Midlands Physics Alliance Graduate School

# Black Holes and Gravitational Waves
# Part I

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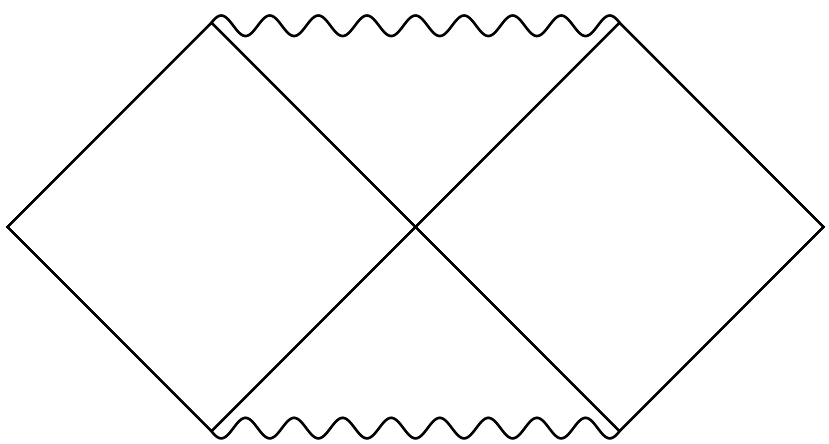

These notes have been prepared for part of a lecture course to be delivered within the 2019 MPAGS program: `https://warwick.ac.uk/fac/sci/physics/mpags`.

The following textbooks were used during the preparation of these notes:

1. Michele Maggiore, "Gravitational Waves. Volume 1: Theory and Experiment". Oxford University Press; 2008.

2. Bernard F. Schutz, "A First Course in General Relativity". Cambridge University Press; 1985.

3. Sean M. Carroll, "An Introduction to General Relativity: Spacetime and Geometry". Addison-Wesley; 2003.

4. Robert M. Wald, "General Relativity". University of Chicago Press; 1984.

The course is divided into three parts which will be lectured by Christopher Moore, Davide Gerosa and Patricia Schmidt respectively. These lecture notes cover only the first part of the course. Part of these notes has evolved from an earlier document prepared in collaboration with David Hilditch.

This course is being taught for the first time this year and therefore these notes will inevitably contain a number of errors; if you spot any please send them to `cmoore@star.sr.bham.ac.uk`. If you have any other comments or questions about the course, then please contact the course lecturers at `cmoore@star.sr.bham.ac.uk`, `dgerosa@star.sr.bham.ac.uk` and `pschmidt@star.sr.bham.ac.uk`.

# Contents

# Chapter 1

# General Relativity

## 1.1 Recapping the general theory of relativity

As this is a graduate course it will be assumed that the reader already has some familiarity with the subject. This section will give only a very brief review and will also serve to introduce some notation.

Let us introduce *coordinates* $x^\mu = (ct, x^i) = (ct, x, y, z)$. Throughout we will take Greek indices $\mu = 0 \ldots 3$, and Latin indices $i = 1 \ldots 3$. The *metric* is an invertible symmetric matrix $g_{\mu\nu}$, each component of which is a function of the coordinates; i.e. $g_{\mu\nu}(x)$. "Symmetric" here means that $g_{\mu\nu} = g_{\nu\mu}$. The metric is taken to have Lorentzian signature, which means that it has one negative and three positive eigenvalues. The metric allows us to measure distances and times; for two points in spacetime separated by some infinitesimal change in coordinates $\mathrm{d}x^\mu$ the proper distance interval between these points is defined to be

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \,. \tag{1.1}$$

The inverse metric is denoted by $g^{\mu\nu}$. We use the *Einstein summation convention*, which means that repeated indices, one upstairs and one downstairs, are understood to be summed over. In other words the expression $g^{\mu\alpha}g_{\alpha\nu}$ corresponds simply to matrix multiplication and so by definition we have $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu{}_\nu$.

In General Relativity (GR) the metric replaces the Newtonian gravitational potential as the unknown, and is required to satisfy *Einstein's equations*:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,. \tag{1.2}$$

The object on the right hand side, $T_{\mu\nu}$, is called the *stress-energy tensor*. $G$ and $c$ are Newton's constant and the speed of light, respectively. The stress-energy tensor is symmetric in $\mu\nu$, and encodes the local energy density, linear momentum and the stresses and strains of any matter content. The object on the left hand side is called the *Einstein tensor*. It is also symmetric, and is constructed from the *Ricci tensor* $R_{\mu\nu}$ as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,, \tag{1.3}$$

with the *Ricci scalar* $R = g^{\mu\nu} R_{\mu\nu}$. A general feature of our notation is that, unless doing so would clash with any previously defined quantity, if the inverse metric is used to trace over two indices of a matrix the resulting object gets the same name but without the indices. The Ricci curvature is in turn defined as a trace of the *Riemann curvature* $R_{\mu\nu\rho}{}^{\sigma}$,

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho} \,, \tag{1.4}$$

$$R_{\mu\nu\rho}{}^{\sigma} = \partial_\nu \Gamma^{\sigma}{}_{\mu\rho} - \partial_\mu \Gamma^{\sigma}{}_{\nu\rho} + \Gamma^{\alpha}{}_{\mu\rho}\Gamma^{\sigma}{}_{\nu\alpha} - \Gamma^{\alpha}{}_{\nu\rho}\Gamma^{\sigma}{}_{\mu\alpha} \,, \tag{1.5}$$

where the *Christoffel symbols* are given by first derivatives of the metric,

$$\Gamma^{\gamma}{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} (\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}) \,. \tag{1.6}$$

With this soup of indices and derivatives it is easy to get lost, but the take home message is that the left-hand side of the field equations contains objects related to the curvature, and the right-hand side stuff to do with matter. This is well-described by Wheeler's famous statement that "Spacetime tells matter how to move; matter tells spacetime how to curve". What's more, since the curvature contains one derivative of the Christoffel, and the Christoffel contains one derivative of the metric, we have

$$G_{\mu\nu}[\partial\partial g, \partial g, g] \approx T_{\mu\nu}[\text{Matter}] \,. \tag{1.7}$$

just like in Newtonian gravity where the potential satisfies *Poisson's equation*;

$$\nabla \phi(\vec{x}) = 4\pi G \rho(\vec{x}) \,. \tag{1.8}$$

It is important to realize however the *crucial* difference that in the latter case the field equations are linear, whereas in GR they are not.

## 1.2 The geodesic equation

In general relativity a freely falling test particle moving in a curved background spacetime follows a *timelike geodesic* in the spacetime metric, $g_{\mu\nu}(x)$. Using a system of coordinates $x^\mu$, the *worldline* of a test particle is the curve $x^\mu(\lambda)$ (see figure 1.1). The parameter $\lambda$ labels points along the curve and is otherwise completely arbitrary; we are free to choose any convenient parameterisation. For two nearby points on the curve, $x^\mu(\lambda)$ and $x^\mu(\lambda + \mathrm{d}\lambda)$, the distance interval between them is calculated using the metric;

$$\mathrm{d}s^2 = g_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \mathrm{d}\lambda^2 \,. \tag{1.9}$$

We will assume that the curve is *timelike*, meaning $\mathrm{d}s^2 < 0$. (Remember, we are using the Minkowski metric sign convention $\eta_{\mu\nu} = \mathrm{diag}[-,+,+,+]$.) The *proper time* interval along the worldline is defined as

$$\mathrm{d}\tau^2 = \frac{-\mathrm{d}s^2}{c^2} < 0\,. \tag{1.10}$$

The proper time is the time that would be measured by a clock moving along the curve $x^\mu(\lambda)$. Because $\mathrm{d}s^2 < 0$ the proper time is always increasing along the curve. Therefore, it is natural to use $\tau$ as a parameter along the curve; we write the position of the particle as a function of its own proper time, $x^\mu(\tau)$. The particle's *four-velocity* is defined as $u^\mu(\tau) = \mathrm{d}x^\mu/\mathrm{d}\tau$; the four-velocity is tangent to the world line of the particle. Notice, that equation (1.9) implies that the magnitude of the four-velocity vector is constant along the worldline of the particle;

$$-c^2 = g_{\mu\nu}\, u^\mu(\tau)\, u^\nu(\tau)\,. \tag{1.11}$$

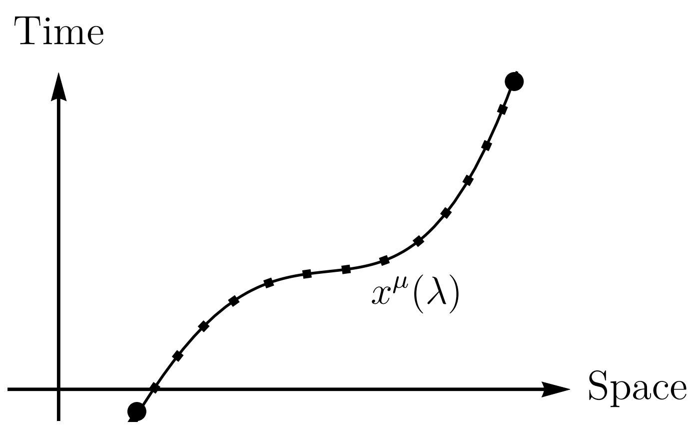


Figure 1.1: One possible particle *worldline* through spacetime linking two fixed endpoints. The parameter $\lambda$ labels points on the curve and is otherwise arbitrary. The tick marks along the curve are intended to indicate equal intervals of $\lambda$.

