2$$

# $\phi^4$- kinks and kink-antikink solutions

If $\phi_+ \neq \phi_-$ we have a kink/antikink in our configuration. We can capture it's topological content by defining a conserved topological charge:

$$N = \frac{\phi_+ - \phi_-}{2m} = \frac{1}{2m}\int_{-\infty}^{\infty} \phi' dx = \int_{-\infty}^{\infty} j^0 dx$$

The corresponding current can be constructed

$$j^\mu = \frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu \phi$$

In this model: $N = \{0, +1, -1\}$ where

$N = 1$ is a kink, $N = -1$ an antikink

# $\phi^4$- kinks and kink-antikink solutions

Note: - “quantized charge” in classical theory?!

- $\phi \to -\phi$ in a kink solution yields an antikink
- $N > 1$ is not compatible with finite energy boundary condition
- a field configuration can contain a mixture

How about the Bogomolny bound?

$$E \geq \left| \int_{\phi_-}^{\phi_+} \sqrt{2\lambda}(m^2 - \phi^2) d\phi \right| = \left| \sqrt{2\lambda} \left[ m^2\phi - \frac{1}{3}\phi^3 \right]_{\phi_-}^{\phi_+} \right| = \frac{4}{3} m^3 \sqrt{2\lambda} |N|$$

Demanding equality:

$$\phi' = \sqrt{2\lambda}(m^2 - \phi^2)$$

# $\phi^4$- kinks and kink-antikink solutions

Defining equation $\phi' = \sqrt{2\lambda}(m^2 - \phi^2)$ can be solved by integrating:

$$\phi(x) = m \tanh(\sqrt{2\lambda} m(x-a))$$

The energy corresponding to the rest mass of the kink is given by:

$$E = \int_{-\infty}^{\infty} \mathcal{E}\, dx = \int_{-\infty}^{\infty} \frac{1}{2}\phi'^2 + \lambda(m^2 - \phi^2)^2\, dx \; = 2\lambda m^4 \operatorname{sech}^4\left(\sqrt{2\lambda} m(x-a)\right)$$

How does the kink resemble a particle?

- Energy density is concentrated over a finite region
- It can be Lorentz-boosted to any velocity smaller than the speed of light
- One can consider solutions with several kinks (antikinks) which can move with different speeds and scatter

# $\phi^4$- kinks and kink-antikink solutions

Show plot of a single kink and antikink

# $\phi^4$- kinks and kink-antikink solutions

Lorentz boosting this solutions the static kink is promoted to a dynamical one:

$$\phi(t,x) = m \tanh\left(\sqrt{2\lambda}\, m\gamma(x - vt - a)\right)$$

In non-relativistic limit, $\gamma \to 1$. Then $\dot{\phi} = -v\phi'$ and

$$T = \frac{1}{2}v^2 \int_{-\infty}^{\infty} \phi' \, dx = \frac{1}{2}Mv^2$$

Examples in Physics:

- Wave mechanics
- False vacua decay

# $\phi^4$- kinks and kink-antikink solutions

How to simulate a kink? = How to simulate the Klein-Gordon equation?

→ Discretize energy and time

→ Make sure this discretization leaves energy conservation untouched

→ Apply an implicit-explicit time stepping scheme which is time reversible and approximately conserves energy

$$\partial_\mu \partial^\mu \phi - 2(1-\phi^2)\phi = 0$$

$$\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi = 2(\phi - \phi^3)$$

$$\xrightarrow{\text{yields}} \frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{(\delta t)^2} - \Delta \frac{\phi^{n+1} + 2\phi^n + \phi^{n-1}}{4} = 2\left[\frac{\phi^{n+1} + 2\phi^n + \phi^{n-1}}{4} - {\phi^n}^3\right]$$

# $\phi^4$- kinks and kink-antikink solutions

Given $\frac{\phi^{n+1}-2\phi^{n}+\phi^{n-1}}{(\delta t)^2} - \Delta\frac{\phi^{n+1}+2\phi^{n}+\phi^{n-1}}{4} = 2\left[\frac{\phi^{n+1}+2\phi^{n}+\phi^{n-1}}{4} - {\phi^{n}}^3\right]$ how do we implement the spatial derivatives?

