MPAGS: Black Holes and Gravitational Waves
Part I: Exercises

Christopher J. Moore

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If you spot any mistakes please send them to `cmoore@star.sr.bham.ac.uk`.

1 Trace-Reversed Einstein Equations

The Einstein field equations are usually written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \,. \tag{1}$$

Starting from (1) show that the equations may be written in trace-reversed form

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) , \tag{2}$$

where $T = T^{\mu}_{\mu}$.

Extension: The trace-reversed equations (2) apply in a 4 dimensions (4 = time + 3 space). How would these equations differ in a D-dimensional spacetime?

2 Linearised Metric Perturbation

For weak gravitational fields, using the notation from the lectures, the spacetime metric can be written as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \,, \tag{3}$$

where $|h_{\mu\nu}| \ll 1$. Show that to leading order in h the inverse metric is given by

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\alpha\beta}(x) \,, \tag{4}$$

where $h^{\mu\nu}(x) \equiv \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}(x)$.

It is sometimes convenient to introduce the *trace-reversed* metric perturbation $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. Show that $\bar{h} = -h$. Also show that if you take the trace-reverse of a tensor twice you get back to where you start; i.e. $\bar{\bar{h}}_{\mu\nu} = h_{\mu\nu}$.

Using the transformation law for the metric tensor, show that under an infinitesimal coordinate transformation $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$ the metric perturbation transforms as

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \,. \tag{5}$$

3 Conserved Quantities Along Geodesics

In a general spacetime suppose that there exists a (co)vector field ξ_μ satisfying

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \,. \tag{6}$$

This equation is called Killing's equation and ξ_μ (if it exists) is called a Killing (co)vector field. A test particle moves along a geodesic in the spacetime with four-velocity u^μ. Show that the quantity $u^\nu \xi_\nu$ is conserved along the worldline of the particle; i.e. show that $u^\mu \nabla_\mu (u^\nu \xi_\nu) = 0$.

Hint: Use the form of the geodesic equations in Box 1.1 (vi) of the lecture notes and write the tensor $\nabla_\mu \xi_\nu$ as a sum of symmetric and anti-symmetric parts.

Extension: Show that $(\partial/\partial t)$ and $(\partial/\partial\phi)$ are Killing vector fields for the Schwarzschild metric.

Hint: Expand (6) explicitly in Schwarzschild coordinates (t, r, θ, ϕ) using the Christoffel symbols $\Gamma^\mu_{\nu\rho}$ found in exercise 3.1 of the lecture notes.

Extension: Using the geodesic equations 3.1 and 3.2 from the lecture notes, evaluate the two conserved quantities associated with these Killing vector fields and show that these are the constants of motion E and L_z.