Among all possible curves linking two fixed endpoints, $x^\mu(\tau_A)$ and $x^\mu(\tau_B)$, a test particle will follow the *geodesic* which extremises the total proper time along the curve. The total proper time is given by

$$S\big[x^\mu(\tau)\big] = \int_{\tau_A}^{\tau_B} \mathrm{d}\tau = \int_{\tau_A}^{\tau_B} \frac{\mathrm{d}\tau}{c}\, \sqrt{-g_{\mu\nu}\, u^\mu(\tau)\, u^\nu(\tau)}\,. \tag{1.12}$$

The calculus of variations can be used to find the equations that must be satisfied by the curve (see box 1.1) and the resultant *geodesic equations* are

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\nu\rho}(x)\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0\,, \tag{1.13}$$

$$\text{or}\quad \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\nu\rho}(x)\,u^\mu(\tau)\,u^\nu(\tau) = 0\,, \tag{1.14}$$

where the following quantities (know as the *connection coefficients*, or the *Christoffel symbols*) have been defined,

$$\Gamma^\mu{}_{\nu\rho}(x) = \frac{1}{2}g^{\mu\alpha}(x)\big[\partial_\nu g_{\alpha\rho}(x) + \partial_\rho g_{\alpha\nu}(x) - \partial_\alpha g_{\mu\nu}(x)\big]\,. \tag{1.15}$$

Equations (1.13) or (1.14) are the equations of motion for a test particle in general relativity. Although the geodesic equations have been derived here for timelike curves, these equations also hold for *spacelike* ($\mathrm{d}s^2 > 0$) and *null* ($\mathrm{d}s^2 = 0$) geodesics.

**Box 1.1: The geodesic equation from calculus of variations**

The integrand, or *Lagrangian*, of the functional in equation (1.12) is given by

$$\mathcal{L}\big(x^\mu, u^\mu\big) = \frac{1}{c}\sqrt{-g_{\mu\nu}\,u^\mu(\tau)\,u^\nu(\tau)}\,. \tag{i}$$

The Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg(\frac{\partial\mathcal{L}}{\partial u^\mu}\bigg) = \frac{\partial\mathcal{L}}{\partial x^\mu}\,. \tag{ii}$$

Evaluating the partial derivatives gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{g_{\mu\nu}(x)\,u^\nu(\tau)}{\sqrt{-g_{\alpha\beta}(x)\,u^\alpha(\tau)\,u^\beta(\tau)}}\right) = \frac{\frac{1}{2}\partial_\mu g_{\gamma\delta}(x)\,u^\gamma(\tau)\,u^\delta(\tau)}{\sqrt{-g_{\alpha\beta}(x)\,u^\alpha(\tau)\,u^\beta(\tau)}}\,. \tag{iii}$$

Using the result in equation (1.11) simplifies the denominator of this equation. Evaluating the total derivative with respect to $\tau$ gives

$$g_{\mu\nu}(x)\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau} + \partial_\alpha g_{\mu\beta}(x)\,u^\alpha(\tau)\,u^\beta(\tau) = \frac{1}{2}\partial_\mu g_{\alpha\beta}(x)\,u^\alpha(\tau)\,u^\beta(\tau)\,, \tag{iv}$$

$$g_{\mu\nu}(x)\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau} + \frac{1}{2}\big[\partial_\alpha g_{\mu\beta}(x) + \partial_\beta g_{\mu\alpha}(x) - \partial_\mu g_{\alpha\beta}(x)\big]\,u^\alpha(\tau)\,u^\beta(\tau) = 0\,. \tag{v}$$

In the last step dummy indices have been relabelled. Multiplying through by the inverse of $g_{\mu\nu}(x)$ gives the result in equation (1.13).

The geodesics equations look a bit messy when written in this form. If you are familiar with the concept of a covariant derivative then they can be written more elegantly as,

$$u^\alpha \nabla_\alpha u^\mu = 0\,. \tag{vi}$$

Written in this form reveals that the tangent vector $u^\mu(\tau)$ is parallel transported along the worldline.

**Exercise 1.1:**
Show that the same geodesic equations may be obtained instead from the Lagrangian

$$\mathcal{L}(x^\mu, u^\mu) = \frac{1}{2} g_{\mu\nu} u^\mu(\tau) u^\nu(\tau)\,. \tag{i}$$

In practice, this modified Lagrangian is often easier to work with than that given in equation 1.12 due to the absence of the square root.

# Chapter 2

# Gravitational Waves

## 2.1 The linearised field equations

The full non-linear field equations in (1.3) are impossible to solve in most circumstances. To try and make progress we can use perturbation theory and analyse the behaviour of small (linear) perturbations about a known background solution.

In the absence of any nontrivial gravitational field we can introduce Cartesian-like coordinates in which the *background metric* becomes,

$$\eta_{\mu\nu} = \mathrm{diag}[-,+,+,+]\,, \tag{2.1}$$

just as in special relativity. Let us now consider solutions with a metric of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with the small *metric perturbation* $h_{\mu\nu}$. The inverse metric (see exercise 2.1) can be written as,

$$g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta} + O(h^2). \tag{2.2}$$

Henceforth we will use the background metric, $\eta^{\mu\nu}$ or $\eta_{\mu\nu}$, to raise and lower indices (i.e. $h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$). Any corrections to this approximation enter at higher order in $h$.

**Exercise 2.1: The linearised inverse metric**
Verify that, up to linear order in $h$, the linearised inverse metric is indeed given by equation (2.2).

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There is a point of view in which linearised GR is a theory of a rank 2 tensor field $h_{\mu\nu}$ propagating on a fixed Minkowski background metric $\eta_{\mu\nu}$. This point of view makes the analogy between gravity and electromagnetism strongest; after all, electromagnetism is the theory of a rank 1 tensor field $A_\mu$ propagating on a fixed background.

Using our ansatz $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ for the metric we can proceed to calculate the Christoffel symbols, the Riemann tensor, and the Ricci tensor and scalar. This is a somewhat lengthy but conceptually trivial calculation resulting in the following expressions:

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}\left(\partial_\nu h_{\sigma\rho} + \partial_\rho h_{\sigma\nu} - \partial_\sigma h_{\nu\rho}\right), \tag{2.1}$$

$$R_{\mu\nu\lambda\rho} = \frac{1}{2}\left(\partial_\mu\partial_\rho h_{\lambda\nu} - \partial_\mu\partial_\lambda h_{\rho\nu} - \partial_\nu\partial_\rho h_{\lambda\mu} + \partial_\nu\partial_\lambda h_{\rho\mu}\right), \tag{2.2}$$

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma\partial_\nu h^{\sigma}{}_{\mu} + \partial_\sigma\partial_\mu h^{\sigma}{}_{\nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right), \tag{2.3}$$

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h\,, \tag{2.4}$$

where $h = \eta^{\mu\nu}h_{\mu\nu} = h^{\mu}{}_{\mu}$. It is sometimes convenient to introduce the *trace-reversed* metric perturbation $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. Plugging these expressions into the field equations and keeping only terms linear in $h_{\mu\nu}$ we obtain, the linearised Einstein equations,

$$-\frac{1}{2}\Box\bar{h}_{\mu\nu} + \partial^\lambda\partial_{(\mu}\bar{h}_{\nu)\lambda} - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\mu\nu}\,. \tag{2.5}$$

The wave, or d'Alembert, operator is denoted by $\Box \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ and round brackets around two indices denotes symmetrization $X_{(\mu\nu)} \equiv (X_{\mu\nu} + X_{\nu\mu})/2$. For self-consistency of the approximation we must also assume that the stress-energy tensor is small. On the left-hand side we see that the first term is of a simple form; solutions to that operator can be treated in a standard way by using Green's functions. Unfortunately, the second and third terms are more complicated. We might therefore wish they were absent. Fortunately, this can be arranged.

Finally, we remark that at linear order in the metric perturbation the law of conservation of energy and momentum becomes

$$\partial_\mu T^{\mu\nu} = 0\,. \tag{2.6}$$

## 2.2 The choice of gauge

Consider the expression for the linearised Riemann curvature tensor in equation (2.2). Now consider a particular metric perturbation of the form $h_{\alpha\beta} = \partial_\alpha\phi_\beta + \partial_\beta\phi_\alpha$; it is possible to show that $R_{\mu\nu\lambda\rho} = 0$ (see exercise 2.2). This shows that not all metric perturbations give rise to any curvature, rather some just represent a poor choice of coordinates.

Since the field equations are built from contractions of the curvature, this means that we have a *gauge-freedom*. This is analogous to the freedom to add a constant to the Newtonian gravitational potential or (more closely) to the gauge freedoms encountered in Electromagnetism. We will exploit this freedom to see if we can make our ugly equations (2.5) prettier. Under an infinitesimal coordinate transformation $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$ the metric perturbation transforms as (see problem sheet),

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu. \tag{2.7}$$

The trace-reverse metric perturbation transforms as

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \partial^\alpha \xi_\alpha \eta_{\mu\nu}. \tag{2.8}$$

By the observation above, this will take a solution of the field equations to new solution; physically the two solutions should be identified. Choosing the *Lorenz-gauge* with

$$\Box \xi_\mu = \partial^\nu \bar{h}_{\mu\nu}\,, \tag{2.9}$$

we find that the modified metric perturbation must satisfy,

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}\,, \tag{2.10}$$

$$\partial^\nu \bar{h}_{\mu\nu} = 0\,. \tag{2.11}$$

This is choice is named from the resemblance with the Lorenz-gauge in Electromagnetism. We have succeeded in getting rid of all of the messy looking terms in equation (2.5)!

In fact when we are in vacuum we can do even better, by noting that if we make yet another adjustment, but this time taking $\Box \bar{\xi}_\mu = 0$, our solution will still still satisfy the Lorenz-gauge condition. Using this we can arrive at the *Transverse-Traceless* (TT) gauge

$$h = 0\,, \qquad h_{0j} = 0\,. \tag{2.12}$$

To achieve this we just choose $\bar{\xi}_\mu$ such that,

$$\partial_\mu \bar{\xi}^\mu = \frac{1}{2} h\,, \qquad \partial_0 \bar{\xi}_j + \partial_j \bar{\xi}_0 = h_{0j}\,. \tag{2.13}$$

By imposing the additional four equations (2.12) we have used up the remaining gauge freedom. Threrefore we expect that there are

$$2 = 10 \text{ \# (Field equations)} - 4 \text{ \#(Lorenz gauge conditions)} - 4 \text{ \#(TT gauge conditions)}$$

remaining physical degrees of freedom; these are the *gravitational waves* (GWs)!

