Nested Sampling

An efficient and robust Bayesian inference tool for astrophysics and cosmology

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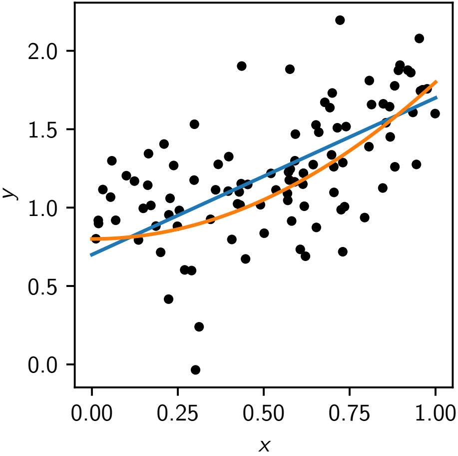
Motivating example

Fitting lines to data

- We have noisy data D
- We wish to fit a model M
- Functional form $y = f_M(x;\theta)$
- For example:

$$f_{\text{linear}}(x;\theta) = ax + b$$
$$f_{\text{quadratic}}(x;\theta) = ax^2 + b$$

- Model parameters $\theta = (a, b)$



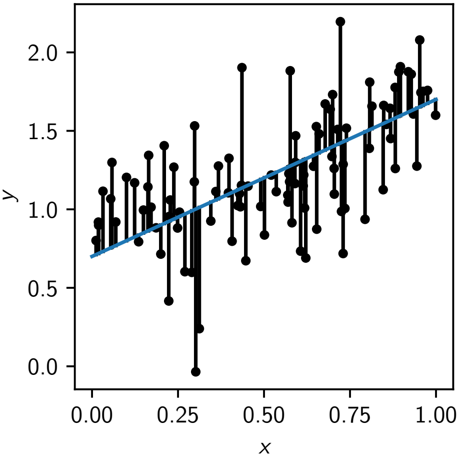
χ^2 best-fit

Fitting lines to data

- For each parameter set θ:

$$\chi^2(\theta) = \sum_i |y_i - f(x_i;\theta)|^2$$

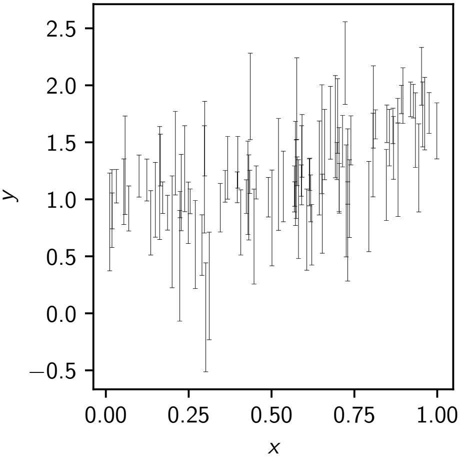
- Minimise χ^2 wrt θ



χ^2 with non-uniform data errors

- If data have non-uniform errors:

$$\chi^2(\theta) = \sum_i \frac{|y_i - f(x_i;\theta)|^2}{\sigma_i^2}$$



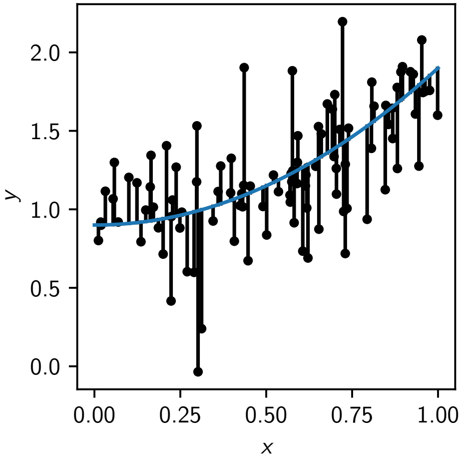
Problems with χ^2

Fitting lines to data

- How do we differentiate between models
- Why square the errors? – could take absolute:

$$\psi^2(\theta) = \sum_i \frac{|y_i - f(x_i;\theta)|}{\sigma_i}$$

- Where does this approach even come from?



Multivariate probability

- Marginalisation:
$$P(x) = \int P(x,y)dy$$
- Conditioning:
$$P(y|x) = \frac{P(x,y)}{P(x)} = \frac{P(x,y)}{\int P(x,y)dy}$$
- De-Conditioning:
$$P(x|y)P(y) = P(x,y)$$
- Bayes theorem:
$$P(y|x) = \frac{P(x|y)P(y)}{P(x)}$$

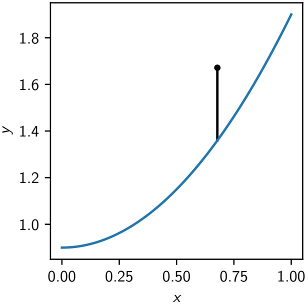
"To flip a conditional $P(x|y)$, you first de-condition on y, and then re-condition on x."

The probability of observing a datum:

$$P(y_i|\theta, M) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{|y_i - f(x_i;\theta)|^2}{2\sigma_i^2}\right)$$

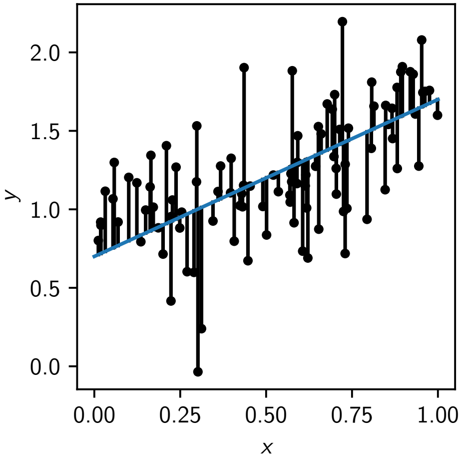
The probability of observing the data:

$$\begin{aligned} P(D|\theta, M) &= \prod_i \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{|y_i - f(x_i;\theta)|^2}{2\sigma_i^2}\right) \\ &= \frac{1}{\prod_i \sqrt{2\pi}\sigma_i} \exp \sum_i -\frac{|y_i - f(x_i;\theta)|^2}{2\sigma_i^2} \\ &\propto e^{-\chi^2(\theta)/2} \end{aligned}$$



Maximum likelihood

Fitting lines to data



- Minimising $\chi^2(\theta)$ is equivalent to maximising $P(D|\theta, M) \propto e^{-\chi^2(\theta)/2}$
- $P(D|\theta, M)$ is called the Likelihood $L = L(\theta)$ of the parameters θ
- "Least squares" $\equiv$ "maximum likelihood" (if data are gaussian).

Bayesian inference

- Likelihood $L = P(D|\theta, M)$ is undeniably correct.
- Frequentists construct inference techniques purely from this function.
- The trend is cosmology is to work with a Bayesian approach.
- What we want are things like $P(\theta|D, M)$ and $P(M|D)$.
- To invert the conditionals, we need Bayes theorem:

$$P(\theta|D, M) = \frac{P(D|\theta, M)P(\theta|M)}{P(D|M)}$$

$$P(M|D) = \frac{P(D|M)P(M)}{P(D)}$$

Terminology

Bayesian inference

$$P(\theta|D,M) = \frac{P(D|\theta,M)P(\theta|M)}{P(D|M)}$$

$$\text{Posterior} = \frac{\text{Likelihood} \times \text{Prior}}{\text{Evidence}}$$

$$P(M|D) = \frac{P(D|M)P(M)}{P(D)}$$

$$\text{Model probability} = \frac{\text{Evidence} \times \text{Model Prior}}{\text{Normalisation}}$$

The prior

Example: Biased coins

- Need to define the **Prior** $P(\theta)$ — probability of the bias, given no data
- Represents our knowledge of parameters before the data – subjective
- Frequentists view this as a flaw in Bayesian inference.
- Bayesians view this as an advantage
- Fundamental rule of Inference:

The prior

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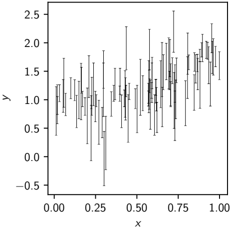
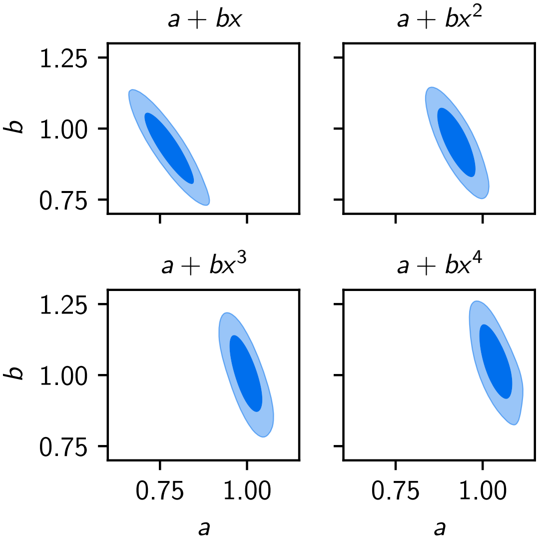
- Bayesians view this as an advantage
- Fundamental rule of Inference:

You cannot extract information from data
without making assumptions

- All Bayesians do is make them explicit
- Any method that claims it is "objective" is simply hiding them

- We may use $P(\theta|D, M)$ to inspect whether a model looks reasonable



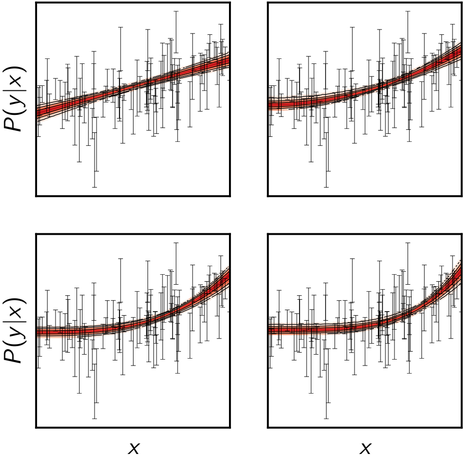


Predictive posterior

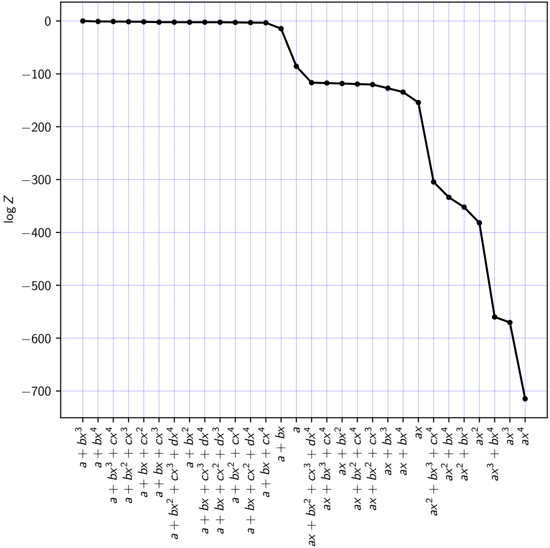
More useful to plot:

$$P(y|x) = \int P(y|x,\theta)P(\theta)d\theta$$

(all conditioned on D, M)



- We may use the Bayezian evidence Z to determine whether a model is reasonable.
- $Z = P(D|M) = \int P(D|M,\theta)P(\theta|M)d\theta$
- Normally assume uniform model priors $Z \propto P(M|D)P(M)$.



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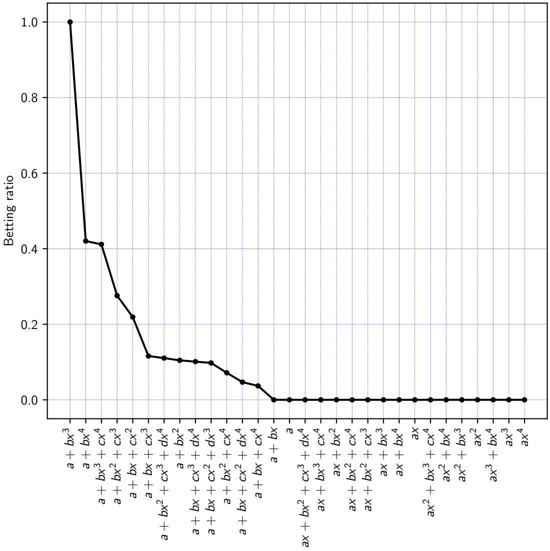
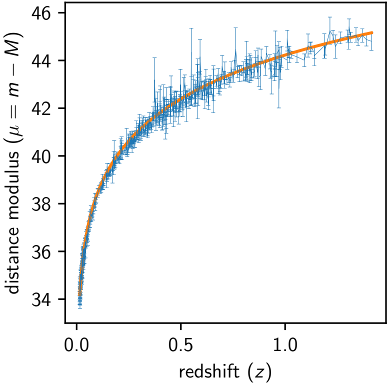


Line fitting (context)

- Whilst this model seems a little trite. . .
- . . . determining polynomial indices $\equiv$ determining cosmological material content:



$$\left(\frac{H}{H_0}\right)^2 = \Omega_\mathrm{r}\left(\frac{a_0}{a}\right)^4 + \Omega_\mathrm{m}\left(\frac{a_0}{a}\right)^3 + \Omega_k\left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda$$

Quantifying error with Probability

- As scientists, we are used to seeing error bars on results.
- Age of the universe (*Planck*):

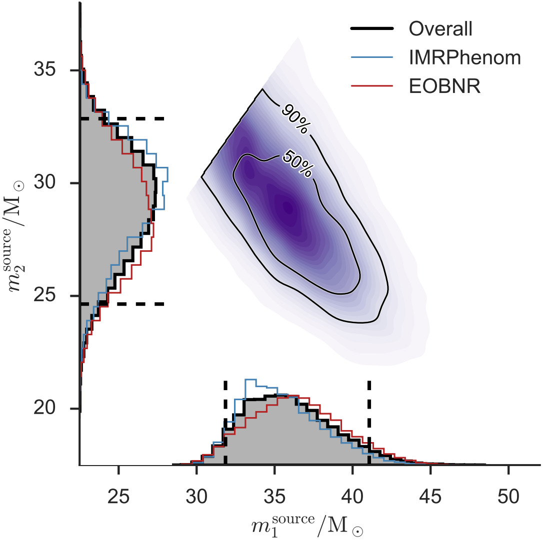
$$13.73 \pm 0.12 \,\text{billion years old.}$$

- Masses of LIGO GW150914 binary merger:

$$m_1 = 39.4^{+5.5}_{-4.9} \, M_\odot, \qquad m_2 = 30.9^{+4.8}_{-4.4} \, M_\odot$$

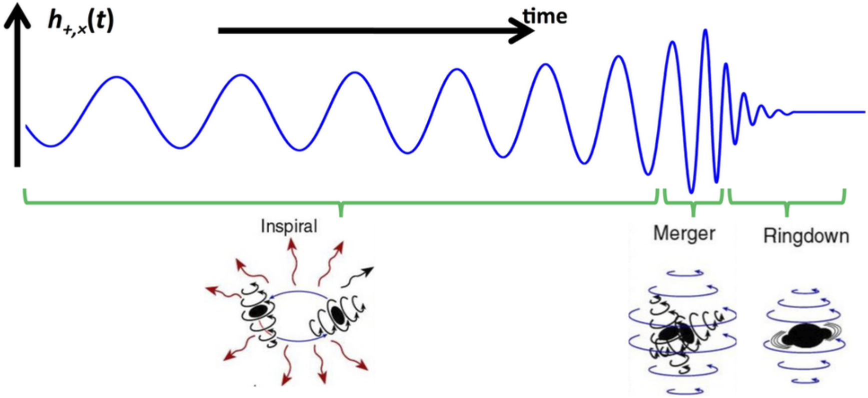
- These are called *credible intervals*, state that we are e.g. 90% confident of the value lying in this range.
- More importantly, these are *summary statistics*.

