THE 1ST INTERNATIONAL SUMMER SCHOOL ON ADVANCED SOIL PHYSICS

MODELING WATER FLUXES IN THE SOIL-PLANT SYSTEM

MODELLING WATER FLUXES IN ROOT SYSTEMS - Theory

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International Soil Modeling Consortium

ENVITAM

Plan

1) Finite difference solution to the root water flow equations

Doussan et al., 1998

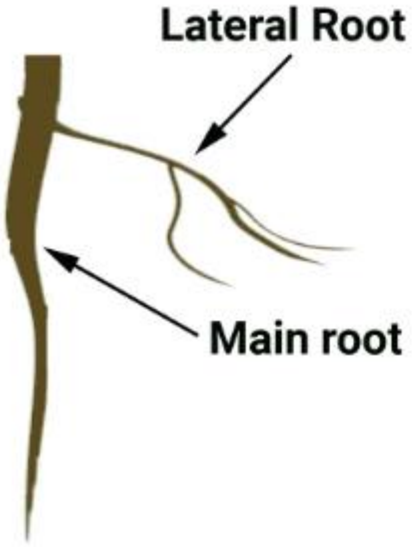
1) Exact solutions to the root water flow equations

Meunier et al., 2018 / Landsberg and Fowkes, 1978

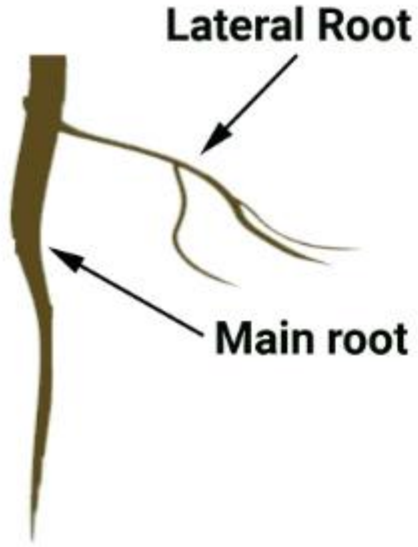
1) A macroscopic approximation of the root water flow equations

