Modern Bayesian Inference

Theory and Practice

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Introduction

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- How to extract information about scientific models from data.
- Most cosmologists work in a *Bayesian* framework of inference, although *Frequentist* methods are also sometimes used.
- **Bayesians use Probability Distributions to quantify uncertainty.**

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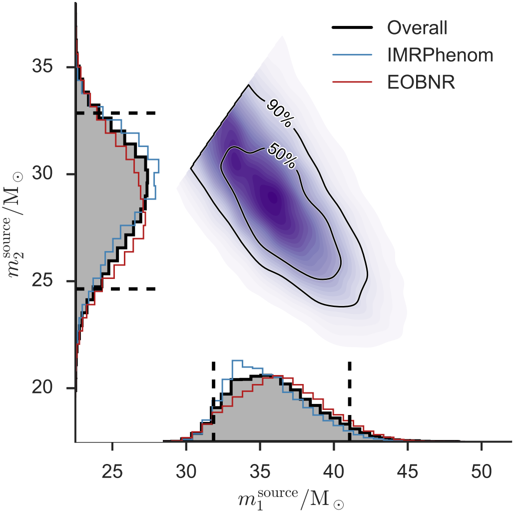
- More importantly, these are *summary statistics*.

LIGO binary merger



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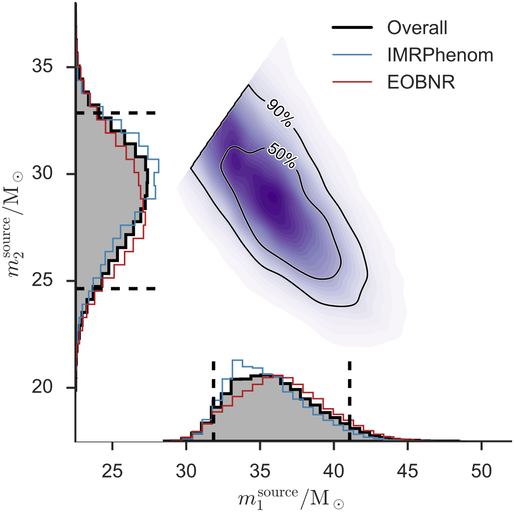

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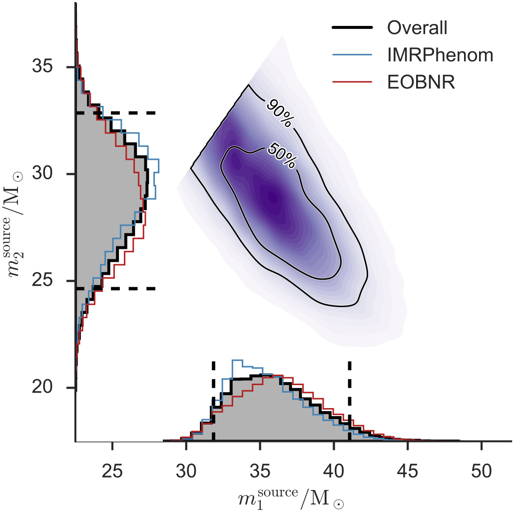

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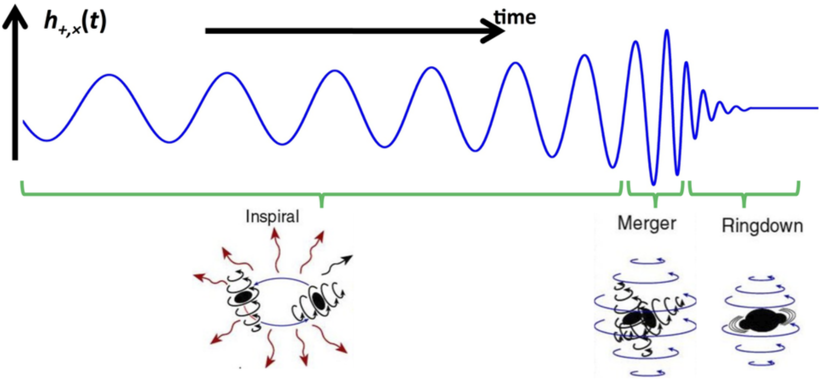
- Summary statistics summarise a full probability distribution.
- One goal of inference is to produce these probability distributions.

Extended example of inference: LIGO

- We will introduce the key concepts by discussing an extended example of the inference process.

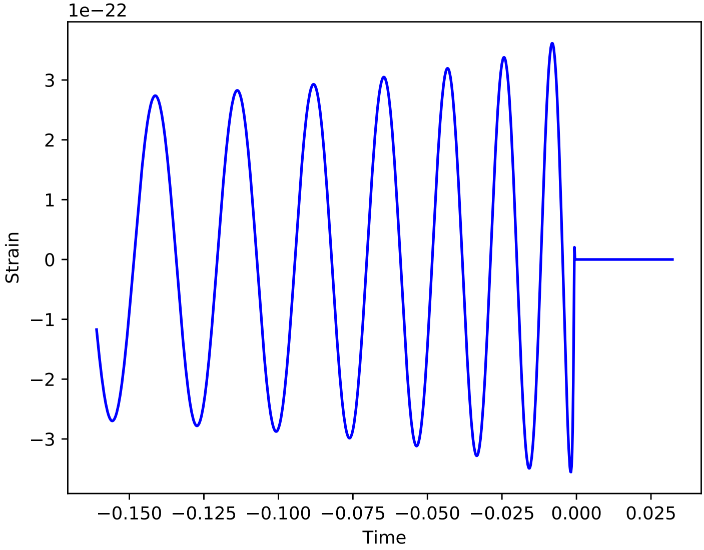
Theory

Extended example of inference: LIGO



The model M

Extended example of inference: LIGO



The parameters Θ of the model M

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- Φ_c, t_c: phase and time of coalescence

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Theoretical signal depends on:

- m_1, m_2: mass of binary
- θ, ϕ: sky location
- r: luminosity distance
- Φ_c, t_c: phase and time of coalescence
- i, θ_{sky}: inclination and angle on sky (orbital parameters)

The data D

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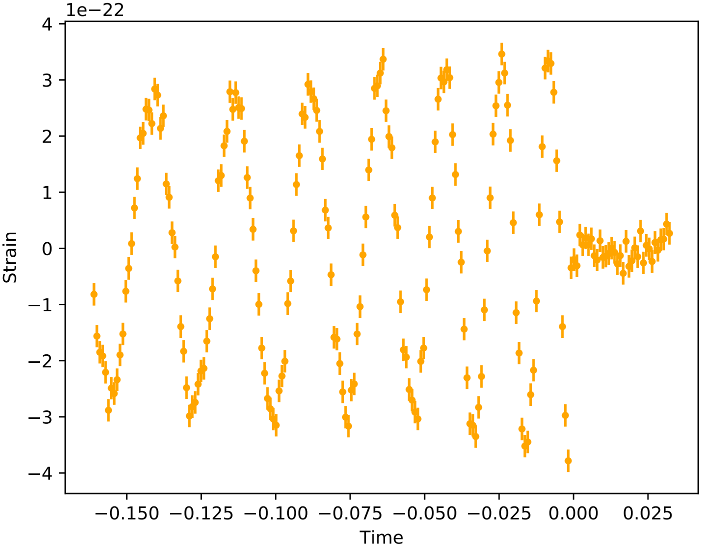

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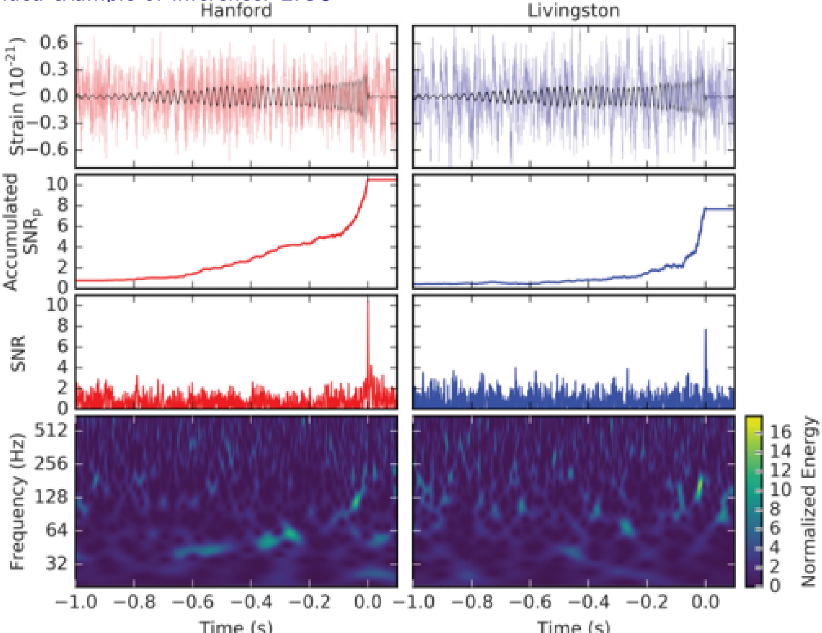

The Likelihood: well matched

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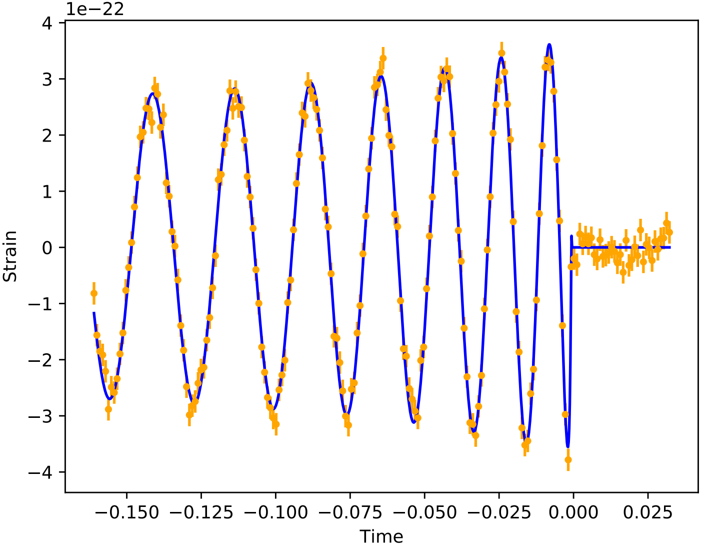

The Likelihood: coalescence off

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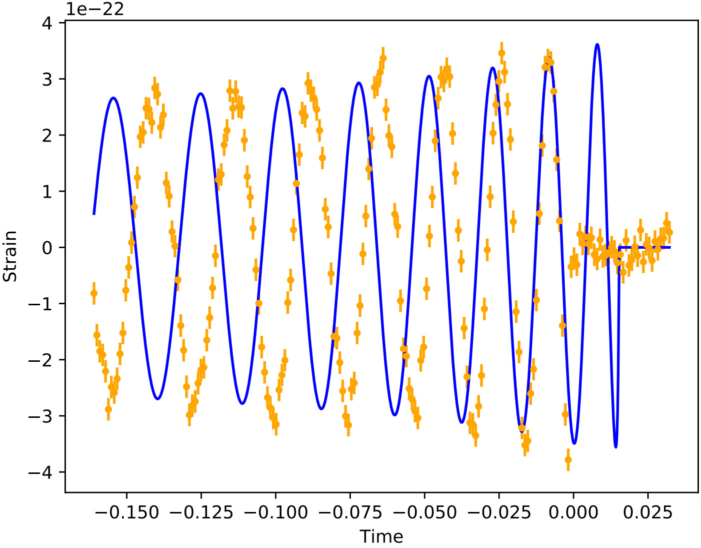

The Likelihood: too large luminosity distance

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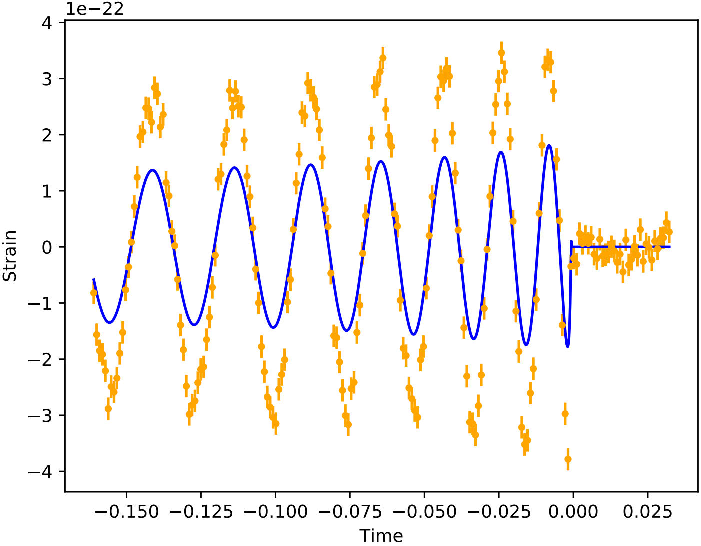