LIGO binary merger



- Summary statistics summarise a full probability distribution.
- One goal of inference is to produce these probability distributions.

Extended example of inference: LIGO



The parameters Θ of the model M

Extended example of inference: LIGO

Theoretical signal depends on:

- m_1, m_2: mass of binary
- θ, ϕ: sky location
- r: luminosity distance
- Φ_c, t_c: phase and time of coalescence
- i, θ_{sky}: inclination and angle on sky (orbital parameters)

- Cannot plot the full posterior distribution:

$$\mathcal{P}(\Theta) \equiv P(m_1, m_2, \theta, \phi, r, \Phi_c, t_c, i, \theta_{\mathrm{sky}}|D, M)$$

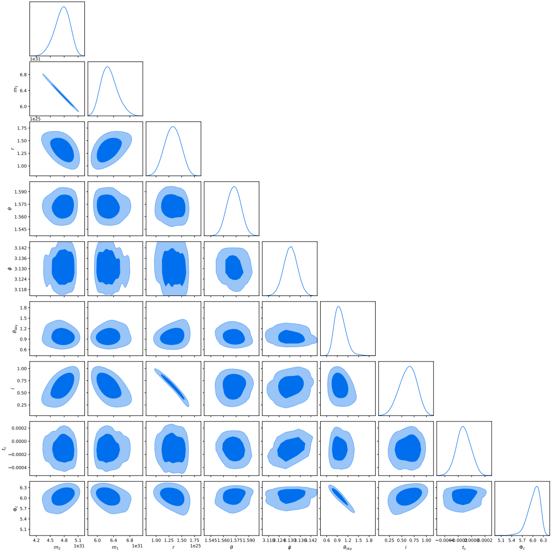
- Can plot 1D and 2D *marginalised* distributions e.g:

$$P(m_1, m_2|D, M) =$$
$$\int P(m_1, m_2, \theta, \phi, r, \Phi_c, t_c, i, \theta_{\mathrm{sky}}|D, M)\, d\theta\, d\phi\, dr\, d\Phi_c\, dt_c\, di\, d\theta_{\mathrm{sky}}$$

- May do this for each pair of parameters
- Generates a *triangle plot*

Posterior $\mathcal{P}$

Extended example of inference: LIGO

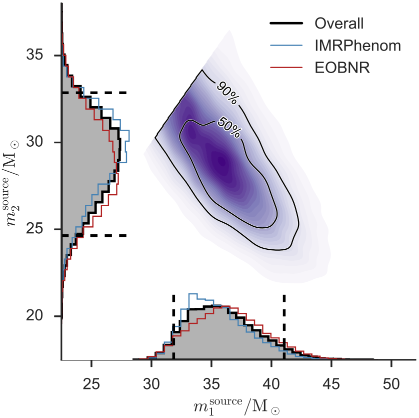


- Does give insight
- Not the full picture

Sampling

How to describe a high-dimensional posterior

- In high dimensions, posterior $\mathcal{P}$ occupies a vanishingly small region of the prior π.
- Gridding is doomed to failure for $D \gtrsim 4$.
- *Sampling* the posterior is an excellent compression scheme.



Why do sampling?

Marginalisation over the posterior

- Set of N samples $S = \{\Theta^{(i)} : i = 1, \ldots N : \ \Theta^{(i)} \sim \mathcal{P}\}$
- Mean mass:
$$\bar{m}_1 \equiv \langle m_1 \rangle_{\mathcal{P}} \equiv \int m_1 P(\theta|D,M) d\theta$$
- Mass covariance:
$$\mathrm{Cov}(m_1, m_2) \equiv \int (m_1 - \bar{m}_1)(m_2 - \bar{m}_2) P(\theta|D,M) d\theta$$
- Marginalised samples: Just ignore the other coordinates.
- N.B. Typically have *weighted* samples

Why do sampling?

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- Set of N samples $S = \{\Theta^{(i)} : i = 1, \ldots N : \; \Theta^{(i)} \sim \mathcal{P}\}$
- Mean mass:

$$\bar{m}_1 \equiv \langle m_1 \rangle_{\mathcal{P}} \approx \frac{1}{N}\sum_{i=1}^{N} m_1^{(i)}$$

- Mass covariance:

$$\mathrm{Cov}(m_1, m_2) \approx \frac{1}{N}\sum_{i=1}^{N}(m_1^{(i)} - \bar{m}_1)(m_2^{(i)} - \bar{m}_2)$$

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Why do sampling?

Marginalisation over the posterior

- Set of N samples $S = \{\Theta^{(i)} : i = 1, \ldots N : \ \Theta^{(i)} \sim \mathcal{P}\}$
- Mean mass:

$$\bar{m}_1 \equiv \langle m_1 \rangle_{\mathcal{P}} \approx \frac{\sum_{i=1}^N w^{(i)} m_1^{(i)}}{\sum_{i=1}^N w^{(i)}}$$

- Mass covariance:

$$\mathrm{Cov}(m_1, m_2) \approx \frac{\sum_{i=1}^N w^{(i)} (m_1^{(i)} - \bar{m}_1)(m_2^{(i)} - \bar{m}_2)}{\sum_{i=1}^N w^{(i)}}$$

- Marginalised samples: Just ignore the other coordinates.
- N.B. Typically have *weighted* samples

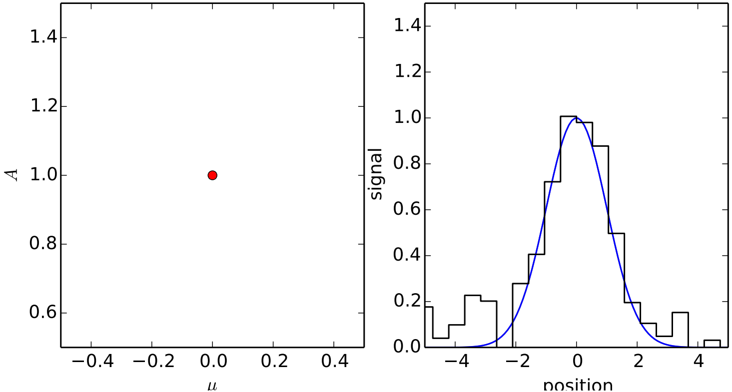
Parameter estimation

- The name of the game is therefore drawing samples S from the posterior $\mathcal{P}$ with the minimum number of likelihood calls.
- Gridding is doomed to failure in high dimensions.
- Enter Metropolis Hastings.

Metropolis Hastings

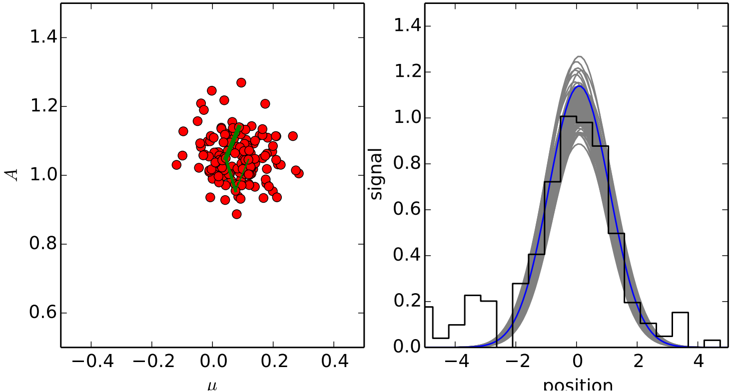
- Turn the N-dimensional problem into a one-dimensional one.
  1. Propose random step
  2. If uphill, make step...
  3. ...otherwise sometimes make step.

Metropolis Hastings



Metropolis Hastings

Metropolis Hastings



Metropolis Hastings

Struggles with...

Metropolis Hastings

Struggles with...

1. Burn in
2. Multimodality
3. Correlated Peaks
4. Phase transitions

Hamiltonian Monte-Carlo

- Key idea: Treat $\log L(\Theta)$ as a potential energy
- Guide walker under “force”:

$$F(\Theta) = \nabla \log L(\Theta)$$

- Walker is naturally “guided” uphill
- Conserved quantities mean efficient acceptance ratios.
- stan is a fully fledged, rapidly developing programming language with HMC as a default sampler.

Ensemble sampling

- Instead of one walker, evolve a set of n walkers.
- Can use information present in ensemble to guide proposals.
- emcee: affine invariant proposals.
- emcee is not the only (or even best) affine invariant approach.

The fundamental issue with all of the above

- They don't give you evidences!

$$
\begin{aligned}
\mathcal{Z} &= \mathrm{P}(D|M) \\
&= \int \mathrm{P}(D|\Theta, M)\mathrm{P}(\Theta|M) d\Theta \\
&= \langle \mathcal{L} \rangle_\pi
\end{aligned}
$$

- MCMC fundamentally explores the posterior, and cannot average over the prior.
- Thermodynamic annealing
  - Suffers from same tuning issues as MCMC
- Nearest neighbor volume estimation (Heavens arXiv:1704.03472)
  - Does not scale to high dimensions $D \gtrsim 15$.

Nested Sampling

John Skilling's alternative to traditional MCMC!

- Nested sampling is a completely different way of sampling.
- Uses ensemble sampling to compress prior to posterior.

New procedure:
Maintain a set S of n samples, which are sequentially updated:

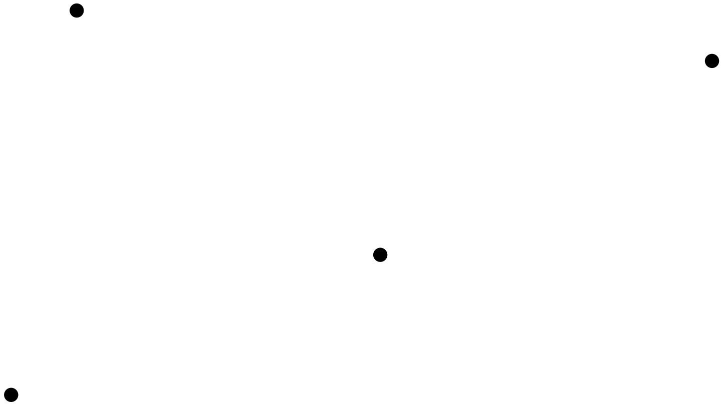
S_0: Generate n samples uniformly over the space (from the prior π).

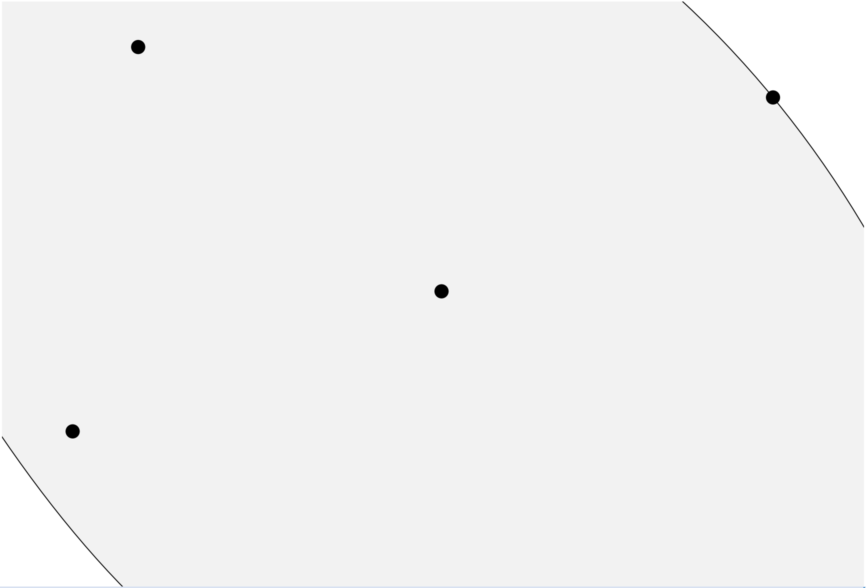
S_{n+1}: Delete the lowest likelihood sample in S_n, and replace it with a new uniform sample with higher likelihood

Requires one to be able to uniformly within a region, subject to a *hard likelihood constraint*.

