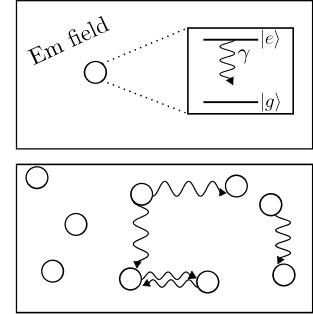


4 Collective light-matter coupling

In this chapter we come back to two-level systems where the levels are low-lying, that is close to the ground state.

We will analyze here how a gas of N atoms of those 2-level atoms interact with the electromagnetic field, following the same steps we did in section 1.3. The result is, though, very different: the atoms interact with each other by exchanging photons and the dissipation (the way the full system of atoms emit photons back into the radiation field) becomes **collective**, with the appearance of **super -** and **subradiant** states.



4.1 Many-body master equation

We are considering here exactly the same we did in section 1.3:

A system (which now has N atoms) interacting with a Markovian environment (again the electromagnetic field). Hence, we can use again the general expression for the master equation:

$$\dot{\rho} = -\frac{1}{\hbar^2} \text{tr}_E \int_0^\infty [H''(t), [H''(t', \rho(t) \otimes \rho_E(0))] dt' \quad (4.1)$$

Where we have assumed that Born and Markov approximations are valid, and where H'' is the interaction Hamiltonian between system and environment in the **interaction picture**. Hence, in order to continue, the first step is to find this Hamiltonian.

4.1.1 Atom-field Hamiltonian

Exactly as in the case of a single atom, the Hamiltonian under the dipole approximation reads:

$$H = H_0 + H_F - \underbrace{\sum_{j=1}^N \vec{d}_j \cdot \vec{E}(\vec{r}_j, t)}_V$$

where, while the field Hamiltonian stays the same:

$$H_F = \sum_{\vec{k}} \hbar \nu_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}} ,$$

the atomic part is modified to consider the N atoms

$$H_0 = \hbar\omega_0 \sum_{j=1}^N \sigma_j^\dagger \sigma_j ,$$

and the interaction part now considers the dipole moment for all atoms being the same for all atoms,

$$\vec{d}_j = \left(|g\rangle_j \langle g| + |e\rangle_j \langle e| \right) \vec{d} \left(|g\rangle_j \langle g| + |e\rangle_j \langle e| \right) = \vec{d}_{eg} \sigma_j + \vec{d}_{eg}^* \sigma_j^\dagger$$

with $\vec{d}_{eg} \langle g| \vec{d} |e\rangle$. Moreover, now we need to consider now the **spatial dependence** of the electric field, which we did not need for the case of one atom. It reads now:

$$\vec{E}(\vec{r}_j, t) = \sum_{\vec{k}} \hat{\epsilon}_{\vec{k}} C_{\vec{k}} \left(a_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_j} + a_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{r}_j} \right) ,$$

where $\hat{\epsilon}_{\vec{k}}$ is the polarization unit vector and $C_{\vec{k}} = \sqrt{\frac{\hbar\nu_k}{2\epsilon_0 V}}$ is the amplitude. Assuming for simplicity that both the dipole moment and the polarization are real, the interaction Hamiltonian becomes:

$$V = -\hbar \sum_{j=1}^N \sum_{\vec{k}} \left(\sigma_j^\dagger + \sigma_j \right) g_{\vec{k}} \left(a_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_j} + a_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{r}_j} \right) ,$$

where $g_{\vec{k}} = \frac{\vec{d}_{eg} \cdot \hat{\epsilon}_{\vec{k}}}{\hbar} C_{\vec{k}}$ is the coupling constant between the atoms and the field.

The next step is again to go into the interaction picture, such that the Hamiltonian reads: (*)

$$H''(t) = -\hbar \sum_{j=1}^N \sum_{\vec{k}} \left(\sigma_j^\dagger e^{i\omega_0 t} + \sigma_j e^{-i\omega_0 t} \right) g_{\vec{k}} \left(a_{\vec{k}} e^{i(\vec{k}\cdot\vec{r}_j - \nu_k t)} + a_{\vec{k}}^\dagger e^{-i(\vec{k}\cdot\vec{r}_j - \nu_k t)} \right) .$$

Note that here we do not perform the RWA just yet. It will come later, but if we do it now we lose terms we will need later!