**Exercise 2.2: Curvature of a pure gauge metric perturbation**
Show that the linearised Riemann tensor (2.2) vanishes for a metric perturbation of the form $h_{\alpha\beta} = \partial_\alpha \phi_\beta + \partial_\beta \phi_\alpha$. Hint: substitute the perturbation into equation 2.2 and use equality of mixed partials.

**Exercise 2.3: linearised Gravity in Lorenz-gauge**
Starting with the stated gauge adjustment, show that equations (2.10) and (2.11) hold true in the Lorenz gauge. You may use the expressions,

$$R_{\mu\lambda} = \partial^\nu \partial_{(\mu} h_{\lambda)\nu} - \frac{1}{2}\Box h_{\lambda\mu} - \frac{1}{2}\partial_\mu \partial_\lambda h \,, \tag{i}$$

$$R = \partial^\nu \partial^\lambda h_{\nu\lambda} - \Box h \,, \tag{ii}$$

for the linearised Ricci tensor and scalar. As in exercise 2.2 the important fact to use is the equality of mixed partial derivatives.

## 2.3 Plane wave solutions

Let us work in vacuum and make a plane wave ansatz,

$$h_{\mu\nu} = \Re\left[\epsilon_{\mu\nu} e^{ik_\lambda x^\lambda}\right] . \tag{2.1}$$

When computing we will just take the real part at the end. The matrix $\epsilon_{\mu\nu}$ is a constant tensor, it is called the polarization tensor. The vector $k_\mu$ is also a constant and is called the wave vector. Plugging this ansatz into the Lorenz-gauge Einstein equations we obtain,

$$k^\mu k_\mu [\text{ Nonvanishing Stuff }] = 0, \tag{2.2}$$

so that the wave-vector is null; $k^\mu k_\mu = 0$. Physically, this means that the solutions all travel at light-speed. This should not be surprising since the field equations reduced to a wave equation which looked very reminiscent of Maxwell's equations.

Not all of these traveling solutions represent physical waves however, as we must still impose the Lorenz-gauge. Substituting our ansatz into (2.11) we obtain,

$$k^\nu \epsilon_{\mu\nu} = \frac{1}{2} k_\mu \epsilon^\nu{}_\nu \,. \tag{2.3}$$

This restricts four of the ten components. Suppose, for simplicity, that the wave is propagating in the $z$ direction of our chosen coordinate system, then $k^\mu = (k, 0, 0, k)$. In the TT gauge $\epsilon^\mu{}_\mu = 0$, so it follows that $\epsilon^\mu{}_\nu k^\nu = 0$. In TT gauge we also have $\epsilon_{0j} = 0$ and hence $\epsilon_{0\mu} = \epsilon_{3\mu} = 0$. The polarization tensor can thus be written,

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2.4}$$

The polarization tensor is a sum of two independent constant matrices, one associated with $h_+$, the other with $h_\times$. These are the two gravitational wave polarization states, commonly called *plus* and *cross*. This will be discussed in more detail in the following lecture.

## 2.4 The interaction of gravitational waves with matter

The first part of this course described weak gravitational fields as small perturbations away from flat Minkowski space. Working in (nearly) Cartesian coordinates $x^\mu = \{ct, x, y, z\}$ the metric is given by $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$. We also saw that GWs admit an especially simple description in the transverse-traceless (TT) gauge. Recall that in the TT gauge a linearly polarised, plane-fronted, monochromatic GW travelling in the $z$-direction can be written as a combination of two linearly independent polarisation states:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \exp\left[\mathrm{i}(\omega t - kz)\right]. \tag{2.5}$$

The GW angular frequency is $\omega$ and the GW travels at the speed of light, so $k = \omega/c$. The constants $h_+$ and $h_\times$ are the amplitudes of the two GW polarisation states and are assumed to be small ($h_{+,\times} \ll 1$). (The metric components are all real numbers; in equation (2.5) and henceforth it is to be understood that the real part is taken implicitly.) With this form of the metric the distance element becomes

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -c^2\mathrm{d}t^2 + \mathrm{d}z^2 + \Big\{1 + h_+ \exp\left[\mathrm{i}(\omega t - kz)\right]\Big\}\mathrm{d}x^2 \\ &\quad + \Big\{1 - h_+ \exp\left[\mathrm{i}(\omega t - kz)\right]\Big\}\mathrm{d}y^2 + 2h_\times \exp\left[\mathrm{i}(\omega t - kz)\right]\mathrm{d}x\mathrm{d}y\,. \end{aligned} \tag{2.6}$$

Given this metric, we can now solve the geodesics equations to see how test particles will move in this metric. The 4 spacetime coordinates of the worldline of a particle are

$$x^\mu(\tau) = \Big\{ct(\tau),\, x(\tau),\, y(\tau),\, z(\tau)\Big\}\,. \tag{2.7}$$

The geodesics equations are four, second order differential equations; therefore, we need to specify the initial spacetime position (4 constraints) and the initial four-velocity ($4+4=8$ constraints) to uniquely solve these equations. We will choose as our initial conditions a particle at rest (i.e. not moving in space) at an arbitrary point in space.

$$\begin{aligned} &x(0) = x_0\,, \qquad y(0) = y_0\,, \qquad z(0) = z_0\,, \\ &\left.\frac{\mathrm{d}x}{\mathrm{d}\tau}\right|_{\tau=0} = 0\,, \quad \left.\frac{\mathrm{d}y}{\mathrm{d}\tau}\right|_{\tau=0} = 0\,, \quad \left.\frac{\mathrm{d}z}{\mathrm{d}\tau}\right|_{\tau=0} = 0\,. \end{aligned} \tag{2.8}$$

We will also initialise our time coordinate such that $t(0) = 0$ But given these choices we are now forced to have our test particle "moving" forwards in time, $\mathrm{d}t/\mathrm{d}\tau\big|_{\tau=0} = c$, in order to satisfy the constraint in equation (1.11).

Having chosen the particle's position and velocity the geodesic equations allow us to calculate the particle acceleration. The $\mu = 1,2,3$ components of equation (1.13) give

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2}\bigg|_{\tau=0} = \left[\Gamma^x{}_{tt}(x)\right]_{\tau=0}, \tag{2.9}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2}\bigg|_{\tau=0} = \left[\Gamma^y{}_{tt}(x)\right]_{\tau=0}, \tag{2.10}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2}\bigg|_{\tau=0} = \left[\Gamma^z{}_{tt}(x)\right]_{\tau=0}. \tag{2.11}$$

Therefore, we need to compute the $\Gamma^x{}_{tt}$, $\Gamma^y{}_{tt}$, and $\Gamma^z{}_{tt}$ *connection coefficients*. This can be done using the definition in equation (1.15) and the metric in equation (2.5). It is a straightford exercise in differentiation to compute these connection coefficients, Taylor expand them to linear order in $h_+$ and $h_\times$, and finally to show that

$$\Gamma^x{}_{tt}(x) = \Gamma^y{}_{tt}(x) = \Gamma^z{}_{tt}(x) = 0\,. \tag{2.12}$$

Just these three components are enough to show that all the spatial components of the acceleration in equation (2.9) are zero. Freely falling particles that start at rest in our coordinates remain at rest. The worldline of our particle must therefore be

$$t(\tau) = \tau\,, \quad x(\tau) = x_0\,, \quad y(\tau) = y_0\,, \quad z(\tau) = z_0\,. \tag{2.13}$$

Don't misunderstand this result as implying that GWs have no effect on free particles. All this result is saying is that the particles remain at rest in our chosen system of coordinates. To understand the true effect of the GW we need to calculate a geometrical quantity (i.e. a scalar) with a value independent of the coordinates used to describe it.

One way to do this is to place many test particles in a circular ring around an observer centred at the origin. As we have seen, the spatial coordinates of the particles and the observer don't change with time. We choose the coordinates of the particles such that

$$z_0 = 0\,, \quad x_0^2 + y_0^2 = R_0^2\,. \tag{2.14}$$

Now compute the proper distance from the origin to the ring of test particles. Although the ring appears to be circular in our coordinate, this distance will nevertheless depend on time and also on which direction we choose. It is natural to introduce 2 dimensional plane polar coordinates $(r,\phi)$ to describe the plane containing the ring;

$$x_0 = r\cos\phi\,, \quad y_0 = r\sin\phi\,. \tag{2.15}$$

Using equation (2.6), the proper distance to the ring is given by (see box 2.1)

$$R(\phi,t)/R_0 = 1 + \frac{h_+}{2}\exp(\mathrm{i}\omega t)(\cos^2\phi - \sin^2\phi) + h_\times \exp(\mathrm{i}\omega t)\cos\phi\sin\phi\,. \tag{2.16}$$

The effect of the GW on the proper distance to the ring is illustrated in figure (2.1).

**Box 2.1: The proper distance**

The proper distance from the origin to the ring of test particles is obtained by integrating the distance element in equation (2.6),

$$R(\phi, t) = \int_{\text{Origin}}^{\text{Ring}} \mathrm{d}s. \tag{i}$$

Integrating along a curve with $\mathrm{d}t = \mathrm{d}z = \mathrm{d}\phi = 0$ gives

$$R(\phi, t) = \int_0^{R_0} \mathrm{d}r \left( \Big\{ 1 + h_+ \exp\big[\mathrm{i}(\omega t - kz)\big] \Big\} \left(\frac{\mathrm{d}x}{\mathrm{d}r}\right)^2 + \right. \tag{ii}$$
$$\left. \Big\{ 1 - h_+ \exp\big[\mathrm{i}(\omega t - kz)\big] \Big\} \left(\frac{\mathrm{d}y}{\mathrm{d}r}\right)^2 + 2h_\times \exp\big[\mathrm{i}(\omega t - kz)\big] \frac{\mathrm{d}x}{\mathrm{d}r}\frac{\mathrm{d}y}{\mathrm{d}r} \right)^{1/2}.$$

Using $\mathrm{d}x/\mathrm{d}r = \cos\phi$ and $\mathrm{d}y/\mathrm{d}r = \sin\phi$ from equation (2.15), and expanding to linear order in the GW amplitude ($h_+$ and $h_\times$) we obtain equation (2.16).