Nested Sampling

Graphical aid





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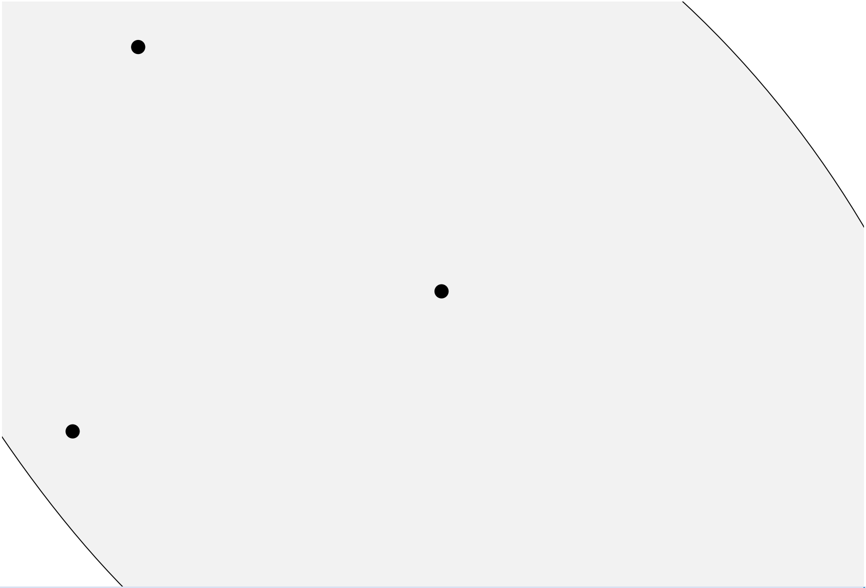

Nested Sampling
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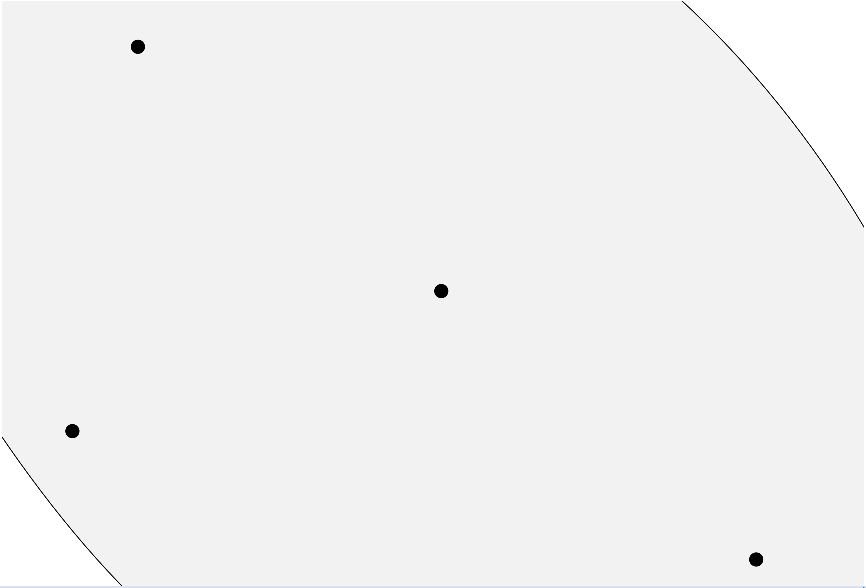

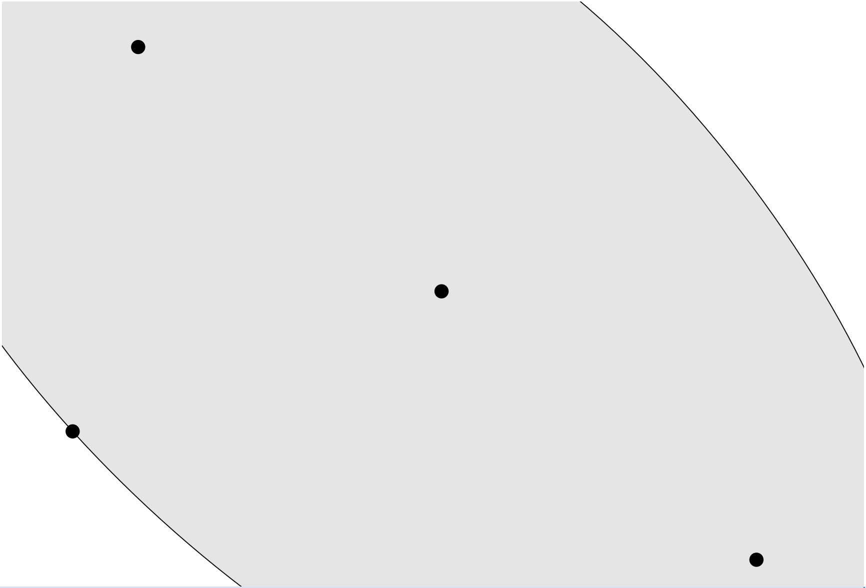

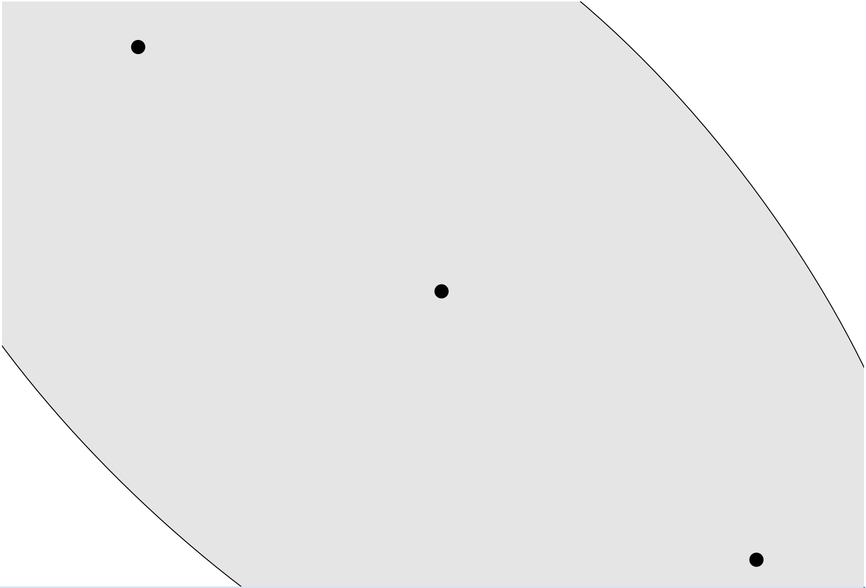

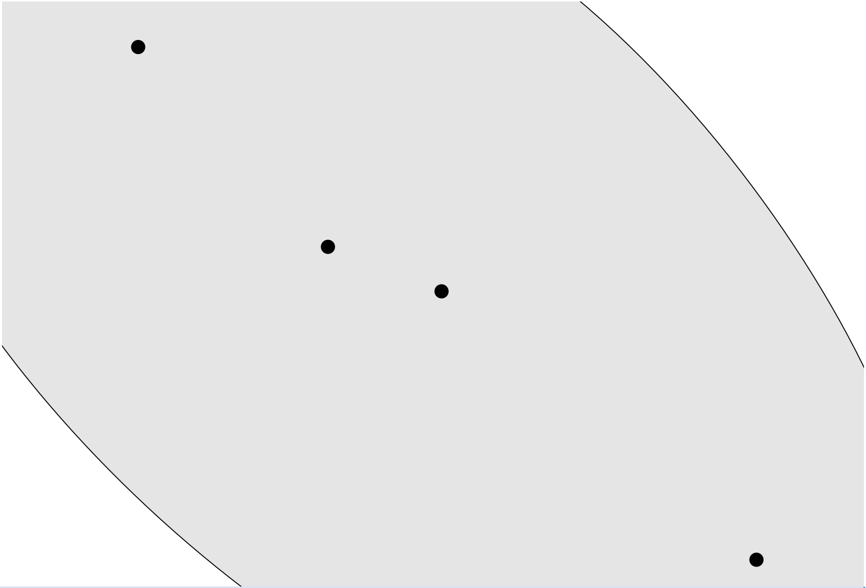

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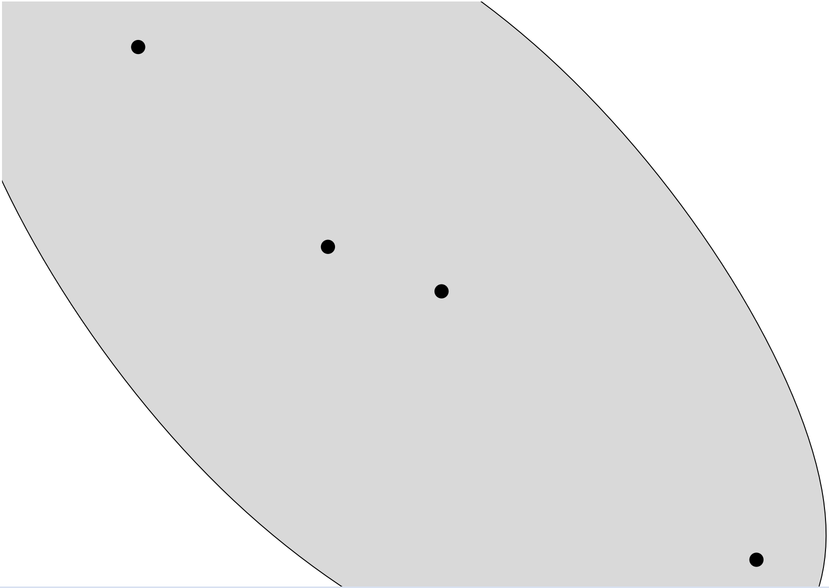

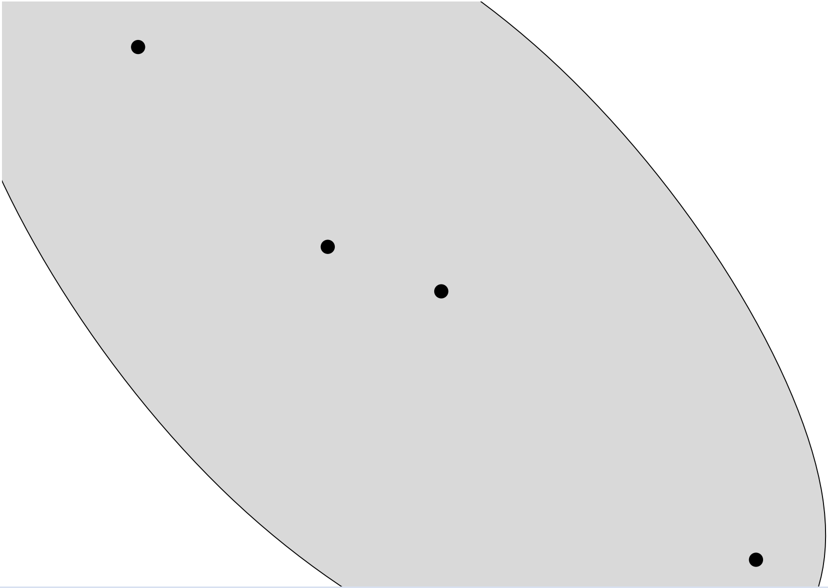

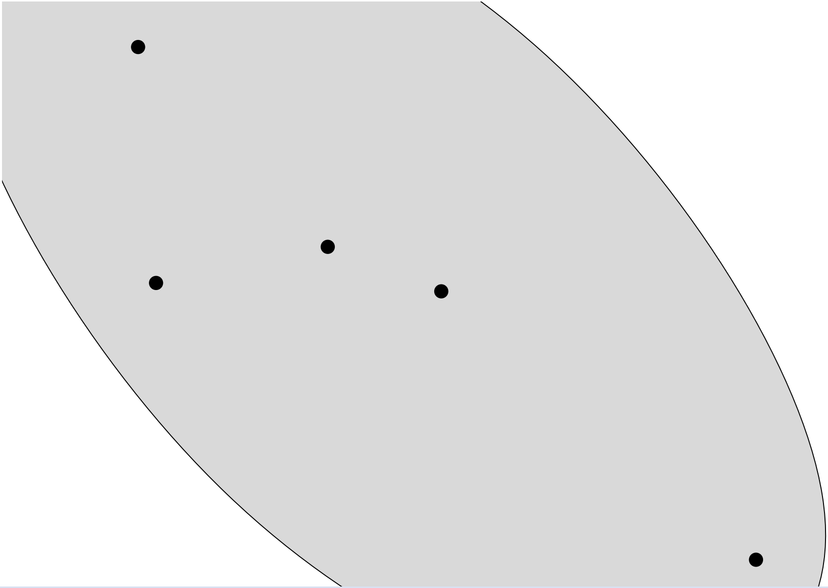

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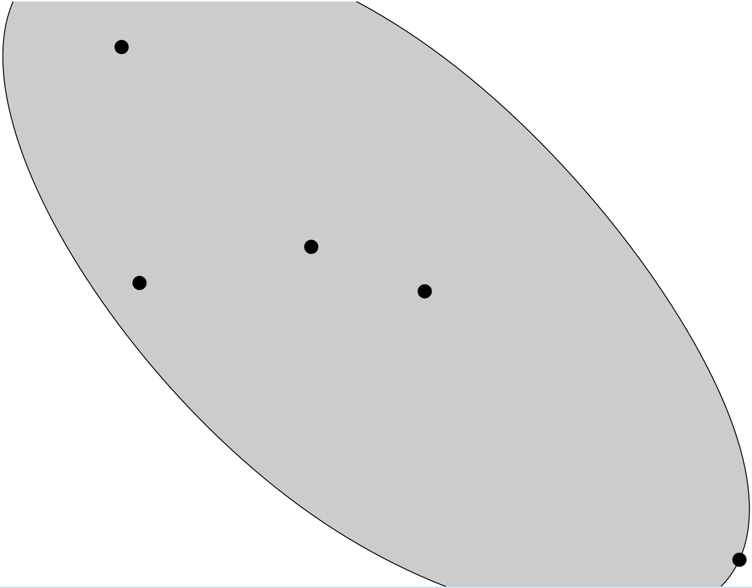

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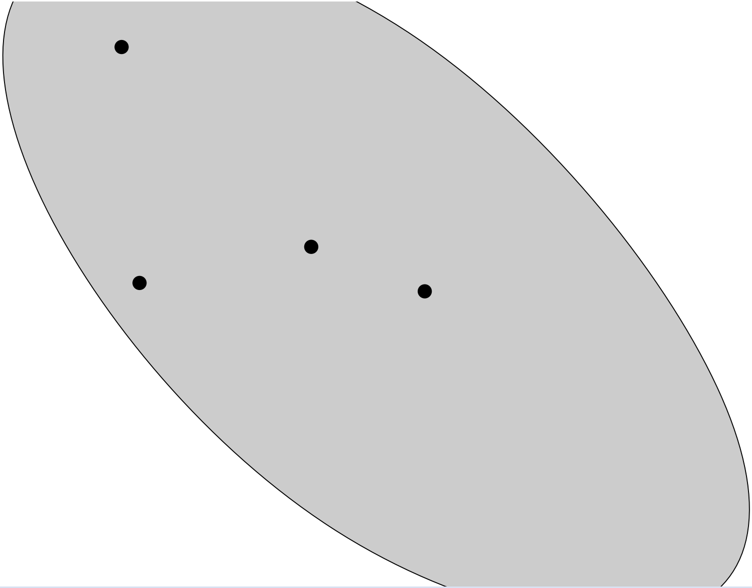

