Statistics

The IFT School on Cosmology Tools

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March 13, 2017

Introduction

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- How to extract information about scientific models from data.

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- How to extract information about scientific models from data.
- Most cosmologists work in a *Bayesian* framework of inference, although *Frequentist* methods are also sometimes used.

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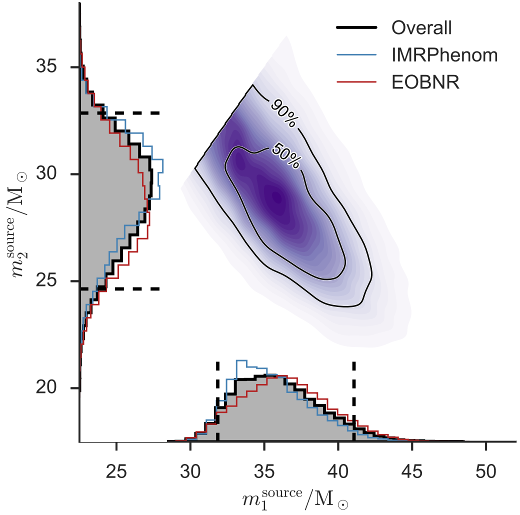
- More importantly, these are *summary statistics*.

LIGO binary merger



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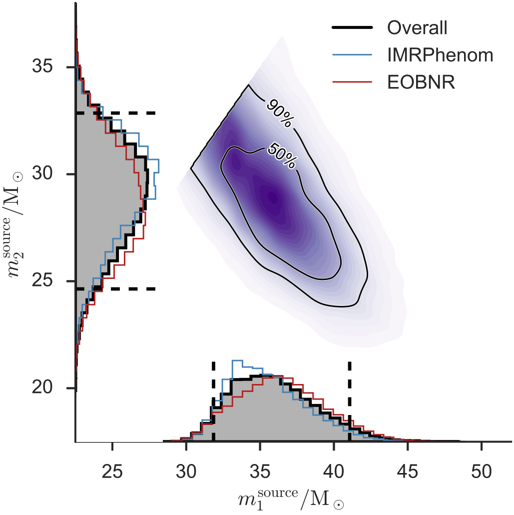

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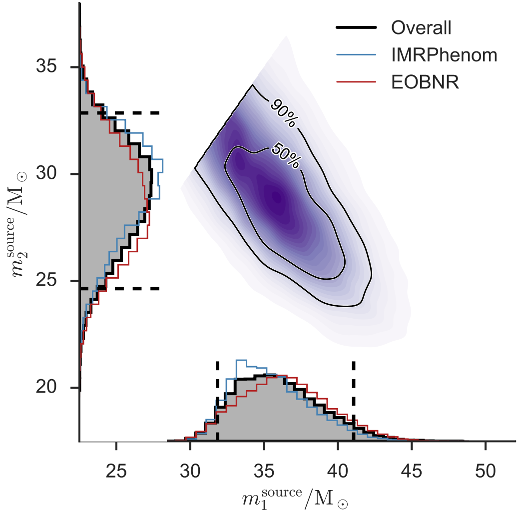


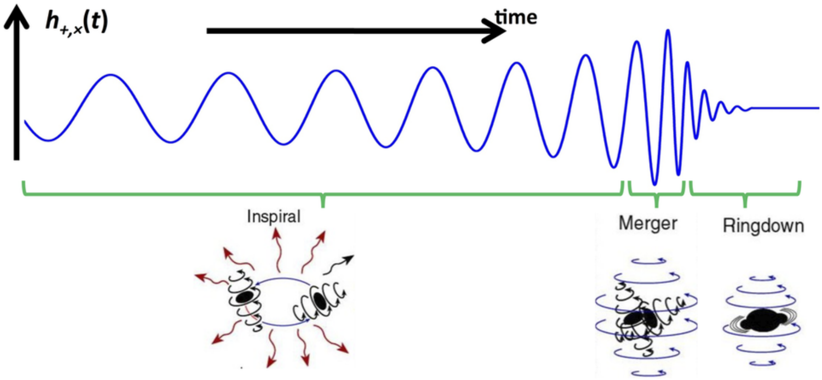
- Summary statistics summarise a full probability distribution.
- One goal of inference is to produce these probability distributions.

Extended example of inference: LIGO

- We will introduce the key concepts by discussing an extended example of the inference process.

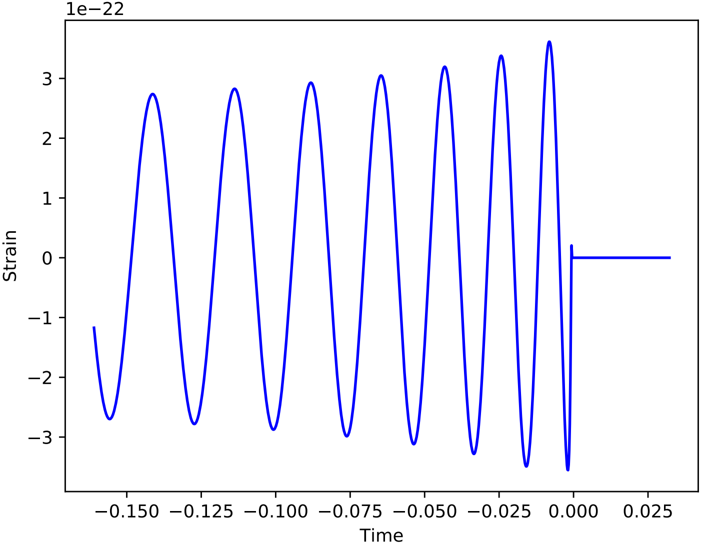
Theory

Extended example of inference: LIGO



The model M

Extended example of inference: LIGO



The parameters Θ of the model M

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- Φ_c, t_c: phase and time of coalescence

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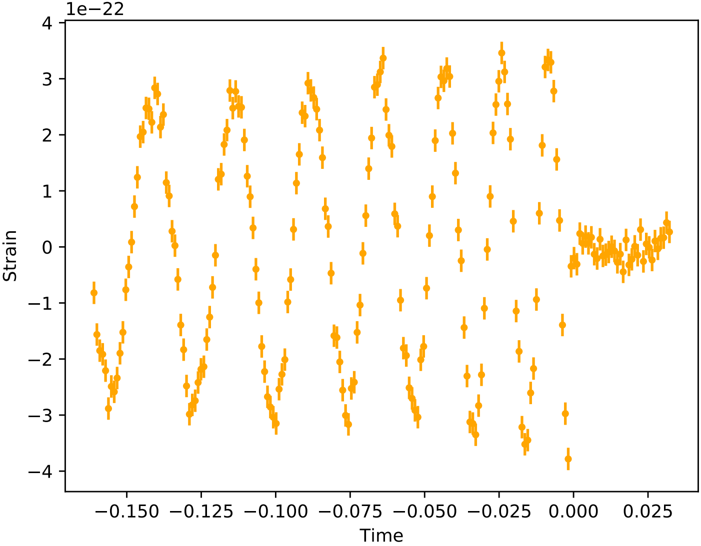
Theoretical signal depends on:

- m_1, m_2: mass of binary
- θ, ϕ: sky location
- r: luminosity distance
- Φ_c, t_c: phase and time of coalescence
- i, θ_{sky}: inclination and angle on sky (orbital parameters)

The data D

Extended example of inference: LIGO



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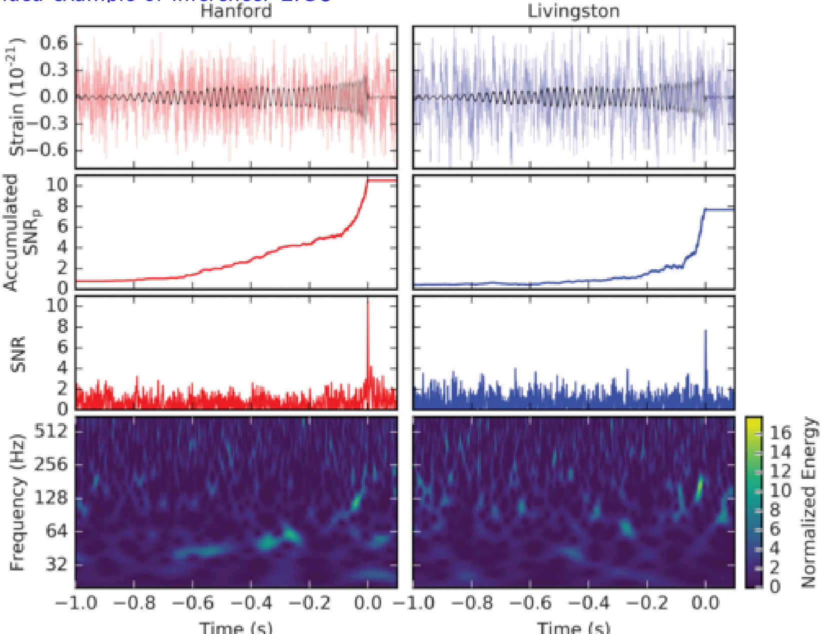

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  - $h(t;\Theta)$: Theoretical strain
- We normally work with log-likelihoods, which turn $\prod \to \sum$.

The Likelihood: well matched

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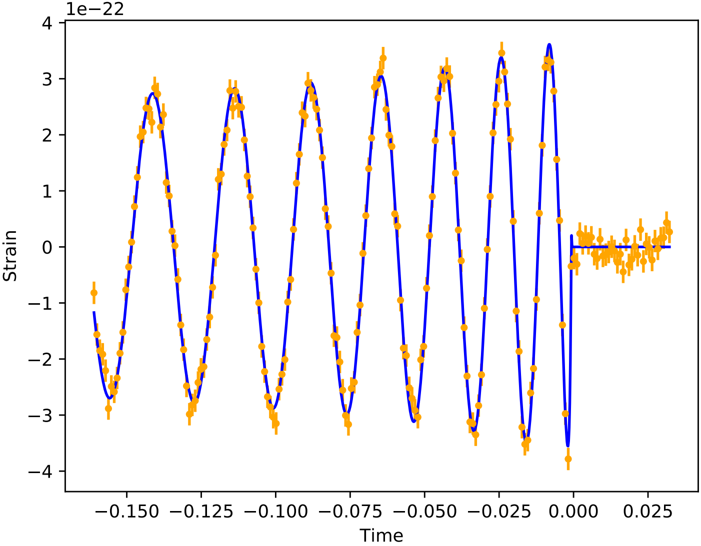

The Likelihood: coalescence off

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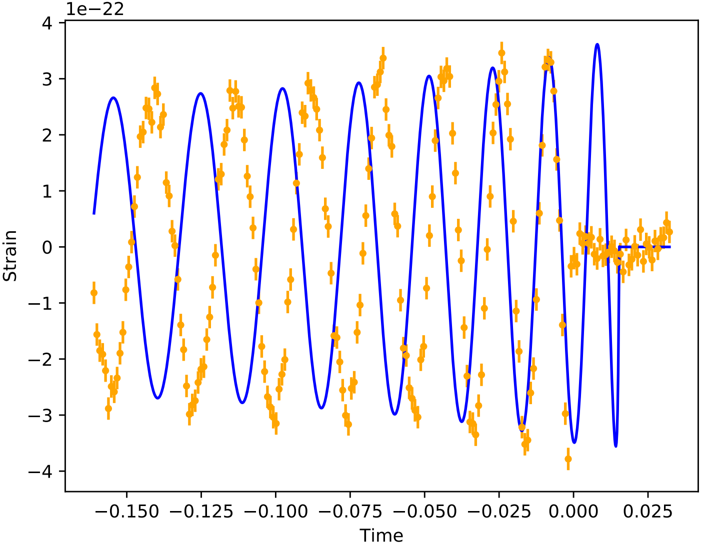

The Likelihood: too large luminosity distance

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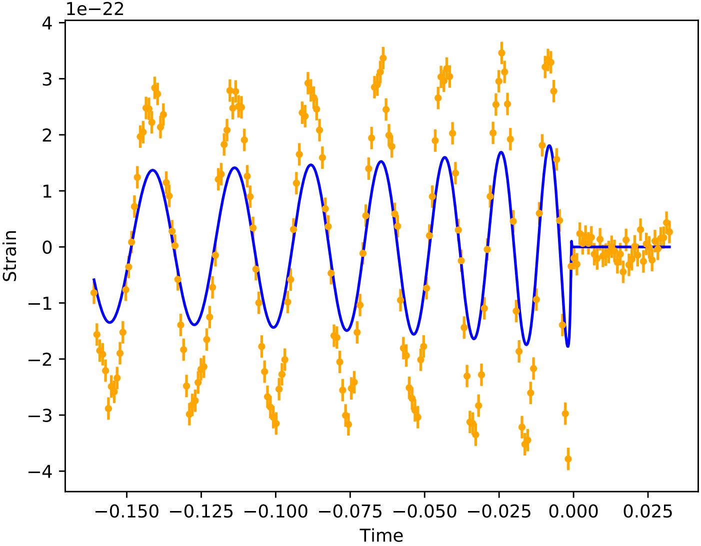

The Likelihood: incorrect inclination

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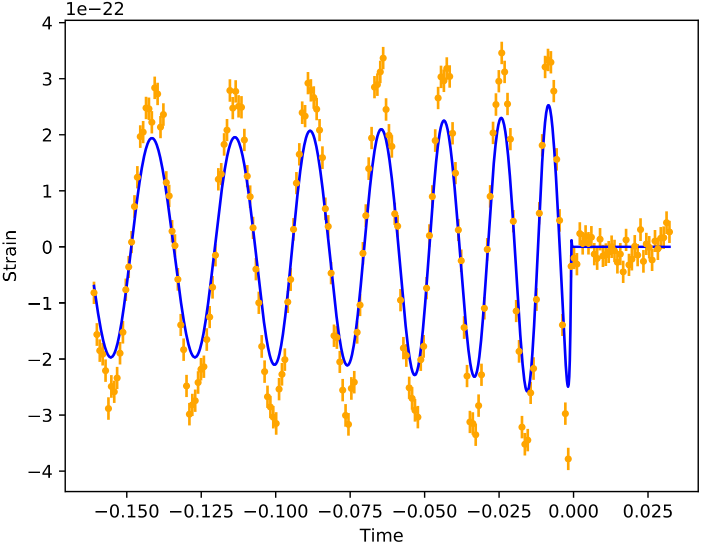

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$$\text{Posterior} = \frac{\text{Likelihood} \times \text{Prior}}{\text{Evidence}}$$

Prior $\pi(\Theta)$

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- e.g. $\pi(\theta,\phi)\, d\theta d\phi = -\frac{1}{2}\sin\theta\; d\theta d\phi$
- Most Bayesian approaches are sensitive to this, and rightly so.