**Exercise 2.4:**
Calculate how the *proper area* inside the ring changes over time.

## 2.5 The generation of gravitational waves

We have seen that in the Lorenz gauge the linearised Einstein equations with sources are,

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}\,, \tag{2.1}$$
$$\partial^\nu \bar{h}_{\mu\nu} = 0\,. \tag{2.2}$$

Working in complete generality by assuming slowly moving, weak, weakly self-gravitating, isolated sources, and evaluating the solution far from the source, the TT part of the metric perturbation can be written as,

$$h_{kl}^{\mathrm{TT}} \approx \frac{2G}{c^4 r} \perp_{kl}^{ij}(n)\, \ddot{\mathcal{I}}_{ij}(t - r/c)\,. \tag{2.3}$$

This is the famous *quadrupole formula*. It determines the gravitational wave emission from the traceless mass quadrupole,

$$\mathcal{I}_{ij}(t) = \int_{\mathbb{R}^3} \rho(t, x'^i)\big(x'_i x'_j - \frac{1}{3}\delta_{ij} r'^2\big) d\vec{x}' \,. \tag{2.4}$$

The isolated system is assumed to be located near the origin. The radial scalar function is defined in the obvious way as $r^2 = x^2 + y^2 + z^2$. The vector $n^i$ is the unit spatial vector in the direction of propagation of the gravitational wave. Since our observer is assumed to lie a large distance from the source, we can set $n^i = x^i/r$. Finally the TT projection operator is,

$$\perp^{ij}_{kl}(n) = q^i{}_{(k} q^j{}_{l)} - \frac{1}{2} q_{kl}\, q^{ij} \,, \tag{2.5}$$

$$q^i{}_j = \delta^i{}_j - n^i n_j \,. \tag{2.6}$$

Here we raise and lower indices with the kronecker delta $\delta^{ij}$, which corresponds to the spatial part of the background metric. The derivation of the quadrupole formula starts with the Green's function form of the solution to the wave equation. The resulting expression is then manipulated using the simplifying assumptions stated above.

**Box 2.2: Gravitational waves from a circular Newtonian binary system**

Let us assume that our matter source consists of a binary system consisting of two point particles of equal mass traveling on Newtonian circular orbits in the $xy$ plane, and compute the TT part of the metric perturbation for an observer on the positive $z$-axis. Those of you who have studied GR may note that this exercise is rather artificial, because the point particles should follow geodesics of flat-space, and should hence not travel on a circular orbit. This shortcoming can be fixed within the Post-Newtonian approximation. This example nevertheless serves as an important first computation.

The positions of the two masses can then be written,

$$x^i_1(t) = R(\cos(\omega t), \sin(\omega t), 0) \,, \quad x^i_2(t) = R(-\cos(\omega t), -\sin(\omega t), 0) \,. \tag{i}$$

The radius of the orbit $R$ is a constant that can be computed from Kepler's third law as $R^3 = GM/\omega^2$ with the total mass of the system $M$.

The mass-density for this system is given by,

$$\rho = \frac{1}{2}M\delta(x - x_1, y - y_1, z) + \frac{1}{2}M\delta(x - x_2, y - y_2, z)\,, \tag{ii}$$

with $\delta$ here denoting the three-dimensional Dirac delta function. Using some trigonometric identities the traceless mass quadrupole can then be computed as,

$$\mathcal{I}_{ij} = \frac{1}{8}MR^2 \begin{pmatrix} \cos(2\omega t) + \frac{1}{3} & \sin(\omega t) & 0 \\ \sin(2\omega t) & -\cos(2\omega t) + \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}. \tag{iii}$$

The TT projection operator $\perp^{ij}_{kl}(n)$ is easily constructed using the fact that for our observer $n^i = (0, 0, 1)$, so that,

$$q^i{}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{iv}$$

In fact however, looking at the form of the traceless mass quadrupole we see that in this particular case we need not apply the TT projection, since the second derivative of $\mathcal{I}_{ij}$ will already be transverse to $z$ and traceless in the $xy$-plane already. Again using Kepler's third law and putting this all together we find that the TT metric perturbation is,

$$h^{TT}_{kl} = -\frac{GM}{c^4 r}(GM\omega)^{2/3} \begin{pmatrix} \cos(2\omega t) & \sin(2\omega t) & 0 \\ \sin(2\omega t) & -\cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{v}$$

The important point to realize here is that the gravitational wave frequency is twice the orbital frequency. This is a reflection of the fact that GR is a spin-2 theory.

**Exercise 2.5: Revisiting the Newtonian binary**
Carefully reproduce the calculation presented in Box 2.2, checking all of the factors throughout. What does the TT metric perturbation look like for observers lying on the $x$-axis?

## 2.6 Energy loss via gravitational waves

It is not trivial to establish, but nevertheless true that gravitational waves carry energy. The difficulty here stems from the fact that the large gauge freedom of even the linearised field equations makes it hard to untangle that which is physical from that which is not. There is however a notion of gravitational wave stress-energy due to Isaacson, who defines,

$$T_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{ij}^{\mathrm{TT}} \partial_\nu h^{ij}_{\mathrm{TT}} \right\rangle . \tag{2.1}$$

The wedge brackets here denote that an average must be taken over a few gravitational wavelengths. The idea of this is that whilst a gauge transformation can eradicate a physical wave at a point, it can not do so over an extended region.

The energy radiated in gravitational waves (the GW luminosity) is determined by integrating the flux of energy momentum over a sphere of large radius,

$$L_{\mathrm{GW}} = \frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}t} = -\int \mathrm{d}A\, T_{0j}^{\mathrm{GW}} n^j = \int \mathrm{d}A\, T_{00}^{\mathrm{GW}} . \tag{2.2}$$

In the quadrupole approximation this becomes,

$$L_{\mathrm{GW}} = \frac{G}{5c^5}\left\langle \dddot{\mathcal{I}}_{ij} \dddot{\mathcal{I}}^{ij} \right\rangle . \tag{2.3}$$

So the system of the worked example in Box 2.2 above should slowly lose energy. The standard picture is that this emission causes the binary orbits to contract so that the objects eventually merge! Throughout the inspiral the approximations that we have used to describe the physics here gradually breakdown, and must be replaced with more sophisticated methods, such as the Post-Newtonian approximation mentioned above, and close to the time of merger full numerical relativity.

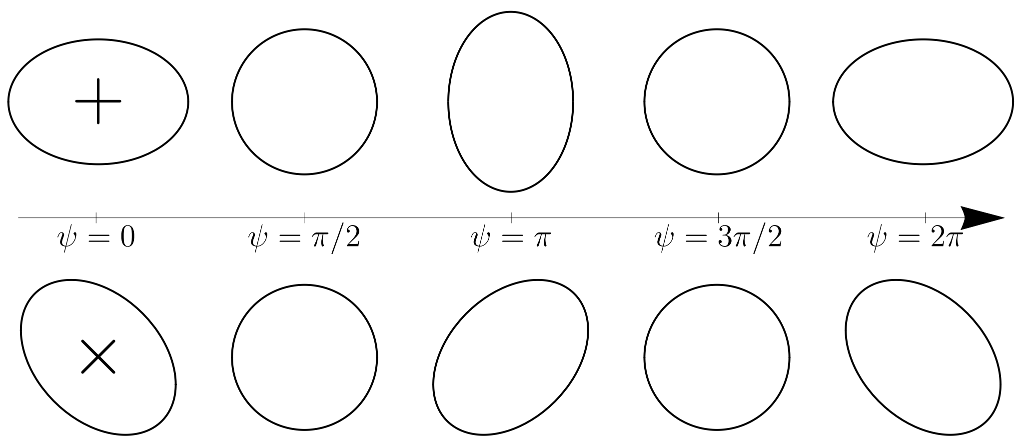


Figure 2.1: The effect of a GW on the proper distance from the origin to an initially circular ring of freely falling test particles. Remember, in our coordinates the particles are not moving, what is plotted here is the *proper distance* $R(\phi, t)$ in equation (2.16). The two rows show the effects of the two linearly independent GW polarisations: the top row shows a *plus polarised* GW with $h_+ \neq 0$ and $h_\times = 0$, and the bottom row shows a *cross polarised* GW with $h_\times \neq 0$ and $h_+ = 0$. Time increases from left to right and five snapshots of the proper distances are shown at equally spaced intervals of the GW phase $\psi = \omega t$.

# Chapter 3

# The Schwarzschild Black Hole

## 3.1 The Schwarzschild solution

The Schwarzschild solution is a static, spherically symmetric solution to the vacuum Einstein field equations (with $T_{\mu\nu} = 0$). In *spherical coordinates* $(t, r, \theta, \phi)$ the metric is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 \mathrm{d}\Omega^2\,, \tag{3.1}$$

where the metric on the unit two-sphere is denoted

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,, \tag{3.2}$$

and

$$\begin{aligned} & f(r) = 1 - \frac{2GM}{c^2 r}\,, \\ \text{or} \quad & f(r) = 1 - \frac{2M}{r} \qquad \text{in natural units (see appendix A).} \end{aligned} \tag{3.3}$$

We will use natural units throughout. The metric depends on a single parameter $M$ which is the mass of the object being described.

Although we will not prove it here, *Birkoff's theorem* states that the Schwarzschild metric is the unique vacuum solution with spherical symmetry. This implies that the Schwarzschild metric describes the exterior gravitational field of any spherically symmetric object, even if interior structure of the object is time dependent. This will be discussed further in Part II of the course where the equations governing the interior structure of a spherical star will be discussed.

## 3.2 Geodesics of the Schwarzschild metric

We are interested in the equations governing the worldlines $x^\mu(\tau)$ of test particles freely falling, or *orbitting*, in the Schwarzschild metric. The geodesics equations were discussed in section 1.2.