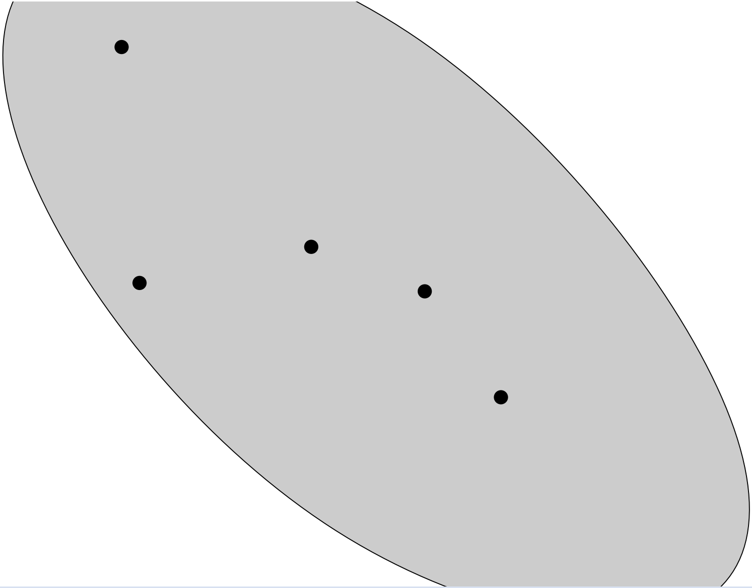

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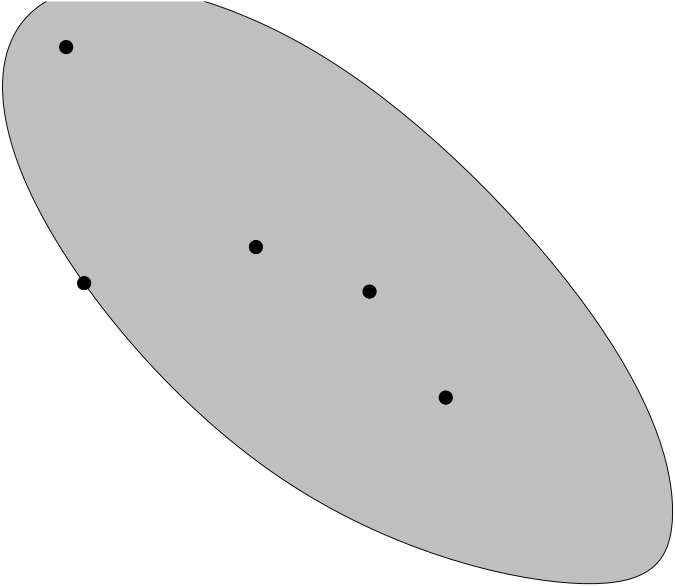

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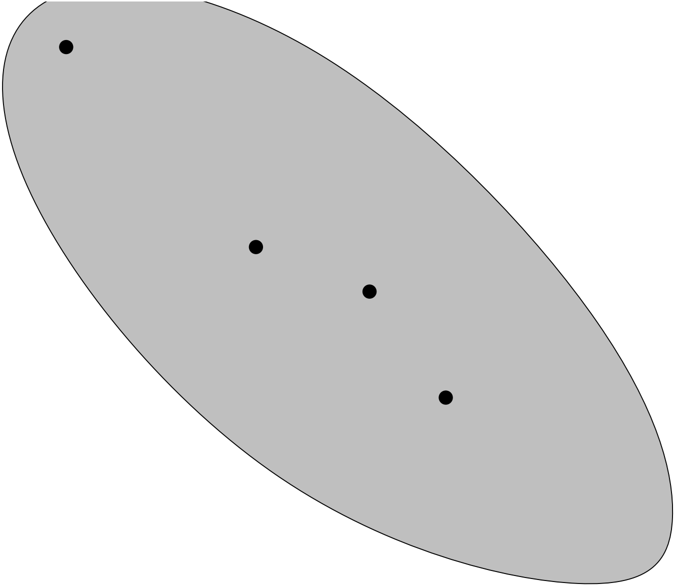

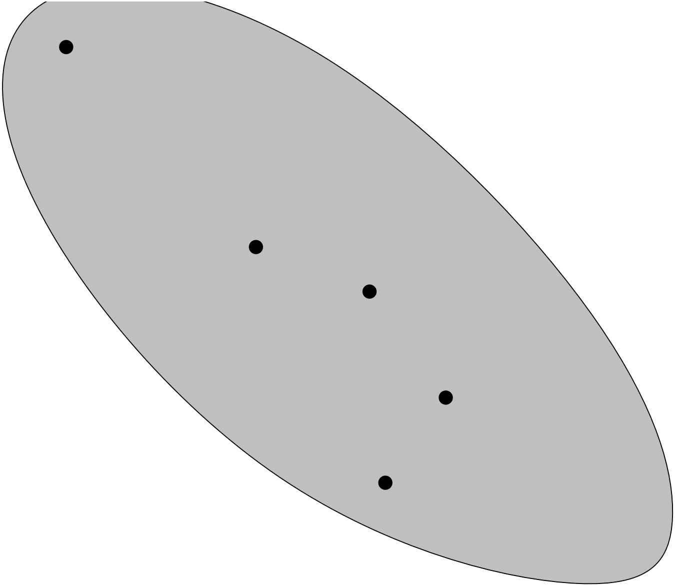

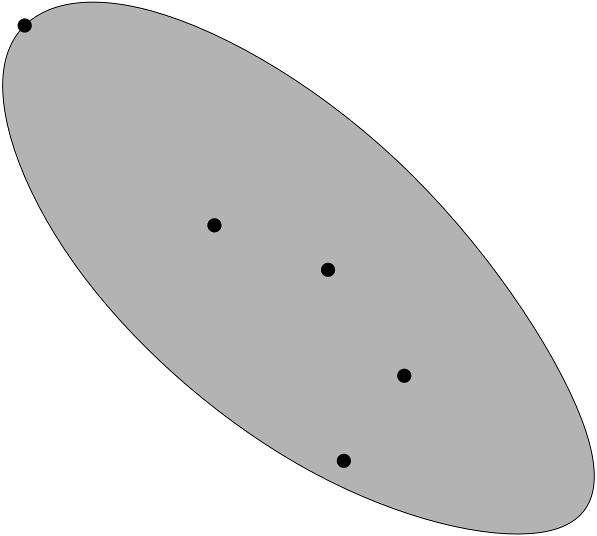

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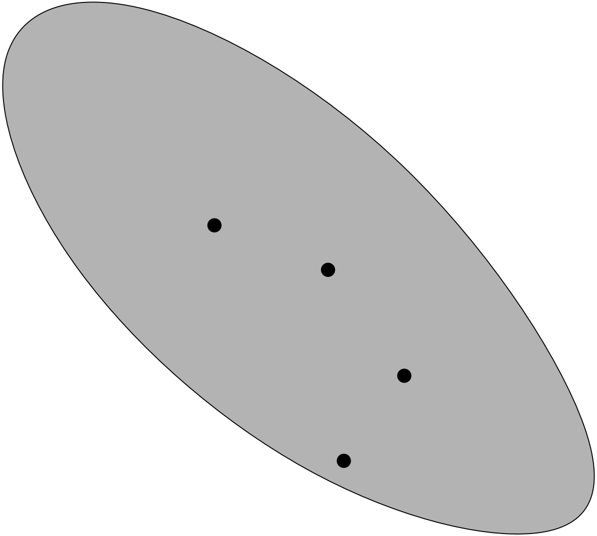

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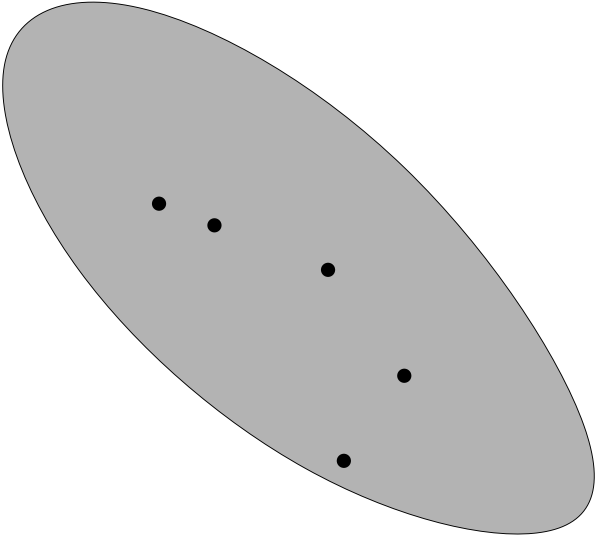

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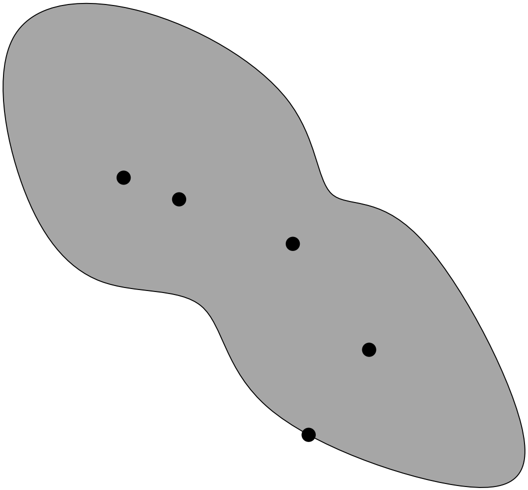

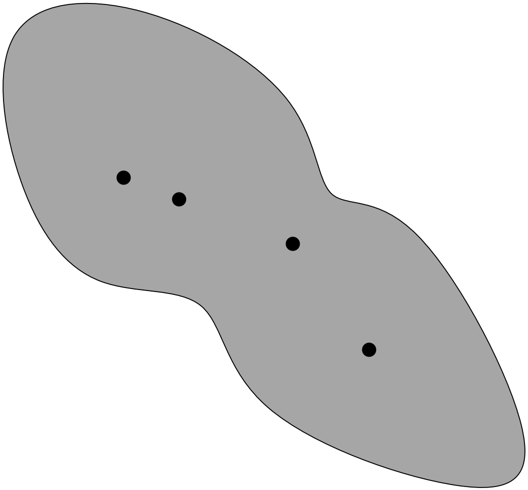

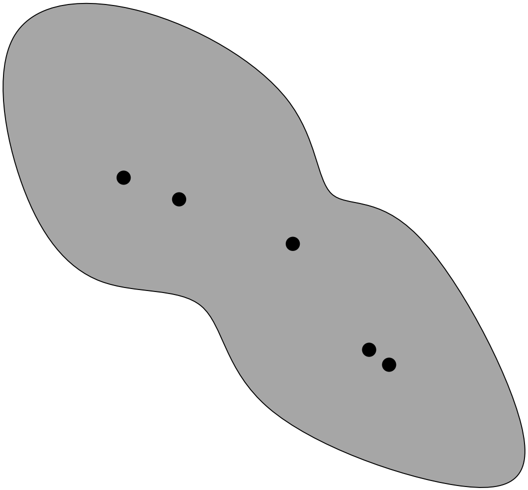

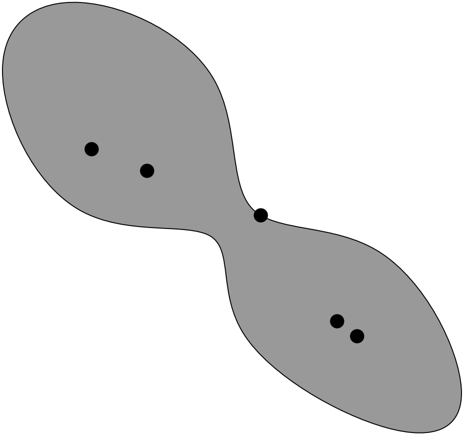

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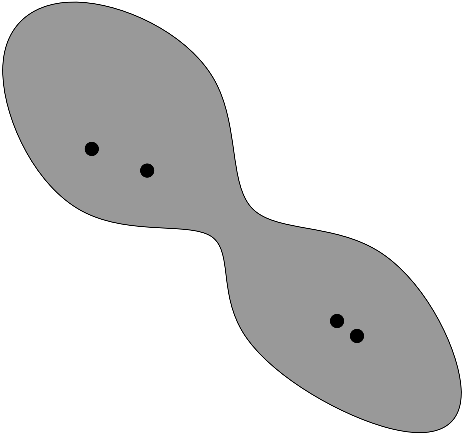

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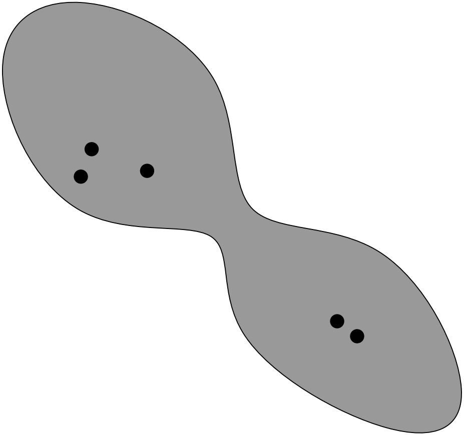
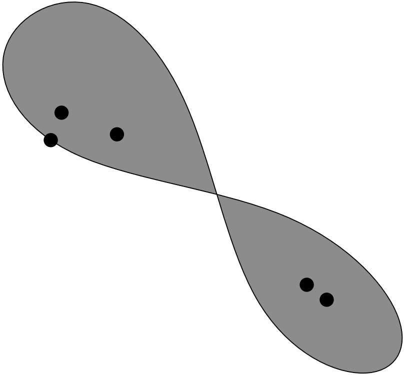
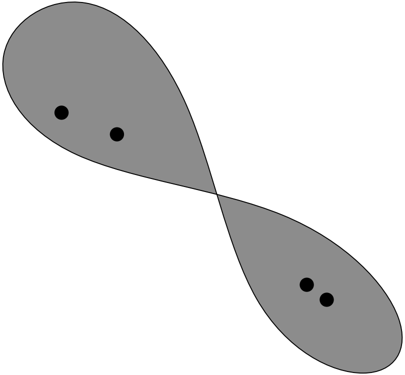
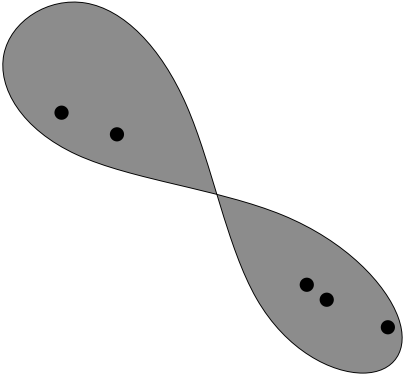