Evidence Z

Extended example of inference: LIGO

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- Normalising constant.
- Difficult to compute.
- Still extremely important.

Posterior $\mathcal{P}$

Extended example of inference: LIGO

- Cannot plot the full posterior distribution:

$$\mathcal{P}(\Theta) \equiv P(m_1, m_2, \theta, \phi, r, \Phi_c, t_c, i, \theta_{\mathrm{sky}} | D, M)$$

- Can plot 1D and 2D *marginalised* distributions e.g:

$$P(m_1, m_2 | D, M) = \int P(m_1, m_2, \theta, \phi, r, \Phi_c, t_c, i, \theta_{\mathrm{sky}} | D, M)\, d\theta\, d\phi\, dr\, d\Phi_c\, dt_c\, di\, d\theta_{\mathrm{sky}}$$

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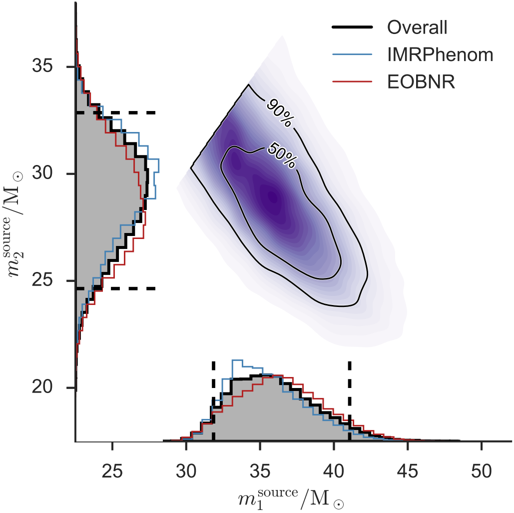

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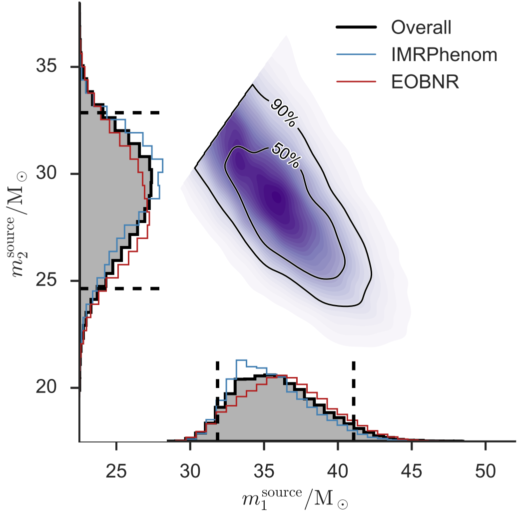

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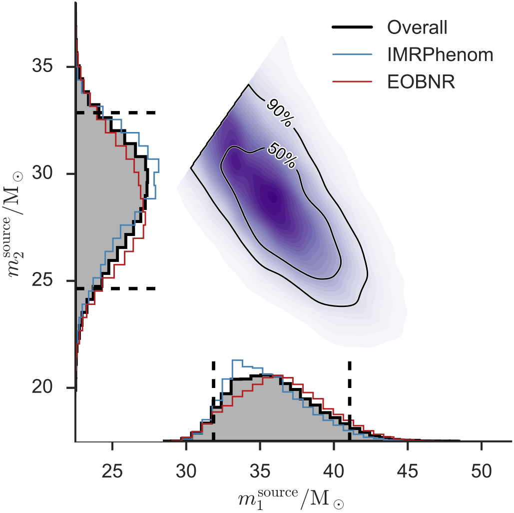

- May do this for each pair of parameters
- Generates a *triangle plot*

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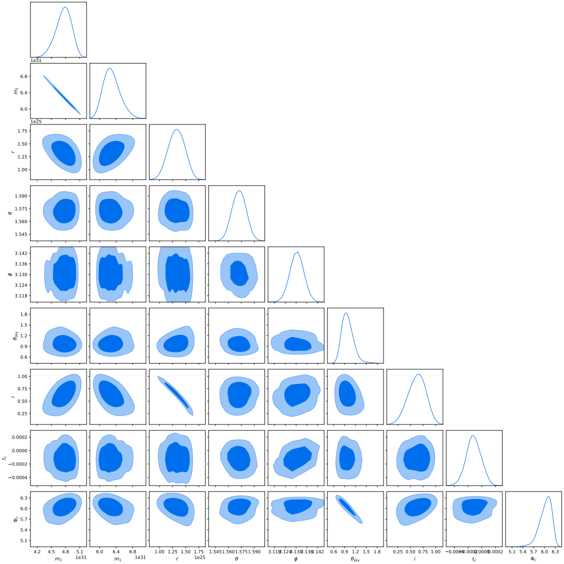

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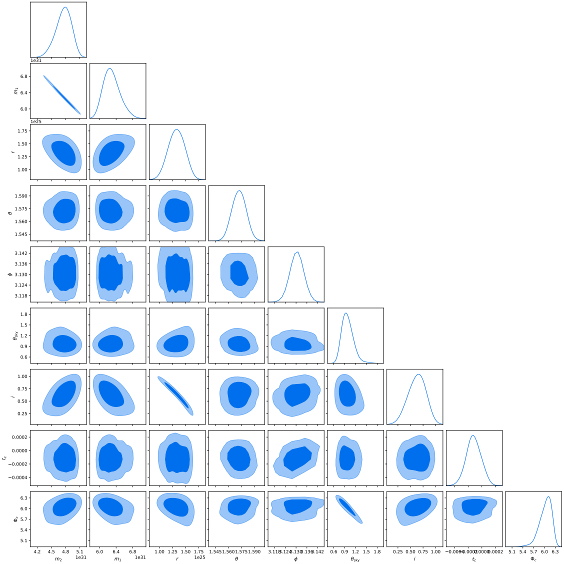

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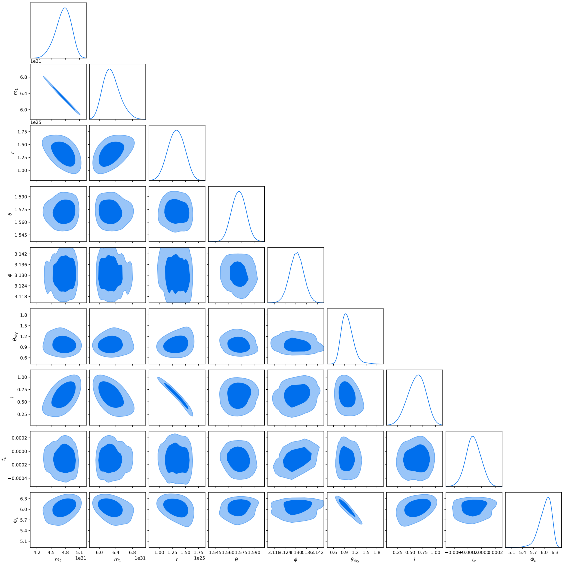

- Does give insight
- Not the full picture

Evidences and model comparison

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- Up until now, we have discussed *Parameter estimation*: inferring what data tell us about parameters Θ of a model M.
- Scientifically speaking, this is only half the story.
- In general, we will have several competing models that describe the data, and we want to know which is the "best".

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Model averaging:

- Multiple models with posterior on the same parameter: $\mathrm{P}(y|M_i, D)$

$$\mathrm{P}(y|D) = \sum_i \mathrm{P}(y|M_i, D)\mathrm{P}(M_i|D)$$

Parameter estimation

Another example.

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\Theta_{\Lambda\mathrm{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10} A_s), n_s)
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\Theta =& \Theta_{\Lambda\mathrm{CDM}} + \Theta_{\mathrm{Planck}} \\
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Parameter estimation

Another example.

$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

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- Parameter estimation: $L, \pi \to \mathcal{P}$: model parameters
- Model comparison: $L, \pi \to Z$: how good model is

Sampling

How to describe a high-dimensional posterior

Sampling

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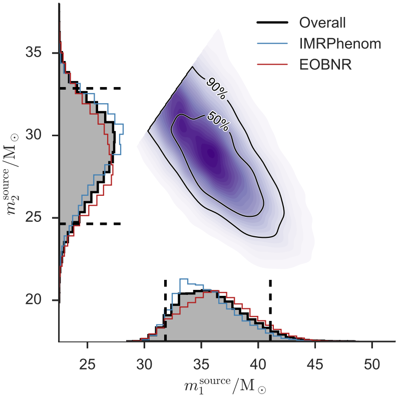
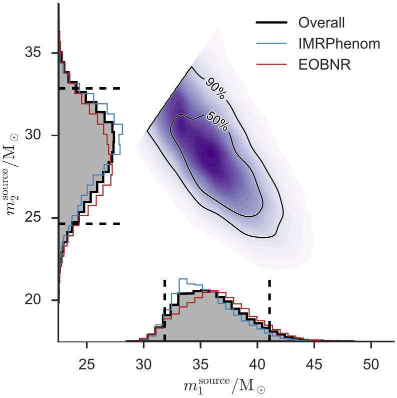


Sampling

How to describe a high-dimensional posterior

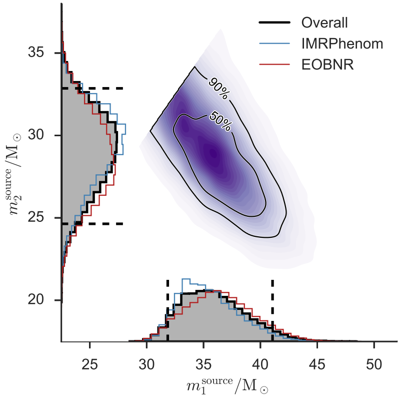
- In high dimensions, posterior $\mathcal{P}$ occupies a vanishingly small region of the prior π.



Sampling

How to describe a high-dimensional posterior

- In high dimensions, posterior $\mathcal{P}$ occupies a vanishingly small region of the prior π.
- *Sampling* the posterior is an excellent compression scheme.



Why do sampling?

Marginalisation over the posterior

- Set of N samples $S = \{\Theta^{(i)} : i = 1, \dots N : \ \Theta^{(i)} \sim \mathcal{P}\}$

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- Gridding is doomed to failure in high dimensions.
- Enter Metropolis Hastings.

Current sampling approaches

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1. Metropolis Hastings.
2. Hamiltonian Monte-Carlo (HMC).
3. Ensemble sampling (e.g. emcee).

Metropolis Hastings

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- Turn the N-dimensional problem into a one-dimensional one.

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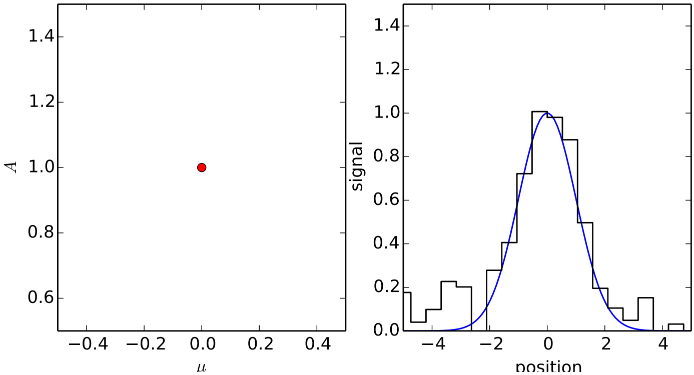
Metropolis Hastings



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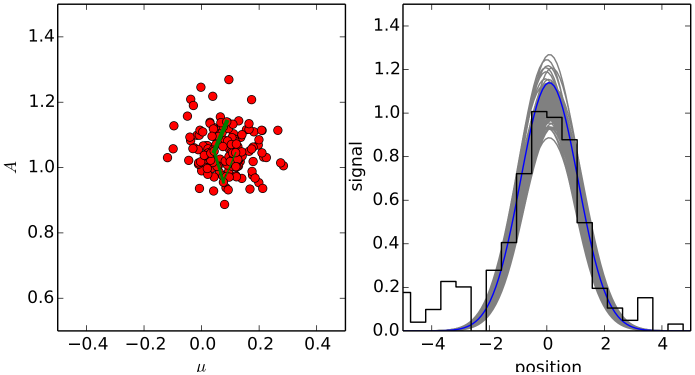

Metropolis Hastings

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Step 0 Choose initial set of parameters $\Theta^{(0)}$, set $w^{(0)} = 1$

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- if accepted, $\Theta^{(i+1)} = \Theta^{(\rm new)}$, $i+ = 1$
- Otherwise, reject, $w^{(i)} + = 1$ and repeat.