4.1.2 Obtaining the master equation

We put H'' back into the master equation (4.1). First, we make a change of variables: $t' \rightarrow t - \tau$.

$$\dot{\rho} = -\frac{1}{\hbar^2} \int_0^\infty d\tau \text{tr}_E \left\{ [H''(t), [H''(t - \tau), \rho(t) \otimes \rho_E(0)]] \right\} .$$

We perform then the trace operation, as this reduces dramatically the amount of terms we have to deal with! (only 1 H'' has 4, the total would be ... 64!). We simplify the problem by assuming we are in the vacuum state (zero temperature) such that

- $\langle a_{\vec{k}} a_{\vec{k}'} \rangle = \langle a_{\vec{k}}^\dagger a_{\vec{k}'}^\dagger \rangle = \langle a_{\vec{k}}^\dagger a_{\vec{k}'} \rangle = 0$
- $\langle a_{\vec{k}} a_{\vec{k}'}^\dagger \rangle = \delta_{\vec{k}\vec{k}'}$

This gets rid of many terms (left with “only’ 16!’) and the $\delta_{\vec{k}\vec{k}'}$ also makes one of the sums over \vec{k} disappear: (*)

$$\dot{\rho} = \int_0^\infty d\tau \sum_{j,l=1}^N \sum_{\vec{k}} \left\{ B_{j\vec{k}}(t) B_{l\vec{k}}^*(t - \tau) [A_l(t - \tau)\rho A_j(t) - A_j(t)A_l(t - \tau)\rho] + \text{h.c.} \right\} ,$$

where

$$B_{j\vec{k}}(t) = g_{\vec{k}} e^{i(\vec{k} \cdot \vec{r}_j - \nu_k t)} \\ A_j(t) = \sigma_j^\dagger e^{i\omega_0 t} + \sigma_j e^{-i\omega_0 t} .$$

Now we simplify further, starting with the terms that contain only atomic operators:

$$A_j(t)A_l(t - \tau)\rho = \left[\underbrace{\sigma_j^\dagger \sigma_l^\dagger e^{i\omega_0(2t-\tau)}}_{\text{RWA}} + \sigma_j^\dagger \sigma_l e^{i\omega_0\tau} + \sigma_j \sigma_l^\dagger e^{-i\omega_0\tau} + \underbrace{\sigma_j \sigma_l e^{-i\omega_0(2t-\tau)}}_{\text{RWA}} \right]$$

... and here we make the RWA, throwing away the terms $\sigma^\dagger \sigma^\dagger$ and $\sigma \sigma$. This in turn means that we are not interested in very “fast” processes, happening on times $\Delta t < 1/\omega_0$ (really tiny!).

Now the other terms that come from the electric field:

$$B_{j\vec{k}}(t)B_{l\vec{k}}^*(t - \tau) = |g_{\vec{k}}|^2 e^{i\vec{k} \cdot \vec{r}_{jl}} e^{-i\nu_k \tau}$$

with $\vec{r}_{jl} = \vec{r}_j - \vec{r}_l$. This leaves the master equation as:

$$\dot{\rho} = \sum_{\vec{k}, j, l} |g_{\vec{k}}|^2 \int_0^\infty d\tau \left(e^{i\vec{k} \cdot \vec{r}_{jl}} \left\{ e^{-i(\nu_k - \omega_0)\tau} [\sigma_l \rho \sigma_j^\dagger - \sigma_j^\dagger \sigma_l \rho] + e^{-i(\nu_k + \omega_0)\tau} [\sigma_l^\dagger \rho \sigma_j - \sigma_j \sigma_l^\dagger \rho] \right\} + \text{h.c.} \right)$$

One can see here that the easiest step to take now is to perform the integral over τ : this is done using the **Heitler** function:

$$\int_0^\infty d\tau e^{-i(\nu_k \pm \omega_0)\tau} = \pi \delta(\nu_k \pm \omega_0) - i P \left(\frac{1}{\nu_k \pm \omega_0} \right)$$

where P represents the principal Cauchy value.