→ Given our program of choice is PYTHON, we can use the integrated function “fast Fourier transformation” (numpy.fft):

- Accessible via the extension package NUMPY, “numpy.fft” provides an implementation for a discrete Fourier transformation.

$$A_k = \sum_{m=0}^{n-1} a_m e^{-2\pi i \frac{mk}{n}}, \qquad k = 0, \dots, n-1$$

- The corresponding inverse Fourier transformation is given by “numpy.ifft”

# $\phi^4$- kinks and kink-antikink solutions

Given this powerful tool and that fact that time is discretized in $\delta t$ we can apply the Fourier transform:

Define $\Phi$ as the Fourier transform of $\phi$ and $\Phi_{new}$ as the Fourier transform of $\phi^{n+1}$($\Phi_{old}$ corresponding to $\phi^{n-1}$), we get

$$\Phi_{new} = \frac{1}{\frac{1}{(\delta t)^2} - \frac{k^2}{4} - \frac{1}{2}} \left[ \frac{1}{2}(2\Phi + \Phi_{old}) + k^2 \frac{2\Phi - \Phi_{old}}{4} - 2\Phi^3 + \frac{1}{(\delta t)^2}(2\Phi - \phi_{old}) \right]$$

This can be computed $\frac{t}{\delta t}$ times to simulate the dynamics governed by the Klein-Gordon equation.

- Show simulation of a single moving kink and error

# $\phi^4$- kinks and kink-antikink solutions

What happens with kink-antikink solutions?

→ Consider the same Lagrangian, set $\lambda = \frac{1}{2}$ and $m = 1$, and calculate the force as the change of the momentum $P = -\int_{-\infty}^{b} \dot{\phi}\phi' \, dx$:

$$F = \dot{P} = -\int_{-\infty}^{b} (\ddot{\phi}\phi' + \dot{\phi}\dot{\phi}')dx = \left[-\frac{1}{2}(\dot{\phi}^2 + \phi'^2) + U(\phi)\right]_{-\infty}^{b}$$

Considering a kink antikink pair with kink at position $a$ and antikink at $-a$ ($-a \ll b \ll a$), insert

$$\phi(x) = \phi_1(x) + \phi_2(x) + 1 = -\tanh(x+a) + \tanh(x-a) + 1.$$

# $\phi^4$- kinks and kink-antikink solutions

$$F = \left[-\frac{1}{2}{\phi_1'}^2 + U(\phi_1) - \phi_1'\phi_2' + (1+\phi_2)\frac{dU}{d\phi}(\phi_1)\right]_{-\infty}^{b} = [-\phi_1'\phi_2' + (1+\phi_2)\phi_1'']_{-\infty}^{b}$$

Since our field configuration is defined such that $\phi' \rightarrow 0$ for $x \rightarrow \infty$ and $-a \ll b \ll a$ insert

$$\phi_1(x) \sim -1 + 2e^{-2(x+a)},\ \phi_2 \sim -1 + 2e^{-2(x-a)},$$

which yields:

$$F = 32e^{-2R} = \frac{dE_{int}}{dR}$$

with $R = 2a$ .

# $\phi^4$- kinks and kink-antikink solutions

Show simulation of kink-antikink attraction

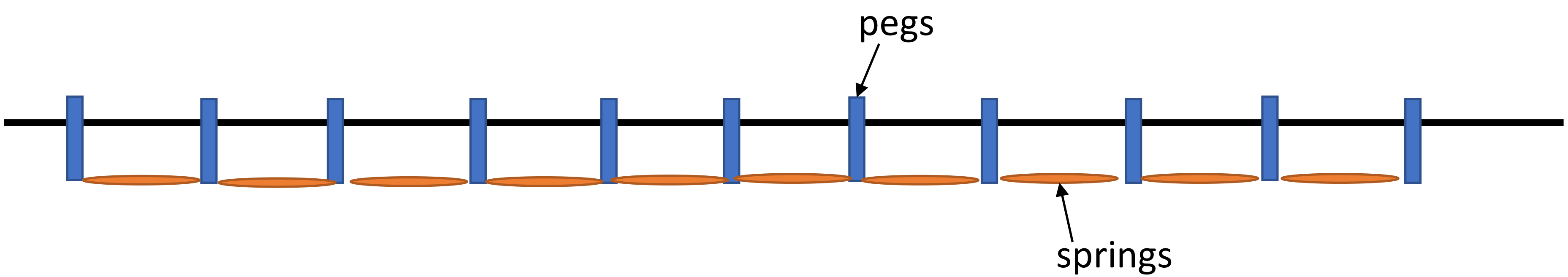
# Sine-Gordon kinks

Consider

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - (1 - \cos(\phi))$$

Which results in multiple zero energy vacua $\phi = 2\pi n$, with $n \in \mathbb{Z}$, in other words $\pi_0(\mathcal{V}) = \mathbb{Z}$.