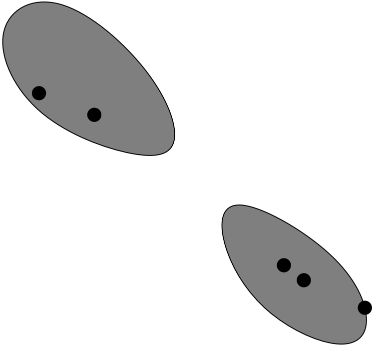


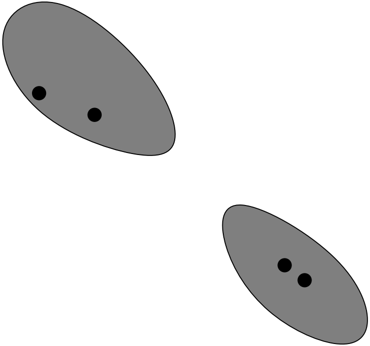




Nested Sampling

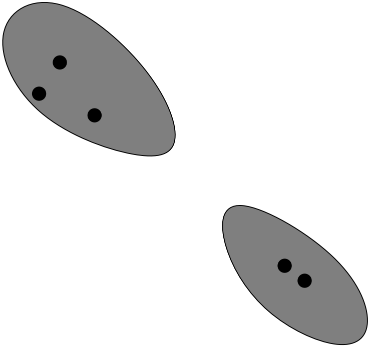
Graphical aid





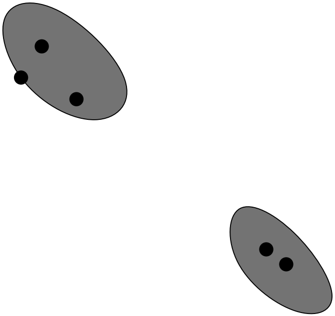
Nested Sampling

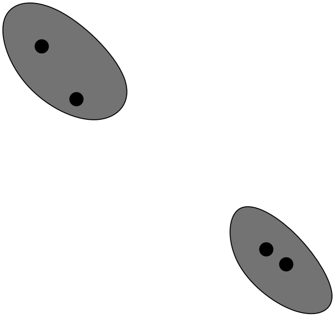
Graphical aid



Nested Sampling

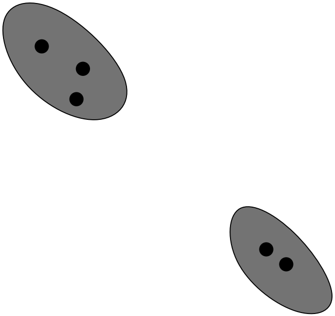
Graphical aid





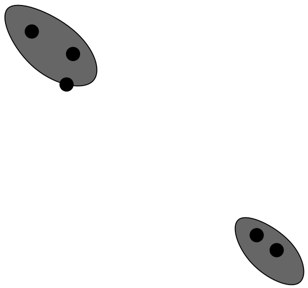
Nested Sampling

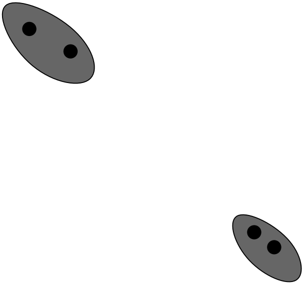
Graphical aid

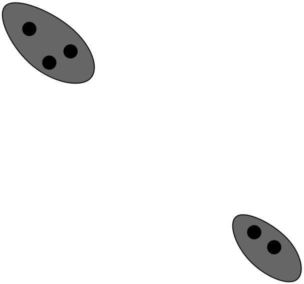


Nested Sampling

Graphical aid

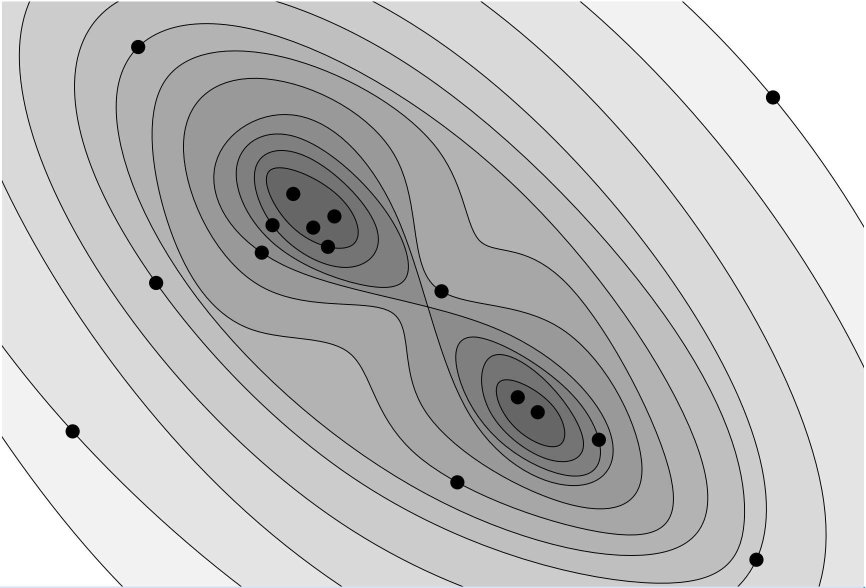






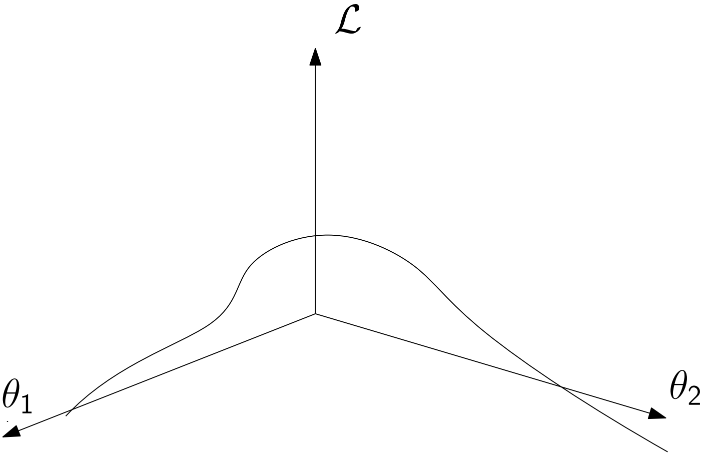
Nested Sampling

Graphical aid



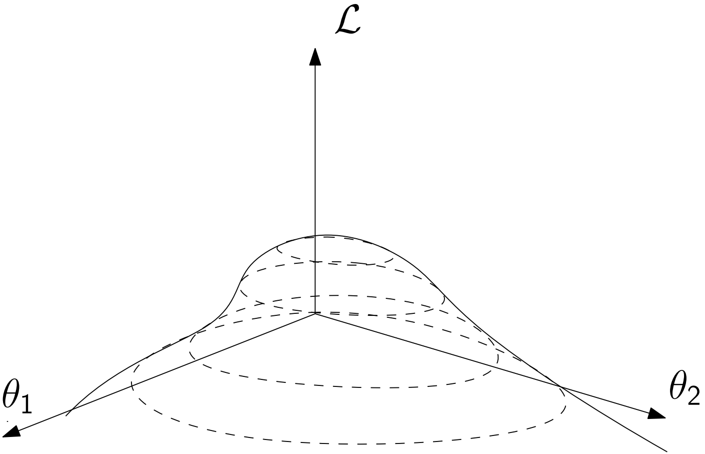
Nested sampling

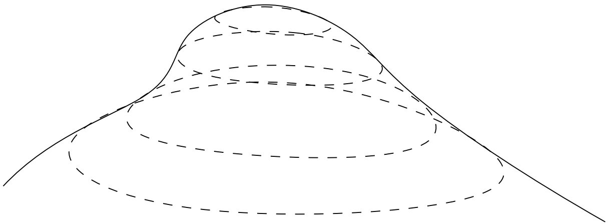
- The set of dead points are posterior samples with an appropriate weighting factor
- They can also be used to calculate evidences, since it sequentially updates the priors.

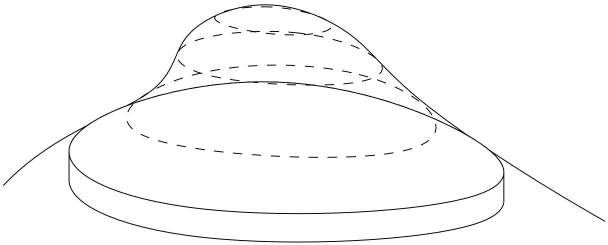


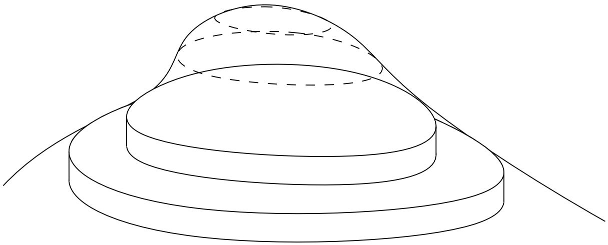
$\mathcal{L}$
θ_1
θ_2

Nested Sampling
Calculating evidences



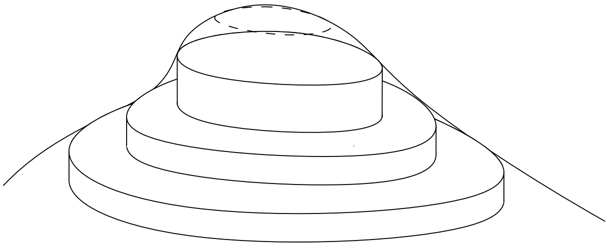


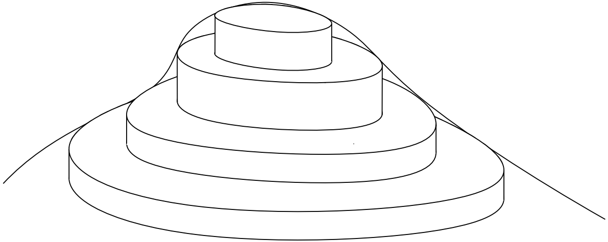


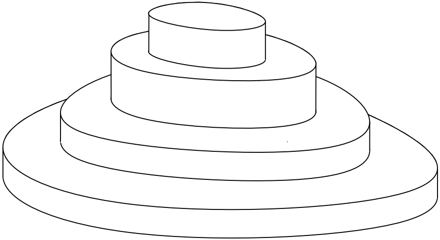


Nested Sampling

Calculating evidences

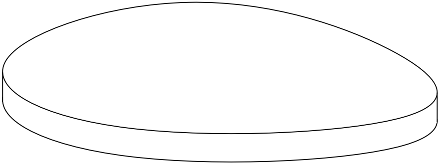






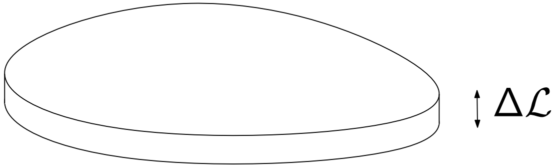
Nested Sampling

Calculating evidences



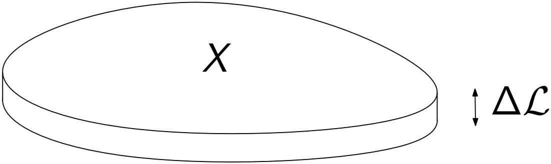
Nested Sampling

Calculating evidences



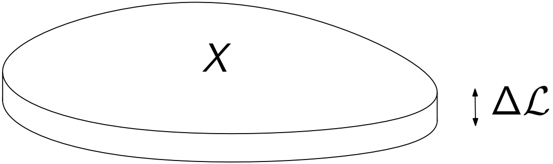
Nested Sampling

Calculating evidences

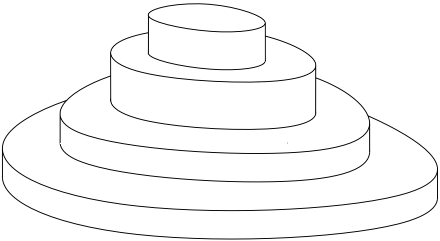


Nested Sampling

Calculating evidences

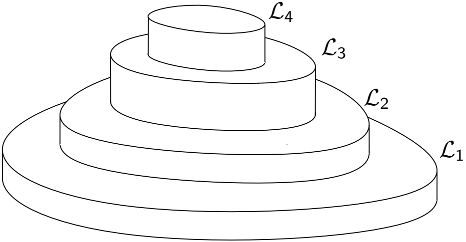


$$\text{Volume} = X\Delta\mathcal{L}$$



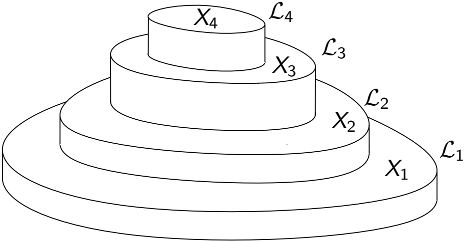
Nested Sampling

Calculating evidences

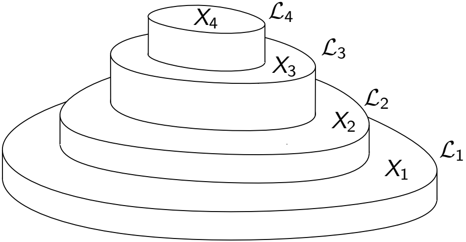


Nested Sampling

Calculating evidences



$$\mathcal{Z} \approx \sum_i X_i \Delta\mathcal{L}_i$$



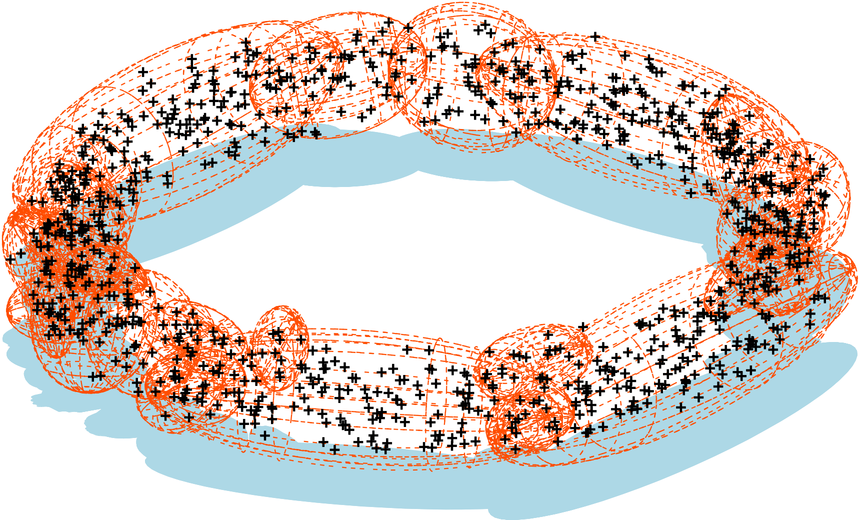
Sampling from a hard likelihood constraint

> *"It is not the purpose of this introductory paper to develop the technology of navigation within such a volume. We merely note that exploring a hard-edged likelihood-constrained domain should prove to be neither more nor less demanding than exploring a likelihood-weighted space."*
>
> — *John Skilling*

- Most of the work in NS to date has been in attempting to implement a hard-edged sampler in the NS meta-algorithm.