The most direct way to study geodesics in the Schwarzschild metric is to begin by evaluating the Christoffel symbols $\Gamma^\mu{}_{\nu\rho}$ for the metric in equation 3.1 and to substitute these into the geodesic equations 1.13. However, it is easier to instead start from the Lagrangian defined in exercise 1.1;

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \,. \tag{3.4}$$

The four Euler-Lagrange equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \,, \tag{3.5}$$

where an overdot denotes a derivative with respect to $\tau$. These may be expanded to give the geodesic equations in second order form:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \frac{2M}{r(r-2M)} \frac{\mathrm{d}r}{\mathrm{d}\tau} \frac{\mathrm{d}t}{\mathrm{d}\tau} = 0 \,, \tag{3.6}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} + \frac{M(r-2M)}{r^3} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \frac{M}{r(r-2M)} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = 0 \,, \tag{3.7}$$

$$\frac{\mathrm{d}^2 \theta}{\mathrm{d}\tau^2} + \frac{2}{r} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \frac{\mathrm{d}r}{\mathrm{d}\tau} - \sin\theta\cos\theta \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 = 0 \,, \tag{3.8}$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}\tau^2} + \frac{2}{r} \frac{\mathrm{d}\phi}{\mathrm{d}\tau} \frac{\mathrm{d}r}{\mathrm{d}\tau} + 2\cot\theta \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = 0 \,. \tag{3.9}$$

**Exercise 3.1: Christoffel symbols**
By comparing equations 3.6-3.9 with equation 1.13, show that the only non-zero Christoffel symbols for the Schwarzschild metric are

$$\begin{array}{llll} \Gamma^t{}_{tr} = \Gamma^t{}_{rt} \,, & & & \\ \Gamma^r{}_{tt}, & \Gamma^r{}_{rr}, & \Gamma^r{}_{\theta\theta}, & \Gamma^r{}_{\phi\phi} \,, \\ \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r}, & \Gamma^\theta{}_{\phi\phi} \,, & & \\ \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r}, & \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} \,, & & \end{array} \tag{i}$$

and find explicit expressions for them.

This is usually a more efficient way of finding the Christoffel symbols than directly evaluating the sums and derivatives in equation 1.15.

---

These equations appear to be fairly complicated set of second order differential equations. However, we have not yet used the symmetries of the metric. Firstly, the metric is static which means that the Lagrangian $\mathcal{L}$ does not depend explicitly on the time coordinate $t$. From the Euler-Lagrange equations 3.5 we can rewrite the $t$-geodesic equation in 3.6 in first order form with a constant of motion $E$;

$$\left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} = E\,. \tag{3.1}$$

Secondly, the metric is spherically symmetric. This means that the motion will always occur in a plane which we can choose to be the equatorial plane; we see immediately that choosing $\theta(\tau) = \text{constant} = \pi/2$ satisfies the $\theta$-geodesic equation 3.8. Additionally, the Lagrangian $\mathcal{L}$ does not depend explicitly on the azimuthal coordinate $\phi$. From the Euler-Lagrange equations 3.5 we can rewrite the $\phi$-geodesic equation in 3.9 in first order form with a constant of motion $L_z$;

$$\begin{aligned} r^2 \sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= L_z\,, \\ r^2 \frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= L_z\,. \end{aligned} \tag{3.2}$$

Finally, instead of using equation 3.7 as a second order equation for $r$ we can instead use the one remaining constant of motion to get a first order equation for $r$. The quantity $g_{\mu\nu}u^\mu u^\nu = \epsilon$, where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, is constant along the geodesic. For a massive test particle following a timelike geodesic $\epsilon = -1$, whereas for a null geodesic $\epsilon = 0$. Writing out this quantity using $\mathrm{d}\theta/\mathrm{d}\tau = 0$ and equations 3.1 and 3.2 gives a first order equation for $r$;

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\text{eff}}(r) = \frac{E^2}{2}\,, \tag{3.3}$$

where

$$V_{\text{eff}}(r) = \frac{1}{2}\left(\frac{L_z^2}{r^2} - \epsilon\right) f(r)\,. \tag{3.4}$$

Written in this way, the complicated system of second order equations in 3.6 to 3.9 has been transformed into to a single first order equation which resembles the 1 dimensional motion of a particle in an external potential $V_{\text{eff}}$. Many of the important properties of geodesics in the Schwarzschild metric can be understood by examining the shape of $V_{\text{eff}}(r)$ which is plotted in figure 3.1.

**Exercise 3.2:**
By considering the metric and the form of the geodesic equations 3.1, 3.2 and 3.3 at large distances ($r \gg M$) show that $E$ and $L_z$ have the interpretation of the energy and angular momentum per unit mass of the test particle.

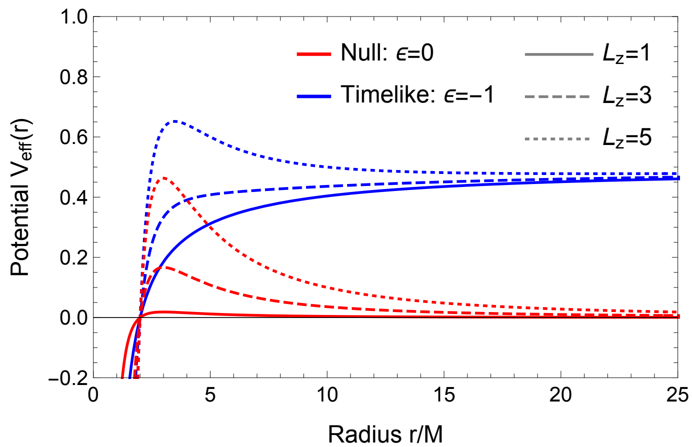


Figure 3.1: The effective potential in equation 3.4 for radial motion in the Schwarzschild metric. The colour (red or blue) lines indicates the causal type of geodesics (null or timelike) while the line style indicates the angular momentum.

We are often more interested in the shape of the orbital path (i.e. $r$ as a function of $\phi$) than exactly where the particle is at any particular instant. An equation that allows the shape to be found more easily is obtained by using the definition of $L_z$ to eliminate $\tau$-derivatives in favour of $\phi$-derivatives;

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{L_z}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi}\,. \tag{3.1}$$

Substituting this into equation 3.3 gives

$$\frac{1}{2}\left(\frac{L_z}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + V_{\text{eff}}(r) = \frac{E^2}{2}\,. \tag{3.2}$$

Finally, making the substitution $u = 1/r$ and differentiating again with respect to $\phi$ gives the *shape equation*,

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{M}{L_z^2} + 3Mu^2\,. \tag{3.3}$$

**Box 3.1: Circular orbits around a Schwarzschild black hole**

We will search for solutions of the timelike geodesic equations describing circular orbits. Substituting $u = 1/r = \text{const}$ into equation 3.3 and rearranging gives

$$Mr^2 - L_z^2 r + 3ML_z^2 = 0\,. \tag{i}$$

This equation has two solutions for $r$,

$$r = \frac{L_z^2 \pm \sqrt{L_z^4 - 12M^2L_z^2}}{2M}\,. \tag{ii}$$

These two solutions correspond to the two stationary points in the effective potential plotted in figure 3.1. When both of the roots are real, the smaller describes an unstable orbit (this can be seen by examining the sign of $\mathrm{d}^2V_{\text{eff}}(r)/\mathrm{d}r^2$) while the larger is stable. For the stable orbit the angular momentum may be found as a function of $r$ by rearranging (i) while the energy as a function of $r$ may be found from equation 3.3 remembering that $r = \text{const}$:

$$L_z = \sqrt{\frac{Mr^2}{r-3M}} \quad \text{and} \quad E = \sqrt{2V_{\text{eff}}(r)}\,. \tag{iii}$$

Examining the discriminant of (ii) reveals that when $L_z < \sqrt{12}M$ there are no longer any circular orbits. The smallest orbit occurs when $L_z = \sqrt{12}M$, using equation (i) shows that this happens at a radius of $r = 6M$.

The existence of an *innermost stable circular orbit (ISCO)* has important astrophysical consequences. For example, it is partly responsible for setting the efficiency with which accretion disks around black holes convert mass into radiation. Some of the most luminous objects in the universe are powered in this way.

## 3.3 Eddington-Finkelstein coordinates

We have until now been ignoring the fact that the Schwarzschild metric in equation 3.1 has some rather glaring problems. Several of the components of the metric are singular at $r = 0$ and $r = 2M$. The singularity at $r = 0$ is "real"; for example, the scalar quantity $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \propto r^{-6}$ diverges as $r \to 0$ and this is something that does not depend on the choice of coordinates. However, we will see in this section that the apparent singularity at $r = 2M$ is different and can be removed by a suitable redefinition of coordinates. The surface $r = 2M$ is an example of a *coordinate singularity*.

**Box 3.2: Causal Structure**

The causal structure of special relativity, in flat Minkowski spacetime, should be familiar. At each spacetime point $p$ there is a light cone which divides spacetime into three regions: the "future", points that can be reached along future directed timelike curves; the "past", points which can be reached along past directed timelike curves; and the remainder of spacetime which is out of causal contact with $p$. The light cone itself is all the null geodesics passing through $p$.

The causal structure in GR is locally the same as that of SR; if you zoom in on a small spacetime patch it looks like Minkowski spacetime. However, globally the causal structure can be different as curvature "twists" the lightcones around. In general it is not always possible to distinguish the past and future light cones. Here we will always be working with *time orientable* spacetimes in which timelike curves can unambiguously be labelled as either "future" or "past" directed. We can then define future/past of $p$ as being the region that can be reached by travelling along a future/past directed timelike curve.

These ideas will be important when we come to defining the concept of a black hole. However, they are also important in other areas such as the *initial value problem* in GR (see part III of the course).