Metropolis Hastings

Struggles with. . .

Metropolis Hastings

Struggles with...

1. Burn in

Metropolis Hastings

Struggles with. . .

1. Burn in
2. Multimodality

Metropolis Hastings

Struggles with...

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Metropolis Hastings

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Hamiltonian Monte-Carlo

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- stan is a fully fledged, rapidly developing programming language with HMC as a default sampler.

Ensemble sampling

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- Can use information present in ensemble to guide proposals.
- emcee: affine invariant proposals.
- emcee is not the only (or even best) affine invariant approach.

The fundamental issue with all of the above

- They don't give you evidences!

$$\begin{aligned}\mathcal{Z} &= \mathrm{P}(D|M) \\ &= \int \mathrm{P}(D|\Theta, M)\mathrm{P}(\Theta|M) d\Theta \\ &= \langle \mathcal{L} \rangle_\pi\end{aligned}$$

- MCMC fundamentally explores the posterior, and cannot average over the prior.
- Simulated annealing gives one possibility for computing evidences.
  - Inspired by thermodynamics.
  - Suffers from similar issues to MCMC.
  - Unclear how to choose correct annealing schedule

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- Nested sampling is an alternative way of sampling posteriors.
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- In doing so, it circumvents many issues (dimensionality, topology, geometry) that beset standard approaches.

Nested Sampling

John Skilling's alternative to traditional MCMC!

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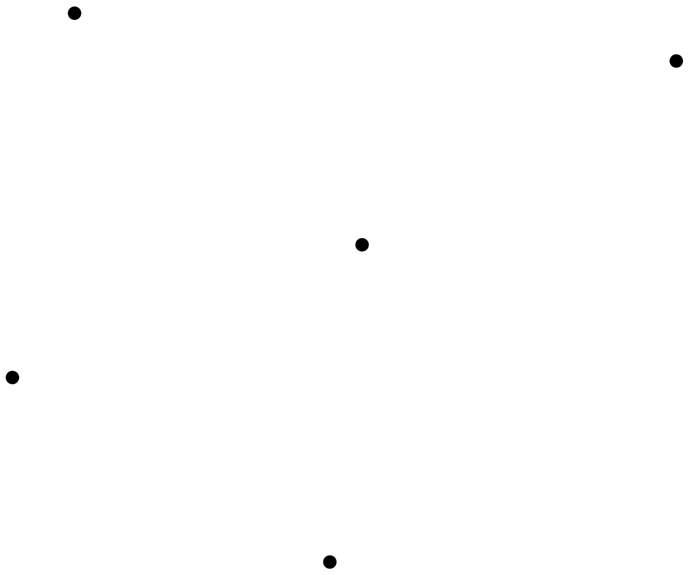
Requires one to be able to uniformly within a region, subject to a *hard likelihood constraint*.

Nested Sampling

Graphical aid



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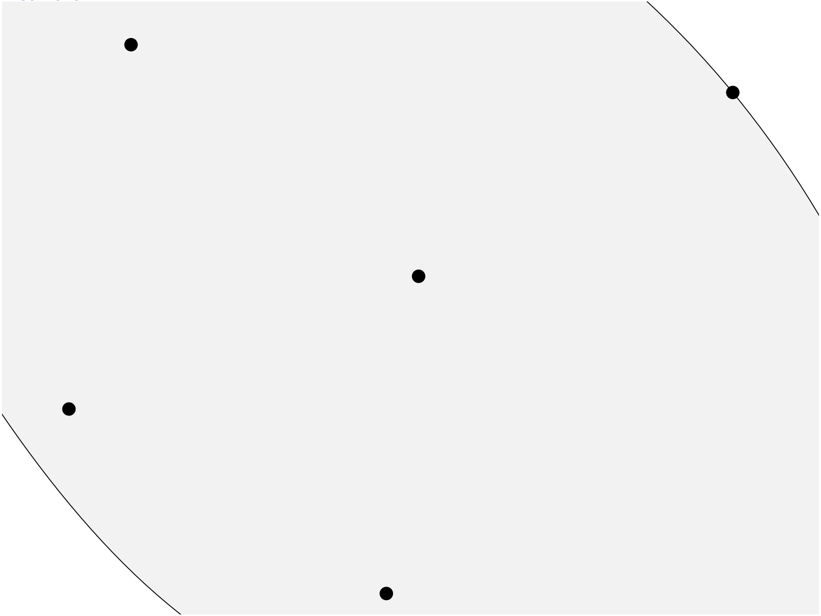

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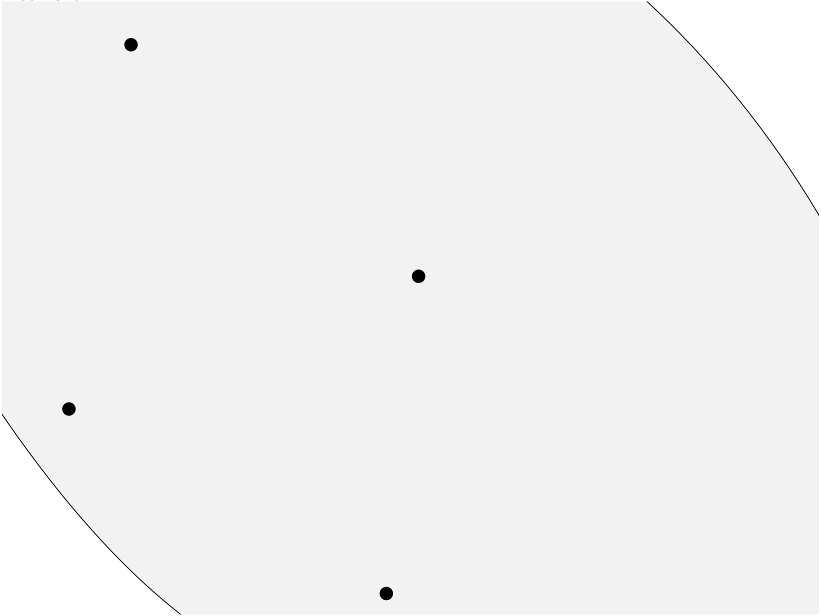

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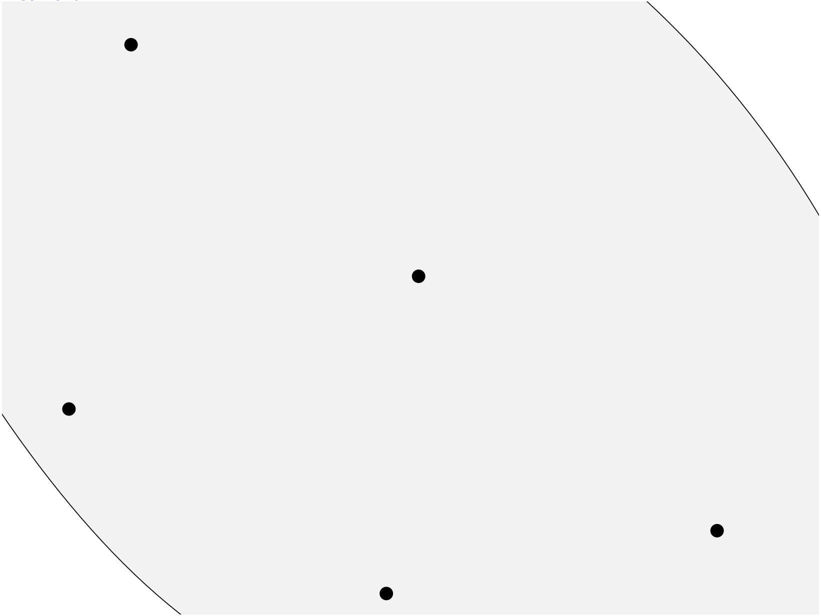

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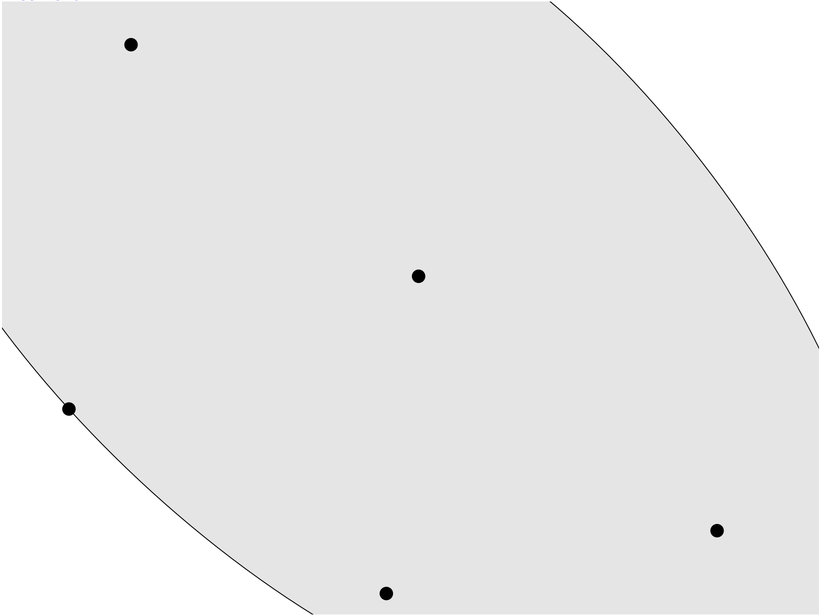

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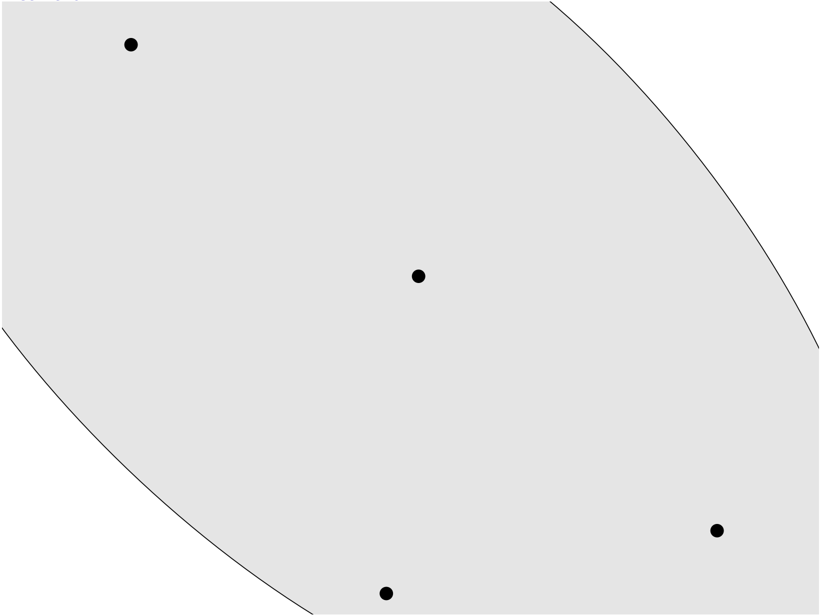

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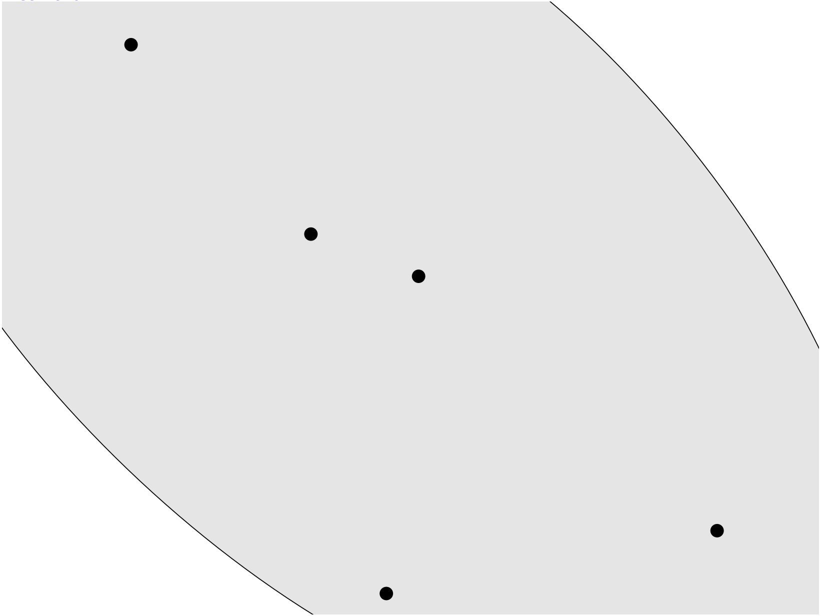