Physical example? → Pegs on a clothesline



# Sine-Gordon kinks

In the same manner as in $\phi^4$ a field configuration is characterized by $(\phi_-, \phi_+)$.

Setting $\phi_- = 0$, since $\mathcal{L}' = \mathcal{L}$ for $\phi \to \phi \pm 2\pi$, then $N = \frac{\phi_+ - \phi_-}{2\pi}$ counts the number of kinks and

$$E \geq \int_0^{2\pi \mathrm{N}} 2 \left| \sin\left(\frac{\phi}{2}\right) \right| d\phi = 4|N| \left[ -\cos\left(\frac{\phi}{2}\right) \right]_0^{2\pi} = 8|N|$$

Equality is attained with $\phi' = \pm 2\sin(\frac{\phi}{2})$ which results in

$$\phi(x) = 4\arctan(e^{x-a}) + c$$

Here, $a$ again is the position of the kink which is confirmed by

$$\mathcal{E} = 4\,\mathrm{sech}^2(x-a)$$

# Sine-Gordon kinks

Show plot of Sine-Gordon kink

# Sine-Gordon kinks

Also analogously to $\phi^4$ we can calculate the interaction energy (for two kinks!) and get:

$$E_{int} = 32\, e^{-R}$$

→ Repulsive force

Note: There is no static multi-soliton solution in Sine-Gordon! (no multi-kink solutions of the Bogomolny equation)

Can we/How can we construct time-dependent multi-soliton solutions?

→Bäcklund transformation

# Bäcklund Transformation

For the following steps, let‘s introduce lightcone coordinates $x_\pm = \frac{1}{2}(x \pm t), \partial_\pm = \partial/\partial x_\pm$.

Then, the eq.o.m. is given by

$$\partial_- \partial_+ \phi = \sin(\phi)$$

Consider

$$\partial_+ \psi = \partial_+ \phi - 2\beta \sin\left(\frac{\phi + \psi}{2}\right), \partial_- \psi = -\partial_- \phi + \frac{2}{\beta} \sin\left(\frac{\phi - \psi}{2}\right)$$

and the compatibility condition $\partial_- \partial_+ \psi = \partial_+ \partial_- \psi$. Inserting the first in the latter we again get the eq.o.m.

Similarly $\partial_- \partial_+ \phi = \partial_+ \partial_- \phi$ yields the Sine-Gordon field equation for $\psi$.

→ Bäcklund transformation allows mapping between solutions of the Sine-Gordon equation

→ Generate new solutions from known ones

# Bäcklund Transformation

E.g. start with $\phi = 0$:

$$\partial_+\psi = -2\beta \sin\left(\frac{\psi}{2}\right),\ \partial_-\psi = -\frac{2}{\beta}\sin\left(\frac{\psi}{2}\right)$$

This can be integrated to

$$\psi(x_+, x_-) = 4\arctan\left(e^{-\beta x_+ - \frac{x_-}{\beta} + \alpha}\right),$$

where $\alpha$ is an integration constant. Defining

$$v = \frac{1-\beta^2}{1+\beta^2},\ \gamma = \frac{1}{\sqrt{1-v^2}} = -\frac{1+\beta^2}{2\beta},\ a = \frac{2\,\beta\alpha}{1+\beta^2},$$

we get

$$\psi(t,x) = 4\arctan\left(e^{\gamma(x-vt-a)}\right)$$

# Bäcklund Transformation – multi-kink

E.g. start with $\phi = \psi_0$ and calculate $\psi_1, \psi_2$, where we simply used two different Bäcklund parameter $\beta_1, \beta_2$.

Now use

$$\partial_+\psi_{12} = \partial_+\psi_1 - 2\beta_2 \sin\left(\frac{\psi_1 + \psi_{12}}{2}\right), \qquad \partial_-\psi_{12} = -\partial_-\psi_1 + \frac{2}{\beta_2}\sin\left(\frac{\psi_1 - \psi_{12}}{2}\right)$$

and

$$\partial_+\psi_{21} = \partial_+\psi_2 - 2\beta_1 \sin\left(\frac{\psi_2 + \psi_{21}}{2}\right), \qquad \partial_-\psi_{21} = -\partial_-\psi_2 + \frac{2}{\beta_1}\sin\left(\frac{\psi_2 - \psi_{21}}{2}\right)$$