The Likelihood: incorrect inclination

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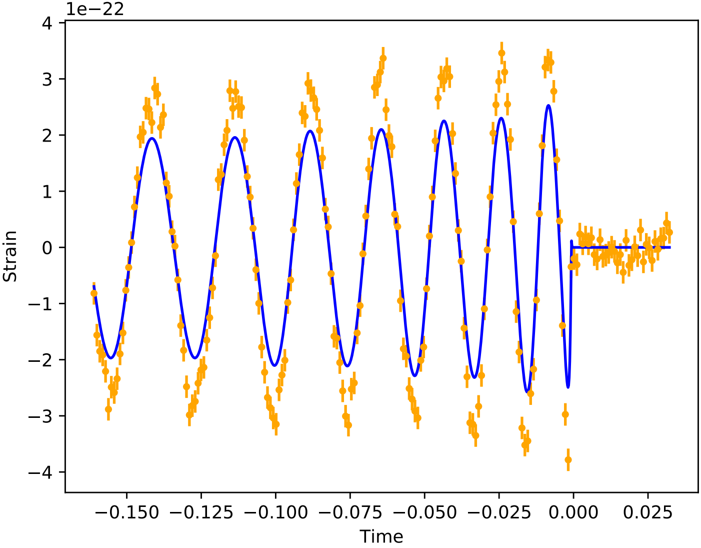

The Likelihood: 'Correct parameters'

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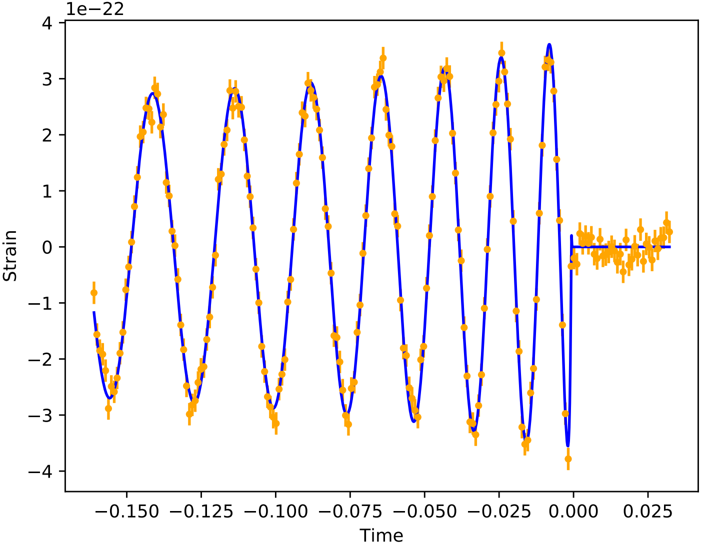

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- We normally work with log-likelihoods, which turn $\prod \to \sum$.

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$$\text{Posterior} = \frac{\text{Likelihood} \times \text{Prior}}{\text{Evidence}}$$

Prior $\pi(\Theta)$

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- e.g. $\pi(\theta, \phi)\, d\theta d\phi = -\frac{1}{2}\sin\theta\, d\theta d\phi$
- Most Bayesian approaches are sensitive to this, and rightly so.

Evidence Z

Extended example of inference: LIGO

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- Normalising constant.
- Difficult to compute.
- Still extremely important.

Posterior $\mathcal{P}$

Extended example of inference: LIGO

- Cannot plot the full posterior distribution:

$$\mathcal{P}(\Theta) \equiv P(m_1, m_2, \theta, \phi, r, \Phi_c, t_c, i, \theta_{\mathrm{sky}}|D, M)$$

- Can plot 1D and 2D *marginalised* distributions e.g:

$$P(m_1, m_2|D, M) =$$
$$\int P(m_1, m_2, \theta, \phi, r, \Phi_c, t_c, i, \theta_{\mathrm{sky}}|D, M)\, d\theta\, d\phi\, dr\, d\Phi_c\, dt_c\, di\, d\theta_{\mathrm{sky}}$$

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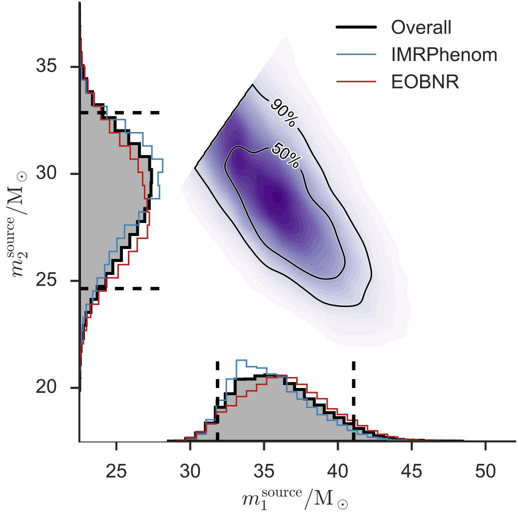

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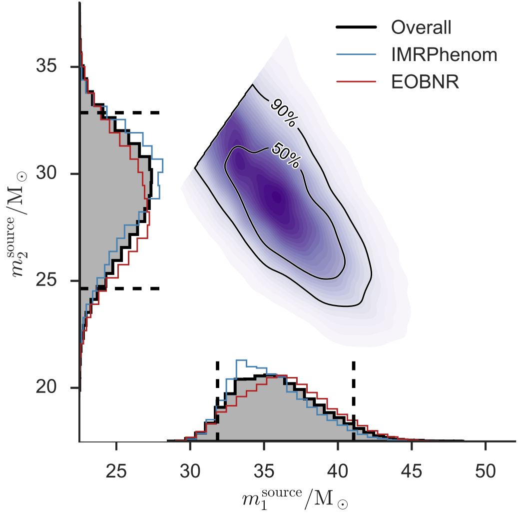

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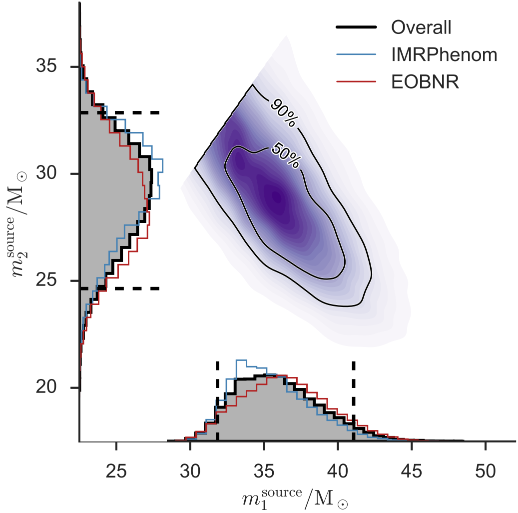

- May do this for each pair of parameters
- Generates a *triangle plot*

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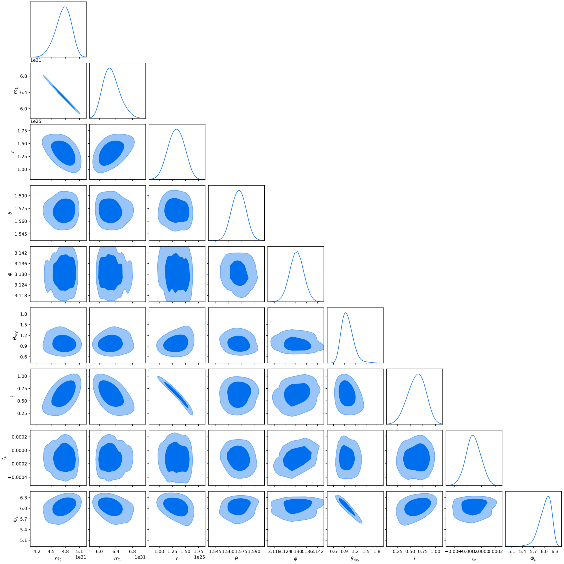

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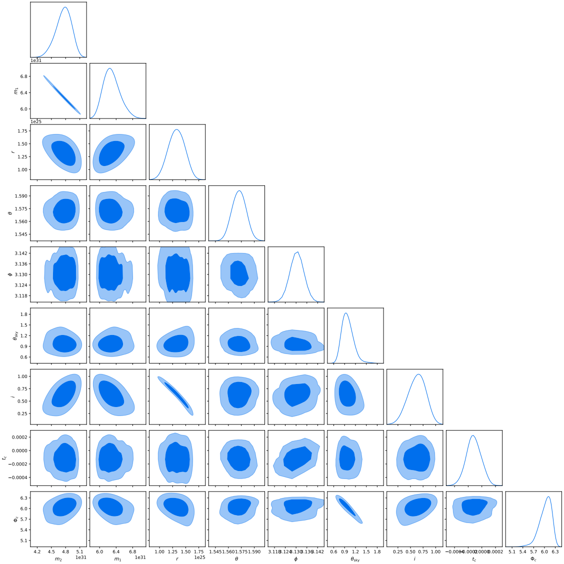

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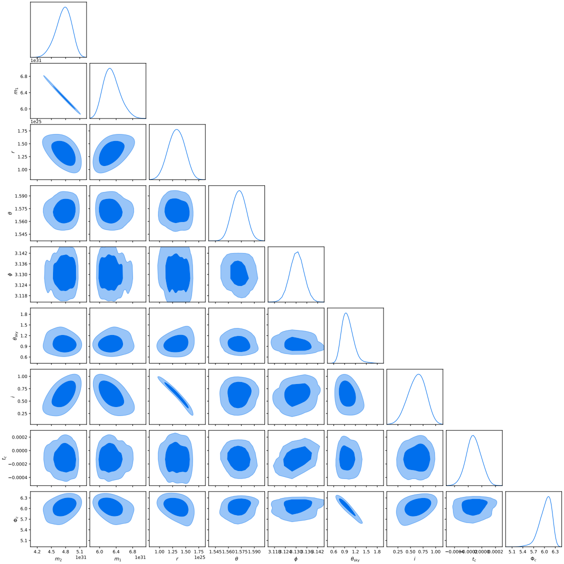

- Does give insight
- Not the full picture

Evidences and model comparison

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- Up until now, we have discussed *Parameter estimation*: inferring what data tell us about parameters Θ of a model M.
- Scientifically speaking, this is only half the story.
- In general, we will have several competing models that describe the data, and we want to know which is the "best".

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Model averaging:

- Multiple models with posterior on the same parameter: $\mathrm{P}(y|M_i, D)$

$$\mathrm{P}(y|D) = \sum_i \mathrm{P}(y|M_i, D)\mathrm{P}(M_i|D)$$

Parameter estimation

Another example.

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$$\begin{aligned}
D =& \{C_\ell\} \\
M =& \Lambda\mathrm{CDM} \\
\Theta =& \Theta_{\Lambda\mathrm{CDM}} \\
\Theta_{\Lambda\mathrm{CDM}} =& (\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, \ln(10^{10} A_s), n_s)
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- Parameter estimation: $L, \pi \to \mathcal{P}$: model parameters
- Model comparison: $L, \pi \to Z$: how good model is

Sampling

How to describe a high-dimensional posterior

Sampling

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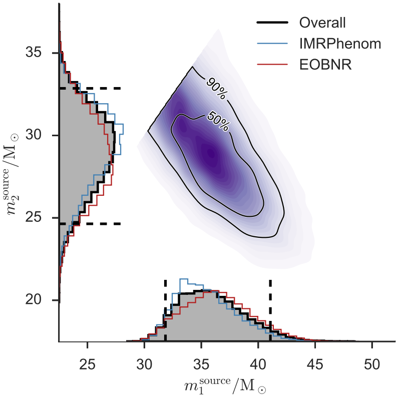
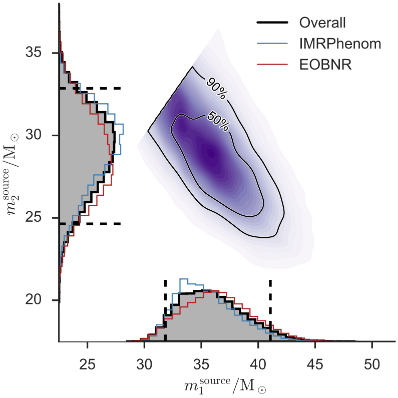


Sampling

How to describe a high-dimensional posterior

- In high dimensions, posterior $\mathcal{P}$ occupies a vanishingly small region of the prior π.



Sampling

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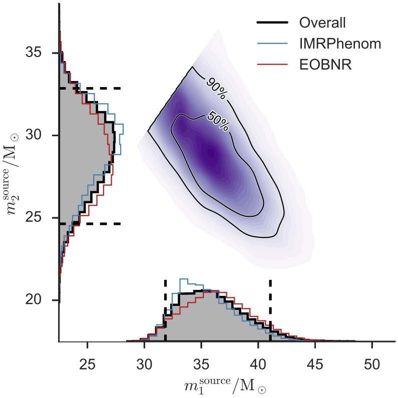
- In high dimensions, posterior $\mathcal{P}$ occupies a vanishingly small region of the prior π.
- *Sampling* the posterior is an excellent compression scheme.