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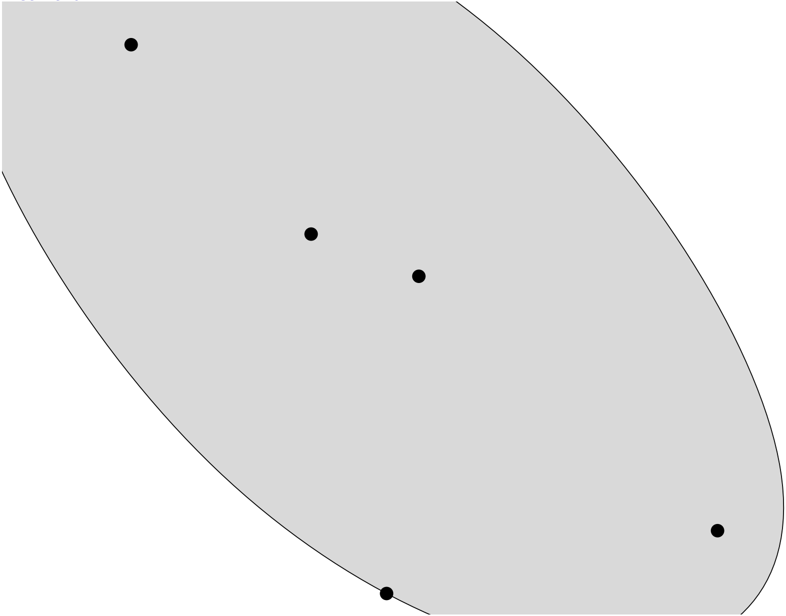

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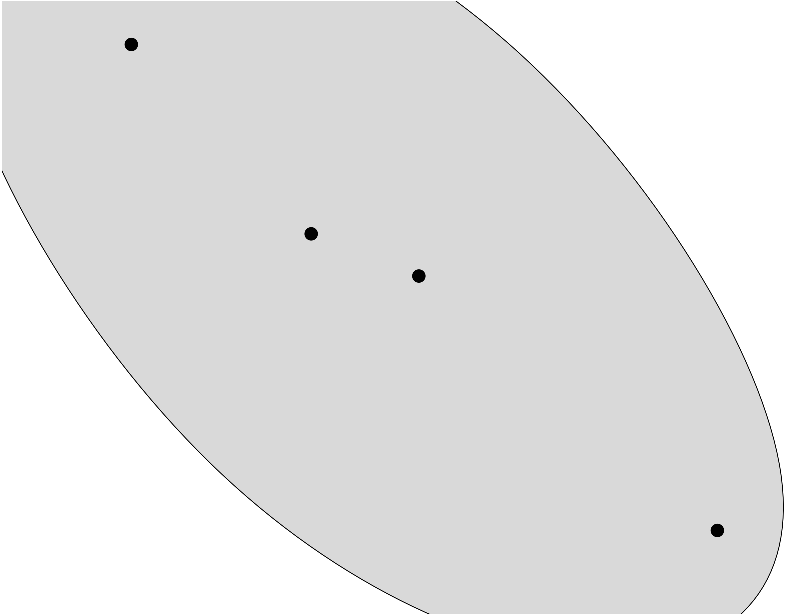

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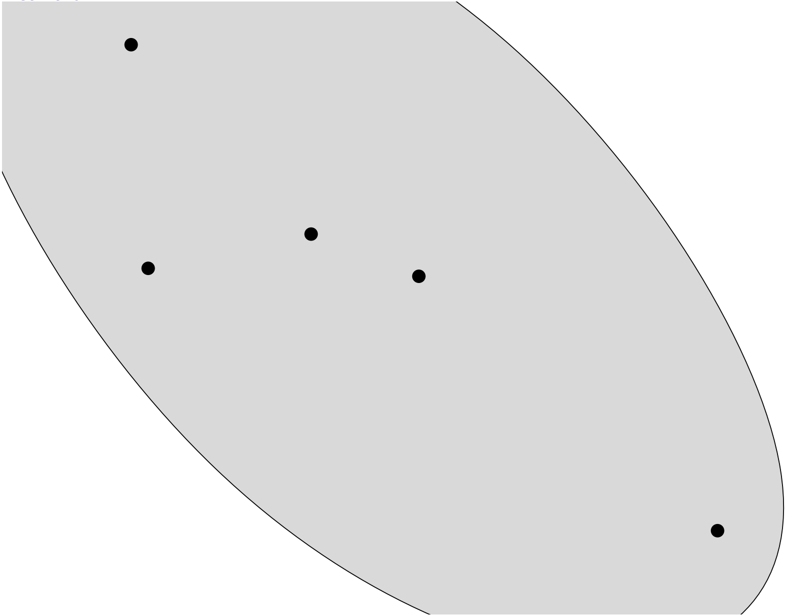

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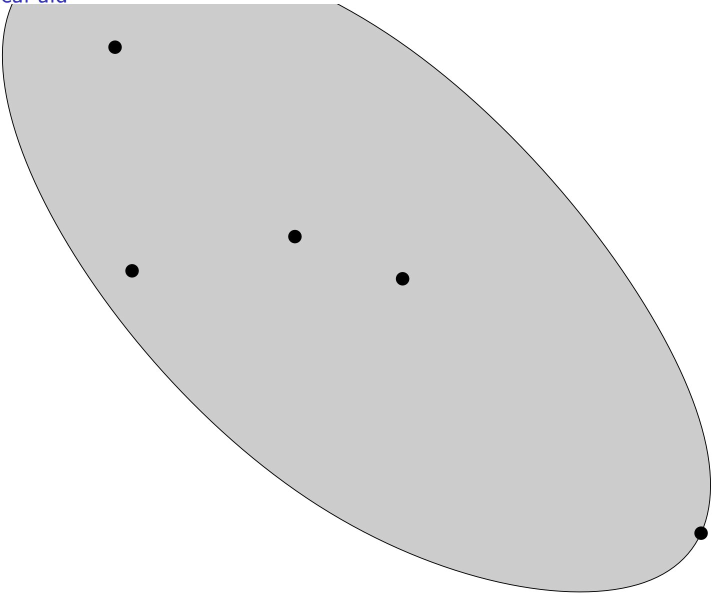

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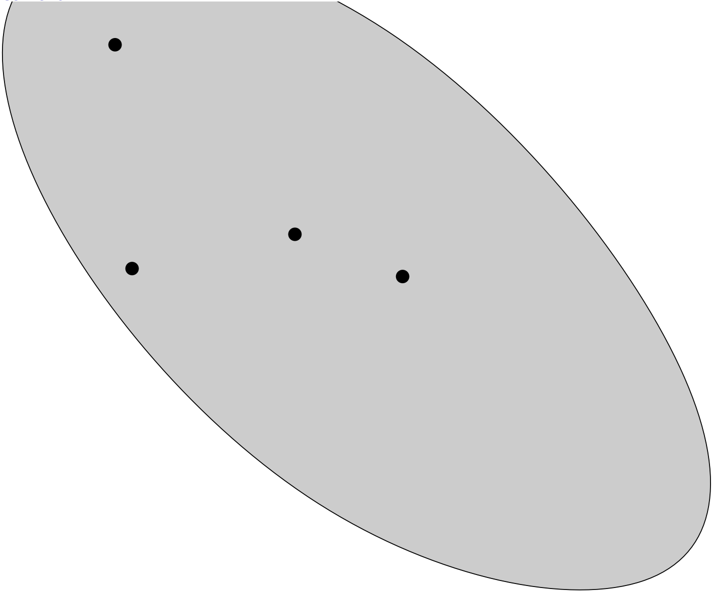

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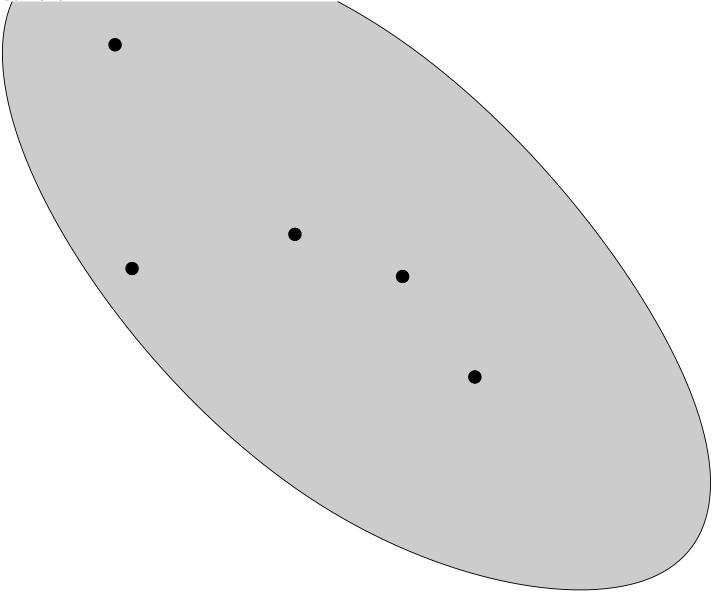

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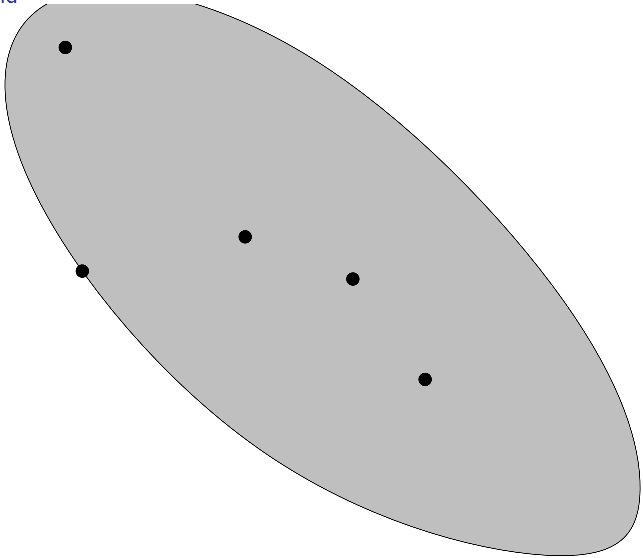

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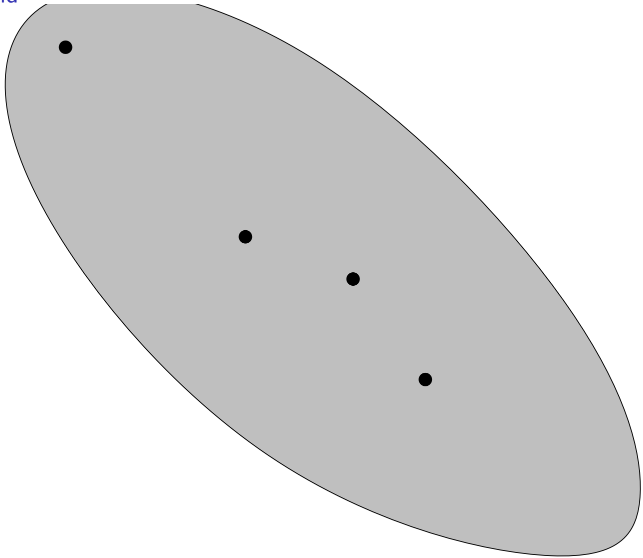

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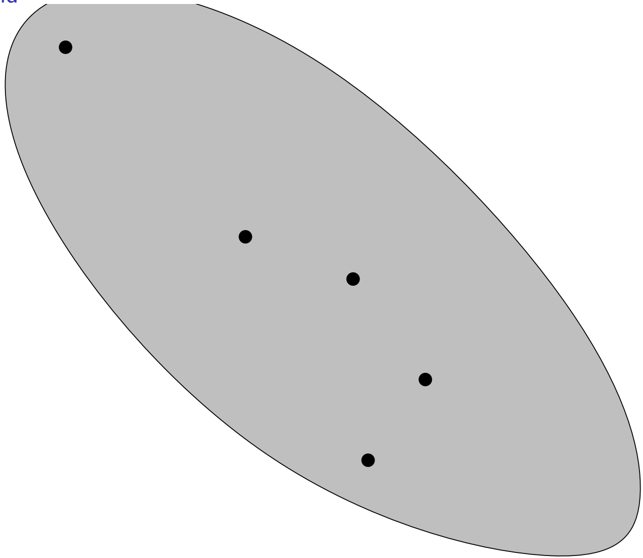

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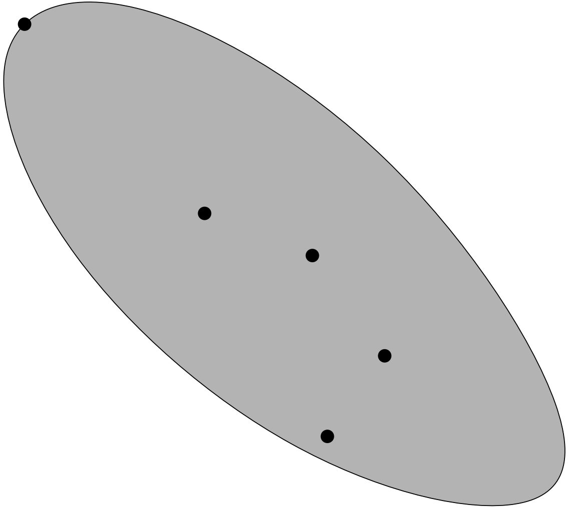

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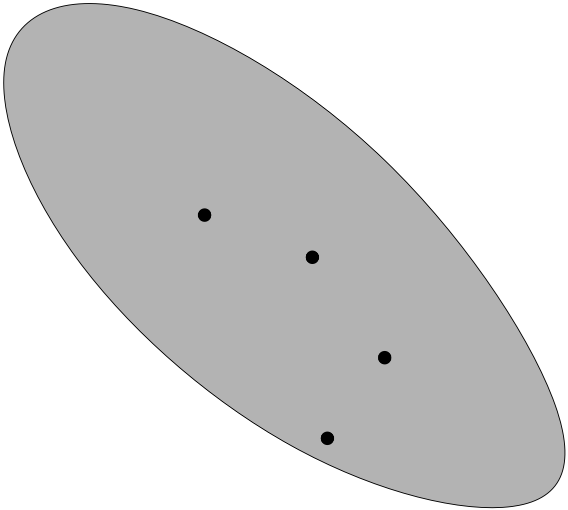