Couvreur et al., 2012

Finite difference solution



Finite difference solution



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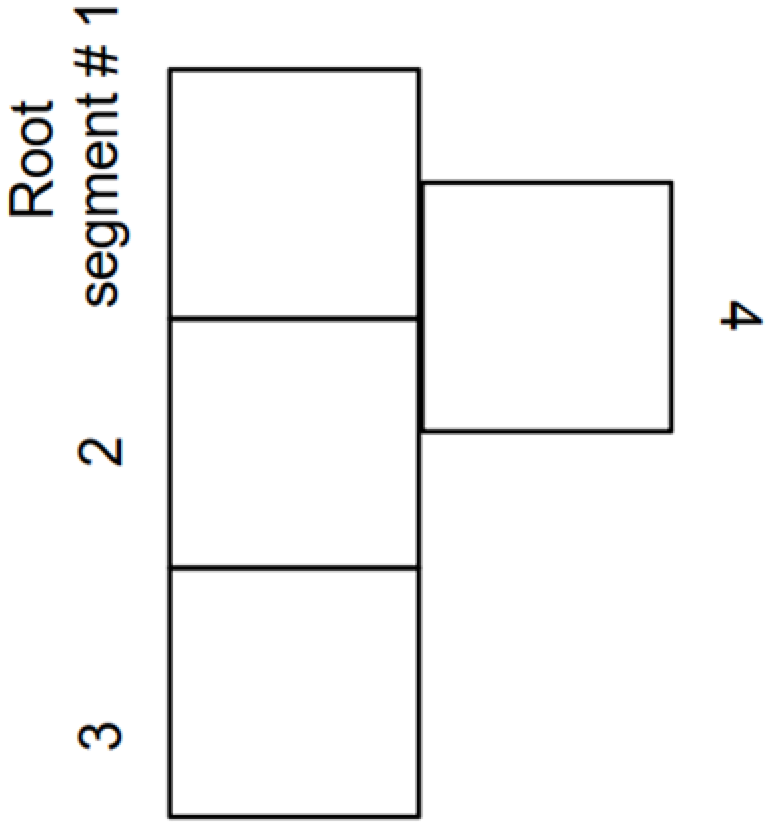
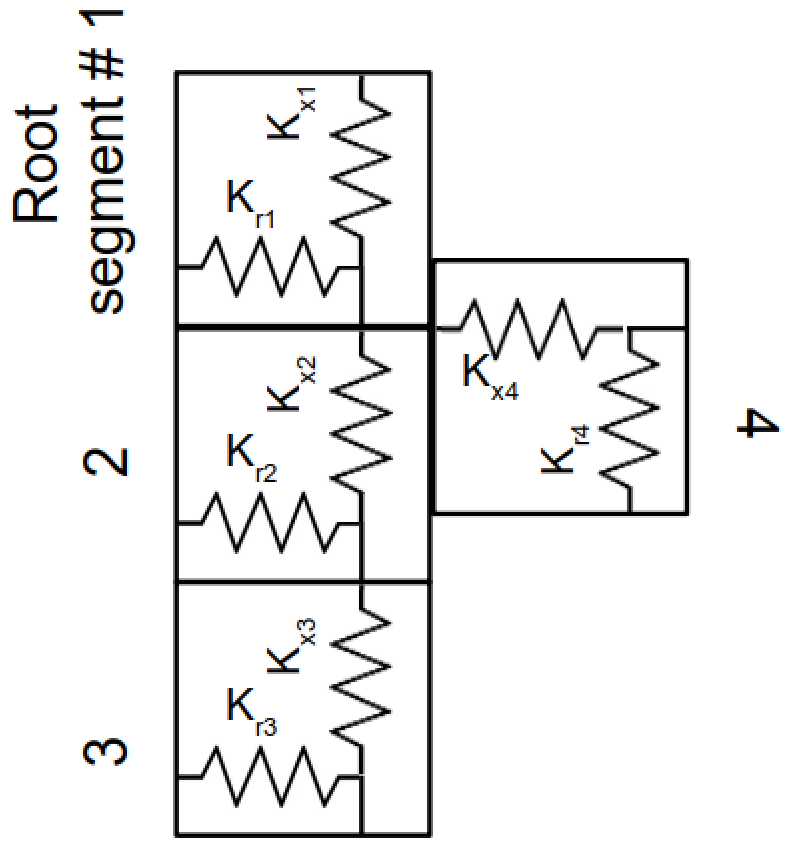


Finite difference solution

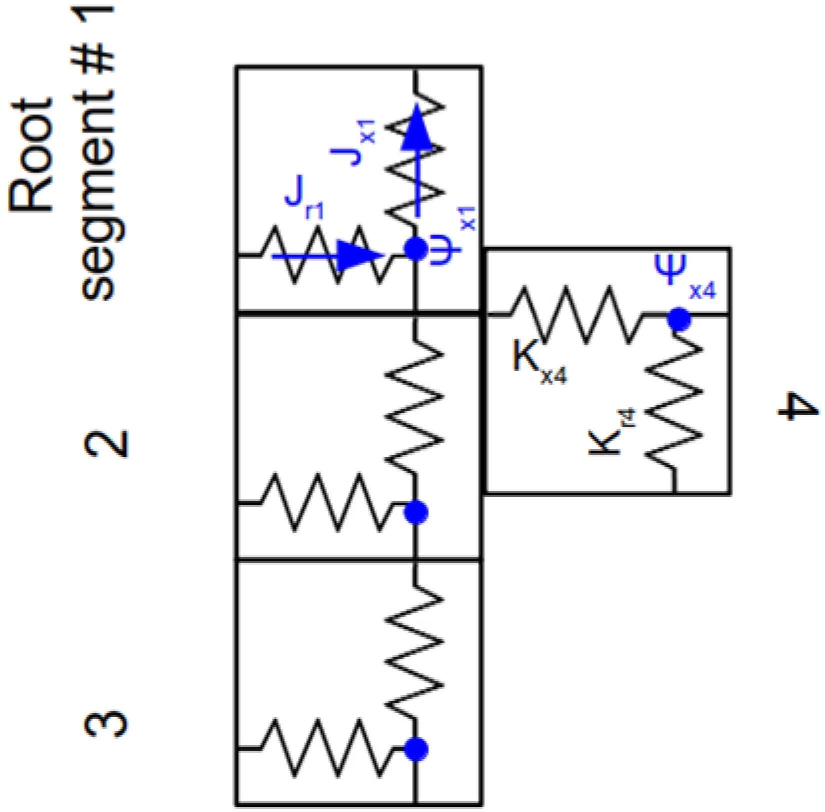


where

- K_{ri} [L^3 P^{-1} T^{-1}] is the segment radial conductance
- K_{xi} [L^3 P^{-1} T^{-1}] is the segment axial conductance