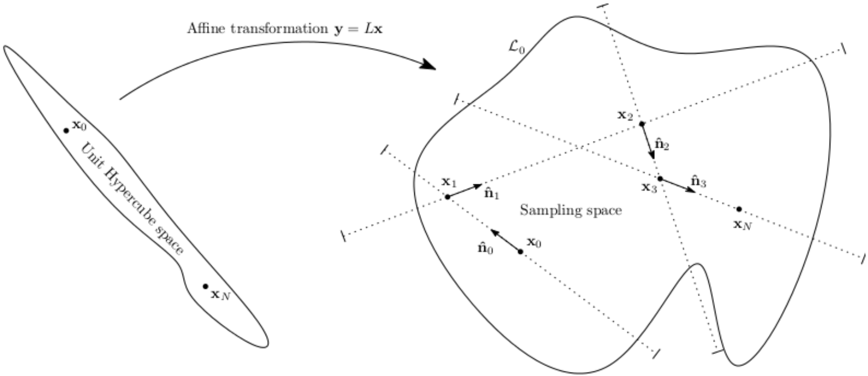
MultiNest

arXiv:0809.3437 arXiv:0704.3704 arXiv:1306.2144



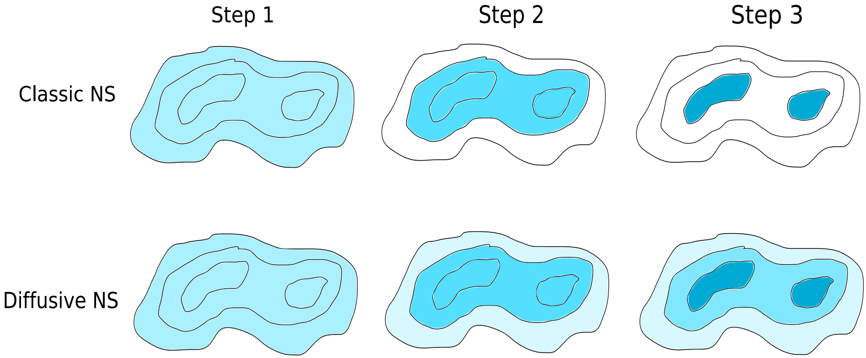
PolyChord

arXiv:1502.01856 arXiv:1506.00171



Diffusive nested sampling

arXiv:0912.2380

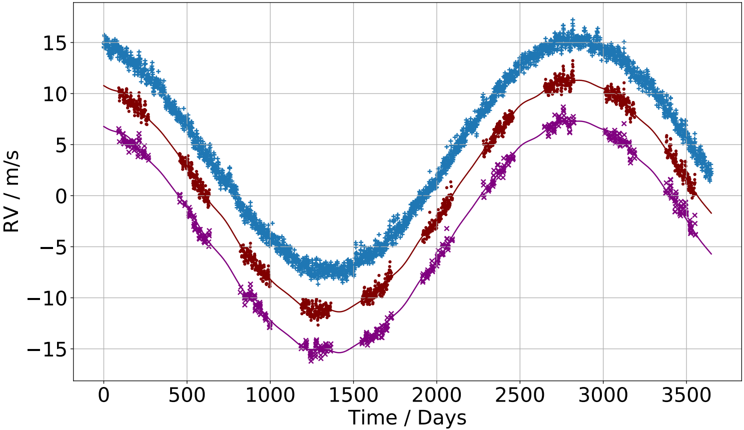


PolyChord vs MultiNest

- MultiNest excels in low dimensions $D < 10 - 20$.
- PolyChord can go up to ~ 150.
- Crossover is problem dependent
- PolyChord can also exploit fast-slow hierarchy

Exoplanets

Nested sampling in action



- Simple radial velocity model

$$\nu(t;\theta) = \sum_{p=1}^{N} K_p \sin(\omega_p t + \phi_p)$$

- Fit each model to data.
- Posteriors on model parameters $[(K_p, \omega_p, \phi_p), p = 1 \cdots N]$ quantify knowledge of system characteristics.
- Evidences of models determine relative likelihood of number of planets in system

Cosmology

Another example.

$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

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$$D = \{C_\ell\}$$

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$$\begin{aligned} D =& \{C_\ell\} \\ M =& \Lambda\mathrm{CDM} \end{aligned}$$

$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

$$\begin{aligned} D =& \{C_\ell\} \\ M =& \Lambda\mathrm{CDM} \\ \Theta =& \Theta_{\Lambda\mathrm{CDM}} \end{aligned}$$

Cosmology

Another example.

$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

$$\begin{aligned}
D =& \{C_\ell\} \\
M =& \Lambda\mathrm{CDM} \\
\Theta =& \Theta_{\Lambda\mathrm{CDM}} \\
\Theta_{\Lambda\mathrm{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10}A_s), n_s)
\end{aligned}$$

$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

$$\begin{aligned}
D =& \{C_\ell^{(\text{Planck})}\} \\
M =& \Lambda\text{CDM} \\
\Theta =& \Theta_{\Lambda\text{CDM}} \\
\Theta_{\Lambda\text{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10} A_s), n_s)
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Cosmology

Another example.

$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

$$\begin{aligned}
D =& \{C_\ell^{(\text{Planck})}\} \\
M =& \Lambda\text{CDM} \\
\Theta =& \Theta_{\Lambda\text{CDM}} + \Theta_{\text{Planck}} \\
\Theta_{\Lambda\text{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10} A_s), n_s)
\end{aligned}$$

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D =& \{C_\ell^{(\mathrm{Planck})}\} \\
M =& \Lambda\mathrm{CDM} \\
\Theta =& \Theta_{\Lambda\mathrm{CDM}} + \Theta_{\mathrm{Planck}} \\
\Theta_{\Lambda\mathrm{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10} A_s), n_s) \\
\Theta_{\mathrm{Planck}} =& (y_{\mathrm{cal}}, A_{217}^{CIB}, \xi^{tSZ-CIB}, A_{143}^{tSZ}, A_{100}^{PS}, A_{143}^{PS}, A_{143\times 217}^{PS}, A_{217}^{PS}, \\
& A^{kSZ}, A_{100}^{\mathrm{dust}\,TT}, A_{143}^{\mathrm{dust}\,TT}, A_{143\times 217}^{\mathrm{dust}\,TT}, A_{217}^{\mathrm{dust}\,TT}, c_{100}, c_{217})
\end{aligned}$$

$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

$$\begin{aligned}
D =& \{C_\ell^{\text{(Planck)}}\} \\
M =& \Lambda\text{CDM} + \text{extensions} \\
\Theta =& \Theta_{\Lambda\text{CDM}} + \Theta_{\text{Planck}} \\
\Theta_{\Lambda\text{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10}A_s), n_s) \\
\Theta_{\text{Planck}} =& (y_{\mathrm{cal}}, A_{217}^{CIB}, \xi^{tSZ-CIB}, A_{143}^{tSZ}, A_{100}^{PS}, A_{143}^{PS}, A_{143\times217}^{PS}, A_{217}^{PS}, \\
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\end{aligned}$$

$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

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D =& \{C_\ell^{(\mathrm{Planck})}\} \\
M =& \Lambda\mathrm{CDM} + \text{extensions} \\
\Theta =& \Theta_{\Lambda\mathrm{CDM}} + \Theta_{\mathrm{Planck}} + \Theta_{\mathrm{extensions}} \\
\Theta_{\Lambda\mathrm{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10} A_s), n_s) \\
\Theta_{\mathrm{Planck}} =& (y_{\mathrm{cal}}, A_{217}^{CIB}, \xi^{tSZ-CIB}, A_{143}^{tSZ}, A_{100}^{PS}, A_{143}^{PS}, A_{143\times217}^{PS}, A_{217}^{PS}, \\
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\end{aligned}$$

Cosmology

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$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

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\Theta =& \Theta_{\Lambda\text{CDM}} + \Theta_{\text{Planck}} + \Theta_{\text{extensions}} \\
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\Theta_{\text{Planck}} =& (y_{\mathrm{cal}}, A_{217}^{CIB}, \xi^{tSZ-CIB}, A_{143}^{tSZ}, A_{100}^{PS}, A_{143}^{PS}, A_{143\times 217}^{PS}, A_{217}^{PS}, \\
& A^{kSZ}, A_{100}^{\mathrm{dust}\,TT}, A_{143}^{\mathrm{dust}\,TT}, A_{143\times 217}^{\mathrm{dust}\,TT}, A_{217}^{\mathrm{dust}\,TT}, c_{100}, c_{217}) \\
\Theta_{\text{extensions}} =& (n_{\mathrm{run}})
\end{aligned}$$

Cosmology

Another example.

$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

$$\begin{aligned}
D =& \{C_\ell^{(\text{Planck})}\} \\
M =& \Lambda\text{CDM} + \text{extensions} \\
\Theta =& \Theta_{\Lambda\text{CDM}} + \Theta_{\text{Planck}} + \Theta_{\text{extensions}} \\
\Theta_{\Lambda\text{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10} A_s), n_s) \\
\Theta_{\text{Planck}} =& (y_{\mathrm{cal}}, A^{CIB}_{217}, \xi^{tSZ-CIB}, A^{tSZ}_{143}, A^{PS}_{100}, A^{PS}_{143}, A^{PS}_{143\times217}, A^{PS}_{217}, \\
& A^{kSZ}, A^{\mathrm{dust}\,TT}_{100}, A^{\mathrm{dust}\,TT}_{143}, A^{\mathrm{dust}\,TT}_{143\times217}, A^{\mathrm{dust}\,TT}_{217}, c_{100}, c_{217}) \\
\Theta_{\text{extensions}} =& (n_{\mathrm{run}}, n_{\mathrm{run,run}})
\end{aligned}$$

$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

$$\begin{aligned}
D =& \{C_\ell^{(\text{Planck})}\} \\
M =& \Lambda\text{CDM} + \text{extensions} \\
\Theta =& \Theta_{\Lambda\text{CDM}} + \Theta_{\text{Planck}} + \Theta_{\text{extensions}} \\
\Theta_{\Lambda\text{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10}A_s), n_s) \\
\Theta_{\text{Planck}} =& (y_{\mathrm{cal}}, A_{217}^{CIB}, \xi^{tSZ-CIB}, A_{143}^{tSZ}, A_{100}^{PS}, A_{143}^{PS}, A_{143\times217}^{PS}, A_{217}^{PS}, \\
& A^{kSZ}, A_{100}^{\mathrm{dust}\,TT}, A_{143}^{\mathrm{dust}\,TT}, A_{143\times217}^{\mathrm{dust}\,TT}, A_{217}^{\mathrm{dust}\,TT}, c_{100}, c_{217}) \\
\Theta_{\text{extensions}} =& (n_{\mathrm{run}}, n_{\mathrm{run,run}}, w)
\end{aligned}$$

$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

$$\begin{aligned}
D =& \{ C_\ell^{\text{(Planck)}} \} \\
M =& \Lambda\text{CDM} + \text{extensions} \\
\Theta =& \Theta_{\Lambda\text{CDM}} + \Theta_{\text{Planck}} + \Theta_{\text{extensions}} \\
\Theta_{\Lambda\text{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10} A_s), n_s) \\
\Theta_{\text{Planck}} =& (y_{\mathrm{cal}}, A_{217}^{CIB}, \xi^{tSZ-CIB}, A_{143}^{tSZ}, A_{100}^{PS}, A_{143}^{PS}, A_{143\times217}^{PS}, A_{217}^{PS}, \\
& A^{kSZ}, A_{100}^{\mathrm{dust}\,TT}, A_{143}^{\mathrm{dust}\,TT}, A_{143\times217}^{\mathrm{dust}\,TT}, A_{217}^{\mathrm{dust}\,TT}, c_{100}, c_{217}) \\
\Theta_{\text{extensions}} =& (n_{\mathrm{run}}, n_{\mathrm{run,run}}, w, \Sigma m_\nu, m_{\nu,\mathrm{sterile}}^{\mathrm{eff}})
\end{aligned}$$

Another example.

$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

$$\begin{aligned}
D =& \{C_\ell^{(\text{Planck})}\} + \{\text{LSS}\} \\
M =& \Lambda\text{CDM} + \text{extensions} \\
\Theta =& \Theta_{\Lambda\text{CDM}} + \Theta_{\text{Planck}} + \Theta_{\text{extensions}} \\
\Theta_{\Lambda\text{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10} A_s), n_s) \\
\Theta_{\text{Planck}} =& (y_{\mathrm{cal}}, A_{217}^{CIB}, \xi^{tSZ-CIB}, A_{143}^{tSZ}, A_{100}^{PS}, A_{143}^{PS}, A_{143\times 217}^{PS}, A_{217}^{PS}, \\
& A^{kSZ}, A_{100}^{\mathrm{dust}\,TT}, A_{143}^{\mathrm{dust}\,TT}, A_{143\times 217}^{\mathrm{dust}\,TT}, A_{217}^{\mathrm{dust}\,TT}, c_{100}, c_{217}) \\
\Theta_{\text{extensions}} =& (n_{\mathrm{run}}, n_{\mathrm{run,run}}, w, \Sigma m_\nu, m_{\nu,\mathrm{sterile}}^{\mathrm{eff}})
\end{aligned}$$

Cosmology

Another example.

