2019-2020 MIDLANDS PHYSICS ALLIANCE GRADUATE SCHOOL

Black Holes and Gravitational Waves – Part 2

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Exercise 1: Gravitational radiation of the simple harmonic oscillator

Assume a gravitational system with reduced mass μ oscillating along the z direction with equation of motion

$$z_0(t) = a\cos\omega_s t\,. \tag{1}$$

The time-time component of the stress energy tensor is

$$T^{00}(t,\mathbf{x}) = \mu\delta(x)\delta(y)\delta(z - z_0(t))\,. \tag{2}$$

1. Predict the scalings of the radiated power P with μ, a, and ω_s using only the quadrupole formula $P \propto \dddot{Q}^2$.
2. Compute the second mass moment $M^{ij}(t)$.
3. Compute the gravitational-wave polarization $h_+(t,\theta,\phi)$ and $h_\times(t,\theta,\phi)$ where θ and ϕ are suitable spherical coordinates.
4. What is the frequency of the emitted waves?
5. Does the emission pattern reflect the cylindrical symmetry of the source?
6. The emission vanishes along the z axis. How is this related to the TT gauge projection tensor Λ_{ijkl}?
7. Compute the radiated power per unit angle $dP/d\Omega$.
8. Compute the total radiated power P. Does this agree with the estimate of point 1?
9. Compute the energy emitted during a single oscillation period $2\pi/\omega_s$.
10. Compute the back-reaction force acting on the source (hint: first predict the scalings).

Exercise 2: Black hole binaries in elliptic orbits

Consider two masses m_1 and m_2 in elliptic orbits. As usual, the system is reduced to a particle of reduced mass $\mu = m_1 m_2/(m_1+m_2)$ subject to an acceleration $\ddot{\mathbf{r}} = -m/r^2\hat{\mathbf{r}}$ where $m = m_1 + m_2$. Recall that Kelper's orbits are given by

$$r = \frac{a(1-e^2)}{1+e\cos\psi} \tag{3}$$

where a is the semi-major axis, e is the eccentricity, and ψ is the true anomaly. The motion along the orbit is described by

$$\dot{\psi} = \left(\frac{m}{a}\right)^{1/2}(1-e^2)^{-3/2}(1+e\cos\psi)^2. \tag{4}$$

The orbital period is set by Kepler's law

$$T = 2\pi\sqrt{\frac{a^3}{m}} \tag{5}$$

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1. Compute the second mass moment $M^{ij}(t)$.
2. Compute the total power radiated P as a function of the orbital position ψ.
3. Average the result over the orbital period T. You should get

$$P = \frac{32\mu^2 m^3}{5a^5} \frac{1}{(1-e^2)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right) \tag{6}$$

   This is a classic result by Peters and Mathews (1963).
4. Does this expression reduces to the circular case correctly?
5. What happens if $e \to 1$?
6. Hang on: I thought that an elliptic orbit reduces to a parabolic one for $e \to 1$. Does the power emitted during a parabolic encounter diverges? The answer is no. What's going on? (hint think about the limit we are doing).