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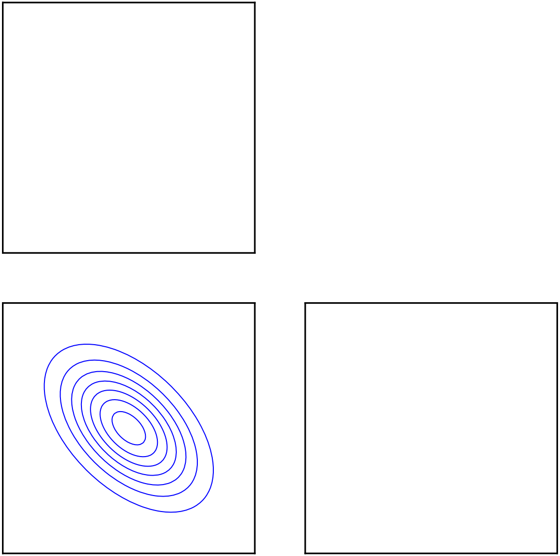

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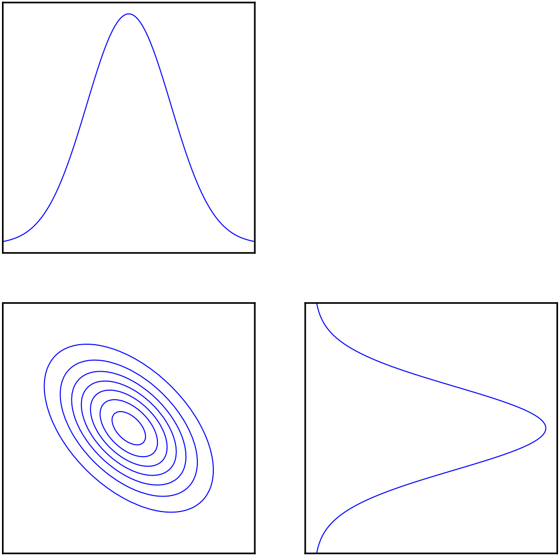

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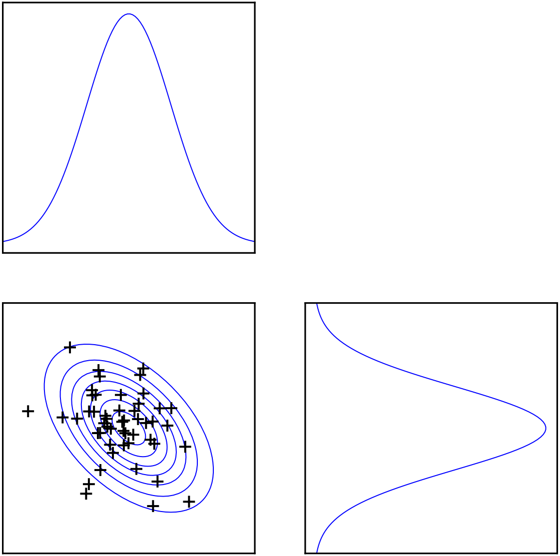

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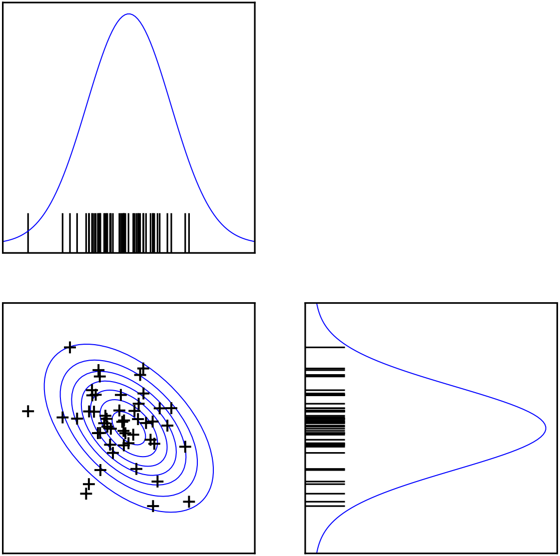

Why do sampling?

Marginalisation over the posterior

- Set of N samples $S = \{\Theta^{(i)} : i = 1, \dots N : \ \Theta^{(i)} \sim \mathcal{P}\}$

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- Gridding is doomed to failure in high dimensions.
- Enter Metropolis Hastings.

Current sampling approaches

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1. Metropolis Hastings.
2. Hamiltonian Monte-Carlo (HMC).
3. Ensemble sampling (e.g. emcee).

Metropolis Hastings

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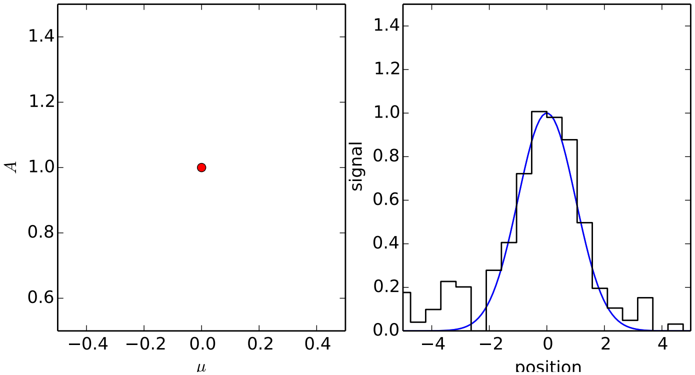
Metropolis Hastings



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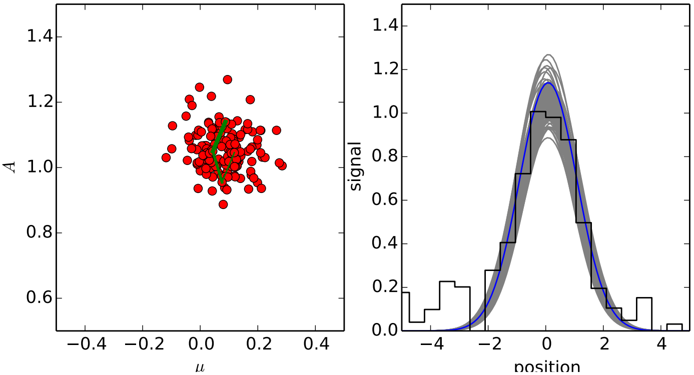

Metropolis Hastings

Struggles with. . .

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Struggles with...

1. Burn in

Metropolis Hastings

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Metropolis Hastings

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- stan is a fully fledged, rapidly developing programming language with HMC as a default sampler.

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- Can use information present in ensemble to guide proposals.
- emcee: affine invariant proposals.
- emcee is not the only (or even best) affine invariant approach.

The fundamental issue with all of the above

- They don't give you evidences!

$$
\begin{aligned}
\mathcal{Z} &= \mathrm{P}(D|M) \\
&= \int \mathrm{P}(D|\Theta, M)\mathrm{P}(\Theta|M) d\Theta \\
&= \langle \mathcal{L} \rangle_\pi
\end{aligned}
$$

- MCMC fundamentally explores the posterior, and cannot average over the prior.
- Simulated annealing gives one possibility for computing evidences.
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  - Suffers from similar issues to MCMC.
  - Unclear how to choose correct annealing schedule

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- Nested sampling is an alternative way of sampling posteriors.
- Uses ensemble sampling to compress prior to posterior.
- In doing so, it circumvents many issues (dimensionality, topology, geometry) that beset standard approaches.

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John Skilling's alternative to traditional MCMC!

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- S_{n+1}: Delete the lowest likelihood sample in S_n, and replace it with a new uniform sample with higher likelihood

Nested Sampling

John Skilling's alternative to traditional MCMC!

New procedure:

Maintain a set S of n samples, which are sequentially updated:

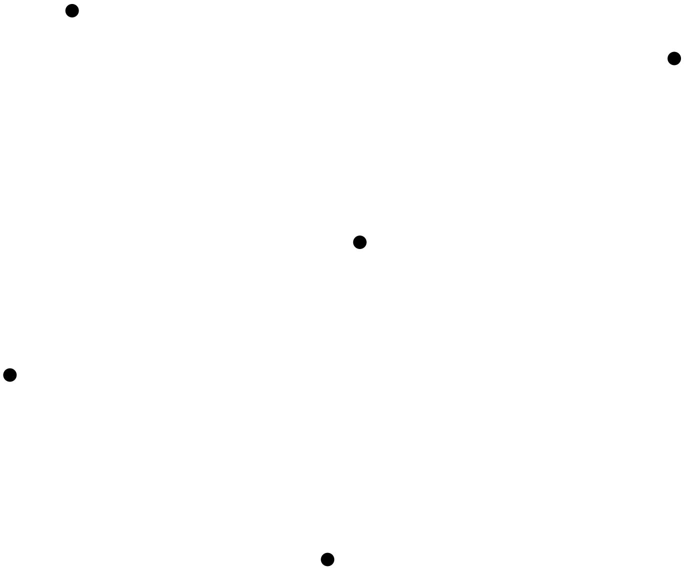
S_0: Generate n samples uniformly over the space (from the prior π).

S_{n+1}: Delete the lowest likelihood sample in S_n, and replace it with a new uniform sample with higher likelihood

Requires one to be able to uniformly within a region, subject to a *hard likelihood constraint*.

Nested Sampling

Graphical aid



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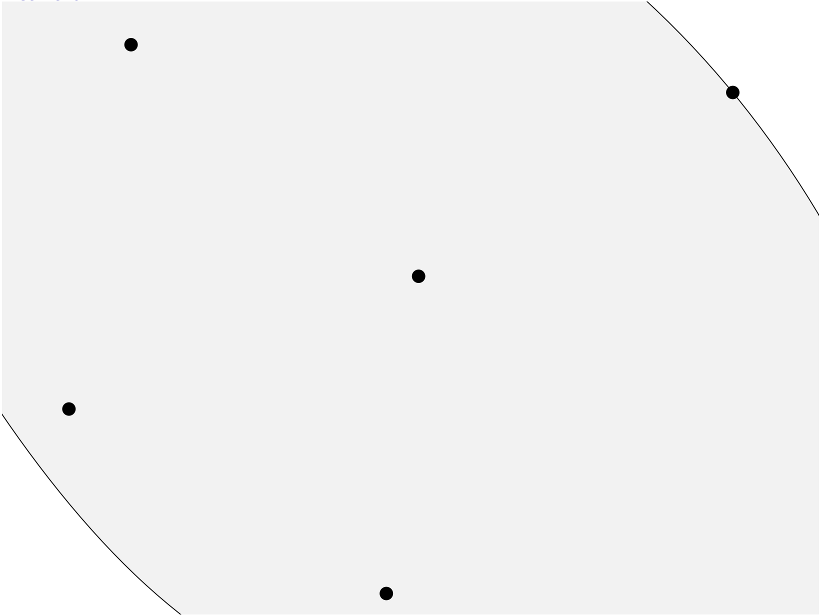

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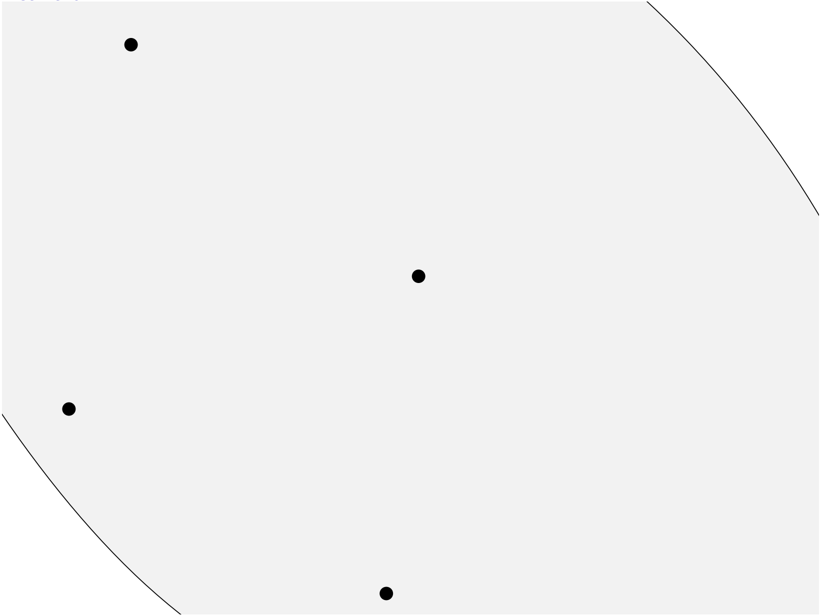

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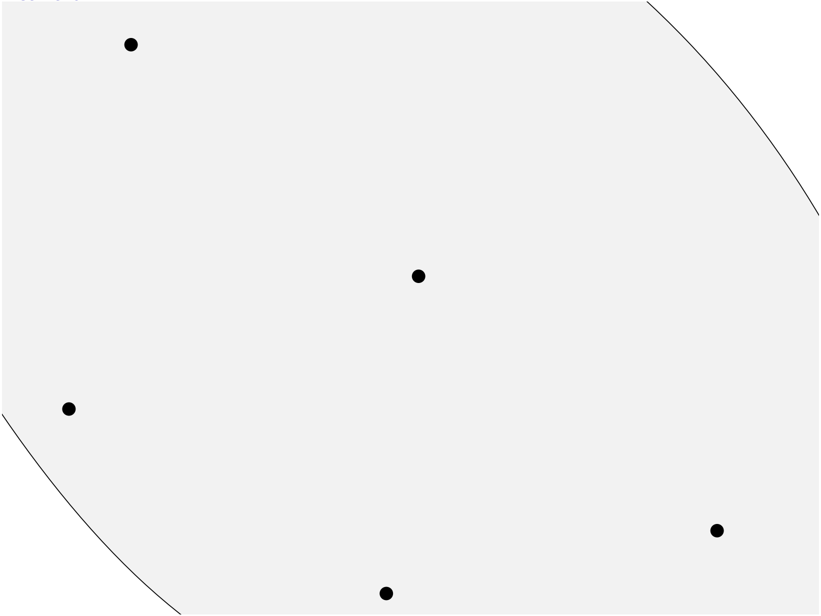