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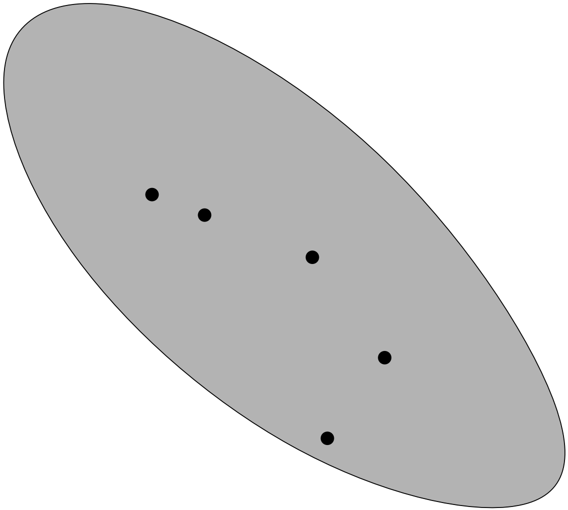

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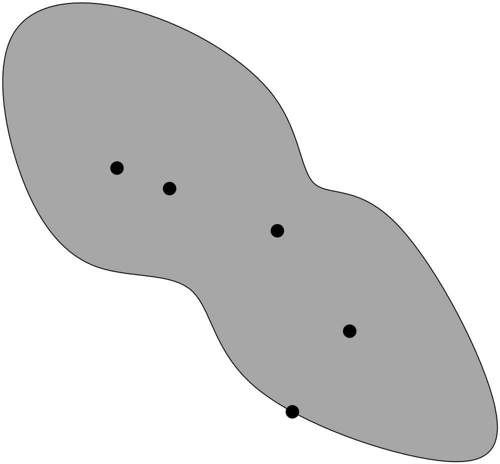

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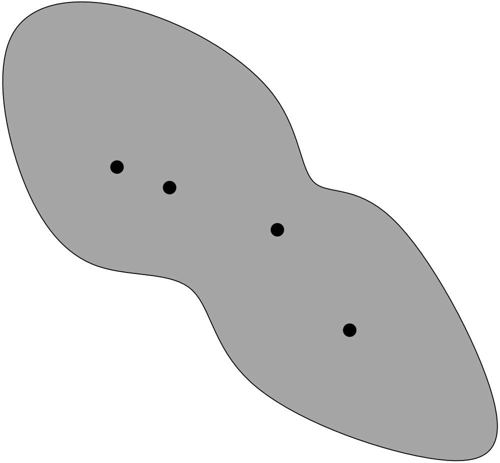

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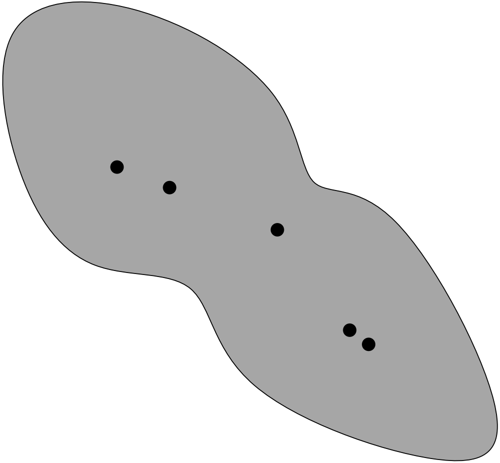

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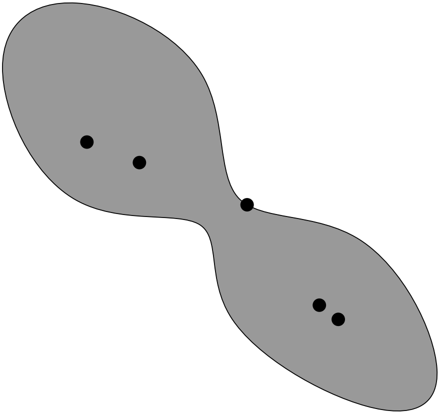

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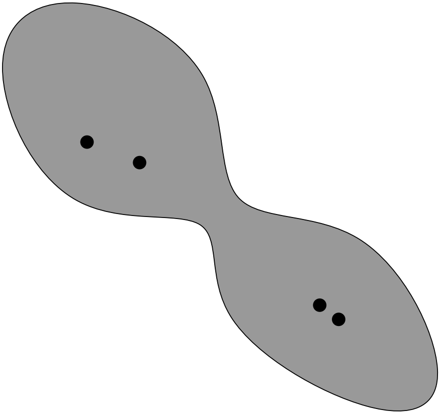

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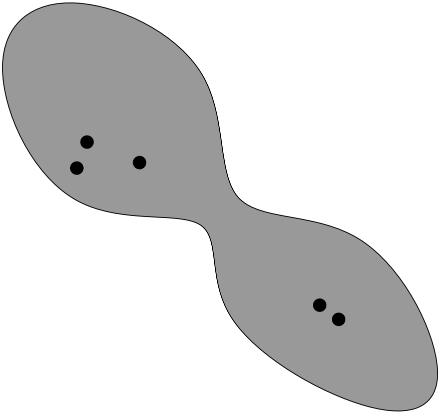

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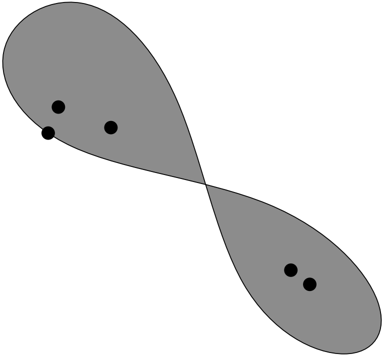

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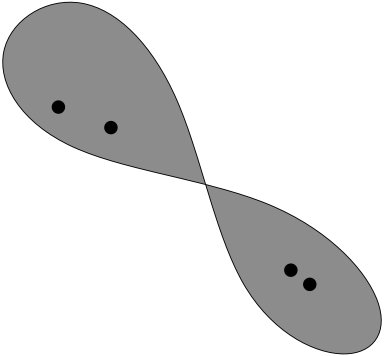

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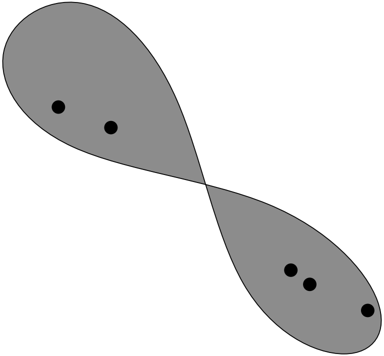

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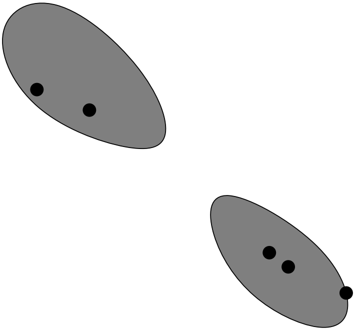

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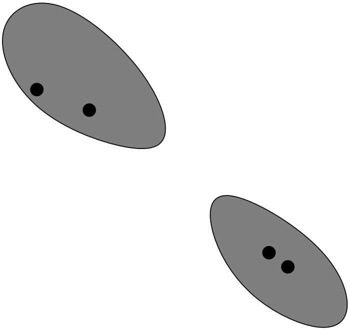

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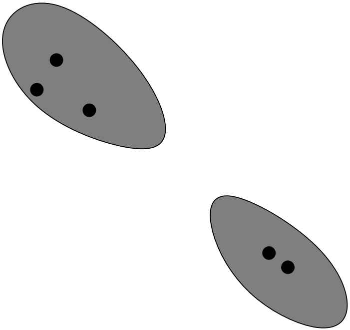

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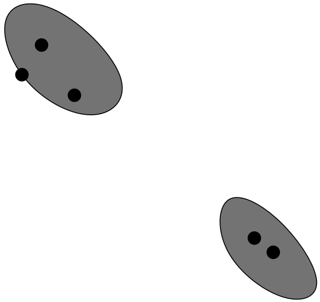

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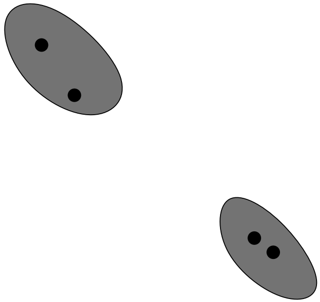

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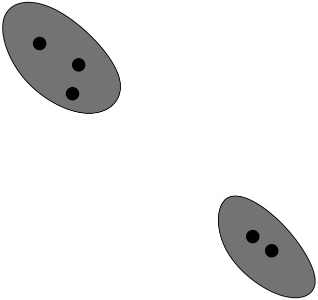

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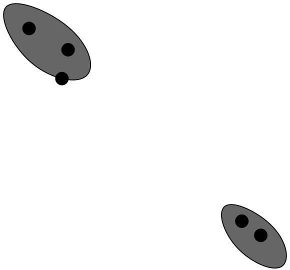

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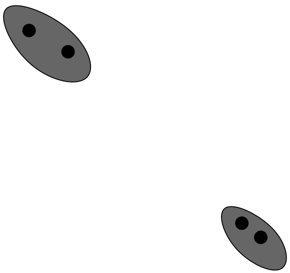

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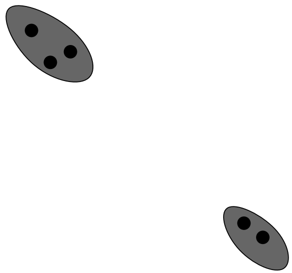

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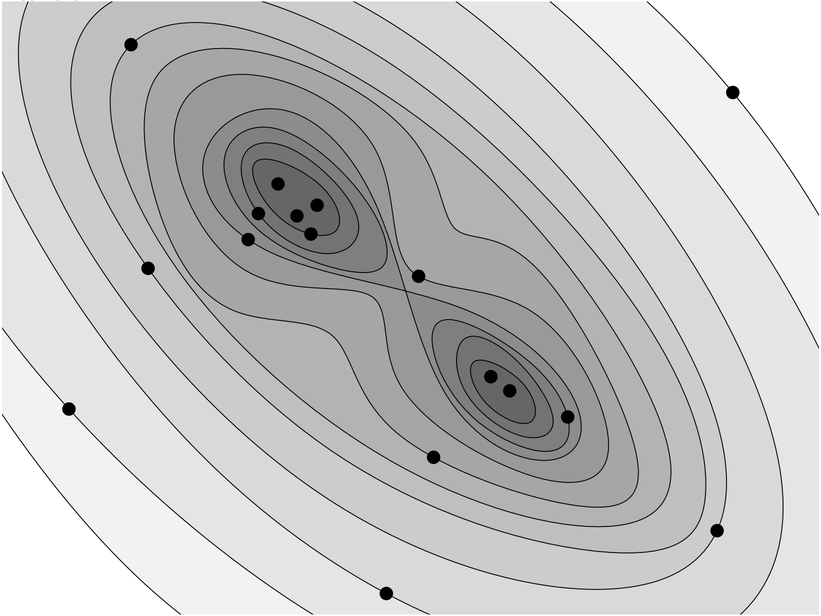

Nested sampling

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- They can also be used to calculate evidences, since it sequentially updates the priors.

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- Most of the work in NS to date has been in attempting to implement a hard-edged sampler in the NS meta-algorithm.

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Previous attempts

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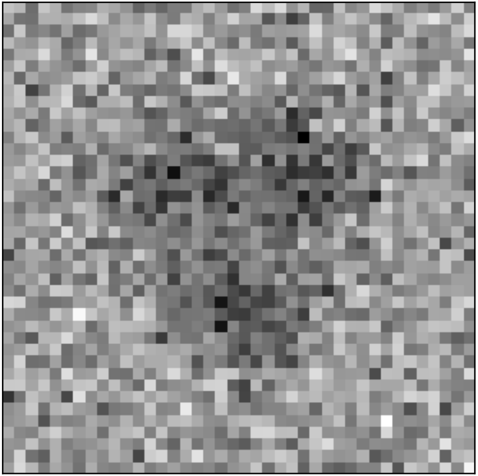
- PolyChord 2.0 imminent.