= Model parameters

Finite difference solution

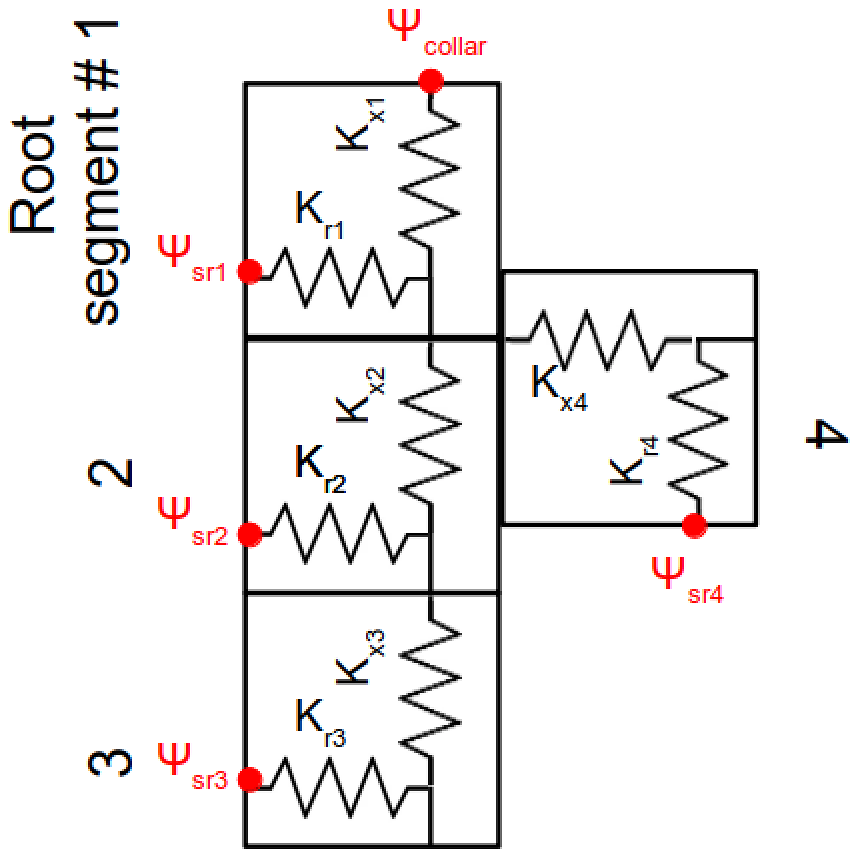


where

- J_{ri} [L^3 T^{-1}] is the segment radial flow
- J_{xi} [L^3 T^{-1}] is the segment axial flow
- Ψ_{xi} [P] is the segment water potential

= Model unknowns (12)