For understanding the *causal structure* (see box 3.2) of the Schwarzschild metric a particularly important family of geodesics are the *radial, null geodesics.* These have $\theta = \phi =$ const and in spherical coordinates are fully described by the function $t(r)$ which satisfies

$$\mathrm{d}s^2 = 0 \;\Rightarrow\; \frac{\mathrm{d}t}{\mathrm{d}r} = \pm f(r)^{-1} \,. \tag{3.4}$$

The choice of sign determines whether the geodesic is ingoing (-) or outgoing (+). This equation is solved by (see exercise 3.3)

$$t = \pm r^* + \mathrm{const}\,, \tag{3.5}$$

where

$$r^* = r + 2M\log\left(\frac{r}{2M} - 1\right) . \tag{3.6}$$

This family of solutions are plotted in figure 3.2. The transformation $r^*(r)$ only makes sense when $r > 2M$. For reasons which we will become clear later, the region $r > 2M$ is called the *exterior* of the Schwarzschild solution and will be denoted here as region 1.

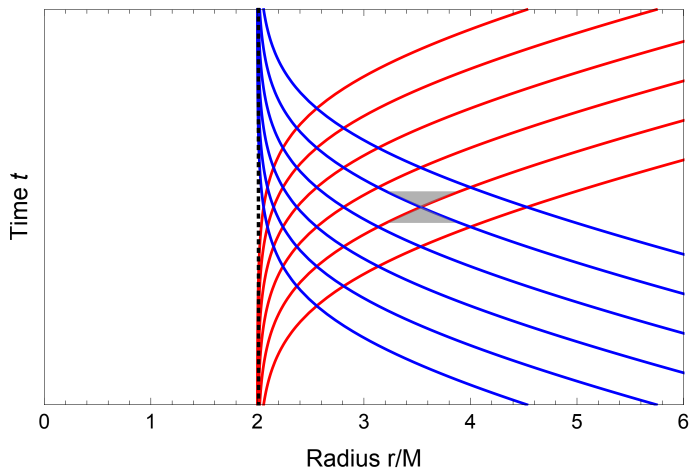


Figure 3.2: The coordinates $t(r)$ of ingoing (blue) and outgoing (red) radial null geodesics. On this plot the space between an intersecting pair of ingoing and outgoing geodesics is the region that can be reached from the intersection point along timelike geodesics and is called the *lightcone* (the light cone is divided into past and future portions). One such lightcone is indicated on the figure, but you should imagine a light cone at every point. Using these coordinates the photon appears to never cross the *Schwarzschild radius* at $r = 2M$.

It will be useful to define a new time coordinate based on these ingoing null geodesics. We will consider two choices, the *advanced* and *retarded* time coordinates:

$$v = t + r^* \,, \tag{3.7}$$

$$u = t - r^* \,. \tag{3.8}$$

First, let us take $(v, r, \theta, \phi)$ as our new coordinates; these are known as *ingoing Eddington-Finkelstein coordinates.* In these coordinates the Schwazschild metric becomes (see exercise 3.4)

$$\mathrm{d}s^2 = -f(r)\mathrm{d}v^2 + 2\mathrm{d}r\mathrm{d}v + r^2\mathrm{d}\Omega^2 \,. \tag{3.9}$$

This metric is no longer singular at $r = 2M$; none of the coordinate components of $g_{\mu\nu}$ diverge at this point. Furthermore, the determinant of this metric equals $g = -r^4 \sin^2\theta$ which is regular for all $r > 0$. In the ingoing Eddington-Finkelstein coordinates the equations for radial null geodesics become

$$\mathrm{d}s^2 = 0 \;\Rightarrow\; \begin{cases} \frac{\mathrm{d}v}{\mathrm{d}r} = 0 & \text{(ingoing)}\,, \\ \frac{\mathrm{d}v}{\mathrm{d}r} = \frac{2}{f(r)} & \text{(outgoing)}\,. \end{cases} \tag{3.10}$$

These equations may be integrated (see exercise 3.3) and the trajectories are plotted in figure 3.3. Although the transformation we wrote down to define $r^*$ in terms of $r$ was only valid in region 1 there is now nothing stopping us from following the path of ingoing null geodesics into the new region 2 with $r < 2M$. Notice that even the *outgoing* trajectories have decreasing $r$ in region 2. This picture makes the nature of the surface $r = 2M$ clearer; as $r$ decreases the lightcones tip over, beyond the surface at $r = 2M$ there is no future directed null (or timelike) curve which can ever reach region 1. We call region 2 a *black hole* and the boundary between regions 1 and 2 an *event horizon*.

At this stage we will loosely define a black hole as the region of spacetime from which you can't escape. Or, slightly more precisely, we say a point $p$ is inside a black hole if the future of $p$ (see box 3.2) does not include any points with large radii. We will return to the definition of a black hole later.

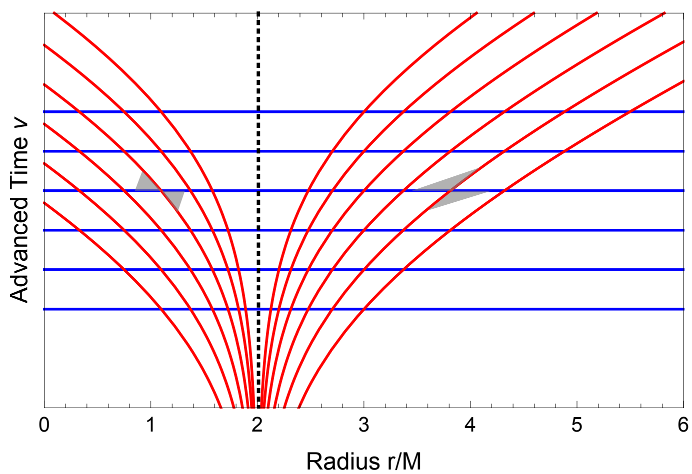


Figure 3.3: The coordinates $v(r)$ of ingoing (blue) and outgoing (red) radial null geodesics. Two lightcones are indicated on the figure, but you should imagine light cones at every point.

**Exercise 3.3:**
Integrate equation 3.10 to find an explicit expression for the the red curves plotted in figure 3.3. Hint: use partial fractions and note the sign of $f(r)$.

We started with the metric in spherical coordinates which only covered region 1. We then changed to ingoing Eddington-Finkelstein coordinates which cover regions 1 and 2. If you find it suspicious that we can simply change coordinates and somehow "discover" a new chuck of spacetime consider that we could equally have started with metric in Eddington-Finkelstein coordinates. If we had done this then both regions 1 and 2 would have been immediately obvious and the surface $r = 2M$ would not have looked so special. A central tenet of GR is that there is no preference among coordinate systems. Therefore, we are forced to treat regions 1 and 2 as being equally valid. Both regions 1 and 2 were there from the beginning, we just happened to chose coordinates that obscured this.

In the above we choose to use $v$ as our time coordinate. The other obvious choice would have been to use $u$. We will now do this and take $(u, r, \theta, \phi)$ as coordinates; these are known as *outgoing Eddington-Finkelstein coordinates.* In these coordinates the Schwazschild metric becomes (see exercise 3.4)

$$\mathrm{d}s^2 = -f(r)\mathrm{d}v^2 - 2\mathrm{d}r\mathrm{d}v + r^2\mathrm{d}\Omega^2 \,. \tag{3.1}$$

The equations for radial null geodesics now become

$$\mathrm{d}s^2 = 0 \ \Rightarrow \begin{cases} \frac{\mathrm{d}u}{\mathrm{d}r} = 0 & \text{(outgoing)}\,, \\ \frac{\mathrm{d}u}{\mathrm{d}r} = \frac{-2}{f(r)} & \text{(ingoing)}\,. \end{cases} \tag{3.2}$$

Again, we can integrate these equations and plot the coordinate expressions for the radial null geodesics. Doing this produces a plot of $u$ against $r$ that looks very like an upside-down version of the previous figure 3.3; now it is the outgoing geodesics which are straight, horizontal lines while the ingoing geodesics resemble inverted ($u \to -u$) versions of the red curves of figure 3.3. Again there is no problem in following the outgoing null geodesics backwards in time into the region $r < 2M$ which we will call region 3. We call region 3 a *white hole.* It is important to note that region 2 is not the same as region 3. To see this pick points in regions 1, 2 and 3 joined by null geodesics. To go from 1 to 2 you can go along a future directed ingoing null geodesic, but to travel from 1 to 3 you must go along a past directed outgoing null geodesic. Region 2 is in the future of 1 while region 3 is in the past. While no causal geodesics can ever leave the black hole the opposite is true of a white hole and ALL causal geodesics are expelled into the exterior region. The white hole looks like a time reversed black hole.

**Exercise 3.4:**
Starting from the metric in equation 3.1 obtain the expressions in equations 3.9 and 3.1 for the metric in ingoing and outgoing Eddington-Finkelstein coordinates.

## 3.4 The maximally extended solution

At this point you might begin to wonder if we can keep playing this game for ever. Can we endlessly invent new coordinate systems and use them to discover new chunks of spacetime? This is not the case. In this section we complete our description of the Schwarzschild black hole by finding the *maximally extended solution*.

We will now use both $u$ and $v$ as coordinates in place of $t$ and $r$; this is sometimes called a *double null* coordinate system. In the coordinates $(u, v, \theta, \phi)$ the metric becomes

$$\mathrm{d}s^2 = -f(r)\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}\Omega^2, \tag{3.1}$$

where $r$ is now to be understood as being a function of $u$ and $v$ defined implicitly via the relation

$$\frac{v-u}{2} = r + 2M\log\left(\frac{r}{2M} - 1\right). \tag{3.2}$$

Just like the original Schwarzschild coordinates, these double null coordinates go bad (the metric determinant vanishes) on the surface $r = 2M$. Instead we can define the modified double null coordinates

$$u' = -\exp\left(\frac{-u}{4M}\right), \tag{3.3}$$

$$v' = \exp\left(\frac{v}{4M}\right), \tag{3.4}$$

in which the metric becomes

$$\mathrm{d}s^2 = \frac{-32M^3}{r}\exp\left(\frac{-r}{2M}\right)\mathrm{d}u'\mathrm{d}v' + r^2\mathrm{d}\Omega^2. \tag{3.5}$$

The new coordinates cover the full range of values $-\infty < u', v' < \infty$.