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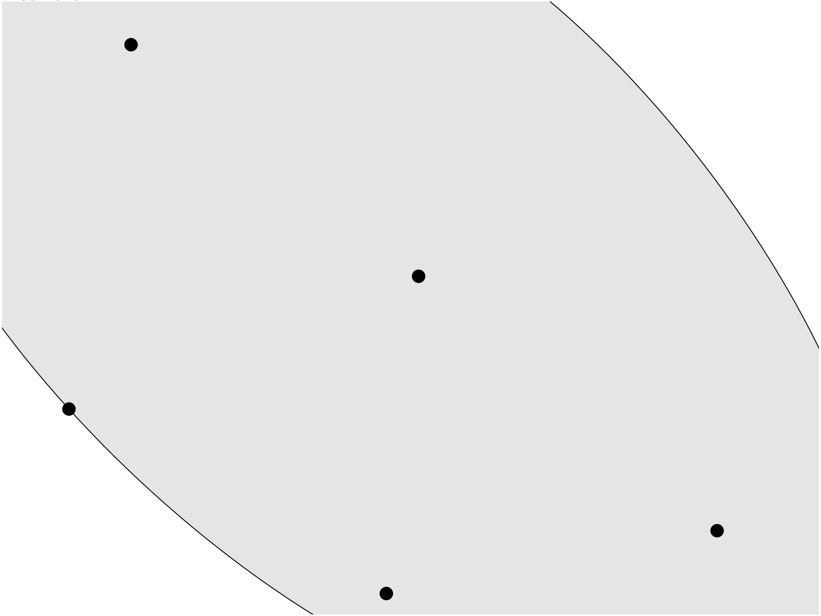

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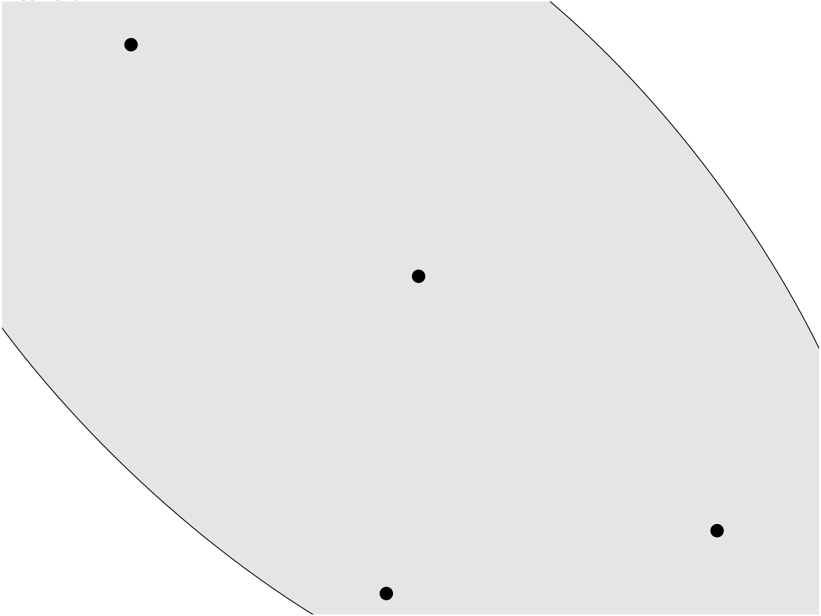

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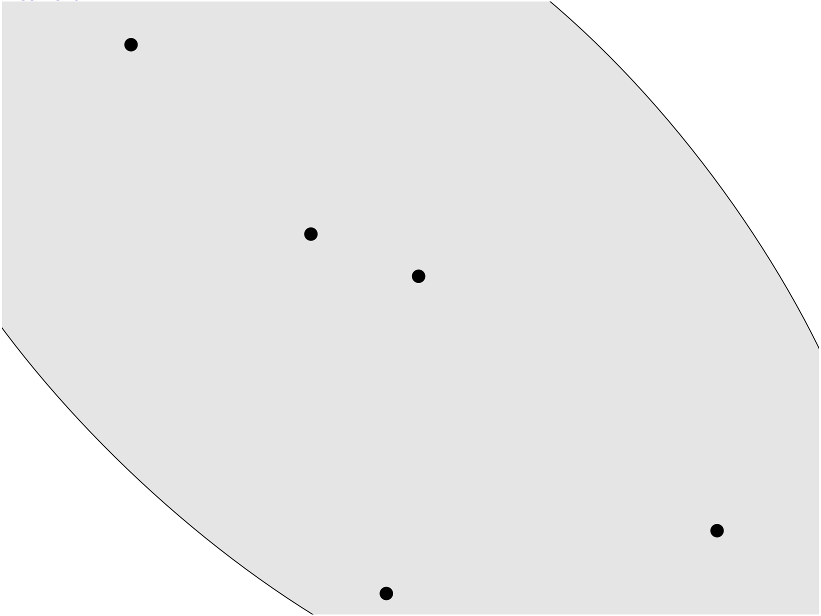

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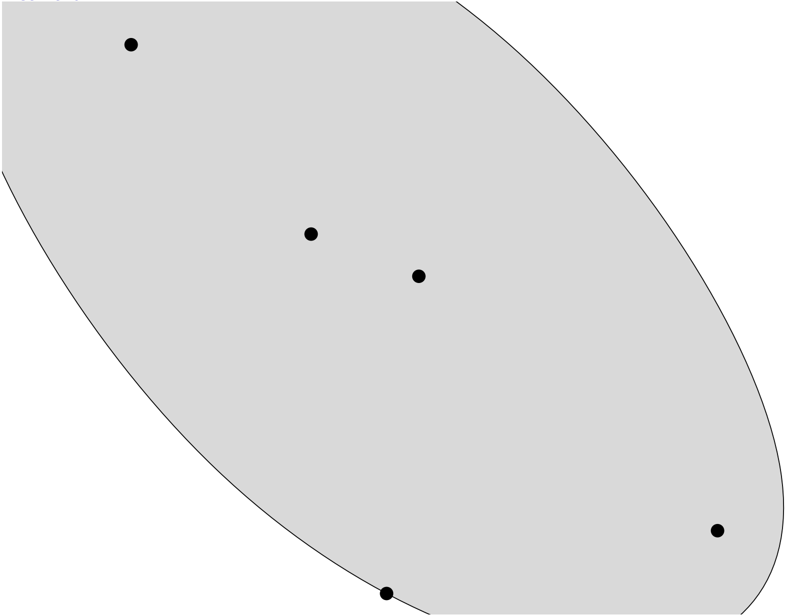

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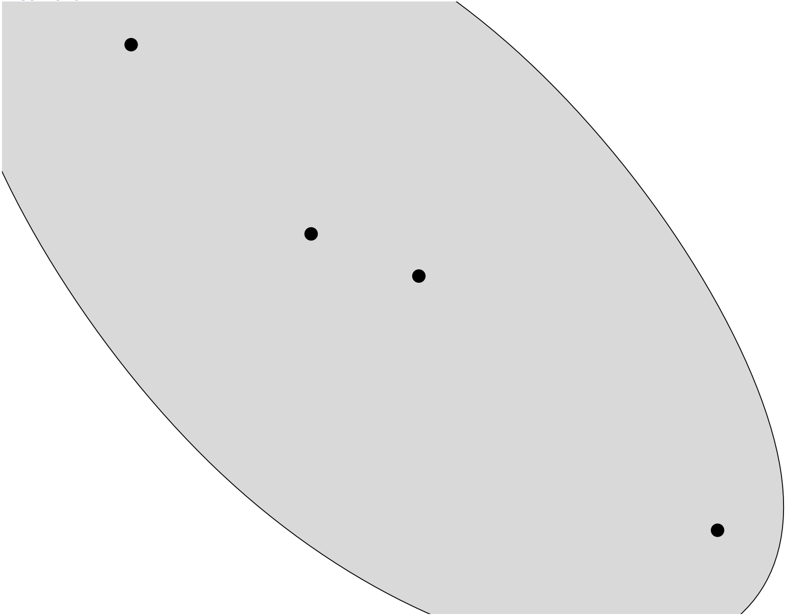

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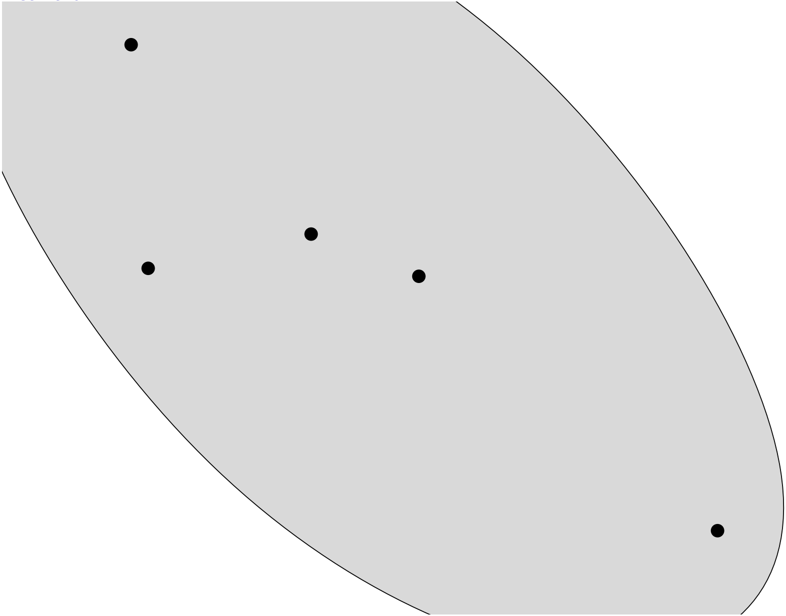

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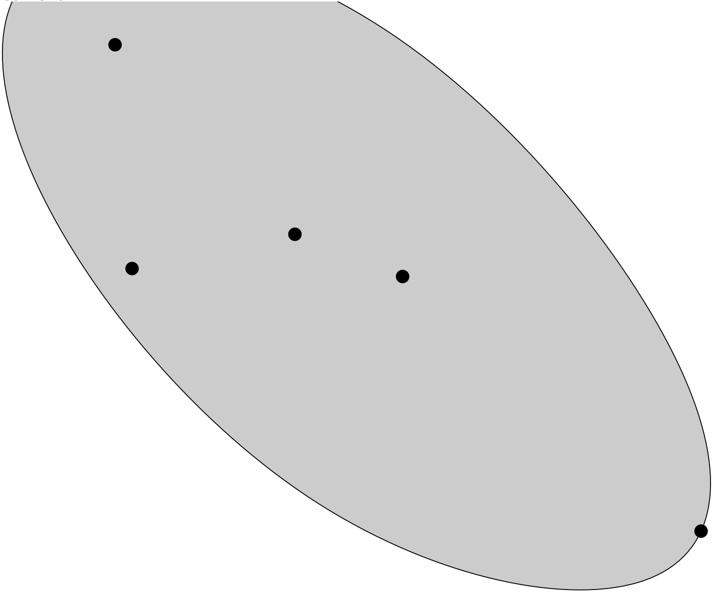

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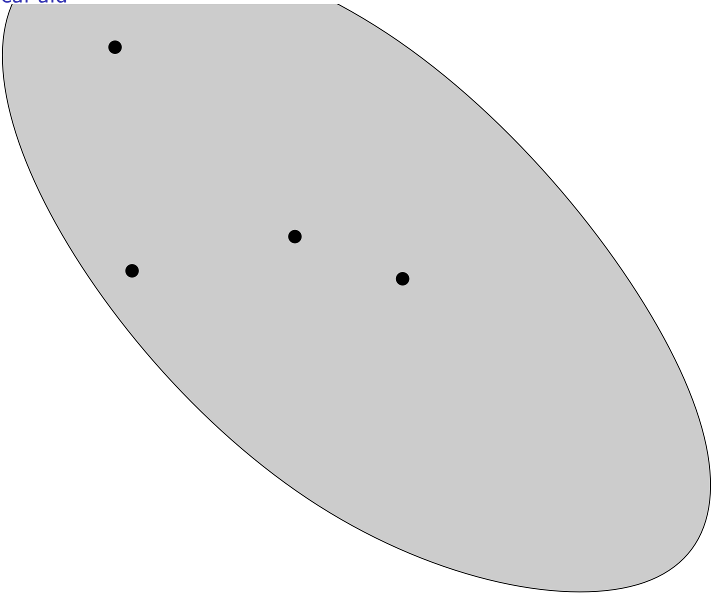