Object detection

Toy problem



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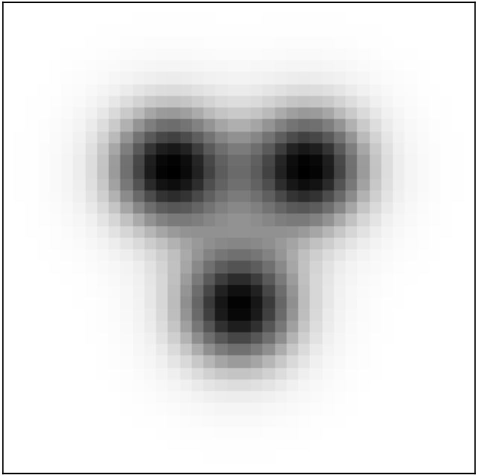
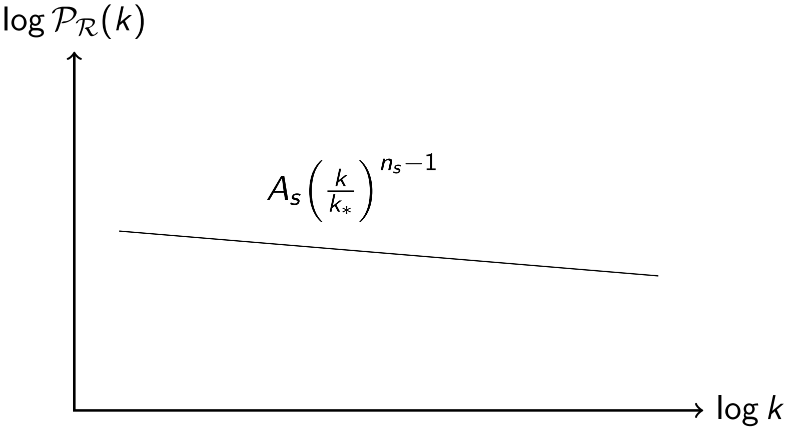


PolyChord in action

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



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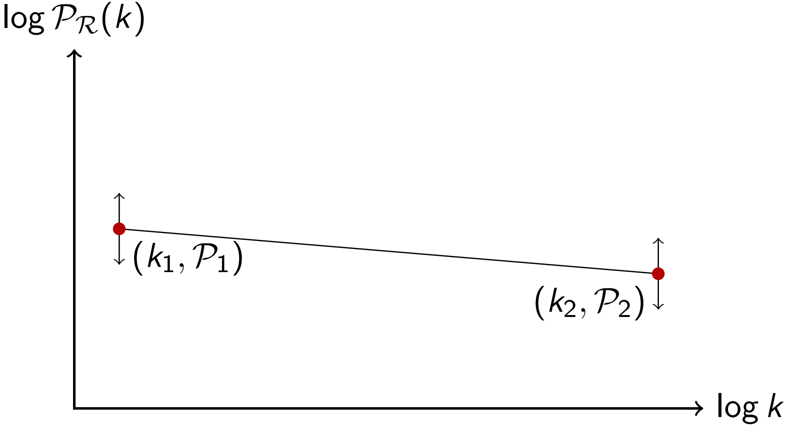

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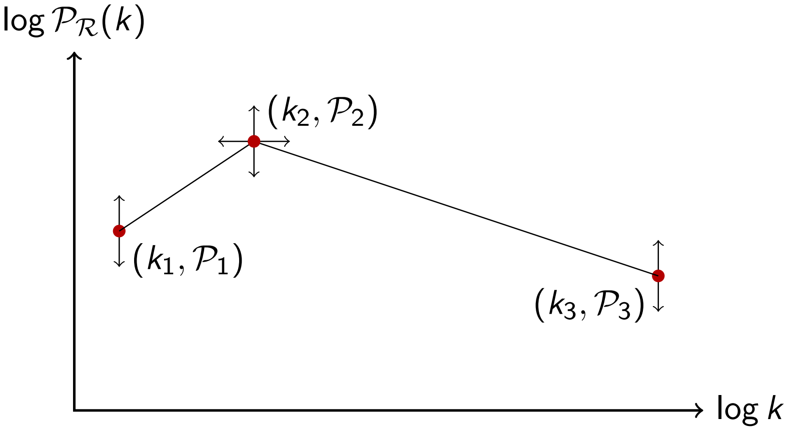

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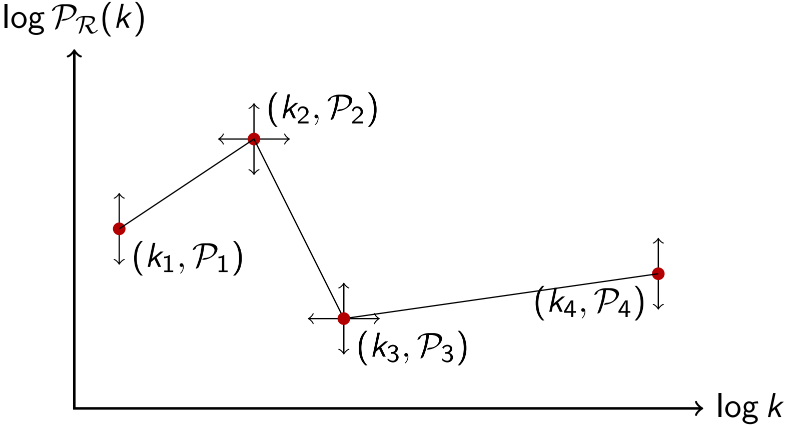

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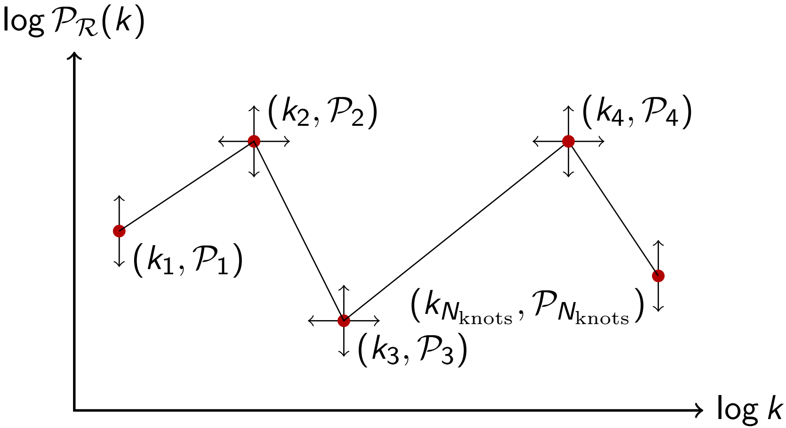
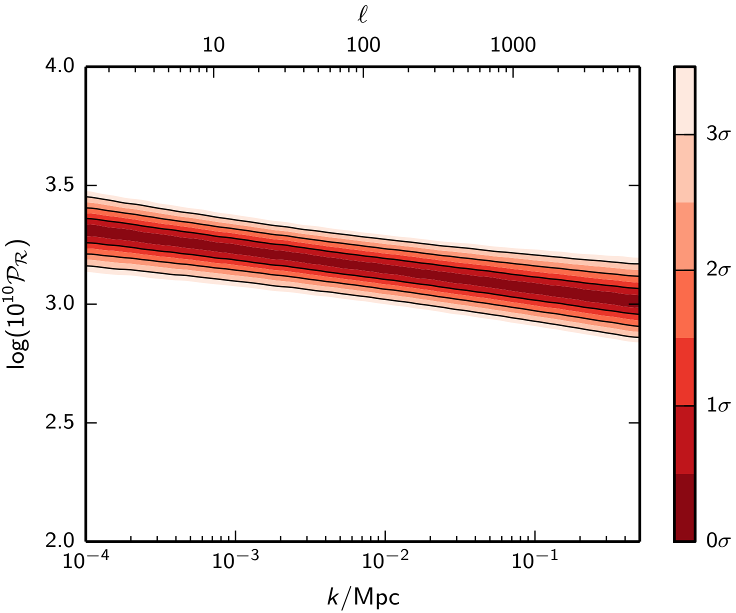


0 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



1 internal knots

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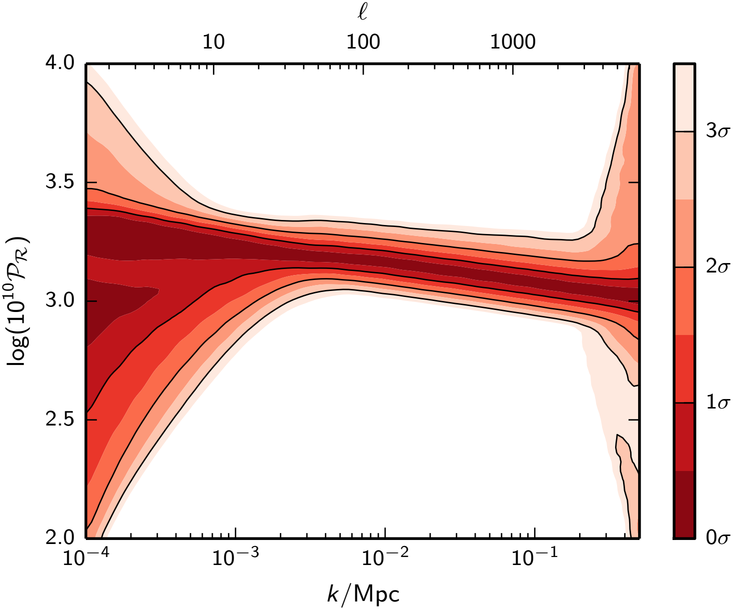

2 internal knots

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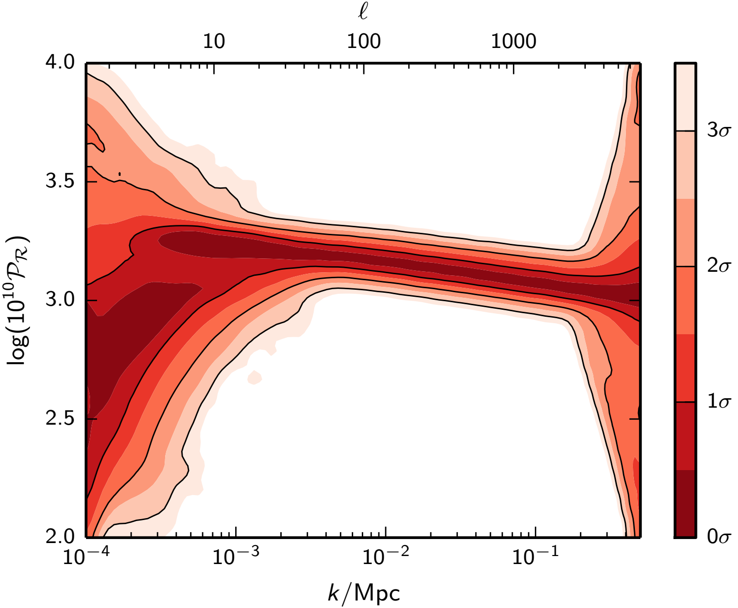

3 internal knots

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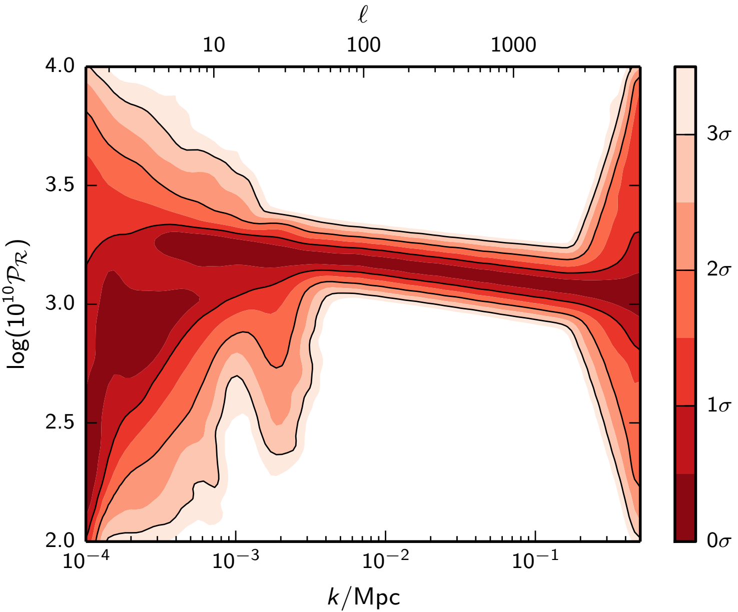

4 internal knots

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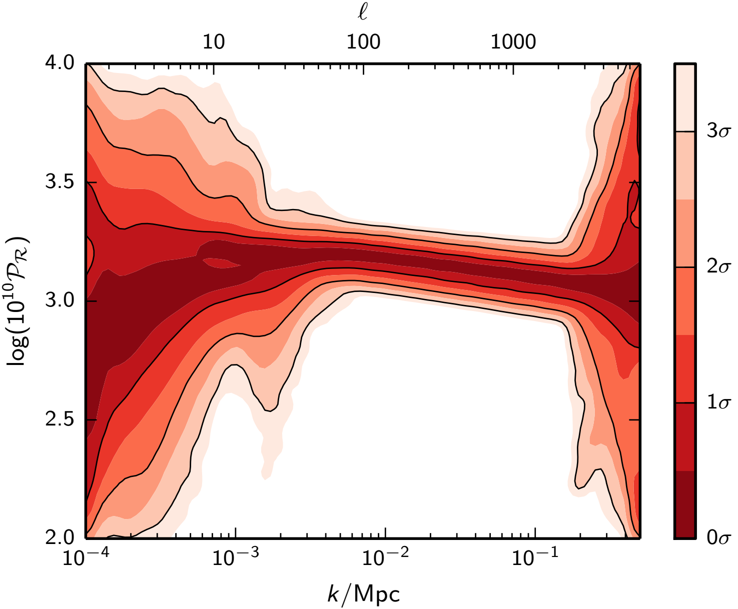
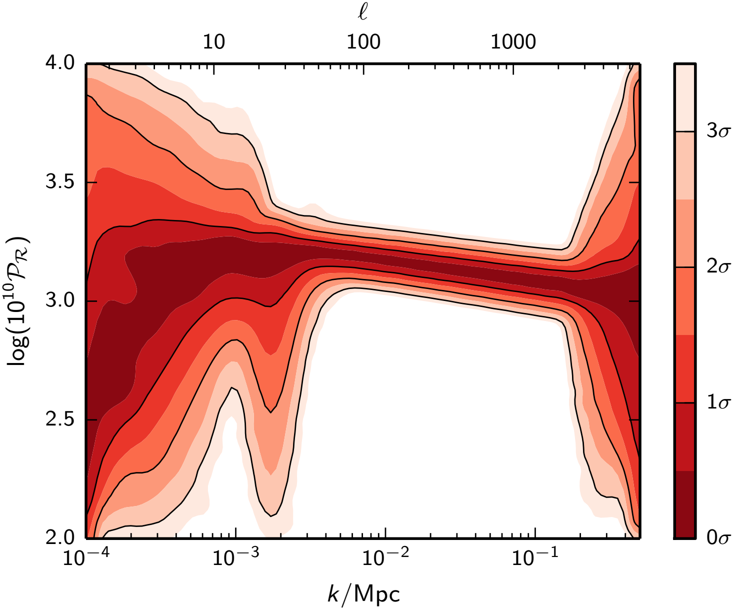


