Monte Carlo Prac
STA Honours: Statistical Computing
University of Cape Town

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Question 1: Accept-Reject Method

1. Plot the following function over the range $-5 < x < 5$:

$$f(x) = e^{-x^2/2} sin(2x)^2$$

2. Find a normal distribution ($h(x)$) that qualifies as a candidate distribution (do this graphically by trial and error), i.e. find a constant C such that $Ch(x)$ satisfies the requirements of a candidate distribution.
3. Generate N values each from the candidate distribution, and from a $U(0, 1)$ distribution.
4. Calculate the ratio

$$g(x) = \frac{f(x)}{Ch(x)}$$

   for each of the candidate values generated above. Substitute your constant for C. x is the value generated from $h(x)$.

5. Accept x with probability g.
6. Plot a histogram of the generated values, and plot the target distribution ($f(x)$) on top to check.
7. Comment on the proportion of values accepted. Is this acceptable?
8. The sample generated using the accept-reject method is a random sample generated from the

a) target distribution,
b) candidate distribution,
c) none of the above?

9. Use simple MCMC (Metropolis algorithm) to sample from $f(x)$.

Question 2: Probability Integral Transform

1. Generate values from an exponential distribution with $\lambda = 2$, using the probability integral transform.
2. Make appropriate plots to check that this has worked.

Question 3: Importance Sampling

Estimate

$$\int_0^1 \frac{e^{-x}}{1+x^2}dx$$

using importance sampling. Compare your estimates and their standard errors using the following importance functions

$$f_0(x) = 1, \qquad 0 < x < 1$$

$$f_1(x) = e^{-x}, \qquad 0 < x < \infty$$

$$f_3(x) = \frac{e^{-x}}{1-e^{-1}}, \qquad 0 < x < 1$$

One can generate from $f_3(x)$ using inverse transform sampling, and from $f_1(x)$ using the `rexp()` random number generator.

Question 4: Random Sums

1. Use Monte Carlo methods to estimate the mean, variance, and $P(S > c)$ for the random sum of exponentials (example in slides). Also find the standard errors (Monte Carlo error) of the mean and the probability.

`rnbinom()` generates values from a negative binomial distribution.

```
p <- 0.5   # use these parameters for the negative binomial NB(p,r)
r <- 20

x0 <- 5    # use these settings
M <- 10000    # number of Monte Carlo simulations
c <- 100
```

Question 5: The Random Number Generator RANDU

$$x_i = 65539 x_{i-1} \mathrm{mod}\ 2^{31}$$

Can you find a problem with the following random number generator? What is it?

```
x0 <- sample(1:1000000, size = 1)   # randomly choose a starting value between
x0                                 # 1 and 1000000

x <- c()         # prepare vector x, into which we are going to put values
x[1] <- 65539 * x0 %% 2^31          # first value

for (i in 2:10000) {
  x[i] <- 65539 * x[i - 1] %% 2^31     # uniform random number generator called RANDU
}

set1 <- seq(1, 10000, by = 3)     # create indices of three sets,
set2 <- seq(2, 10000, by = 3)     # every 3rd, starting from 2
set3 <- seq(3, 10000, by = 3)     # every 3rd, starting from 3


x1 <- x[set1]       # subset 1 (at lag 2)
x2 <- x[set2]       # subset 2 (at lag 1)
x3 <- x[set3]   # subset 3

library(rgl)            # rotating graphics library
```

```
open3d()
plot3d(x1, x2, x3, col = "red", cex = 0.7)
```

Question 6: Integration

Integrate

$$f(x) = \frac{1}{1 + \sinh(2x)\log(x)^2}$$

on the interval $0.8 < x < 3$. What is the Monte Carlo error of your estimate?

Question 7: Estimate π

Find a Monte Carlo estimate of π.

What is the error with $N = 1000$ generated points?

Question 8: Discrete Inverse Transform Sampling

Using only the `runif()` random number generator in R, generate random outcomes from the following distribution

$$p(x) = \begin{cases} 0.4, & x = \text{turn left} \\ 0.5, & x = \text{turn right} \\ 0.1, & x = \text{stay in place} \\ 0, & \text{otherwise} \end{cases}$$

Question 9: Antithetic Sampling

Use Monte Carlo integration to estimate the following integral:

$$\int_0^1 f(x)dx, \qquad \text{where } f(x) = x$$

Find the Monte Carlo error with $N = 1000$. Then use antithetic sampling. Compare the estimates and their errors.

Question 10: MCMC

$$f(x) = 10\exp(-4(x+4)^2) + 3\exp(-0.2(x+1)^2) + \exp(-2(x-5)^2)$$

Use the Metropolis algorithm to sample from $f(x)$ using $g(x) : X \sim U(-1,1)$ as proposal distribution and $N = 5000$.

Illustrate your results.