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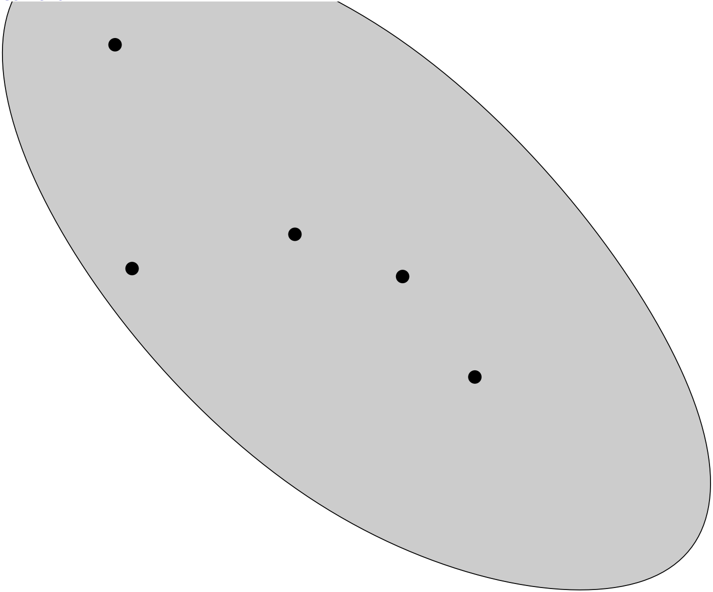

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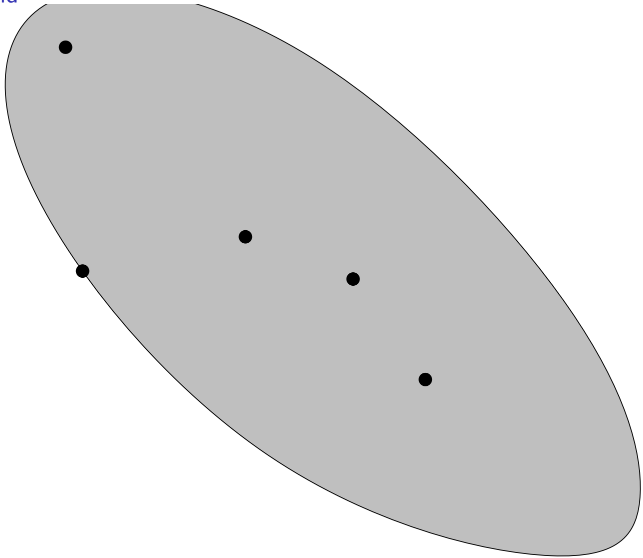

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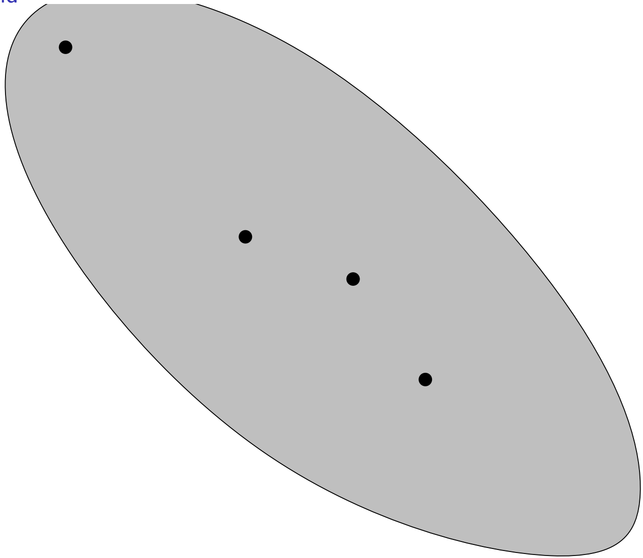

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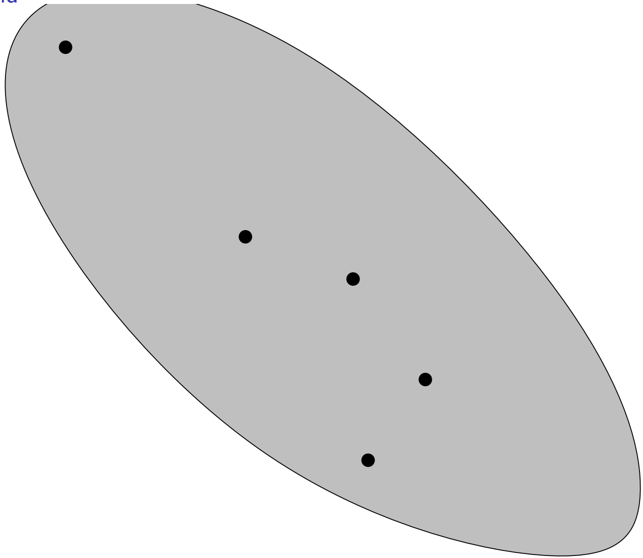

Nested Sampling

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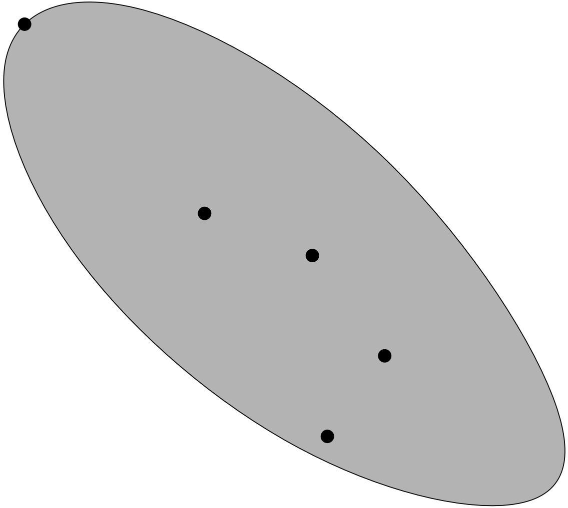

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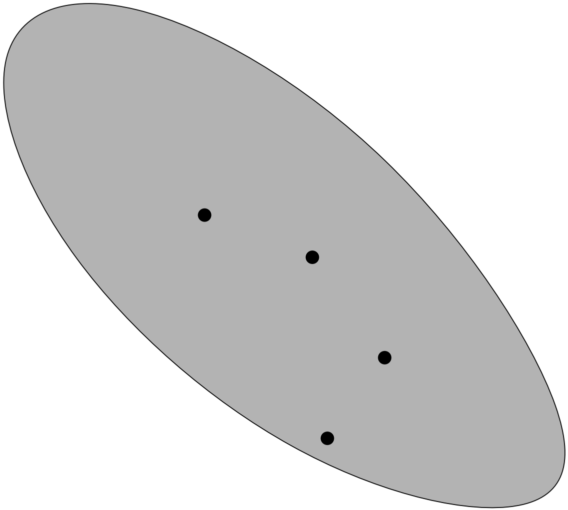

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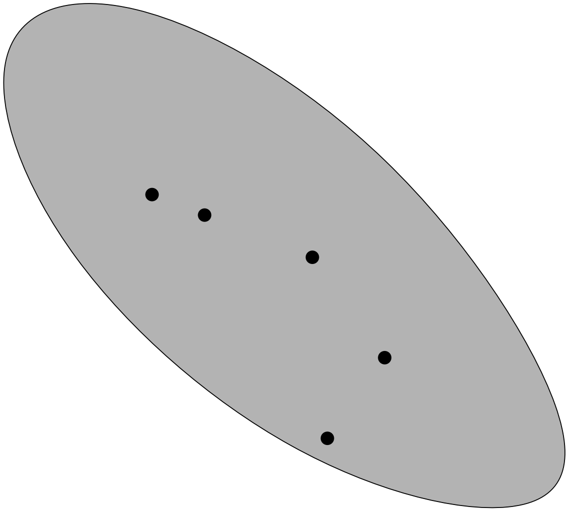

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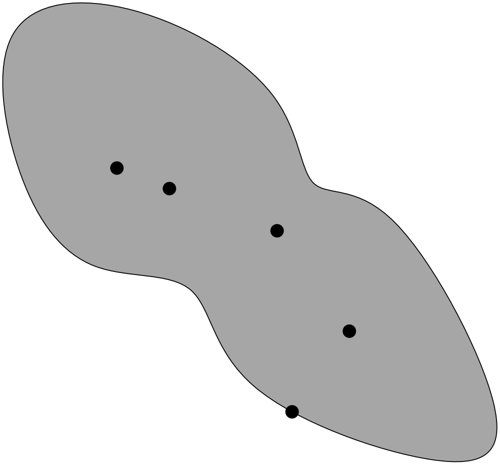

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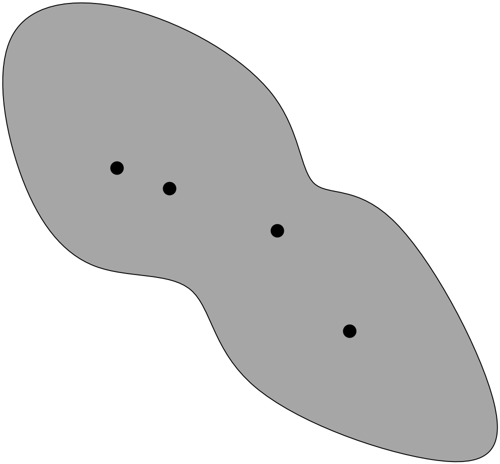

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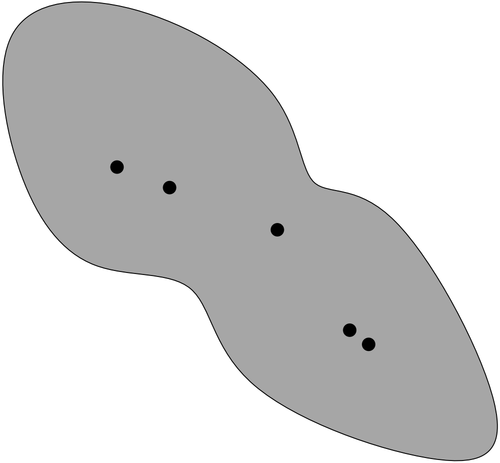

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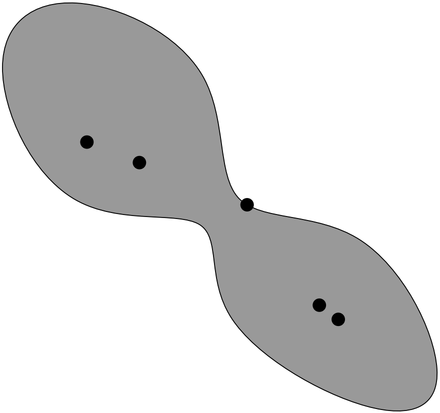

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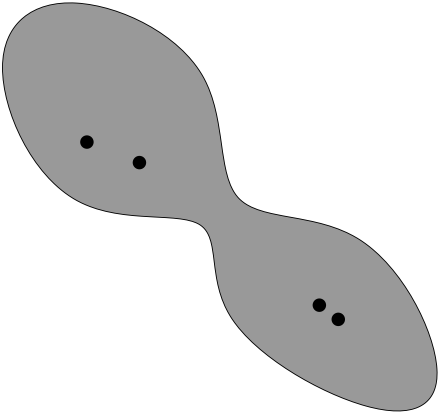

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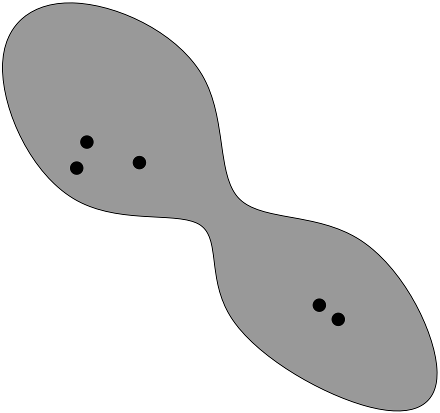

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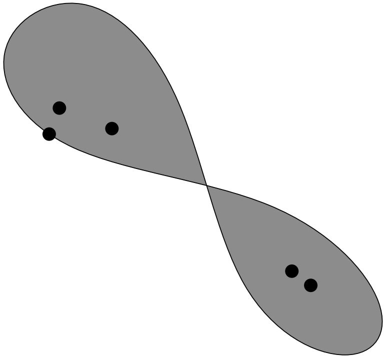

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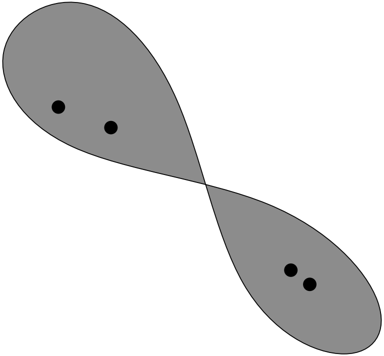

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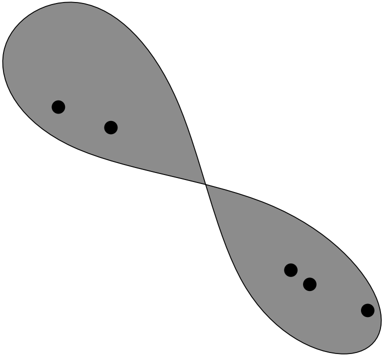