The next step requires us to do the summation over all modes of the field. We have done similar things before: first convert the sum into an integral:

$$\sum_{\vec{k}} \rightarrow \frac{V}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \int_0^\infty dk k^2$$

and use that $|g_{\vec{k}}|^2 = \frac{|\vec{d}_{eg}|^2 \nu_k}{2\hbar\epsilon_0 V} \sin^2 \theta$ to obtain

$$\begin{aligned} \dot{\rho} = & \frac{|\vec{d}_{eg}|^2}{8\pi^2 \hbar \epsilon_0 c^3} \sum_{j, l=1}^N \left(\int_0^\infty \nu_k^3 d\nu_k \int_0^\pi d\theta \sin^3 \theta \int_0^{2\pi} d\varphi e^{i\vec{k} \cdot \vec{r}_{jl}} \times \right. \\ & \left\{ \delta(\nu_k - \omega_0) \left[\sigma_l \rho \sigma_j^\dagger - \frac{1}{2} \{ \sigma_j^\dagger \sigma_l, \rho \} \right] + \underbrace{\delta(\nu_k + \omega_0) \left[\sigma_l^\dagger \rho \sigma_j - \frac{1}{2} \{ \sigma_j \sigma_l^\dagger, \rho \} \right]}_{=0 \text{ since: } \nu_k \geq 0!} + \right. \\ & \left. \left. \frac{i}{2} P \left[\frac{[\sigma_j^\dagger \sigma_l, \rho]}{\nu_k - \omega_0} + \frac{[\sigma_j \sigma_l^\dagger, \rho]}{\nu_k + \omega_0} \right] \right\} \right). \end{aligned}$$

And here is where many atoms enters: the integral over the angles does not simply give

a number ($4\pi/3$) as now we have:

$$\begin{aligned} & \int_0^\pi d\theta \sin^3 \theta \int_0^{2\pi} d\varphi e^{i\vec{k}\cdot\vec{r}_{jl}} \\ &= 4\pi \left[\left[1 - \underbrace{\left(\hat{d} \cdot \hat{r}_{jl} \right)^2}_{\text{angular}} \right] \underbrace{\frac{\sin(kr_{jl})}{(kr_{jl})}}_{\text{dependence}} + \left[1 - 3 \underbrace{\left(\hat{d} \cdot \hat{r}_{jl} \right)^2}_{\text{distance dependence}} \right] \left[\underbrace{\frac{\cos(kr_{jl})}{(kr_{jl})^2}}_{\text{distance dependence}} - \underbrace{\frac{\sin(kr_{jl})}{(kr_{jl})^3}}_{\text{distance dependence}} \right] \right] \end{aligned}$$

After some more mathematical manipulation one obtains the final form of the master equation:

$$\dot{\rho} = -i \underbrace{\sum_{j \neq l} V_{jl} [\sigma_j^\dagger \sigma_l, \rho]}_{\text{"Hamiltonian"}} + \underbrace{\sum_{j,l=1}^N \Gamma_{jl} \left[\sigma_l \rho \sigma_j^\dagger - \frac{1}{2} \left\{ \sigma_j^\dagger \sigma_l, \rho \right\} \right]}_{\text{"Dissipation"}} \quad (4.2)$$

with

$$\begin{aligned} V_{jl} &= -\frac{3}{4}\gamma \left[\left[1 - \left(\hat{d} \cdot \hat{r}_{jl} \right)^2 \right] \frac{\cos(\kappa_{jl})}{\kappa_{jl}} - \left[1 - 3 \left(\hat{d} \cdot \hat{r}_{jl} \right)^2 \right] \left[\frac{\sin(\kappa_{jl})}{\kappa_{jl}^2} + \frac{\cos(\kappa_{jl})}{\kappa_{jl}^3} \right] \right] \\ \Gamma_{jl} &= -\frac{3}{2}\gamma \left[\left[1 - \left(\hat{d} \cdot \hat{r}_{jl} \right)^2 \right] \frac{\sin(\kappa_{jl})}{\kappa_{jl}} + \left[1 - 3 \left(\hat{d} \cdot \hat{r}_{jl} \right)^2 \right] \left[\frac{\cos(\kappa_{jl})}{\kappa_{jl}^2} - \frac{\sin(\kappa_{jl})}{\kappa_{jl}^3} \right] \right] \end{aligned}$$

where $\kappa_{jl} = r_{jl}\omega_0/c = 2\pi r_{jl}/\lambda_0$ is the **reduced distance**, between the j -th and l -th atom (scaled by the wavelength λ of the transition $|g\rangle \rightarrow |e\rangle$).

4.2 Collective behavior

Let's try now to make sense of the master equation (4.2) from a physical point of view. The equation can be rewritten as

$$\dot{\rho} = -\frac{i}{\hbar} [H_{dd}, \rho] + \mathcal{D}(\rho) .$$

Here, the Hamiltonian is

$$H_{dd} = \sum_{j \neq l} \hbar V_{jl} \sigma_j^\dagger \sigma_l$$

and we refer to this as the **coherent dipole-dipole** interaction, which conserves the number of excitations. The second term reads

$$\mathcal{D}(\rho) = \sum_{j,l=1}^N \Gamma_{jl} \left(\sigma_l \rho \sigma_k^\dagger - \frac{1}{2} \{ \sigma_j^\dagger \sigma_l, \rho \} \right),$$

and is responsible for the **collective dissipation** in the system.