$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

$$\begin{aligned}
D =& \{\, C_\ell^{(\text{Planck})} \} + \{\text{LSS}\} + \{\,\text{“Big Data”}\,\} \\
M =& \Lambda\text{CDM} + \text{extensions} \\
\Theta =& \Theta_{\Lambda\text{CDM}} + \Theta_{\text{Planck}} + \Theta_{\text{extensions}} \\
\Theta_{\Lambda\text{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10} A_s), n_s) \\
\Theta_{\text{Planck}} =& (y_{\mathrm{cal}}, A_{217}^{CIB}, \xi^{tSZ-CIB}, A_{143}^{tSZ}, A_{100}^{PS}, A_{143}^{PS}, A_{143\times 217}^{PS}, A_{217}^{PS}, \\
& A^{kSZ}, A_{100}^{\mathrm{dust}\,TT}, A_{143}^{\mathrm{dust}\,TT}, A_{143\times 217}^{\mathrm{dust}\,TT}, A_{217}^{\mathrm{dust}\,TT}, c_{100}, c_{217}) \\
\Theta_{\text{extensions}} =& (n_{\mathrm{run}}, n_{\mathrm{run,run}}, w, \Sigma m_\nu, m_{\nu,\mathrm{sterile}}^{\mathrm{eff}})
\end{aligned}$$

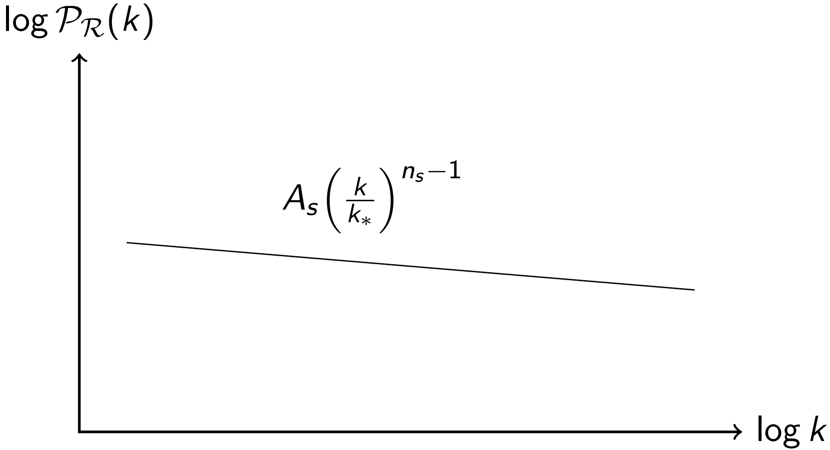
$$\mathcal{L}(\Theta) = P(D|\Theta, M)$$

$$\begin{aligned}
D =& \{C_\ell^{(\text{Planck})}\} + \{\text{LSS}\} + \{\text{“Big Data”}\} \\
M =& \Lambda\text{CDM} + \text{extensions} \\
\Theta =& \Theta_{\Lambda\text{CDM}} + \Theta_{\text{Planck}} + \Theta_{\text{extensions}} \\
\Theta_{\Lambda\text{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10}A_s), n_s) \\
\Theta_{\text{Planck}} =& (y_{\mathrm{cal}}, A_{217}^{CIB}, \xi^{tSZ-CIB}, A_{143}^{tSZ}, A_{100}^{PS}, A_{143}^{PS}, A_{143\times217}^{PS}, A_{217}^{PS}, \\
& A^{kSZ}, A_{100}^{\mathrm{dust}\,TT}, A_{143}^{\mathrm{dust}\,TT}, A_{143\times217}^{\mathrm{dust}\,TT}, A_{217}^{\mathrm{dust}\,TT}, c_{100}, c_{217}) \\
\Theta_{\text{extensions}} =& (n_{\mathrm{run}}, n_{\mathrm{run,run}}, w, \Sigma m_\nu, m_{\nu,\mathrm{sterile}}^{\mathrm{eff}})
\end{aligned}$$

- Parameter estimation: $L, \pi \to \mathcal{P}$: model parameters
- Model comparison: $L, \pi \to Z$: how good model is

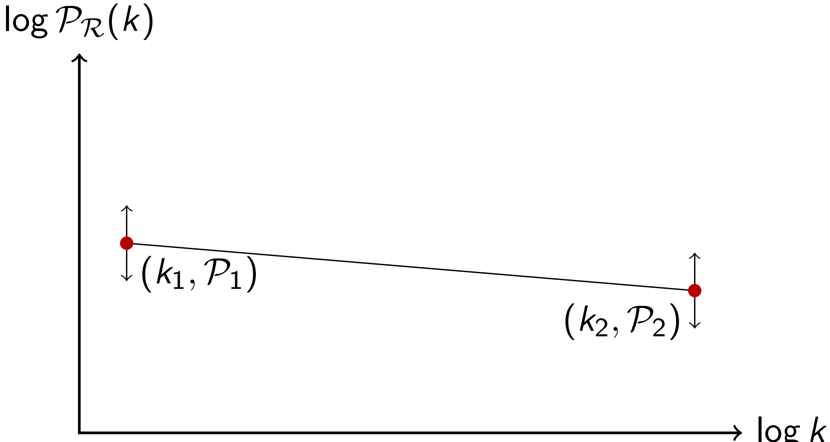
Nested Sampling in action

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



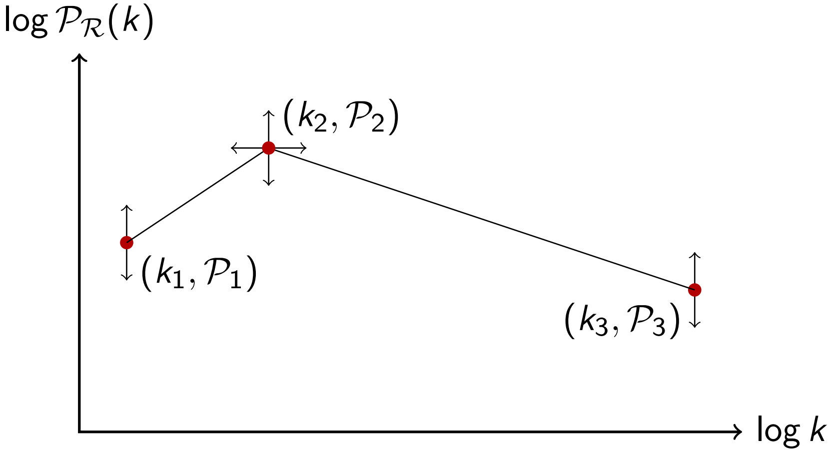
Nested Sampling in action

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



Nested Sampling in action

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



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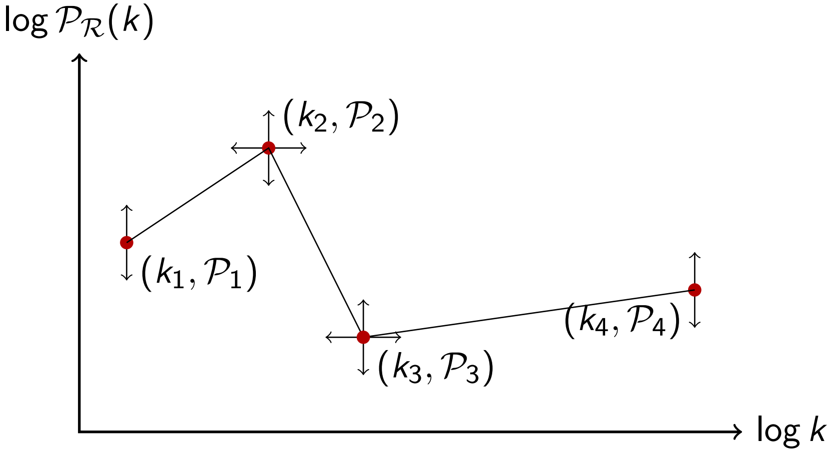
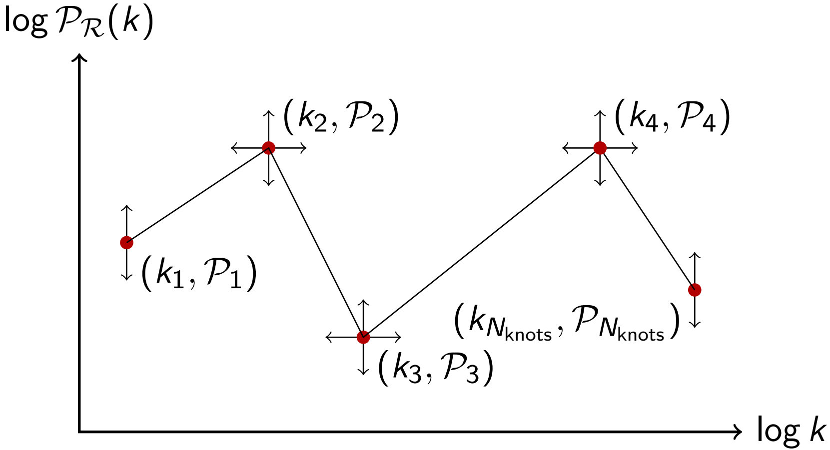


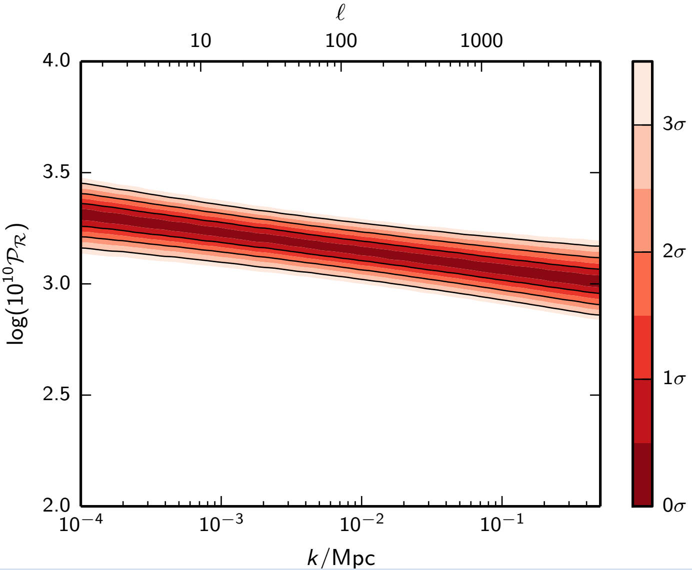
Nested Sampling in action

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



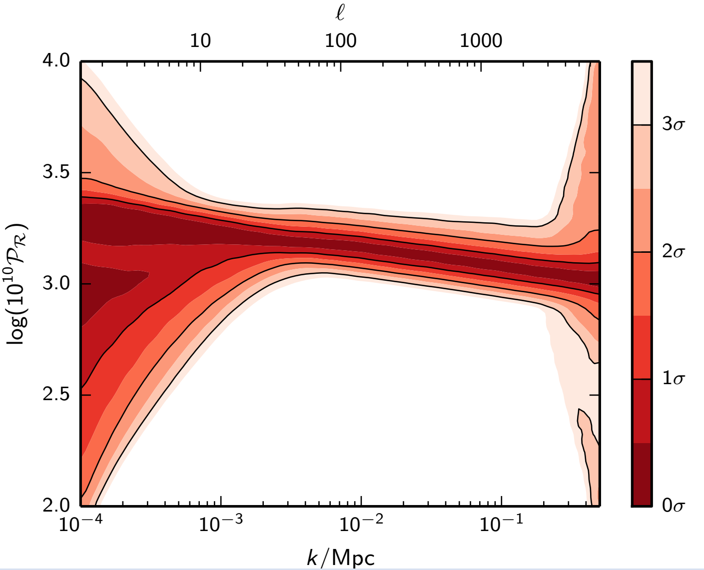
0 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



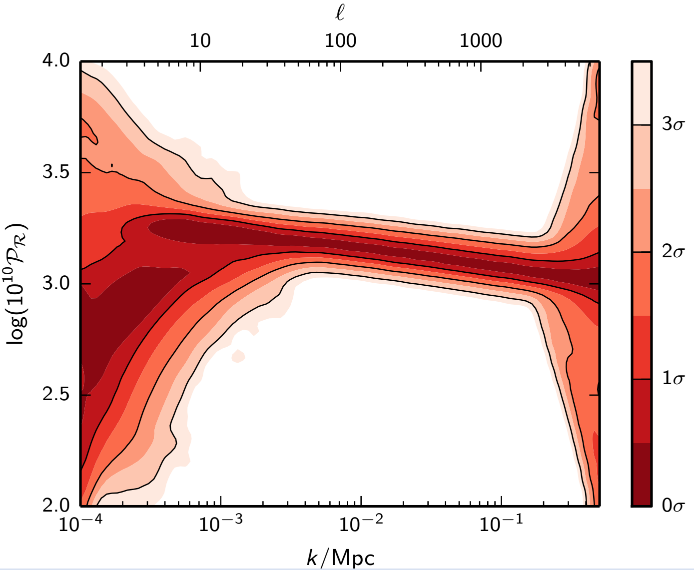
1 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



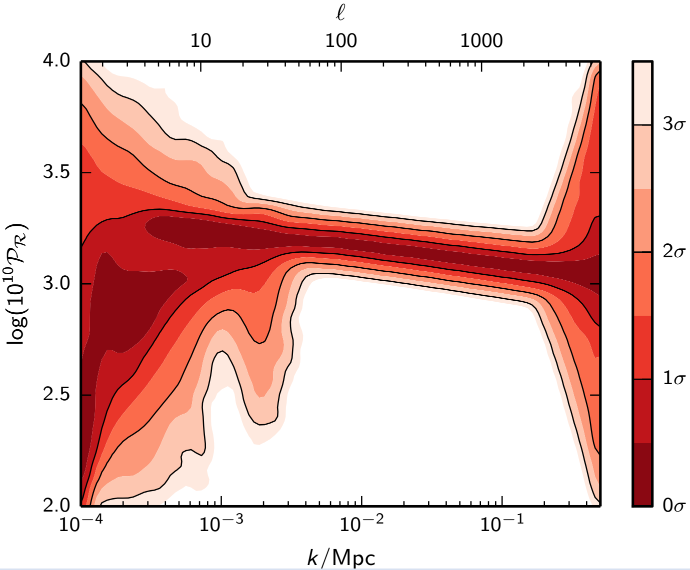
2 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



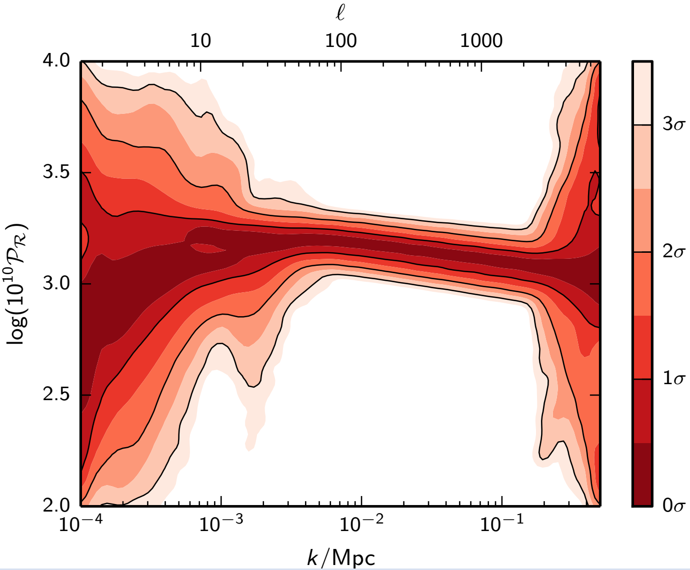
3 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



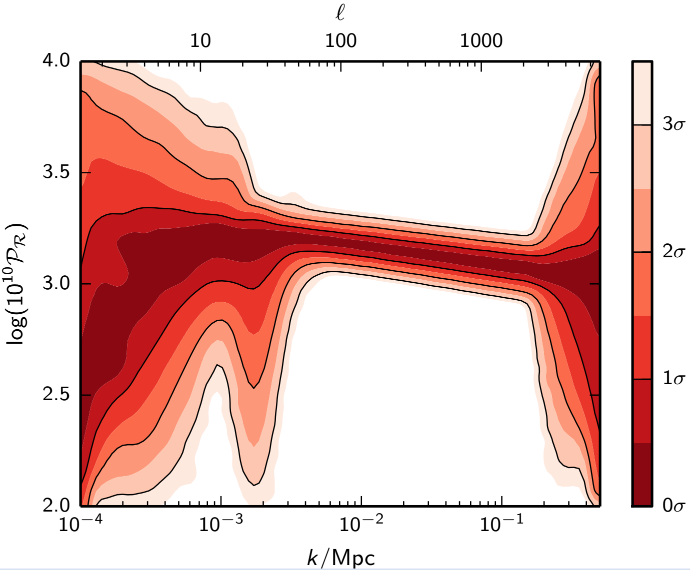
4 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