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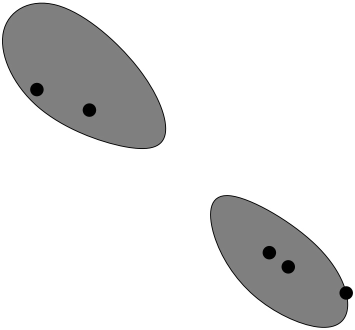

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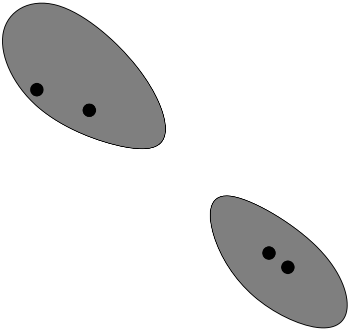

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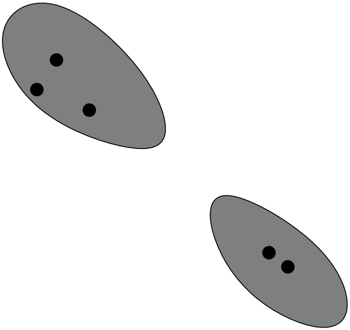

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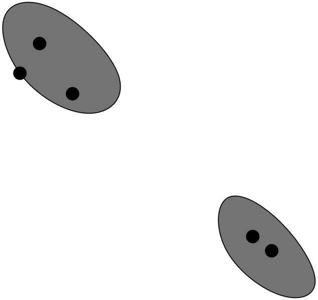

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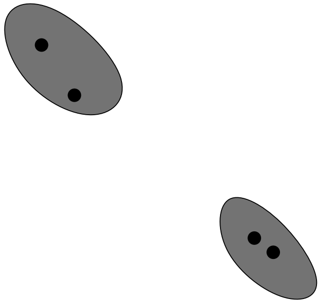

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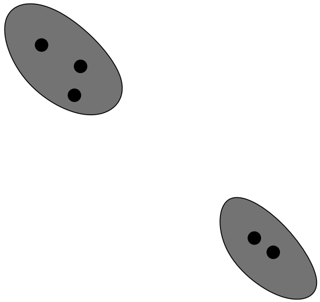

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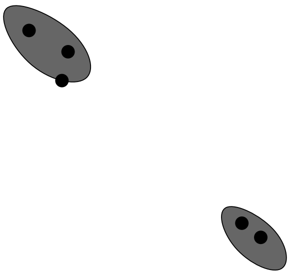

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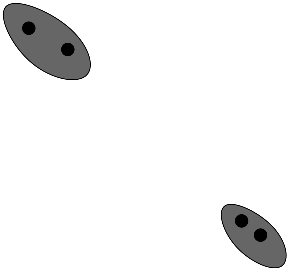

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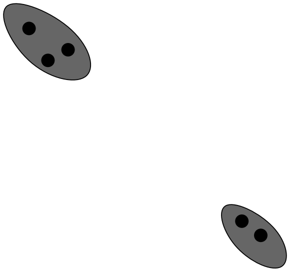

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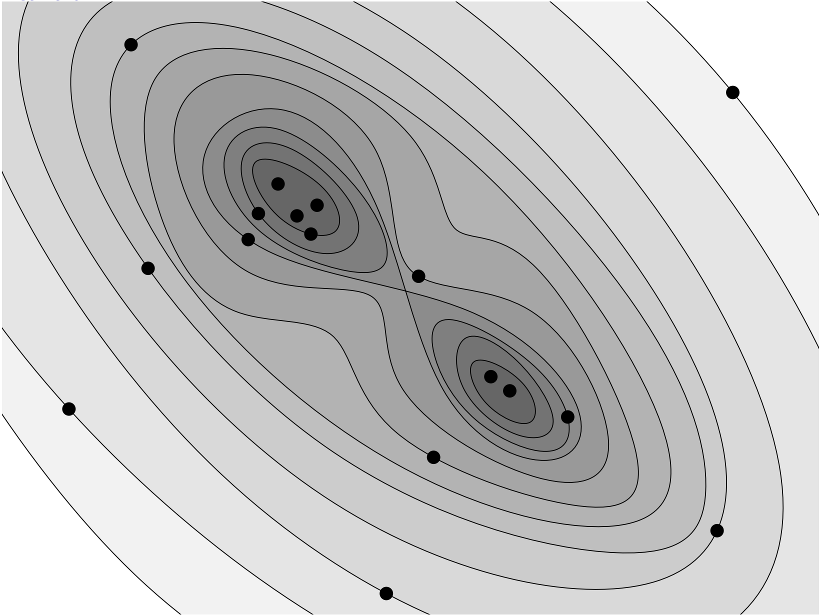

Nested sampling

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- They can also be used to calculate evidences, since it sequentially updates the priors.

Sampling from a hard likelihood constraint

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- Most of the work in NS to date has been in attempting to implement a hard-edged sampler in the NS meta-algorithm.

Sampling within an iso-likelihood contour

Previous attempts

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Rejection Sampling MultiNest; F. Feroz & M. Hobson (2009).

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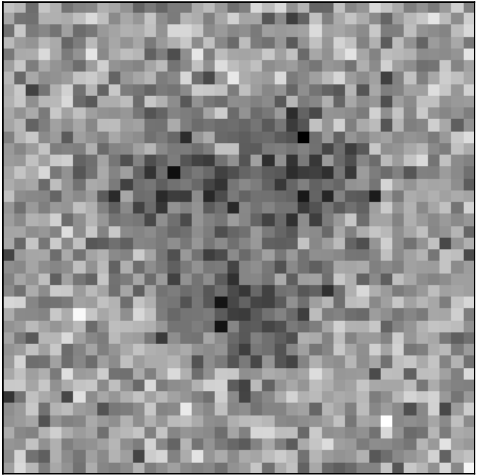
- PolyChord 2.0 imminent.

Object detection

Toy problem



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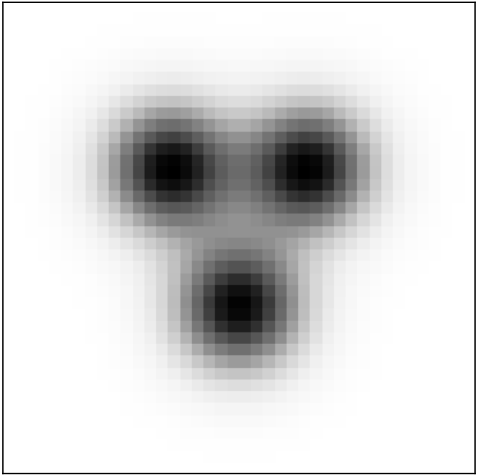
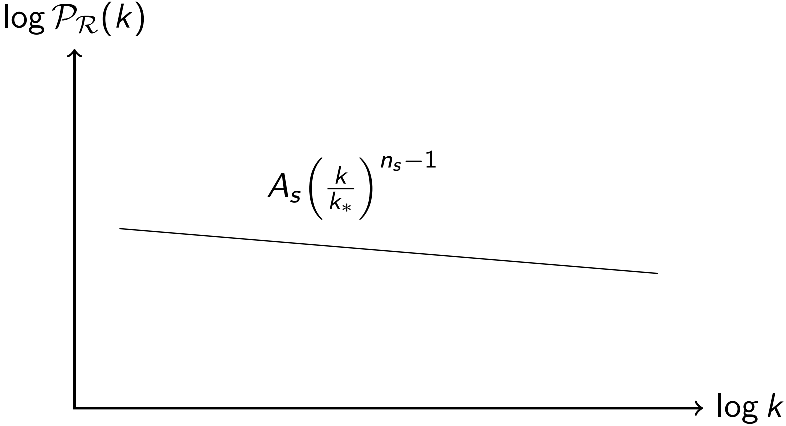


PolyChord in action

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



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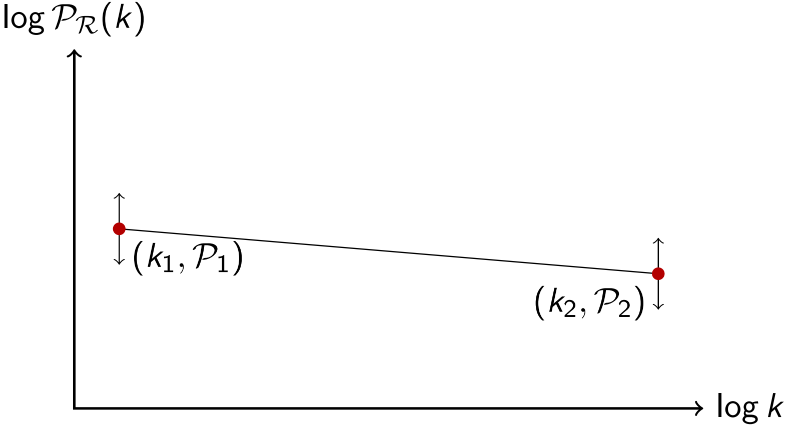

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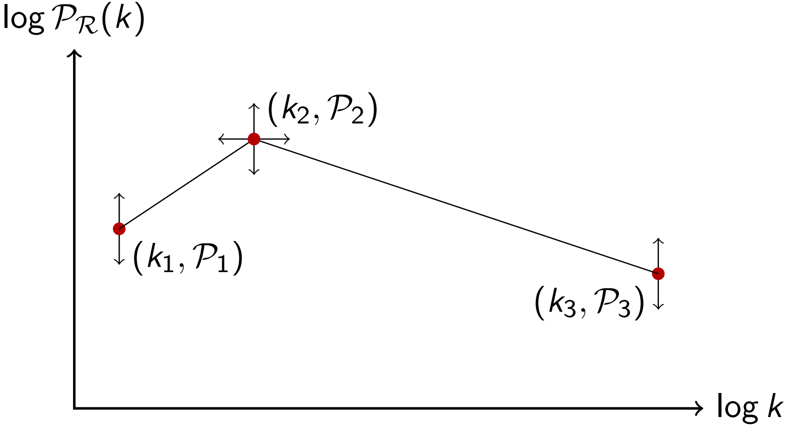

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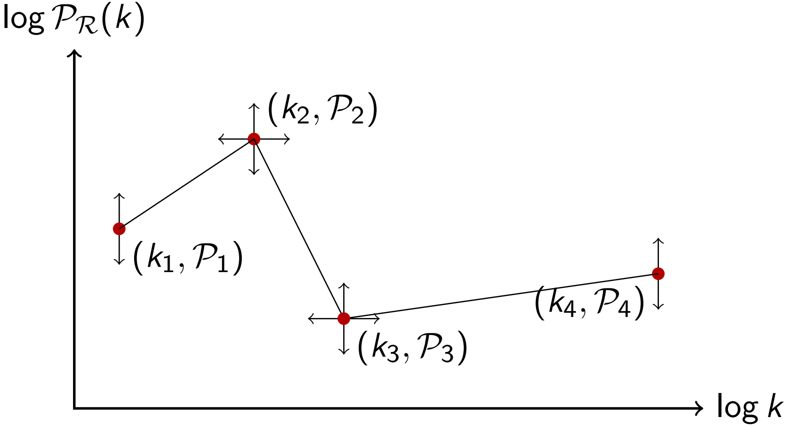

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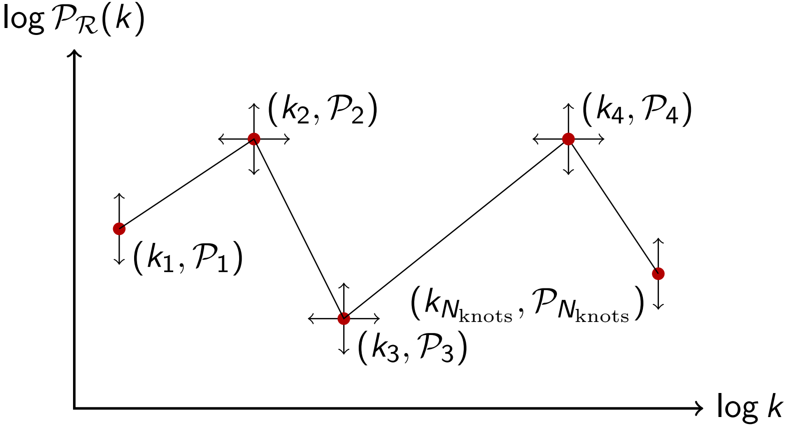
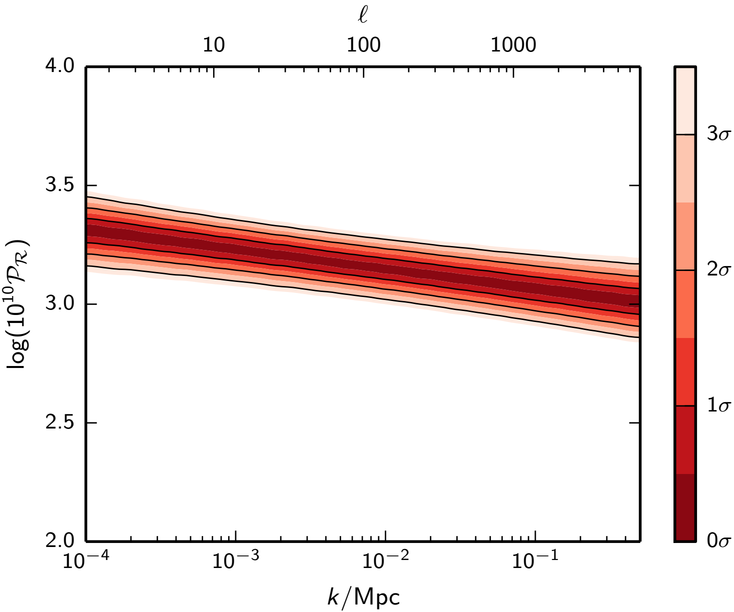