5 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



6 internal knots

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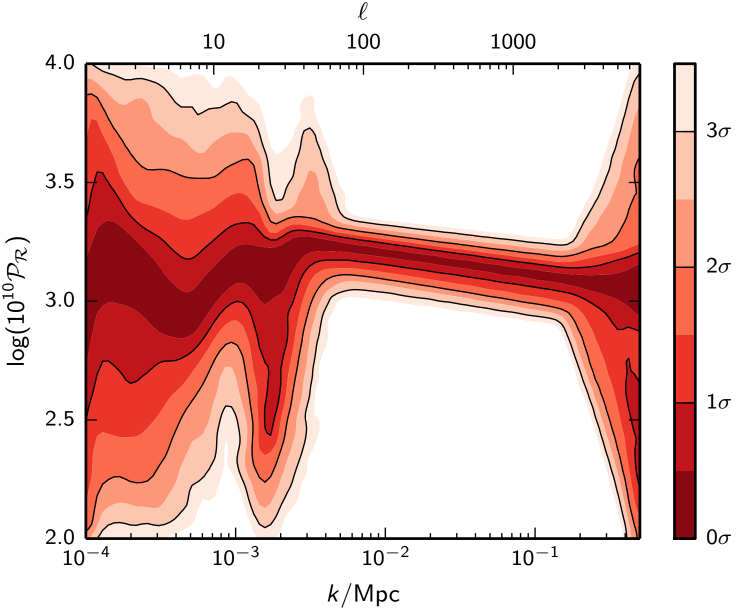
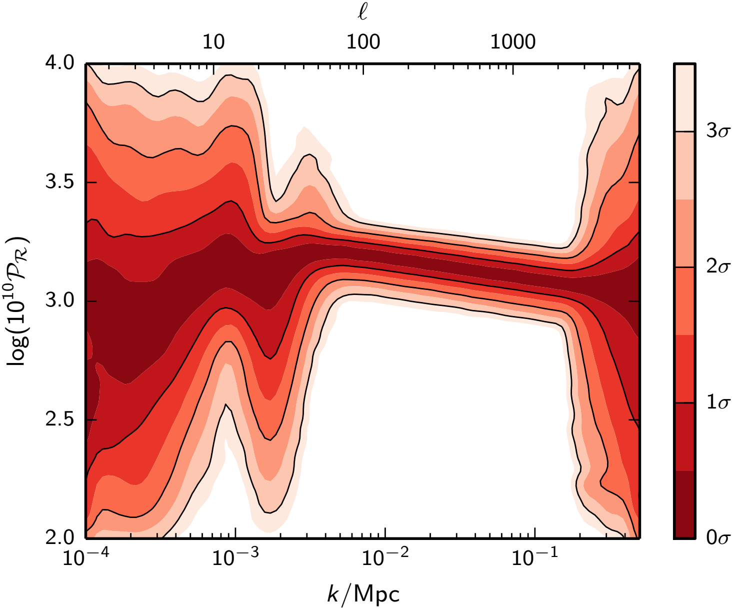


7 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



8 internal knots

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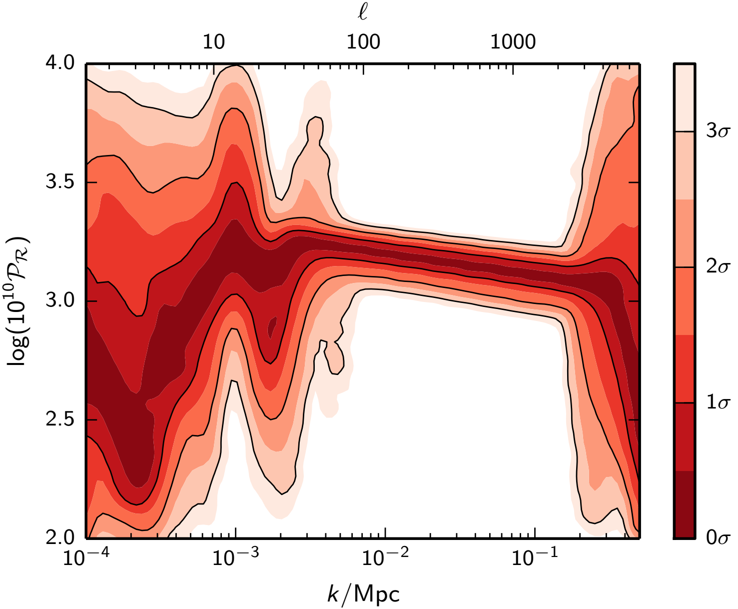
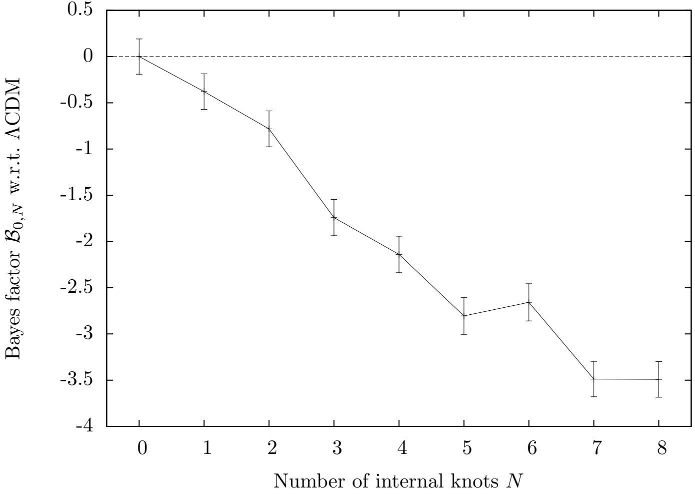


Bayes Factors

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



Marginalised plot

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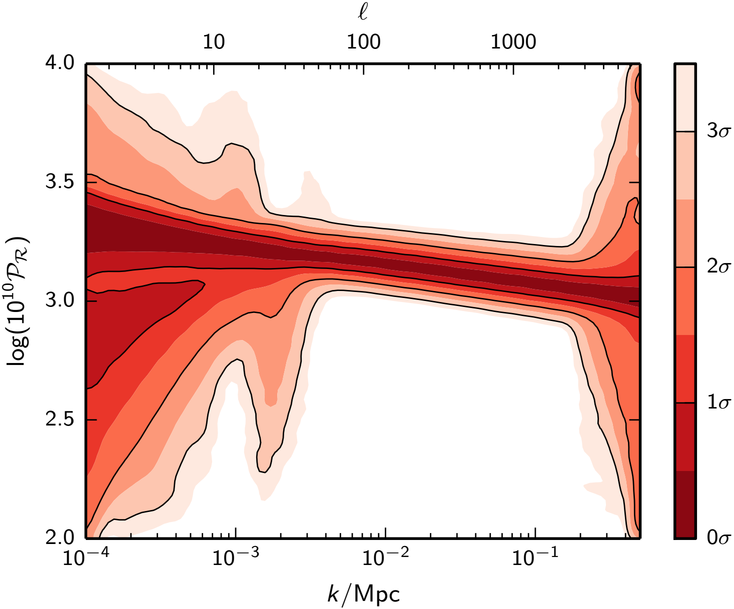

Dark energy equation of state reconstruction

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- Same thing, but for Dark energy equation of state $w(z)$ (quintessence).

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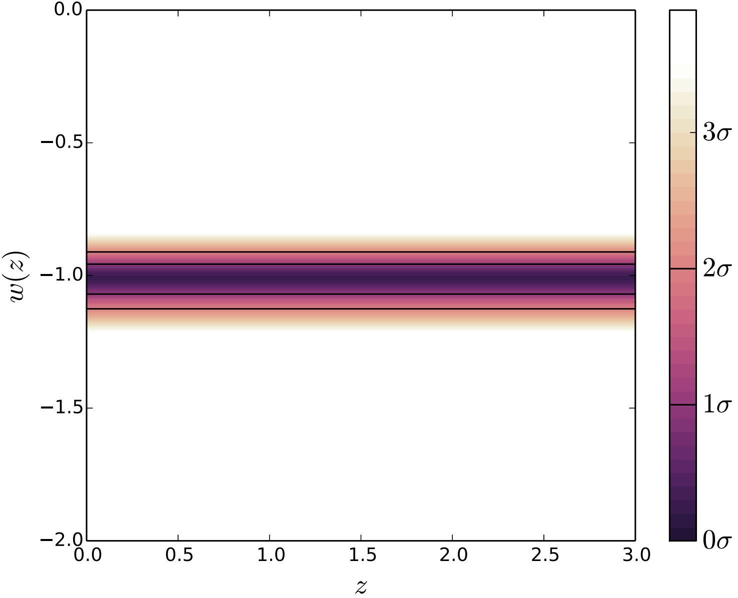
- Data used is Planck 2015, BOSS DR 11, JLA supernovae and BOSS Lyα data