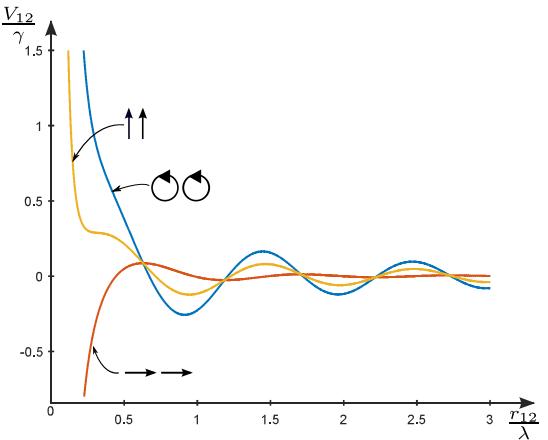
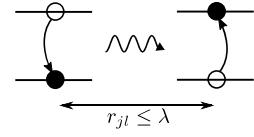
4.2.1 Coherent dipole-dipole interactions

Lets begin by analyzing the coherent dipole-dipole interactions and the rate or strength of the interactions, given by V_{jl} . The de-excitation of one atom creates a **virtual** photon in the electromagnetic field, which can in turn be absorbed by a neighboring atoms. This process is labelled dipole-dipole interaction, or “flip-flop” interactions.

The likelihood of this process occurring is given by V_{jl} , and depends on two key parameters:

- a) The angle formed by the atomic dipole moments (aligned) and the atomic separation vector \vec{r}_{jl} .
- b) The (reduced!) distance between the atoms.

Independently of the dipole orientation, one can see separations r_{jl}/λ , decaying approximately as $1/\kappa_{jl}^3$ and diverging for $|r_{jl}| \rightarrow 0$. Note, that while those interactions are much weaker than the ones between Rydberg atoms, they are still quite strong, although they only become noticeable for distances smaller than the wavelength λ of the atomic transition, typically a few hundred nanometers. As κ_{jl} grows, the interaction strength oscillates and decay close to zero. This in turn means that no interactions will take place.



4.2.2 Collective dissipation

Next we consider the dissipator containing the coefficients Γ_{jl} . The form of the dissipator is similar to the one we have seen for a single atom, but the indices j and l make it difficult to identify a **rate** and **jump operator**. In order to recreate that “diagonal” case, let us diagonalize the matrix Γ_{jl} , as

$$\Gamma_{jl} = \sum_{m=1}^N M_{jm} \Gamma_D^m M_{ml}^\dagger ,$$

where Γ_D is a diagonal matrix that will contain the **collective decay rates**. Using this expression we obtain:

$$\begin{aligned} \mathcal{D}(\rho) &= \sum_{jl} \Gamma_{jl} \left(\sigma_l \rho \sigma_j^\dagger - \frac{1}{2} \{ \sigma_j^\dagger \sigma_l, \rho \} \right) \\ &= \sum_{m=1}^N \Gamma_D^m \left(J_m \rho J_m^\dagger - \frac{1}{2} \{ J_m^\dagger J_m, \rho \} \right) , \end{aligned}$$

where we have defined the **collective jump operators**,

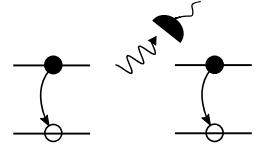
$$J_m = \sum_{l=1}^N \sigma_l M_{lm} .$$

The good thing of this form of $\mathcal{D}(\rho)$ is that it is literally a sum of N terms, each one of the shape

$$\Gamma_D^m \left(J_m \rho J_m^\dagger - \frac{1}{2} \{ J_m^\dagger J_m, \rho \} \right)$$

Where one can now really identify Γ_D^m as a rate associated with the process determined by J_m . Therese processes are the ones that are associated with the emission of a photon from the system: every photon emission is the result of one of these **collective jumps** occurring. The difference with a system where no interactions are present is that there each emission has a decay rate γ , and is associated with a photon emitted from one atom. When all atoms are interacting, as it is the case here, though, the rates are either larger or smaller than γ :

$$\begin{aligned} \Gamma_D^m < \gamma &\longrightarrow \text{superradiant} \\ \Gamma_D^m > \gamma &\longrightarrow \text{subradiant} \end{aligned}$$