0 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



1 internal knots

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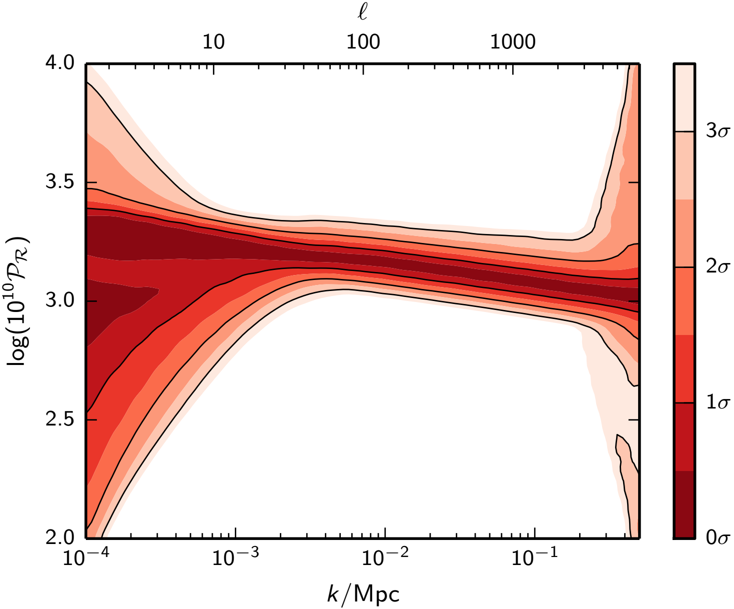

2 internal knots

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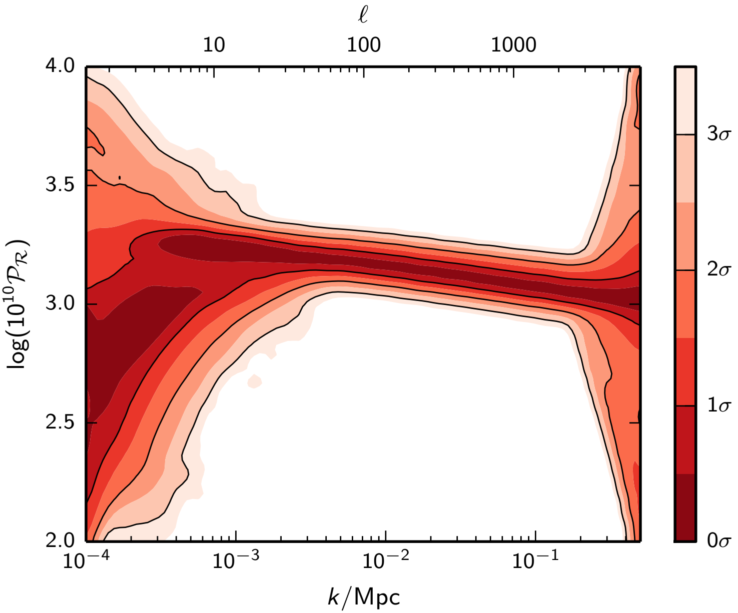

3 internal knots

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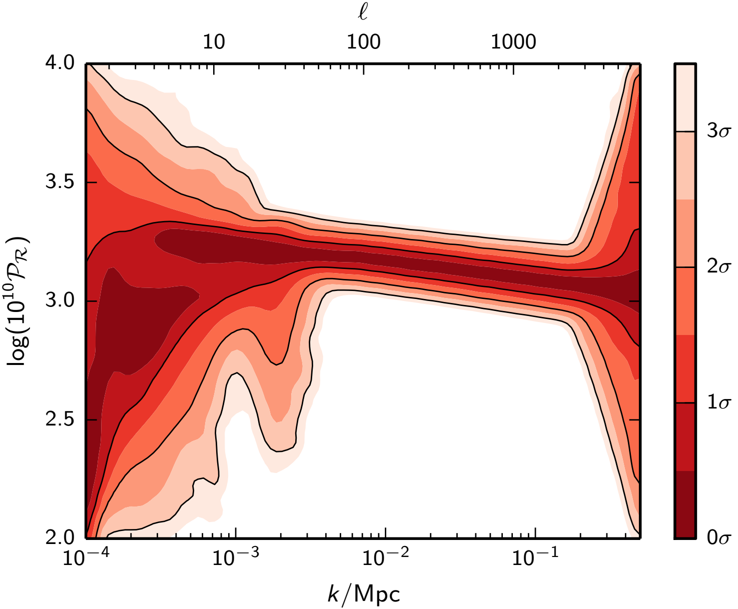
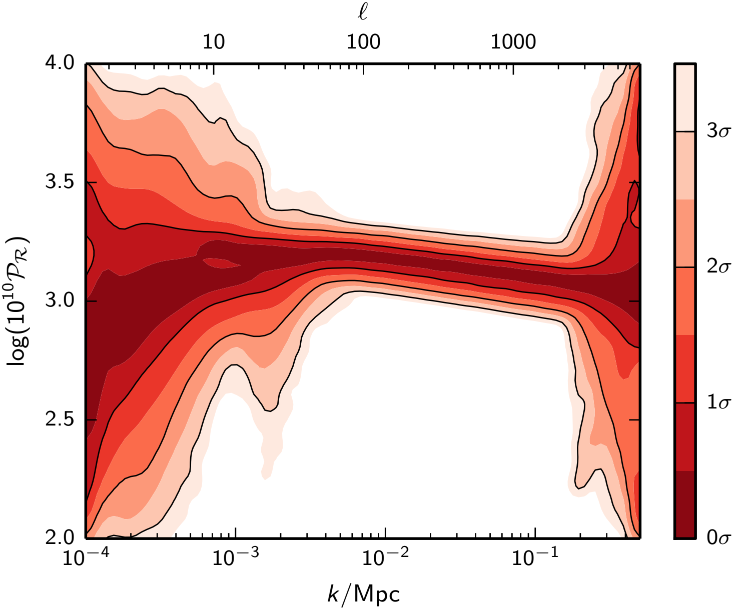


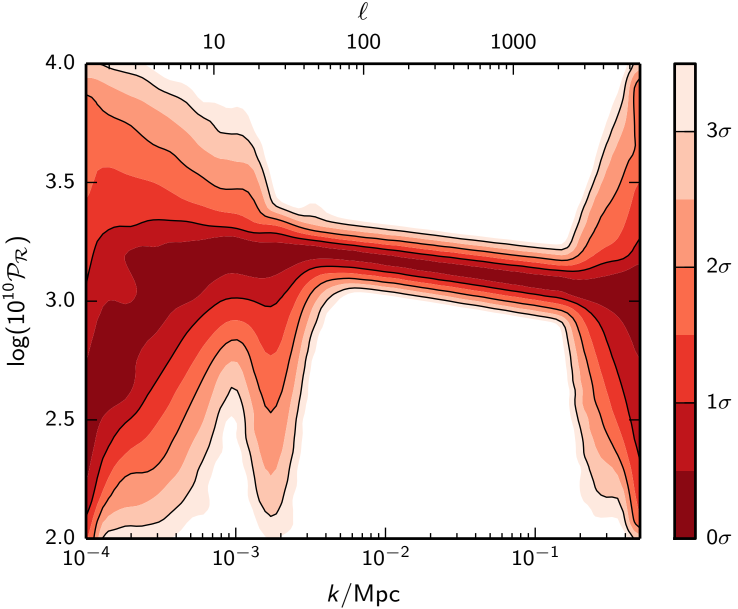
4 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



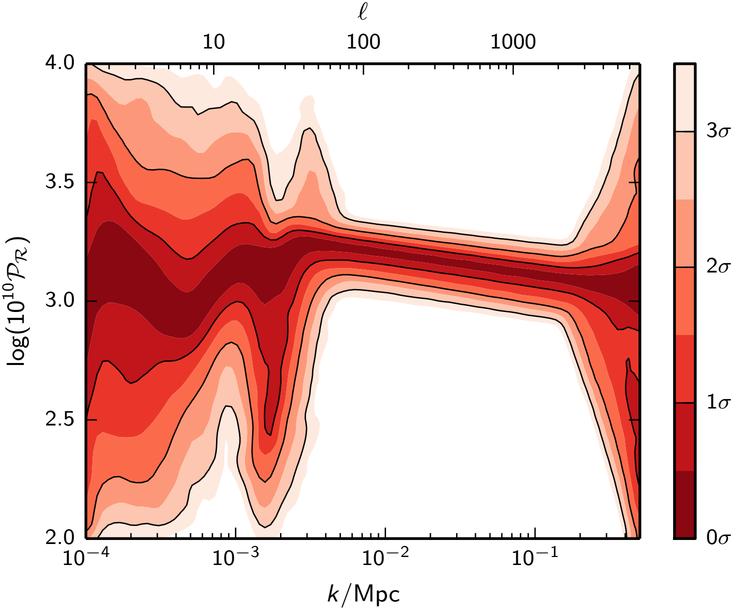
5 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



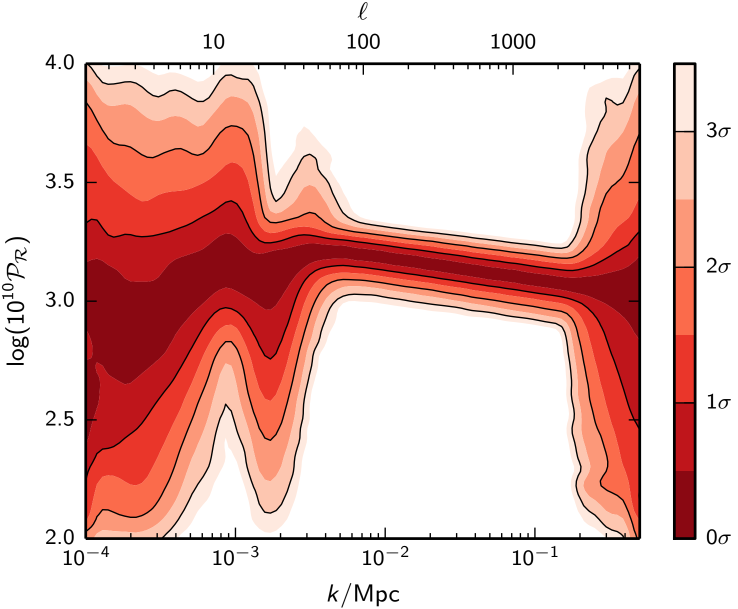
6 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



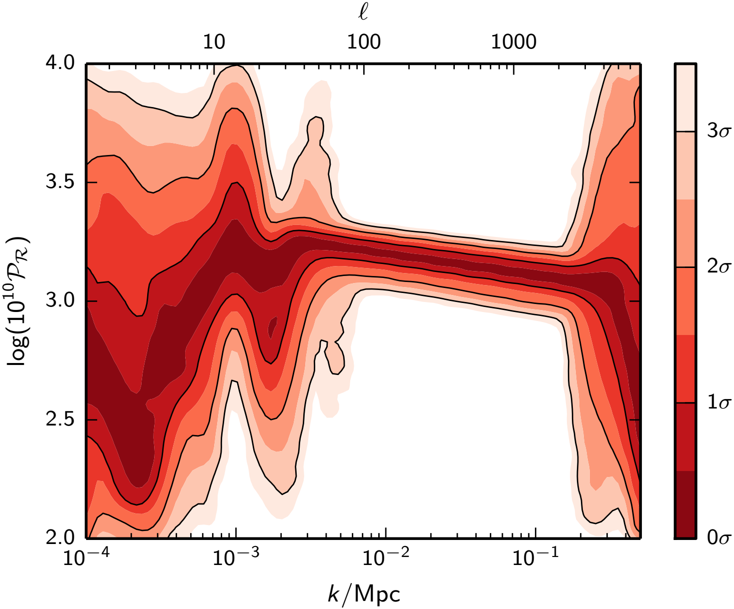
7 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