It is somewhat difficult to visualise double null coordinates. Therefore it is convenient at this point to go back to having a timelike coordinate $T = (v' + u')/2$ and a spacelike radial coordinate $R = (v' - u')/2$. In terms of the *Kruskal coordinates* $(T, R, \theta, \phi)$ the metric becomes

$$\mathrm{d}s^2 = \frac{32M^3}{r}\exp\left(\frac{-r}{2M}\right)\left(-\mathrm{d}T^2 + \mathrm{d}R^2\right) + r^2\mathrm{d}\Omega^2, \tag{3.6}$$

where $r$ is defined implicitly via the relation

$$T^2 - R^2 = f(r)\exp\left(\frac{r}{2M}\right). \tag{3.7}$$

Kruskal coordinates are particularly nice for understanding radial null geodesics and hence the causal structure of the spacetime. In fact, in these coordinates radial null geodesics look exactly the same as they would in flat, Minkowski spacetime, they are simply the curves $T = \pm R + \text{const}$. The location of the event horizon in these new coordinates is $T = \pm R$; this makes clear that the event horizon is a *null surface*. Surfaces of constant $r$ are given by the hyperbolae $T^2 - R^2 = \text{const}$ while surfaces of constant $t$ are given by the straight lines $T = R\tanh(t/4M)$. These straight line extend through the origin of these coordinates into a new region 4. The coordinates $T$ and $R$ are allowed to span the maximum possible ranges without touching the singularity at $r = 0$.

We can now draw a diagram of the Schwarzschild solution using Kruskal coordinates. In this diagram we plot only the $T$ and $R$ coordinates and suppress $\theta$ and $\phi$ (you should think of every point in the plot as representing a sphere). The *Kruskal diagram* is shown in figure 3.4.

The Kruskal coordinates have revealed one final addition region 4. This region is another exterior region similar to region 1. The reason we didn't discover region 4 earlier when using the Eddington-Finkelstein coordinates is that we were only using null geodesics and from region 1 it is impossible to reach 4 along anything other than a spacelike curve.

**Exercise 3.5:**
At this point it is worth taking stock. At various points in the above discussion we have used *Schwarzschild coordinates* $(t, r, \theta, \phi)$, the *ingoing Eddington-Finkelstein coordinates* $(v, r, \theta, \phi)$, the *outgoing Eddington-Finkelstein coordinates* $(u, r, \theta, \phi)$, the *double null* coordinates $(u', v', \theta, \phi)$, and the *Kruskal coordinates* $(T, R, \theta, \phi)$. Which of the regions 1, 2, 3 and 4 are covered by each of these coordinate systems?

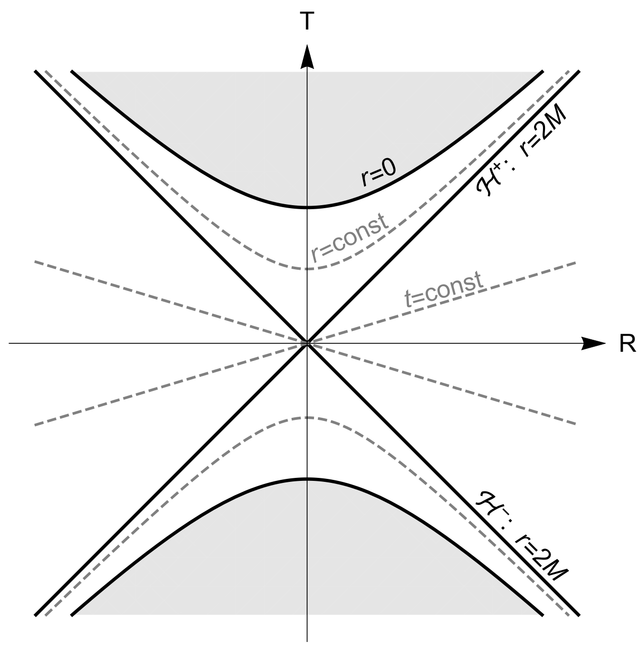


Figure 3.4: The Kruskal diagram of the Schwarzschild black hole.

**Box 3.3: The $r$ coordinate**

This diagram helps to clarify a frequent misunderstanding. When first encountering the Schwarzschild metric it is perhaps natural to think of the singularity at $r = 0$ as somehow being in the "middle" of the spacetime. After all, this is how spherical coordinates work in flat spacetime. It is actually more accurate to instead think of the singularity as being in the future, see figure 3.4.

In general, it is important to remember that the coordinate $r$ is not a distance from any central origin. So what, if anything, is the significance of the Schwarzschild radial coordinate? There is one important geometrical property belonging to the coordinate $r$. Consider a 2-dimensional spherical surface with constant $r$ and $t$, the area of this sphere can be found using the metric in equation 3.1 and is given by $4\pi r^2$. For this reason $r$ is known as the *areal radius.*

## 3.5 The Penrose diagram of the Schwarzschild solution

The Kruskal diagram is very useful, but it is often even more useful to go one step further and draw the *conformal diagram*, also known as a *Penrose diagram.* These diagrams compactify spherically symmetric spacetimes in a way that allows them to be drawn on a finite portion of the 2-dimensional plane. The transformation that achieves this compactification is carefully chosen so that it leaves the causal structure (i.e. null geodesics) unchanged. For this reason conformal diagrams are especially useful for understanding the causal structure of spherically symmetric spacetimes. In this section we will first illustrate the concept of a conformal diagram by applying it to the simple and familiar case of flat Minkowski space. We will then see how the same techniques to the more interesting case of the Schwarzschild black hole.

Before we start compactifying spacetime it will be necessary to define the concept of a *conformal transformation.* A conformal transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu}$ is a position dependent change of scale achieved by multiplying the metric by a non-vanishing function of position;

$$\tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}\,. \tag{3.1}$$

The function $\omega(x) \neq 0$ is known as the *conformal factor.* The two metrics $g_{\mu\nu}$ and $\tilde{g}_{\mu}$ describe different geometries; for example, the Riemann curvature tensor of $\tilde{g}_{\mu\nu}$ is not the same as that of $g_{\mu\nu}$. However, one important aspect of the geometry is left invariant by the conformal transformation. The null curves of $\tilde{g}_{\mu}$ are the same as the null curves of $g_{\mu\nu}$. If $\tilde{\mathrm{d}s^2} = \tilde{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ and $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$, then $\tilde{\mathrm{d}s} = 0 \iff \mathrm{d}s = 0$. This means that as long as we are only interested in the causal structure of the spacetime we can consider all metrics related by conformal transformations to be equivalent.

### 3.5.1 A warm up: Minkowski spacetime

We will start with the line element for flat Minkowski space written in spherical polar coordinates which makes the spherical symmetry explicit;

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \,. \tag{3.2}$$

In the following we will largely ignore the angular coordinates $\theta$ and $\phi$. The remaining coordinates $t$ and $r$ cover the ranges

$$-\infty < t < \infty \,, \quad 0 < r < \infty \,, \tag{3.3}$$

and the line $r = 0$ at the origin is a coordinate singularity. Radial null geodesics in these coordinates are simply the curves $t = \pm r + \text{const.}$

We will now change to double null coordinates: $u = t - r$ and $v = t + r$. In these double null coordinates the flat metric becomes

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v + \left(\frac{v-u}{2}\right)^2 \mathrm{d}\Omega^2 \,. \tag{3.4}$$

Given the ranges of $t$ and $r$ in equation 3.3 we find that the null coordinates $u$ and $v$ must cover the ranges

$$-\infty < u < \infty \,, \quad -\infty < v < \infty \,, \quad \text{with} \quad u \leq v \,. \tag{3.5}$$

Now we compactify the null coordinates. To do this we use the arctan function. This function maps the entire real line into the finite interval between $\pm\pi/2$. We define the new compactified null coordinates $U = \arctan(u)$ and $V = \arctan(v)$ in which the the flat metric becomes

$$\mathrm{d}s^2 = \frac{1}{4\cos^2 U \cos^2 V}\left[-4\mathrm{d}U\mathrm{d}V + \sin^2(V-U)\mathrm{d}\Omega^2\right] \,. \tag{3.6}$$

This metric is conformally equal to the following, much simpler metric,

$$\mathrm{d}\tilde{s}^2 = -4\mathrm{d}U\mathrm{d}V + \sin^2(V-U)\mathrm{d}\Omega^2 \,. \tag{3.7}$$

The null coordinates $U$ and $V$ cover the ranges

$$-\frac{\pi}{2} < U < \frac{\pi}{2} \,, \quad -\frac{\pi}{2} < V < \frac{\pi}{2} \,, \quad \text{with} \quad U \leq V \,. \tag{3.8}$$

Finally, we can achieve a further simplification by transforming back into a timelike coordinate $T = V + U$ and a spacelike radial coordinate $R = V - U$. In these coordinate the conformal metric becomes

$$\mathrm{d}\tilde{s}^2 = -\mathrm{d}T^2 + \mathrm{d}R^2 + \sin^2 R \,\mathrm{d}\Omega^2 \,. \tag{3.9}$$

The coordinates $T$ and $R$ cover the ranges

$$0 \leq R < \pi \,, \quad |T| + R < \pi \,. \tag{3.10}$$

This final expression for the metric looks almost like what we started with in equation 3.2, but the coordinates now only cover finite ranges. This metric is not however not flat (it has a non-vanishing Riemann tensor). This should not be a surprise, this is not Minkowski space after all. But it is conformally related to Minkowski space and so the null geodesics $T = \pm R + \text{const}$ are unchanged.

