Continuous Assessment 3 - Group 3

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Question 1:

Statement

Let $X \sim N_p(\mu, \Sigma)$ with $|\Sigma| > 0$. Using the spectral decomposition of the covariance matrix, prove that:

$$(X - \mu)'\Sigma^{-1}(X - \mu) \sim \chi^2_p.$$

Proof

Since Σ is a positive definite $p \times p$ covariance matrix, it admits the spectral decomposition:

$$\Sigma = Q\Lambda Q'$$

where: - Q is an orthogonal matrix ($Q'Q = QQ' = I$), - Λ is a diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_p$ on its diagonal.

Define the transformation:

$$Z = Q'(X - \mu).$$

Since $X \sim N_p(\mu, \Sigma)$, the mean of Z is:

$$E[Z] = Q'E[X - \mu] = Q'(\mu - \mu) = 0.$$

The covariance of Z is:

$$\mathrm{Var}(Z) = Q'\mathrm{Var}(X)Q = Q'\Sigma Q = Q'(Q\Lambda Q')Q = Q'Q\Lambda Q'Q = I\Lambda I = \Lambda.$$

Thus, $Z \sim N_p(0, \Lambda)$, meaning the components of Z are independent normal variables:

$$Z_i \sim N(0, \lambda_i), \quad i = 1, \dots, p.$$

We compute:

$$(X - \mu)'\Sigma^{-1}(X - \mu).$$

Using the spectral decomposition:

$$\Sigma^{-1} = Q\Lambda^{-1}Q'.$$

Thus:

$$(X - \mu)'\Sigma^{-1}(X - \mu) = (X - \mu)'Q\Lambda^{-1}Q'(X - \mu).$$

Substituting $Z = Q'(X - \mu)$:

$$= Z'\Lambda^{-1}Z.$$

Since $Z = (Z_1, Z_2, \ldots, Z_p)'$ with $Z_i \sim N(0, \lambda_i)$, we expand:

$$Z'\Lambda^{-1}Z = \sum_{i=1}^{p} \lambda_i^{-1} Z_i^2.$$

Since $Z_i \sim N(0, \lambda_i)$, we have:

$$\frac{Z_i}{\sqrt{\lambda_i}} \sim N(0, 1).$$

Squaring both sides:

$$\left(\frac{Z_i}{\sqrt{\lambda_i}}\right)^2 \sim \chi_1^2.$$

Summing over all p terms:

$$\sum_{i=1}^{p} \left(\frac{Z_i}{\sqrt{\lambda_i}}\right)^2 \sim \chi_p^2.$$

Thus,

$$(X - \mu)'\Sigma^{-1}(X - \mu) \sim \chi_p^2.$$

This completes the proof. $\square$

Question 2:

Statement

Consider $X \sim N_5(\mu, \Sigma)$, where:

$$\mu = \begin{bmatrix} 5 \\ 0 \\ -2 \\ 6 \\ 2 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 8 & 3 & -1 & 0 & 5 \\ 3 & 12 & 2 & 2 & -2 \\ -1 & 2 & 9 & 0 & 1 \\ 0 & 2 & 0 & 8 & 2 \\ 5 & -2 & 1 & 2 & 10 \end{bmatrix}$$

Define:

$$X_1 = \begin{bmatrix} X_1 \\ X_2 \\ X_4 \end{bmatrix}, \quad X_2 = \begin{bmatrix} X_3 \\ X_5 \end{bmatrix}.$$

Find the conditional (joint) distribution of $X_1 | X_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Solution

The mean vectors are:

$$\mu_1 = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}, \quad \mu_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

The covariance matrix is partitioned as:

$$\Sigma_{11} = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 12 & 2 \\ 0 & 2 & 8 \end{bmatrix}, \quad \Sigma_{12} = \begin{bmatrix} -1 & 5 \\ 2 & -2 \\ 0 & 2 \end{bmatrix}.$$

$$\Sigma_{22} = \begin{bmatrix} 9 & 1 \\ 1 & 10 \end{bmatrix}.$$

$$\Sigma_{22}^{-1} = \frac{1}{89} \begin{bmatrix} 10 & -1 \\ -1 & 9 \end{bmatrix}.$$

$$\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2).$$

Substituting values:

$$\mu_{1|2} \approx \begin{bmatrix} 4.831 \\ 0.247 \\ 5.978 \end{bmatrix}.$$

$$\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

$$\Sigma_{1|2} \approx \begin{bmatrix} 6.752 & 2.528 & -0.528 \\ 2.528 & 11.507 & 2.562 \\ -0.528 & 2.562 & 7.596 \end{bmatrix}.$$

$$X_1|X_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \sim N\left(\begin{bmatrix} 4.831 \\ 0.247 \\ 5.978 \end{bmatrix}, \begin{bmatrix} 6.752 & 2.528 & -0.528 \\ 2.528 & 11.507 & 2.562 \\ -0.528 & 2.562 & 7.596 \end{bmatrix} \right).$$