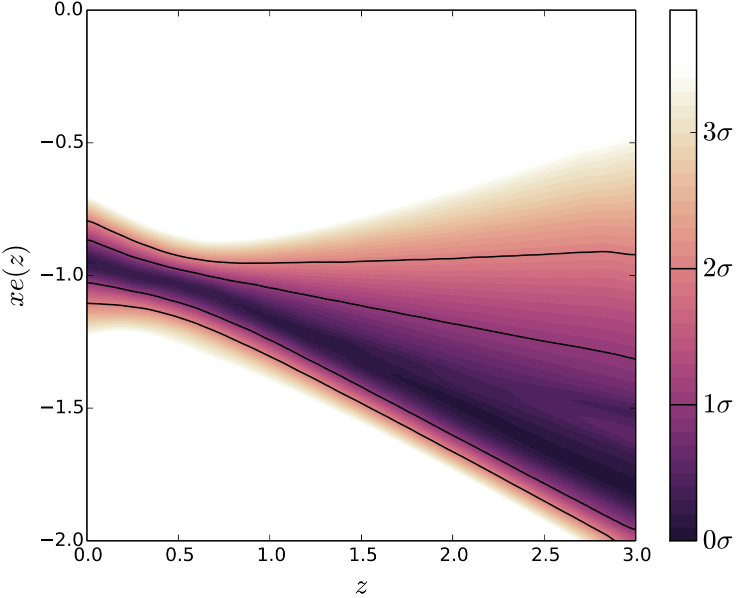
Flat, variable *w*

Dark energy equation of state reconstruction



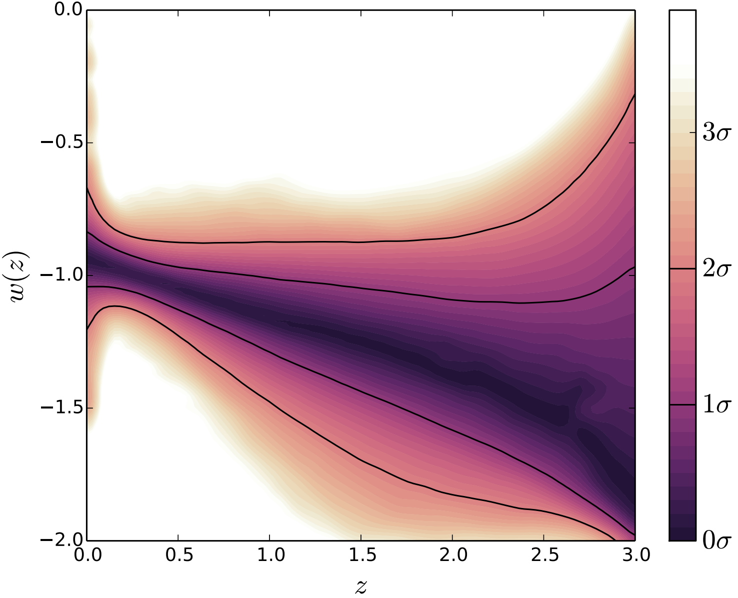
Tilted

Dark energy equation of state reconstruction



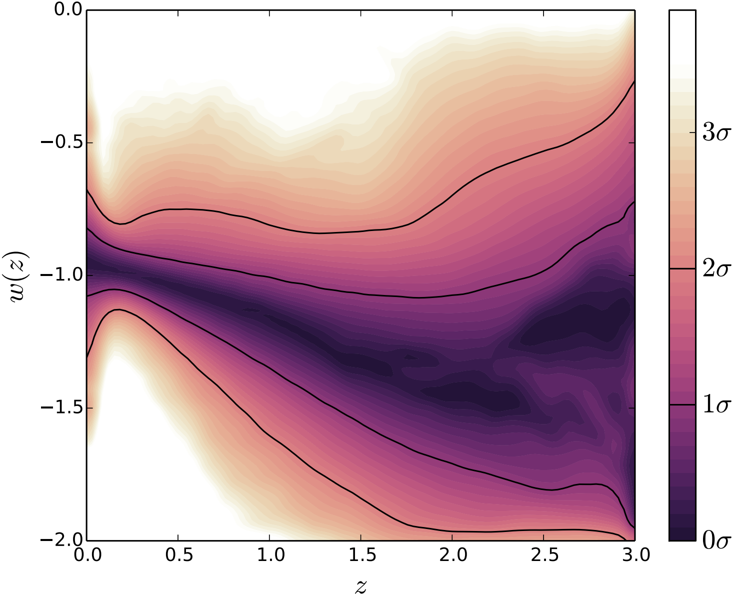
1 internal node

Dark energy equation of state reconstruction



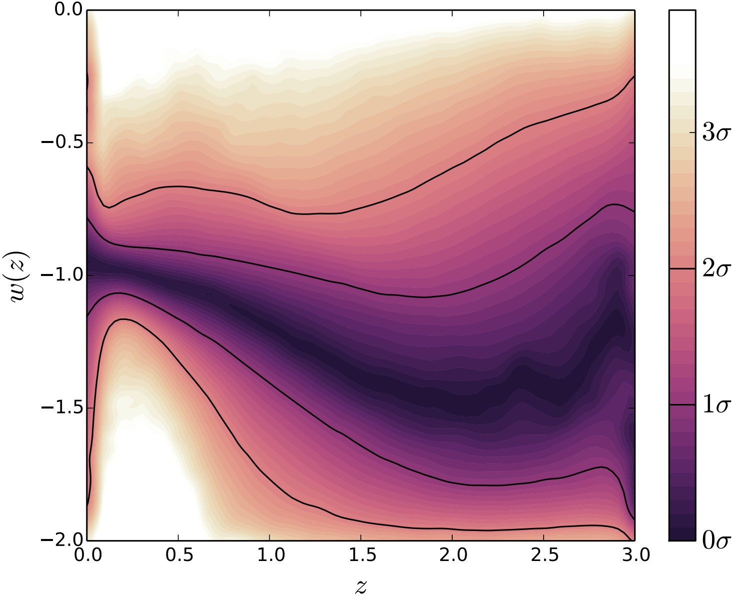
2 internal nodes

Dark energy equation of state reconstruction



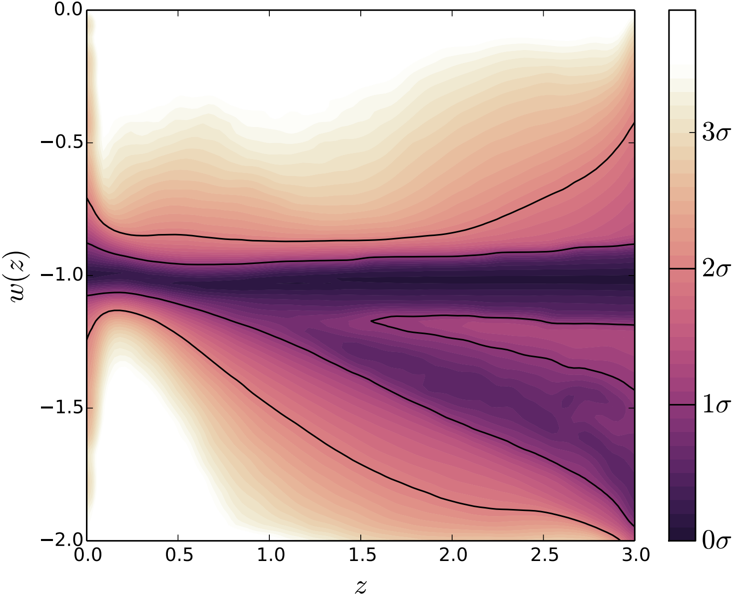
3 internal nodes

Dark energy equation of state reconstruction



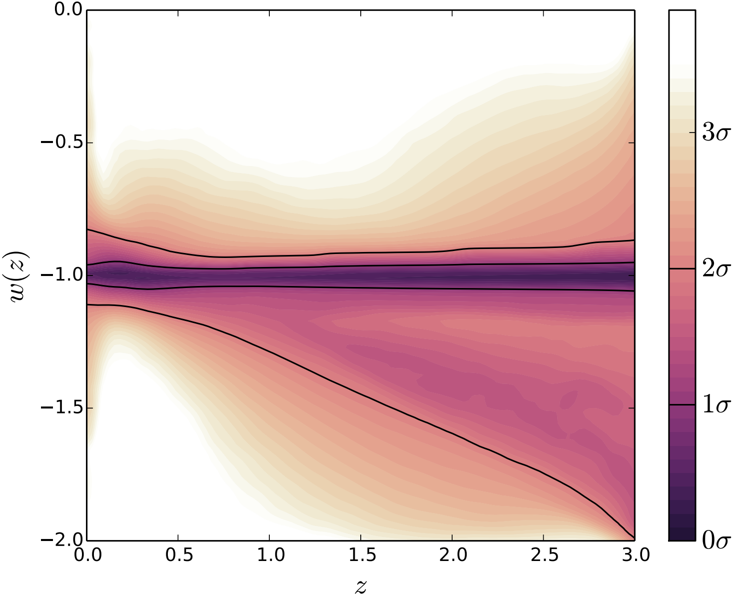
Marginalised plot - just extension models

Dark energy equation of state reconstruction



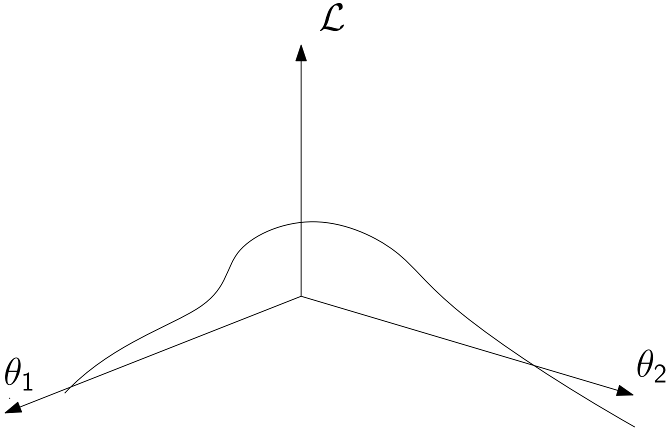
Marginalised plot - including LCDM

Dark energy equation of state reconstruction



Nested Sampling

Calculating evidences



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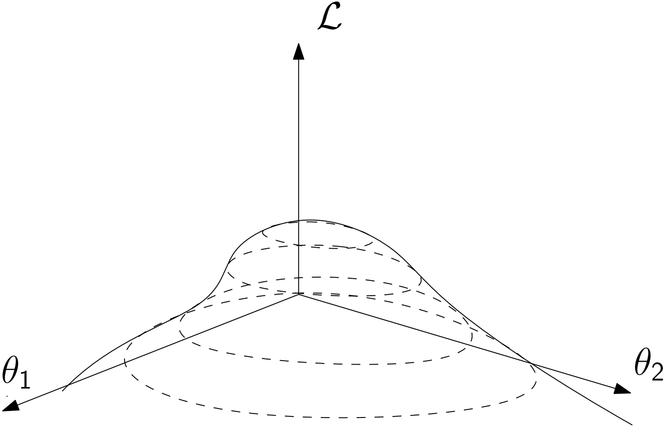

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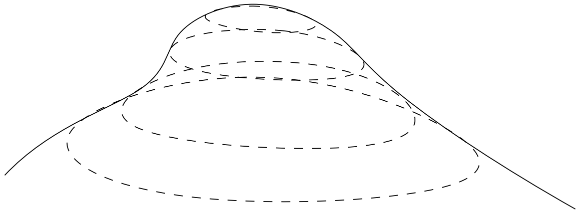

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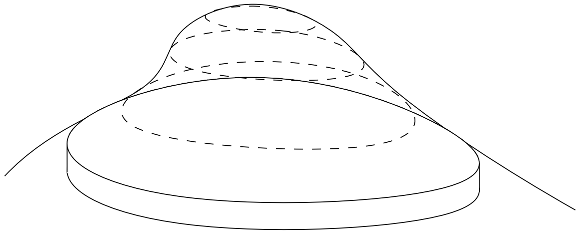

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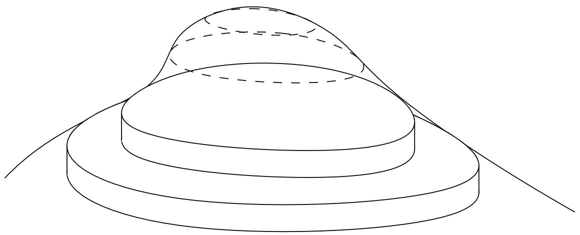

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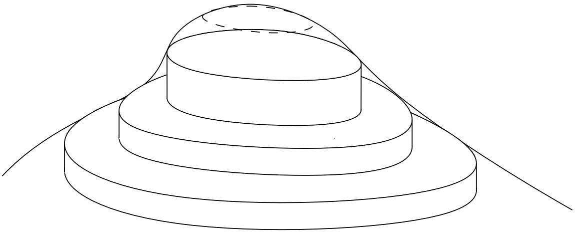

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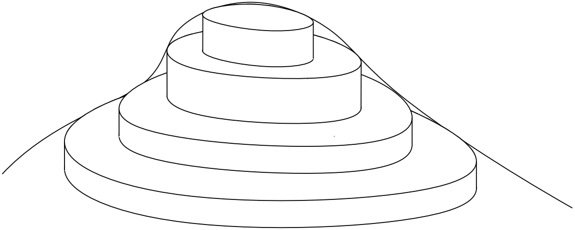

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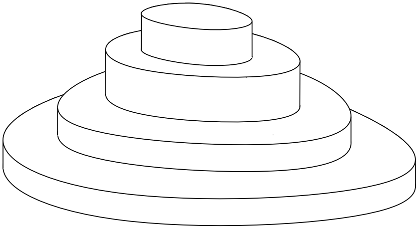

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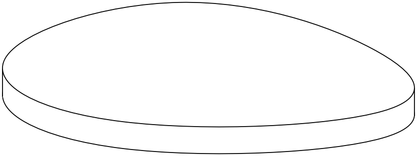

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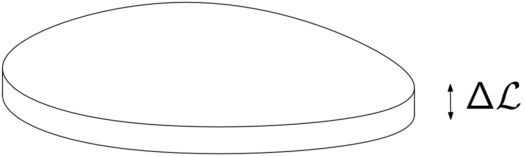

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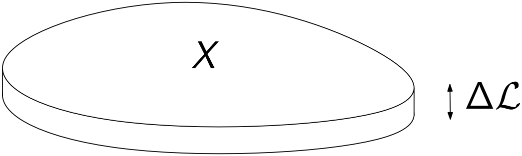

Nested Sampling
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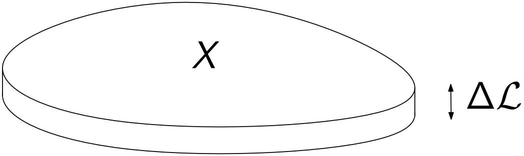
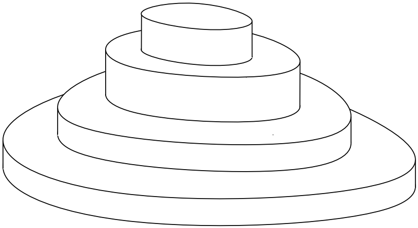


$$\text{Volume} = X\Delta\mathcal{L}$$

Nested Sampling

Calculating evidences



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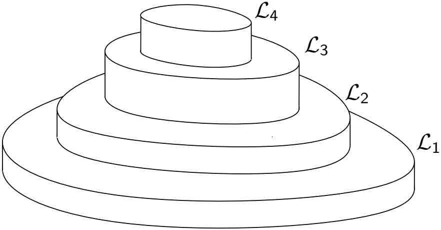

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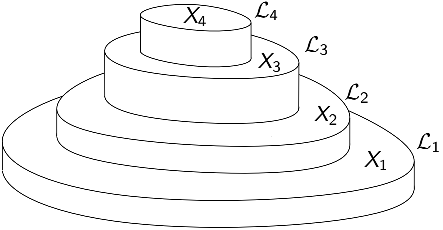
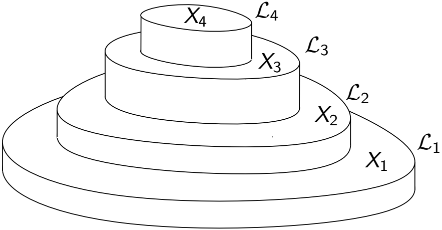


Nested Sampling

Calculating evidences

$$\mathcal{Z} \approx \sum_i X_i \Delta\mathcal{L}_i$$



Nested Sampling

Exponential volume contraction

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- $$\mathcal{Z} \approx \sum_i \Delta\mathcal{L}_i X_i \tag{1}$$
- $$X_{i+1} \approx \frac{n}{n+1} X_i, \qquad X_0 = 1 \tag{2}$$