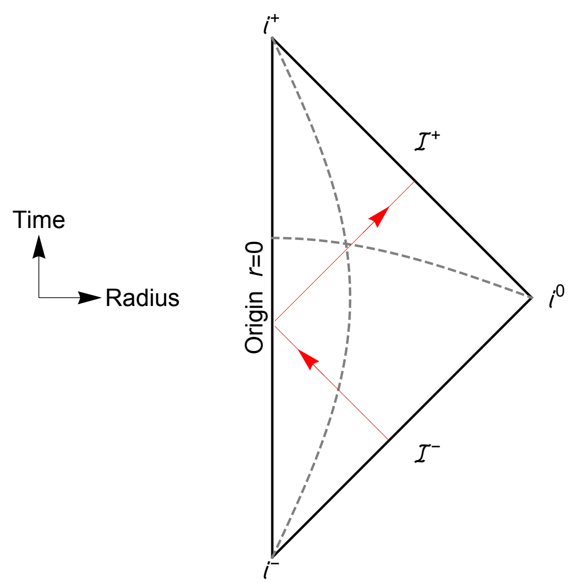


Figure 3.5: The Penrose diagram for Minkowski spacetime. Dashed gray lines indicate surfaces of constant $t$ (horizontal) and $r$ (vertical). The red arrow indicates the path of a photon which enters from $\mathscr{I}^-$ and leaves to $\mathscr{I}^+$ passing through the origin.

We can draw a diagram on the spacetime using these new coordinates, see figure 3.5. If you look closely at the definition of the ranges of the coordinates you will see that the original Minkowski spacetime we started with is only the interior portion of this triangular diagram. However, when looked at in this way it is almost irresistible to think of the boundary of the triangle as somehow representing the concept of being "at infinity" in spacetime. The boundaries are referred to as *conformal infinity*, and the union of the interior and the boundary is known as the *conformal compactification.*

We can go further and even start to talk about different parts of conformal infinity:

- *Future null infinity*, denoted $\mathscr{I}^+$, is the upper right diagonal and is the end point of all future directed null geodesics. (The symbol $\mathscr{I}$ is pronounced "scri".)

- *Past null infinity*, denoted $\mathscr{I}^-$, is the lower right diagonal and is the end point of all past directed null geodesics.

- *Future timelike infinity*, denoted $i^+$, is the uppermost point is the end point of all future directed timelike geodesics. (If you live in Minkowski space then this is where you are headed, like it or not.)

- *Past timelike infinity*, denoted $i^-$, is the lowermost point and is the end point of all past directed timelike geodesics.

- *Spacelike infinity*, denoted $i^0$, is the rightmost point and is the end point of all spacelike geodesics.

- The leftmost boundary of the spacetime is not part of conformal infinity. This is the coordinate singularity $r = 0$ and is not part of the diagram. We already know that it is perfecty possible for an observer (or a photon) to pass unobstructed through the origin of our coordinates. In the diagram this is represented by curves being "reflected" by the coordinate singularity, see figure 3.5.

This is a nice way of looking at the causal structure of Minkowski space, but it hasn't revealed anything that we did not already know about physics in flat spacetime. In the next section we will apply similar techniques to the Schwarzschild spacetime and in that case the conformal diagram will reveal some features that were not initially obvious.

### 3.5.2 Schwarzschild spacetime

A similar series of coordinate manipulations can be performed on the Schwarzschild metric. The procedure would be to start with the metric in the modified double null coordinates $(u', v', \theta, \phi)$ (from equation 3.5) in which we had

$$\mathrm{d}s^2 = \frac{-32M^3}{r} \exp\left(\frac{-r}{2M}\right) \mathrm{d}u'\mathrm{d}v' + r^2\mathrm{d}\Omega^2 \,. \tag{3.11}$$

Just as we did in the case of Minkowski space, we can now define the compactified coordinates

$$u'' = \arctan(u')\,, \tag{3.12}$$
$$v'' = \arctan(u')\,, \tag{3.13}$$

where the new coordinates span the ranges

$$-\frac{\pi}{2} < u'' < \frac{\pi}{2}\,, \quad -\frac{\pi}{2} < v'' < \frac{\pi}{2}\,, \quad \text{with} \quad -\frac{\pi}{2} < u'' + v'' < \frac{\pi}{2}\,. \tag{3.14}$$

In these new coordinate the singularities at $r = 0$ are straight lines. We can plot a diagram of the Schwarzschild solution using these new coordinates, see figure 3.6. This looks very like the Kruskal diagram in figure 3.4 only compactified.

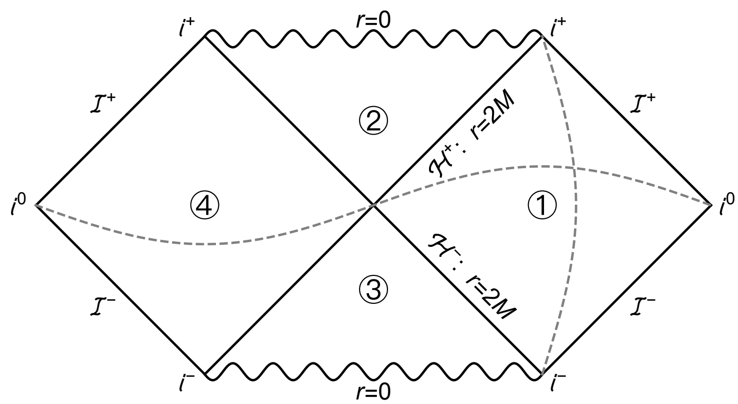


Figure 3.6: The Penrose diagram for the Schwarzschild spacetime. Dashed gray lines indicate surfaces of constant $t$ (horizontal) and $r$ (vertical).

### 3.5.3 A general definition

We have studied the Schwarzschild solution in some detail and seen that it contains a black hole. As promised, we will conclude by giving a general definition of a what a black hole is. This definition applies even when considering more exotic black holes which are rotating or electrically charged. The definition is very mathematical and will rely on the concept of *future null infinity* (which we have seen several examples of but not defined precisely) and on the notion of the causal "past" (a fairly precise definition of which was given in box 3.2). We will also use some terminology from set theory: the *complement* of a set $A$ is all of the elements not in $A$. You should consider the following definition carefully, comparing it against the conformal diagram of the Schwarzschild black hole in figure 3.6 to understand how it works.

**Definition:** A *black hole* is the complement of the past of future null infinity.

# Appendix A

# Natural Units

In SI units

$$
\begin{aligned}
G &= 6.67430(15) \times 10^{-11}\,\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}\,, \\
c &= 2.99792458 \times 10^{8}\ \mathrm{m\,s}^{-1}\,.
\end{aligned} \tag{A.1}
$$

The Schwarzschild black hole is described by a single mass parameter, $M$, which sets the scale of the black hole. It is a remarkable observational fact that we see objects well described by this very simple Schwarzschild (or more generally the Kerr) solution with masses spanning ten orders of magnitude. When describing processes occurring around a black hole it is common to use units adapted to relevant scale.

In general relativistic calculations it is usually convenient to use *natural*, or *geometrised* units where $G = c = 1$. All quantities which in normal units have dimensions of powers of length, $L$, time, $T$ and mass, $M$, are now expressed as a power of length in geometrised units. Around a black hole we can now measure all times, lengths and masses in terms of a single unit $M$ which is naturally adapted to the right scale. Removing the factors of $G$ and $c$ also has the very practical benefit of making many of the equations in these notes more manageable.

In order to convert any formula back into normal units it is necessary to reinsert the correct powers of $G$ and $c$. It is easiest to demonstrate how to do this with an example. Let's say we encounter the following formula (in geometrised units) for the *Hawking temperature* of a Schwarzschild black hole,

$$
T_{\mathrm{Hawking}} = \frac{\hbar}{8\pi k_{\mathrm{B}} M}\,. \tag{A.2}
$$

How should we convert this into normal units? If we first rearrange it as

$$
\frac{k_{\mathrm{B}} T_{\mathrm{Hawking}}}{\hbar} = \frac{1}{8\pi M}\,, \tag{A.3}
$$

then the quantity on the left hand side has units of $T^{-1}$. We need to multiply the inverse mass on the right hand side by $G^{\alpha}c^{\beta}$ where the powers are chosen such that units of the right hand side match the left. The only choice is $\alpha = -1$ and $\beta = 3$. Therefore, the formula for the Hawking temperature in normal units is

$$T_{\text{Hawking}} = \frac{c^3 \hbar}{8\pi G k_{\text{B}} M} \,. \tag{A.4}$$

This can be used to evaluate the temperature of a one solar mass black hole in Kelvin as $\approx 6.17 \times 10^{-8}$ K (cold!).

The following conversion table (reproduced from Wald; 1984) can be a useful aid.

**Conversion Factors to Geometrized Units**

| Quantity | Nongeometrized Dimension | Geometrized Dimension | Conversion Factor |
|---|---|---|---|
| Acceleration | $LT^{-2}$ | $L^{-1}$ | $c^{-2}$ |
| Angular momentum | $L^2T^{-1}M$ | $L^2$ | $G/c^3$ |
| Electric charge (cgs) | $L^{3/2}T^{-1}M^{1/2}$ | $L$ | $G^{1/2}/c^2$ |
| Energy | $L^2T^{-2}M$ | $L$ | $G/c^4$ |
| Energy density | $L^{-1}T^{-2}M$ | $L^{-2}$ | $G/c^4$ |
| Force | $LT^{-2}M$ | 1 | $G/c^4$ |
| Length | $L$ | $L$ | 1 |
| Mass | $M$ | $L$ | $G/c^2$ |
| Mass density | $L^{-3}M$ | $L^{-2}$ | $G/c^2$ |
| Pressure | $L^{-1}T^{-2}M$ | $L^{-2}$ | $G/c^4$ |
| Time | $T$ | $L$ | $c$ |
| Velocity | $LT^{-1}$ | 1 | $c^{-1}$ |

**Exercise A.1:**
What are the natural length, time and mass densities associated with a black hole of mass (i) $M_{\odot}$, and (ii) $4.1 \times 10^6 M_{\odot}$ (the mass of the black hole at the centre of our galaxy).