Finite difference solution

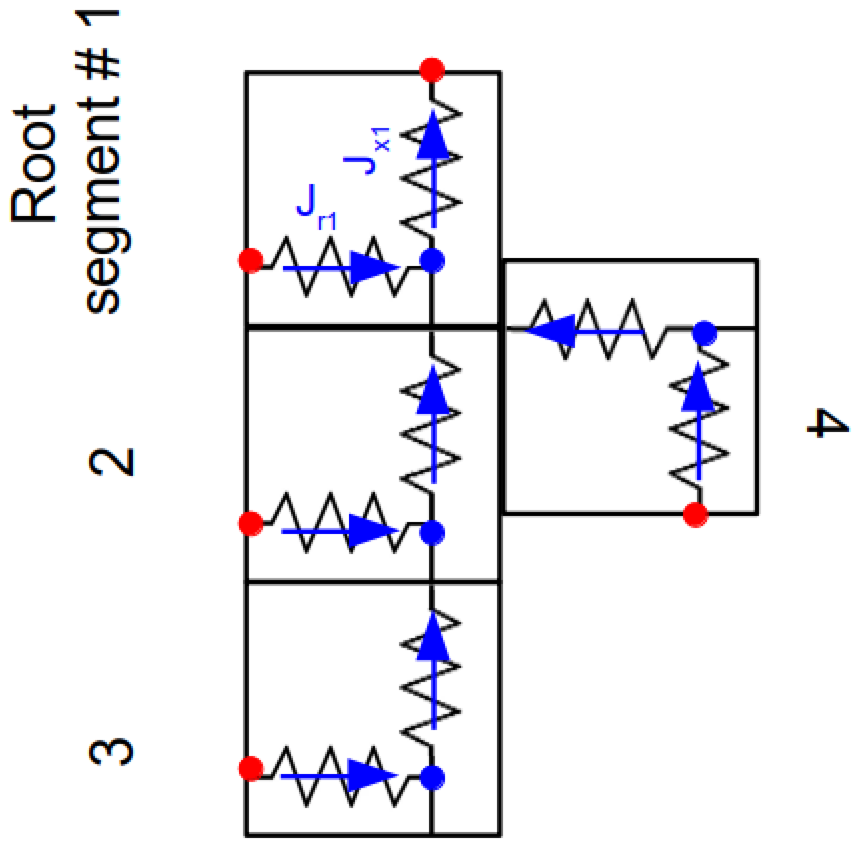


where

- Ψ_{collar} [P] is the root system collar potential
- Ψ_{sri} [P] is the soil-root interface water potential

= Model boundary conditions

Finite difference solution



To solve the water flow equations, we use:

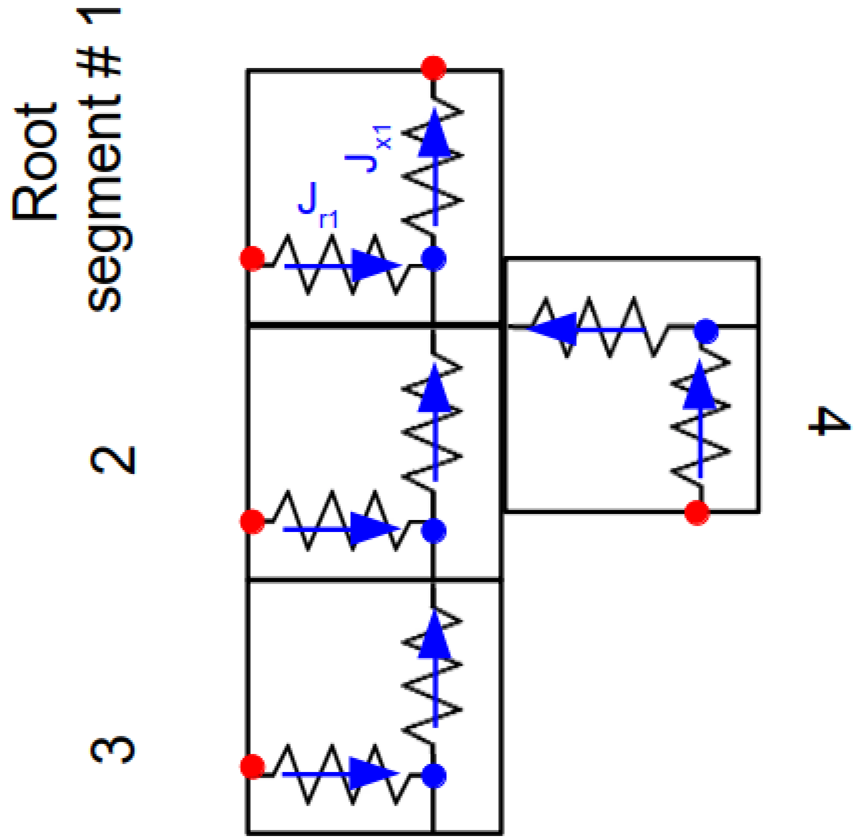
- Kirchhoff's law

$$\sum J_{in} = \sum J_{out}$$

- Ohm's law

$$J = K \Delta \Psi$$

Finite difference solution



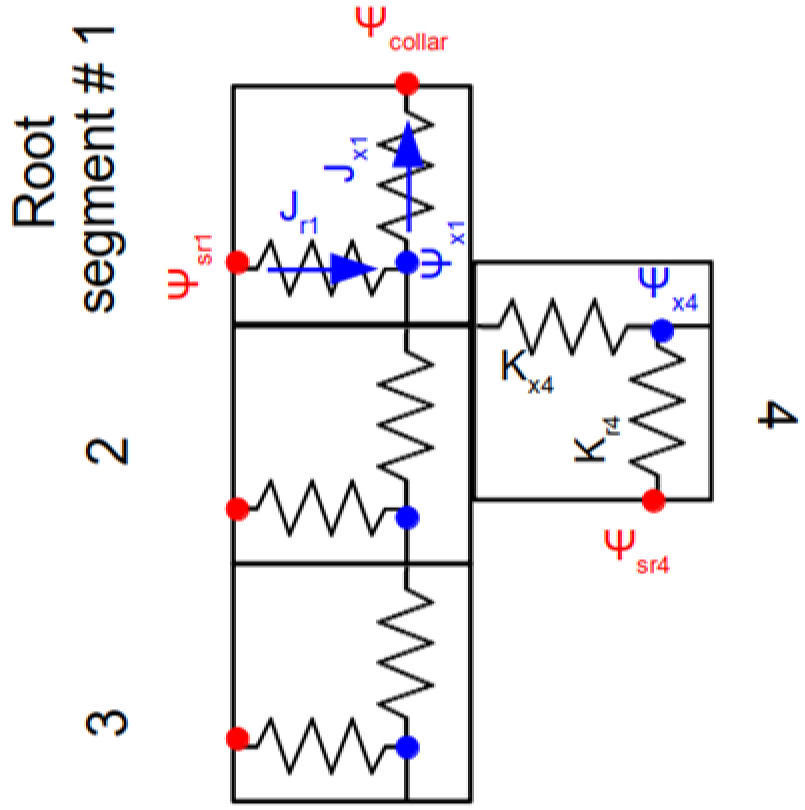
Kirchhoff's law

$$\sum J_{in} = \sum J_{out}$$

1

$$\begin{cases} J_{x1} = J_{r1} + J_{x2} + J_{x4} \\ J_{x2} = J_{r2} + J_{x3} \\ J_{x3} = J_{r3} \\ J_{x4} = J_{r4} \end{cases}$$

Finite difference solution



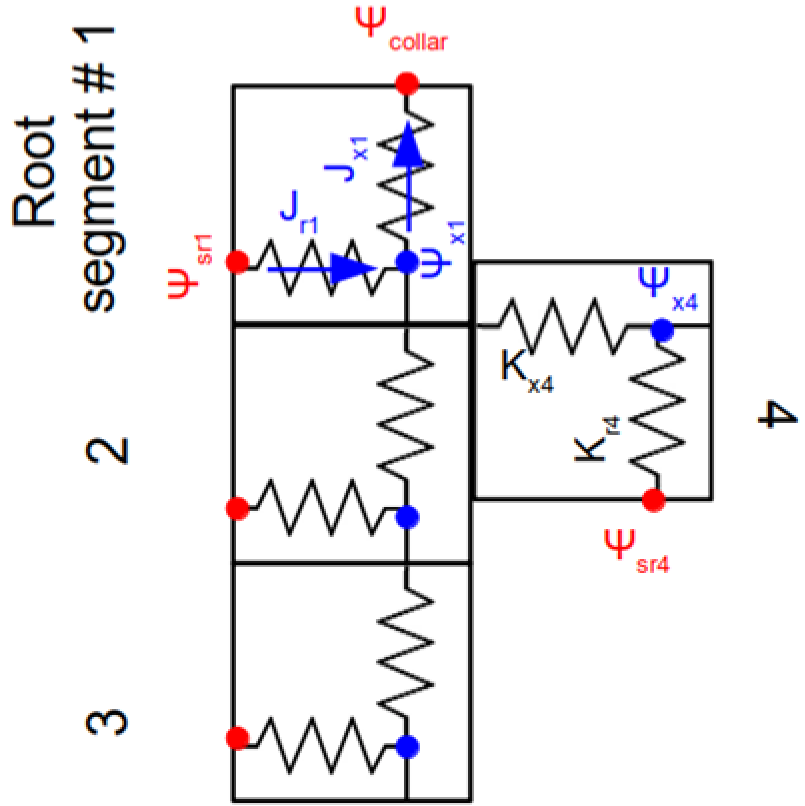
Ohm's law (1)