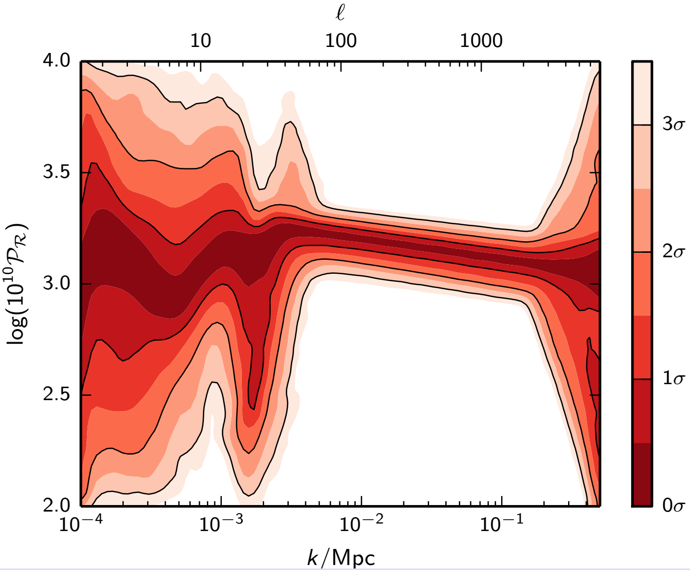
5 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



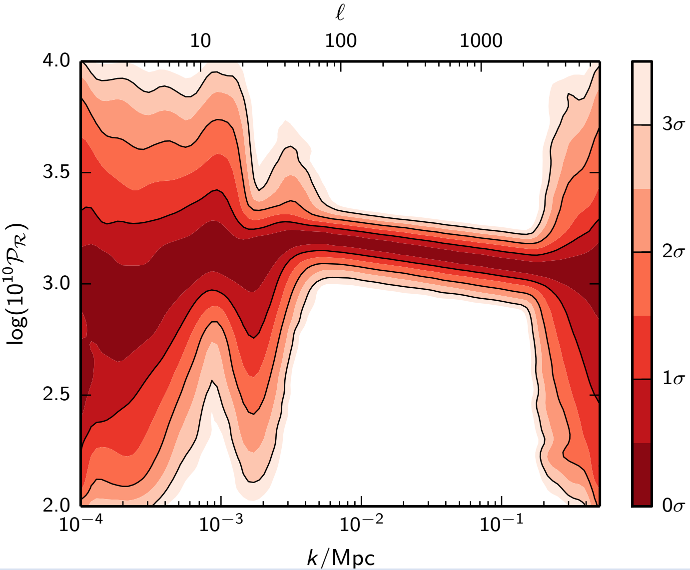
6 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



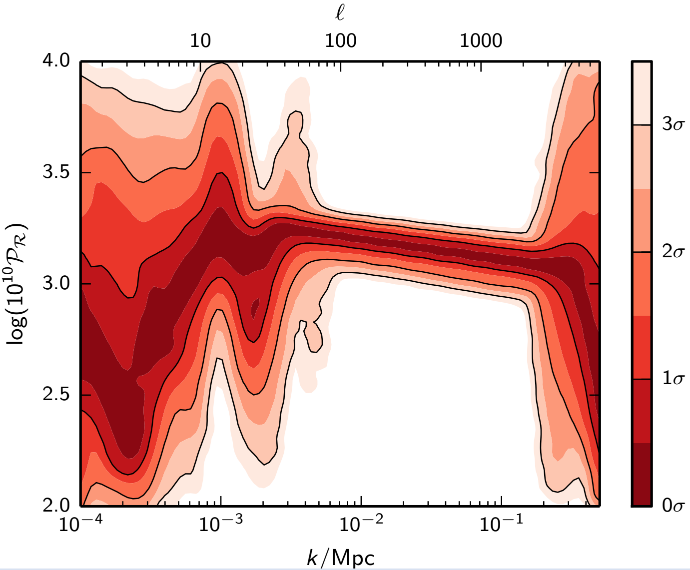
7 internal knots

Primordial power spectrum $\mathcal{P}_{\mathcal{R}}(k)$ reconstruction



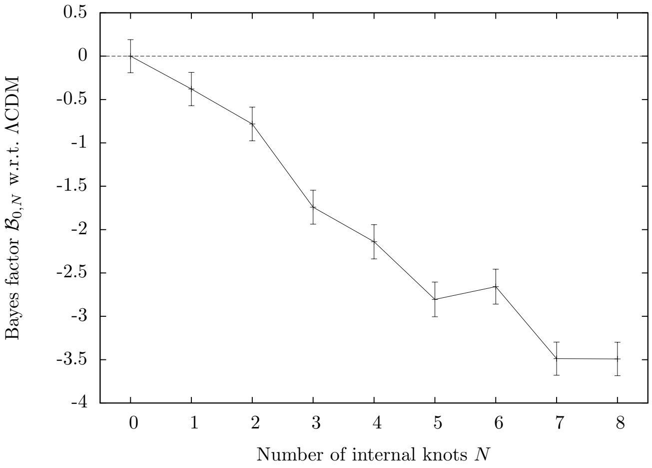
8 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



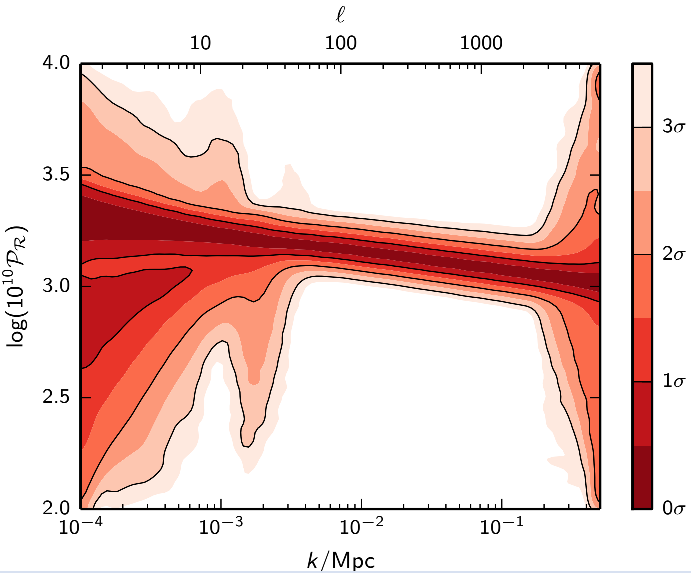
Bayes Factors

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



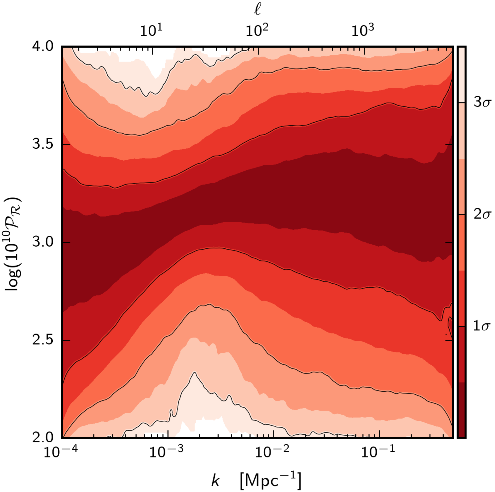
Marginalised plot

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction

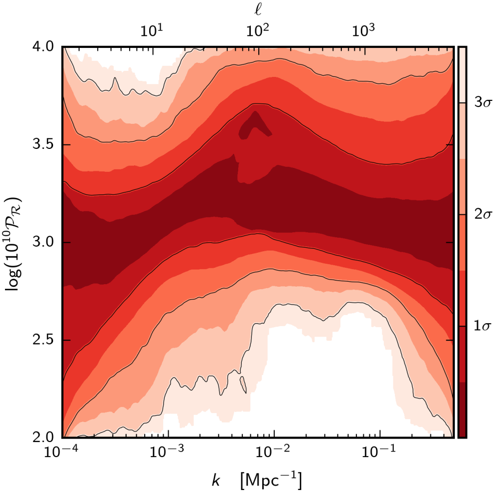


COBE (pre-2002)

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction

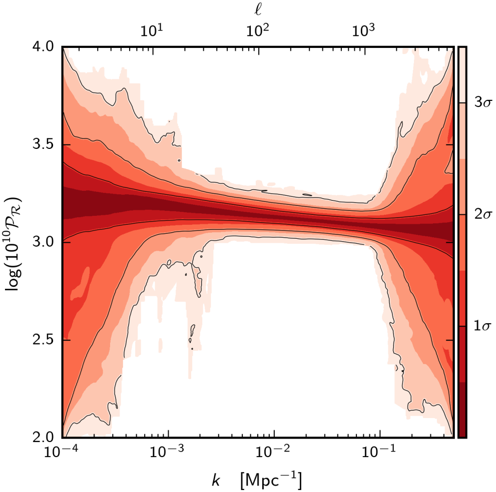


Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



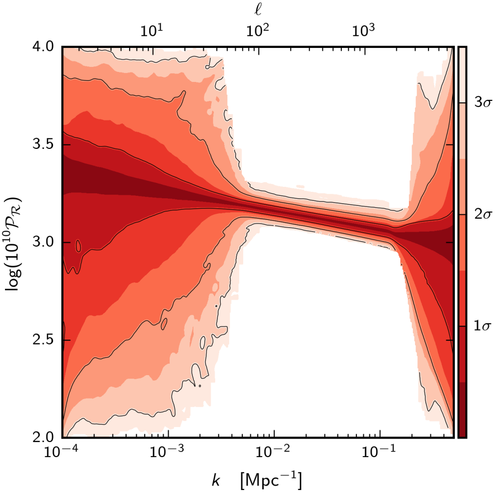
WMAP (2012)

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



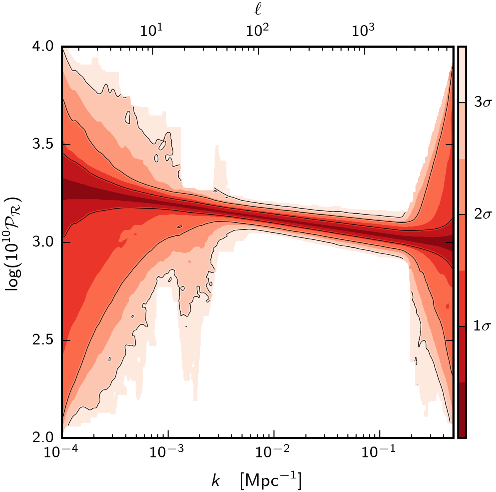
Planck (2013)

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction

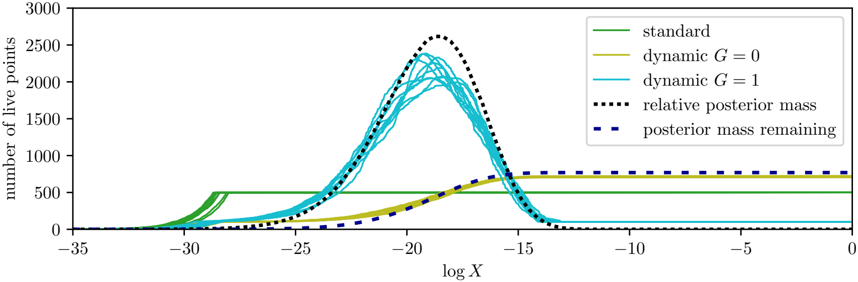


Planck (2015)

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



- John Skilling noted that two nested sampling runs can be combined in likelihood order to produce a valid run with a larger number of live points.
- The reverse is also true (Higson 1704.03459).
- In general, a run with n live points can be “unweaved” into n runs with a single live point.
- Useful for providing convergence diagnostics and better parameter estimation.



The number of live points can be varied dynamically in order to oversample regions of interest

Multi-temperature sampling

Advances in nested sampling

- By compressing from prior to posterior, Nested Sampling's weighted samples are fundamentally different from traditional MCMC.
- Nested sampling tails and peaks equally.
- We can define the "temperature" of a distribution in analogy with thermodnyamics:

$$\log L \sim E \Rightarrow P \propto e^{-\beta E} = e^{-E/kT}, \quad \beta = 1$$

- Sampling at different temperatures can be useful for exploring tails.
- Nested sampling runs give you the full partition function $\log Z(\beta)$.

- Much of the time spent in a nested sampling run is spent "compressing the tails".
- Sometimes we have a-priori good knowledge of the posterior bulk (analagous to an MCMC proposal distribution).

$$
\begin{aligned}
Z_0 &= \int L(\theta)\pi_0(\theta)d\theta, \qquad Z_1 = \int L(\theta)\pi_1(\theta)d\theta \\
&= \int L(\theta)\pi_1(\theta)\frac{\pi_0(\theta)}{\pi_1(\theta)}d\theta = \left\langle \frac{\pi_0(\theta)}{\pi_1(\theta)} \right\rangle_{P_1}
\end{aligned}
$$

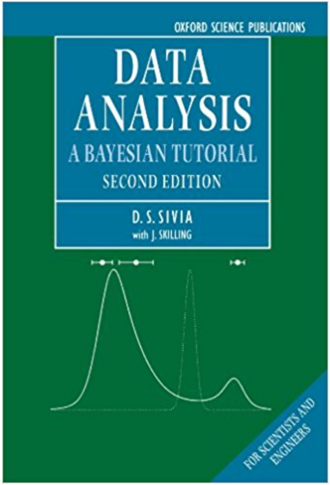
- This importance weighting only works if you have a lot of tail samples.

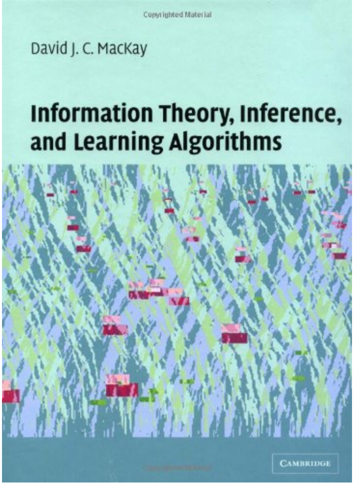
N-σ contours

Future research

- Traditional posterior samples only allow you to plot contours out to 2-3σ.
- Nested sampling fully samples the tails, so in theory one could do 20σ contours.
- Requires further thought in alternatives to kernel density estimation.

Further reading





- Data analysis: A Bayesian Tutorial (Sivia & Skilling)
- Information Theory, Inference and Learning Algorithms (Mackay)