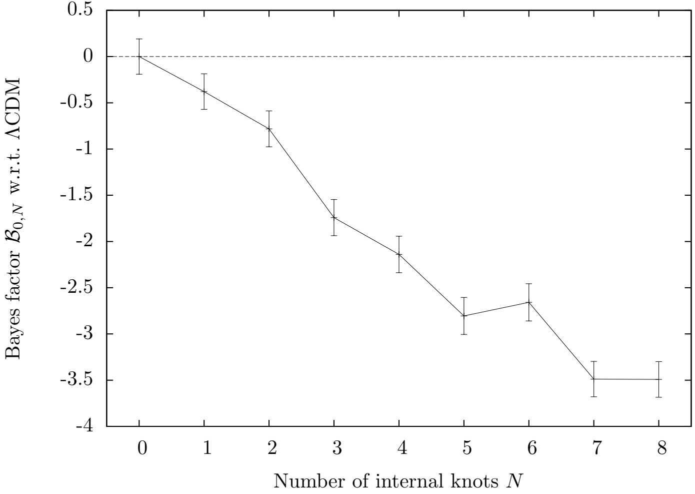
8 internal knots

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



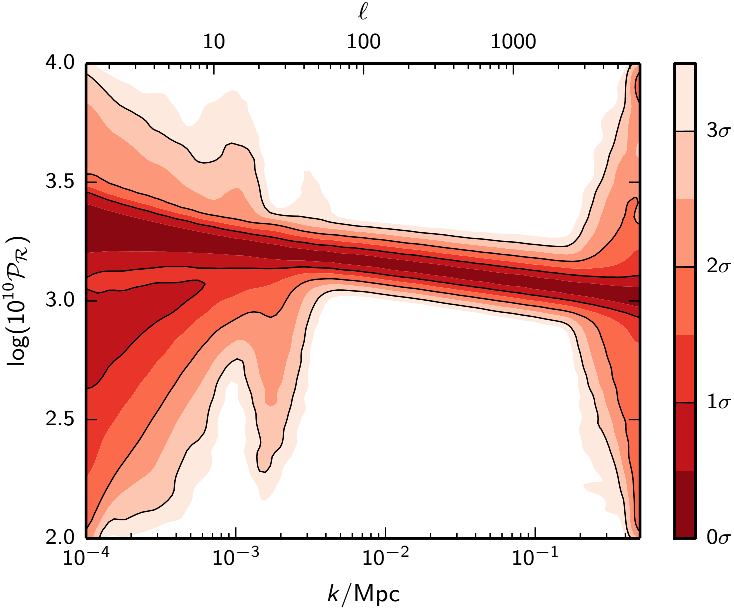
Bayes Factors

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



Marginalised plot

Primordial power spectrum $\mathcal{P}_\mathcal{R}(k)$ reconstruction



Dark energy equation of state reconstruction

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- Same thing, but for Dark energy equation of state $w(z)$ (quintessence).

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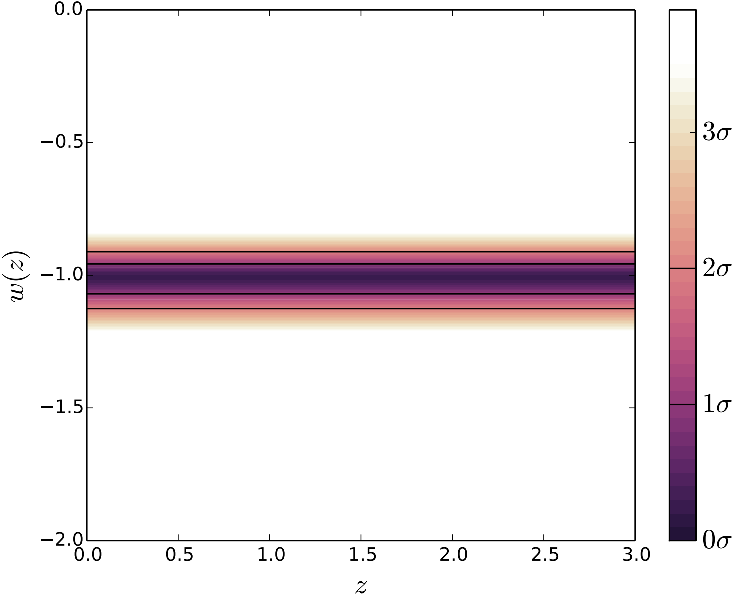
- Data used is Planck 2015, BOSS DR 11, JLA supernovae and BOSS Lyα data

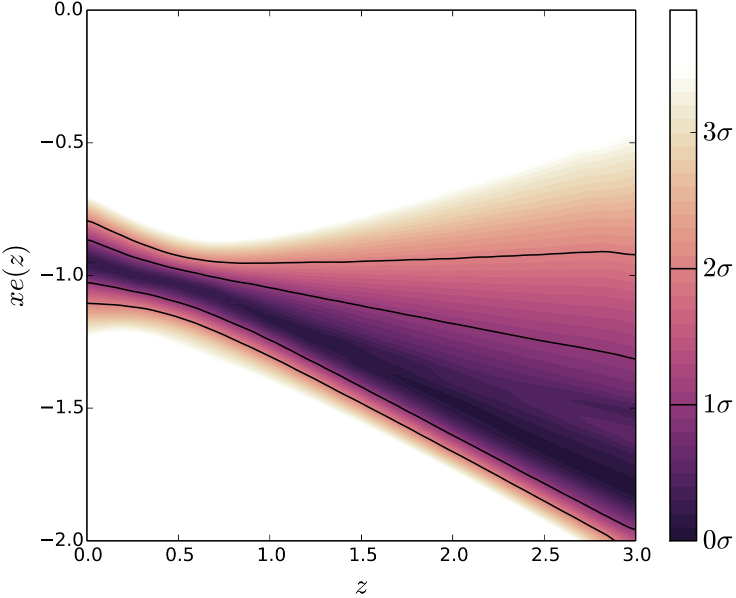
Flat, variable w

Dark energy equation of state reconstruction



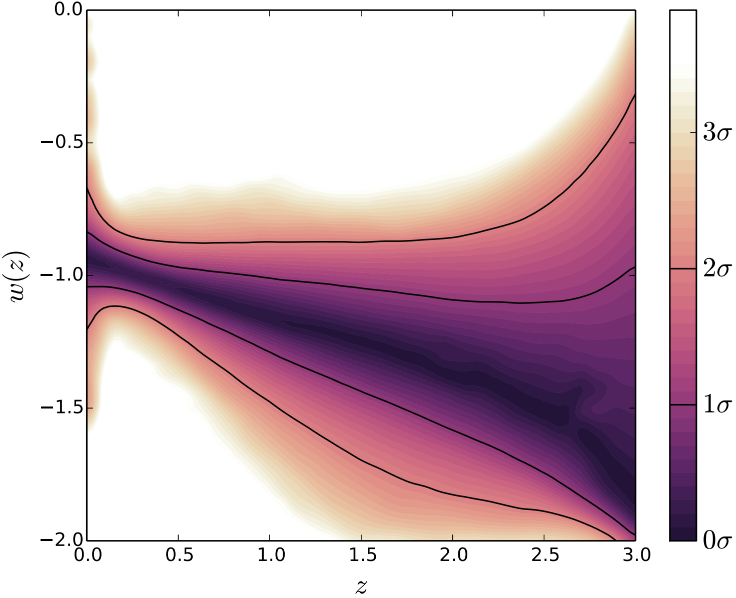
Tilted

Dark energy equation of state reconstruction



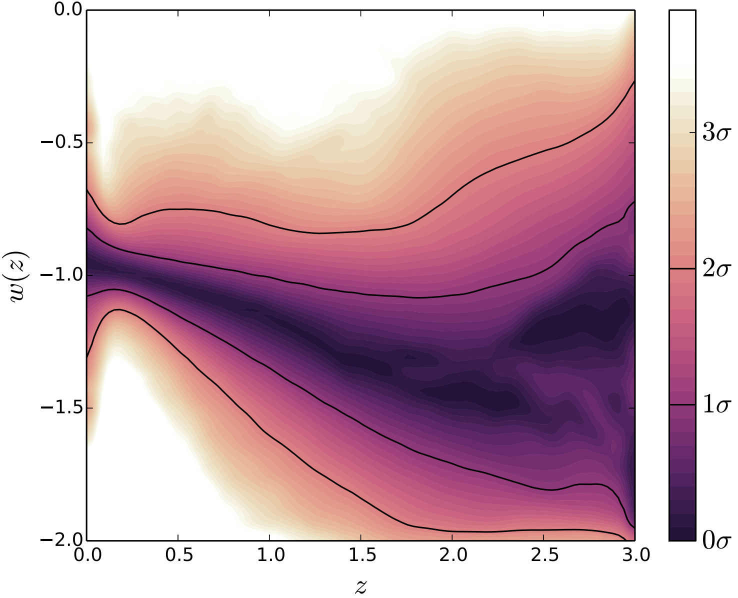
1 internal node

Dark energy equation of state reconstruction



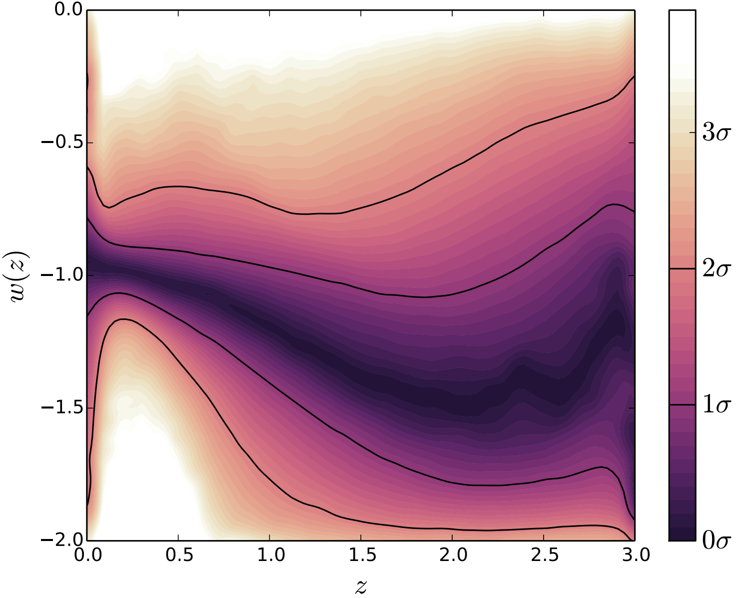
2 internal nodes

Dark energy equation of state reconstruction



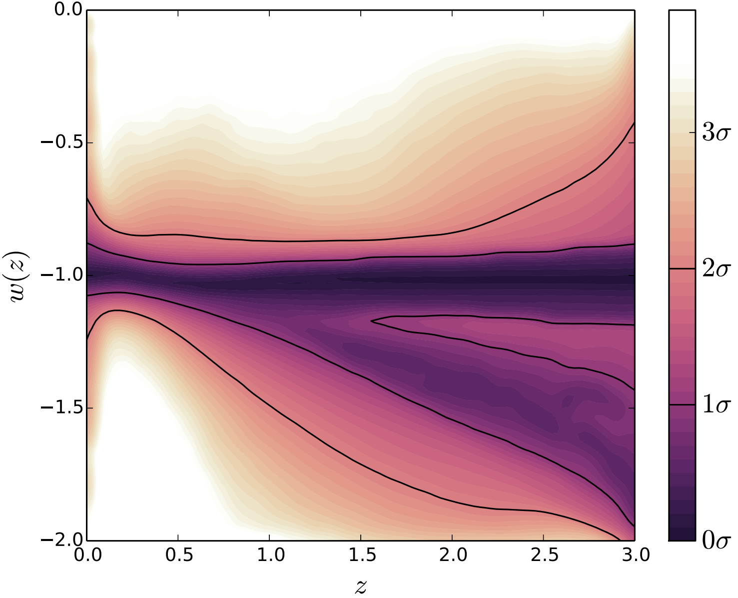
3 internal nodes

Dark energy equation of state reconstruction



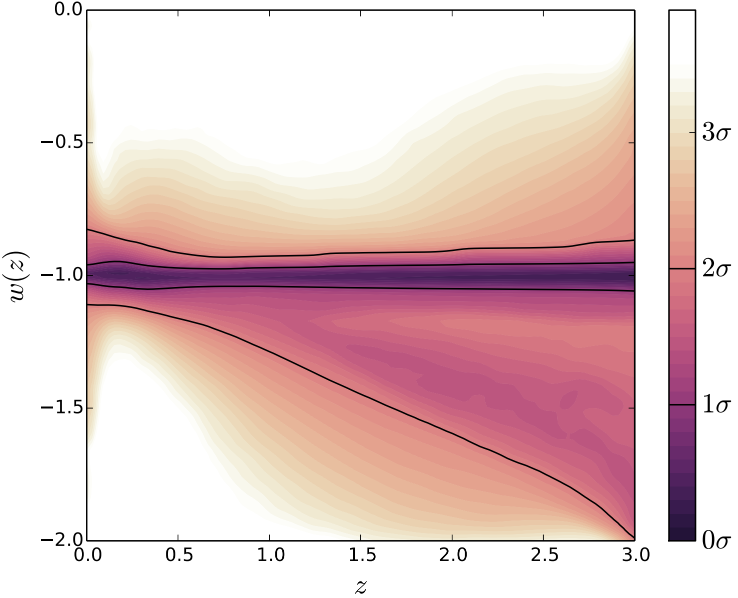
Marginalised plot - just extension models

Dark energy equation of state reconstruction



Marginalised plot - including LCDM

Dark energy equation of state reconstruction



Useful links

My email:	wh260@cam.ac.uk
My room:	Room 104, Tuesday-Thursday this week
PolyChord:	ccpforge.cse.rl.ac.uk/gf/project/polychord
MultiNest:	ccpforge.cse.rl.ac.uk/gf/project/multinest
Stan:	mc-stan.org/
emcee:	dan.iel.fm/emcee/current/