$$J = K \Delta \Psi$$

2

$$\begin{cases} J_{r1} = K_{r1} \left(\Psi_{sr1} - \Psi_{x1} \right) \\ J_{r2} = K_{r2} \left(\Psi_{sr2} - \Psi_{x2} \right) \\ J_{r3} = K_{r3} \left(\Psi_{sr3} - \Psi_{x3} \right) \\ J_{r4} = K_{r4} \left(\Psi_{sr4} - \Psi_{x4} \right) \end{cases}$$

Finite difference solution



Ohm's law (2)

$$J = K \Delta \Psi$$

3

$$\begin{cases} J_{x1} = K_{x1}\left(\Psi_{x1} - \Psi_{collar}\right) \\ J_{x2} = K_{x2}\left(\Psi_{x2} - \Psi_{x1}\right) \\ J_{x3} = K_{x3}\left(\Psi_{x3} - \Psi_{x2}\right) \\ J_{x4} = K_{x4}\left(\Psi_{x4} - \Psi_{x1}\right) \end{cases}$$

Finite difference solution

$$\begin{pmatrix} 1 & & & & -\frac{1}{K_{x1}} & & & \\ -1 & 1 & & & & -\frac{1}{K_{x2}} & & \\ & -1 & 1 & & & & -\frac{1}{K_{x3}} & \\ -1 & & & 1 & & & & -\frac{1}{K_{x4}} \\ K_{r1} & & & & 1 & -1 & & -1 \\ & K_{r2} & & & & 1 & -1 & \\ & & K_{r3} & & & & 1 & \\ & & & K_{r4} & & & & 1 \end{pmatrix} * \begin{pmatrix} \Psi_{x1} \\ \Psi_{x2} \\ \Psi_{x3} \\ \Psi_{x4} \\ J_{x1} \\ J_{x2} \\ J_{x3} \\ J_{x4} \end{pmatrix} = \begin{pmatrix} \Psi_{collar} \\ 0 \\ 0 \\ 0 \\ \Psi_{sr1} K_{r1} \\ \Psi_{sr2} K_{r2} \\ \Psi_{sr3} K_{r3} \\ \Psi_{sr4} K_{r4} \end{pmatrix}$$

Finite difference solution

$$
\begin{pmatrix}
1 & & & & -\frac{1}{K_{x1}} & & & \\
-1 & 1 & & & & -\frac{1}{K_{x2}} & & \\
 & -1 & 1 & & & & -\frac{1}{K_{x3}} & \\
-1 & & & 1 & & & & -\frac{1}{K_{x4}} \\
K_{r1} & & & & 1 & -1 & & -1 \\
 & K_{r2} & & & & 1 & -1 & \\
 & & K_{r3} & & & & 1 & \\
 & & & K_{r4} & & & & 1
\end{pmatrix}
*
\begin{pmatrix}
\Psi_{x1} \\ \Psi_{x2} \\ \Psi_{x3} \\ \Psi_{x4} \\ J_{x1} \\ J_{x2} \\ J_{x3} \\ J_{x4}
\end{pmatrix}
=
\begin{pmatrix}
\Psi_{collar} \\ 0 \\ 0 \\ 0 \\ \Psi_{sr1} K_{r1} \\ \Psi_{sr2} K_{r2} \\ \Psi_{sr3} K_{r3} \\ \Psi_{sr4} K_{r4}
\end{pmatrix}
$$

Unknowns

Finite difference solution

$$
\begin{pmatrix}
1 & & & & -\frac{1}{K_{x1}} & & & \\
-1 & 1 & & & & -\frac{1}{K_{x2}} & & \\
 & -1 & 1 & & & & -\frac{1}{K_{x3}} & \\
-1 & & & 1 & & & & -\frac{1}{K_{x4}} \\
K_{r1} & & & & 1 & -1 & & -1 \\
 & K_{r2} & & & & 1 & -1 & \\
 & & K_{r3} & & & & 1 & \\
 & & & K_{r4} & & & & 1
\end{pmatrix}
*
\begin{pmatrix}
\Psi_{x1} \\ \Psi_{x2} \\ \Psi_{x3} \\ \Psi_{x4} \\ J_{x1} \\ J_{x2} \\ J_{x3} \\ J_{x4}
\end{pmatrix}
=
\begin{pmatrix}
\boxed{\Psi_{collar}} \\ 0 \\ 0 \\ 0 \\ \Psi_{sr1} K_{r1} \\ \Psi_{sr2} K_{r2} \\ \Psi_{sr3} K_{r3} \\ \Psi_{sr4} K_{r4}
\end{pmatrix}
$$

