MPAGS - Black Holes and Gravitational Waves 2020
Assessed Problemsheet Part III

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Problem 1

(a) Show that the covariant derivative D induced on a spatial hypersurface Σ is compatible with the spatial metric γ_{ab}, i.e.,

$$D_c \gamma_{ab} = 0. \tag{1}$$

(b) Show that

$$\mathscr{L}_n \gamma_{ab} = -2K_{ab}. \tag{2}$$

Hint: Use the covariant definition of the Lie derivative and the projection operator $\gamma_a{}^b = \delta_a{}^b + n_a n^b$.

Problem 2

(a) Derive the momentum constraint, i.e., $D_b K_a{}^b - D_a K = 8\pi S_a$, via a contraction of the Codazzi equation, which is given by:

$$\gamma_a{}^p \gamma_b{}^q \gamma_c{}^r n^s {}^{(4)}R_{pqrs} = D_b K_{ac} - D_a K_{bc} \tag{3}$$

Hint: Recall that $G_{ab} = {}^{(4)}R - \frac{1}{2}{}^{(4)}R g_{ab} = 8\pi T_{ab}$.

(b) Derive the time evolution equations for the extrinsic curvature tensor and the spatial metric, i.e. $\mathscr{L}_t K_{ab}$ and $\mathscr{L}_t \gamma_{ab}$, without making use of the foliation-adapted coordinates but rather by making use of the general definition of the time coordinate, $t^a = \alpha n^a + \beta^a$, and the linearity of the Lie derivative, i.e. $\mathscr{L}_t \rightarrow \mathscr{L}_{\alpha n} + \mathscr{L}_\beta$.

Problem 3

Given the Schwarzschild metric in isotropic coordinates,

$$ds^2 = -\frac{\left(1-\frac{m}{2r}\right)^2}{\left(1+\frac{m}{2r}\right)} dt^2 + \left(1+\frac{m}{2r}\right)^2 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2), \tag{4}$$

consider the foliation given by spacelike hypersurfaces with constant time coordinate t. Compute the covector Ω_a, the lapse α of this foliation, the unit normal n^a and the spatial metric γ_{ab}. What are the extrinsic curvature K_{ab} and the mean curvature K of this foliation? Is this foliation maximally sliced?

Note: Indices are dummy indices indicating tensorial notation. Remember that the extrinsic curvature and the spatial metric are purely spatial objects. We use the Einstein summation notation, i.e., repeated co- and contravariant indices are summed over.

Recommended literature for solving the exercises:

1. T. Baumgarte and S. Shapiro: "Numerical Relativity" (Cambridge University Press)
2. M. Alcubierre: "Introduction to $3+1$ Numerical Relativity" (Oxford University Press)

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