# Bäcklund Transformation – multi-kink

The consistancy equation $\psi_{12} = \psi_{21}$ leads to

$$\psi_{12} = \psi_{21} = 4\arctan\left(\left(\frac{\beta_1 + \beta_2}{\beta_2 - \beta_1}\right)\tan\left(\frac{\psi_1 - \psi_2}{4}\right)\right) - \psi_0$$

Starting from $\psi_0 = 0$ we saw that $\psi_i = 4\arctan\left(e^{-\beta_i x_+ - \frac{x_-}{\beta_i} + \alpha_i}\right)$. Inserting this in $\psi_{12}$ leads to

$$\psi(t,x) = 4\arctan\left(\frac{v\sinh(\gamma x)}{\cosh(\gamma \mathrm{v} t)}\right)$$

when $\beta_1 = -\frac{1}{\beta_2} \equiv \beta,\ \alpha_1 = \alpha_2 = 0.$

# Bäcklund Transformation – multi-kink

Show 3 plots ,t<0, t=0, t>0 and ask the audience for N in this solution to wake them up again (maybe simulation if I can do it in time)

# Bäcklund Transformation – multi-kink

How do we interpret this solution?

Rewrite

$$\tan\left(\frac{\psi}{4}\right) = e^{\gamma(x-a)} - e^{-\gamma(x+a)}$$

with $a(t) = \frac{1}{\gamma}\log\left(\frac{2}{v}\cosh(\gamma v t)\right)$.

For $|vt| \gg 1$: - $a \sim |vt|$, and near $x = a$ the second part in the first equation can be neglected, the rest describes a single moving kink

- near $x = -a$ viseversa, $-a \sim -(|vt| + \delta)$

→ two well separated kinks, approaching the origin with $v$ for $t < 0$, and separating for $t > 0$

→ „Scattering solution“

# Bäcklund Transformation – multi-kink

Is it really scattering?

- Either scattering (elastic bounce) or pass through (with position shift $2\delta$)
- No distinction possible
- However, the repulsive force makes backward scattering more „physical“ (at least at slow speed)

How about kink-antikink scattering/annihilation?

- In Sine-Gordon elastic scattering
- Infinite number of conserved quantities prevent annihilation
- However, there are also exact time periodic solutions in $N = 0$ sector with kink-antikink bound states

# Thirring model

Let's have a look at the infinite number of conserved quantities. On may suggest the conclusion that:

- For degenerate vacua there can exist topological currents
- They lead to conserved quantities which are not associated with any manifest symmetry of the lagrangian (unlike Noether)

Is this true?

→ Consider the Thirring model:

$$\mathcal{L} = \frac{1}{2}\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi - \frac{1}{2}g\bar{\psi}\gamma^{\mu}\psi\bar{\psi}\gamma_{\mu}\psi$$

(1+1 dimensional spacetime)

From $\psi \to \psi' = e^{i\alpha}\psi$ we get the conserved current $J^{\mu} = \bar{\psi}\gamma^{\mu}\psi$. In the quantum treatment we get

$$Q = \int_{-\infty}^{\infty} J^0 dx, \qquad [Q, \psi(x)] = -\hbar\psi$$

# Thirring model

Therefore, $\frac{J^{\mu}}{\hbar} = \frac{\bar{\psi}\gamma^{\mu}\psi}{\hbar}$ and $j^{\mu} = \frac{\beta\epsilon^{\mu\nu}\partial_{\nu}\phi}{2\pi}$ each have charges of the spectrum $\mathbb{Z}$. Yet, one is a Noether current, one a topological.

They are even the same if we quantize Sine-Gordon and set

$$\frac{\beta^2\hbar}{4\pi} = \frac{1}{1 + \frac{g\hbar}{\pi}}$$

This condition follows from the demand of equivalent equal-time commutators for the currents.

→ The suggestion made before is in some sense false:

- there is a symmetry connected to the topological current

- this becomes evident when transforming from $\phi \rightarrow \psi$ and comparing Sine-Gordon with Thirring

# Thank you for your attention!

References:

[1]: N. Manton, P. Sutclie, “Topological solitons", Cambridge University Press (2004)
[2]: P. Goddard, D.I. Olive, “Magnetic monopoles in gauge field theories", Rep. Prog. Phys. 41, 1357 (1978)
[3]: https://en.wikibooks.org/wiki/Parallel_Spectral_Numerical_Methods/The_Klein-Gordon_Equation, last visit 20.04.20