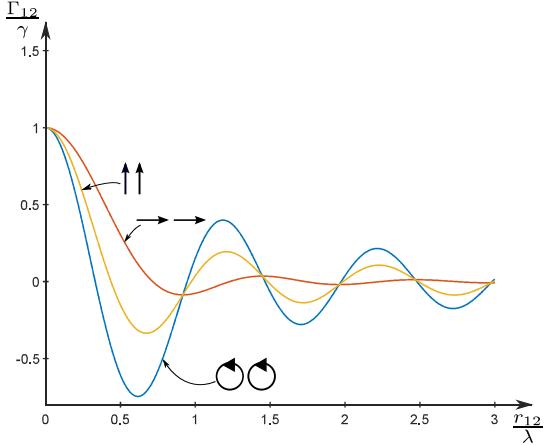
Moreover, the jump operators corresponding to these collective rates are also collective: i.e. the emission of a photon is coming from a **superposition** of atoms in the excited state.

In order to understand this a bit better, let us discuss the case of 2 atoms. Here, the dissipation matrix reads:

$$\begin{pmatrix} \gamma & \Gamma_{12} \\ \Gamma_{12} & \gamma \end{pmatrix}$$

The decay rates here are simply calculated as:

$$\Gamma_D^1 = \gamma + |\Gamma_{12}| \quad ; \quad \Gamma_D^2 = \gamma - |\Gamma_{12}|$$



Now, as we can see in the figure, when the atoms are very far away $\Gamma_{12} \rightarrow 0$, independently of the angle between the dipoles and \hat{r}_{12} . This means that indeed here we recover the “simple” situation where

$$\Gamma_D^1 = \gamma \quad ; \quad \Gamma_D^2 = \gamma$$

which go accompanying jump operators:

$$J_1 = \sigma_1 \quad ; \quad J_2 = \sigma_2 .$$

Now, if the atoms are very close together, $r_{12} \ll \lambda$, then $\Gamma_{12} \propto \gamma$, and the collective rates and jumps become in this limit

- $\Gamma_D^1 \approx 2\gamma \rightarrow J_1 = \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_2)$ **super**
- $\Gamma_D^1 \approx 0 \rightarrow J_2 = \frac{1}{\sqrt{2}} (\sigma_1 - \sigma_2)$ **sub**

This means that the photons coming from this system can come at a rate 2γ , and also that one can excite in the system a state:

$$|\Psi_D\rangle = \frac{1}{\sqrt{2}} (|eg\rangle - |ge\rangle)$$

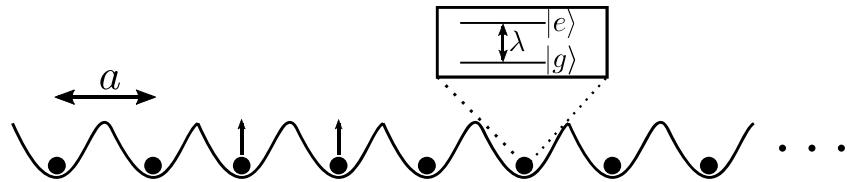
which will not emit photons into the environment as

$$J_1 |\Psi_D\rangle = 0$$

$$J_2 |\Psi_D\rangle = 0$$

This is a special subradiant state, it is completely dark. Usually, the situation is such that one has some states that are superradiant and some subradiant but not always completely dark.

Let us try to understand what happens when more than 2 atoms are in the ensemble. In particular, let's make our life even easier and consider atoms trapped in a one-dimensional periodic chain such as



and vary the nearest-neighbor distance a . We look then at the collective decay rates (slides) for $N = 10$ atoms. One can see there that indeed when the atoms are well separated ($a \gg 1$), one recovers 10 decay rates very close to γ . However, when $a < \gamma$ many collective decay rates become much smaller than γ (subradiant states), and a few superradiant states arise.

Literature for Chapter 4:

- R.H. Lehmberg, Phys. Rev. A 2, 883 (1970)
- D.F.V. James, Rev. Mod. Phys. 47, 1336 (1993)
- M.O. Scully, Phys. Rev. Lett. 115, 243602 (2015)
- M.O. Araújo, I. Kresić, R. Kaiser, and W. Guerin Phys. Rev. Lett. 117, 073002 (2016)