Boundary conditions

Finite difference solution

$$\begin{pmatrix} 1 & & & & -\frac{1}{K_{x1}} & & & \\ -1 & 1 & & & & -\frac{1}{K_{x2}} & & \\ & -1 & 1 & & & & -\frac{1}{K_{x3}} & \\ -1 & & & 1 & & & & -\frac{1}{K_{x4}} \\ K_{r1} & & & & 1 & -1 & & -1 \\ & K_{r2} & & & & 1 & -1 & \\ & & K_{r3} & & & & 1 & \\ & & & K_{r4} & & & & 1 \end{pmatrix} * \begin{pmatrix} \Psi_{x1} \\ \Psi_{x2} \\ \Psi_{x3} \\ \Psi_{x4} \\ J_{x1} \\ J_{x2} \\ J_{x3} \\ J_{x4} \end{pmatrix} = \begin{pmatrix} \Psi_{collar} \\ 0 \\ 0 \\ 0 \\ \Psi_{sr1} K_{r1} \\ \Psi_{sr2} K_{r2} \\ \Psi_{sr3} K_{r3} \\ \Psi_{sr4} K_{r4} \end{pmatrix}$$

Model parameters (hydraulics)

Finite difference solution

$$
\left(\begin{array}{cccc|cccc}
1 & & & & -\frac{1}{K_{x1}} & & & \\
-1 & 1 & & & & -\frac{1}{K_{x2}} & & \\
 & -1 & 1 & & & & -\frac{1}{K_{x3}} & \\
-1 & & & 1 & & & & -\frac{1}{K_{x4}} \\
\hline
K_{r1} & & & & 1 & -1 & & -1 \\
 & K_{r2} & & & & 1 & -1 & \\
 & & K_{r3} & & & & 1 & \\
 & & & K_{r4} & & & & 1
\end{array}\right)
*
\begin{pmatrix} \Psi_{x1} \\ \Psi_{x2} \\ \Psi_{x3} \\ \Psi_{x4} \\ J_{x1} \\ J_{x2} \\ J_{x3} \\ J_{x4} \end{pmatrix}
=
\begin{pmatrix} \Psi_{collar} \\ 0 \\ 0 \\ 0 \\ \Psi_{sr1} K_{r1} \\ \Psi_{sr2} K_{r2} \\ \Psi_{sr3} K_{r3} \\ \Psi_{sr4} K_{r4} \end{pmatrix}
$$

Model parameters (architecture)

Finite difference solution

$$
\left(\begin{array}{cccc|cccc}
1 & & & & -\frac{1}{K_{x1}} & & & \\
-1 & 1 & & & & -\frac{1}{K_{x2}} & & \\
 & -1 & 1 & & & & -\frac{1}{K_{x3}} & \\
-1 & & & 1 & & & & -\frac{1}{K_{x4}} \\
\hline
K_{r1} & & & & 1 & -1 & & -1 \\
 & K_{r2} & & & & 1 & -1 & \\
 & & K_{r3} & & & & 1 & \\
 & & & K_{r4} & & & & 1
\end{array}\right)
*
\begin{pmatrix}
\Psi_{x1} \\ \Psi_{x2} \\ \Psi_{x3} \\ \Psi_{x4} \\ J_{x1} \\ J_{x2} \\ J_{x3} \\ J_{x4}
\end{pmatrix}
=
\begin{pmatrix}
\Psi_{collar} \\ 0 \\ 0 \\ 0 \\ \Psi_{sr1} K_{r1} \\ \Psi_{sr2} K_{r2} \\ \Psi_{sr3} K_{r3} \\ \Psi_{sr4} K_{r4}
\end{pmatrix}
$$

Model parameters

Unknowns

Boundary conditions

Finite difference solution

$$
\underbrace{\begin{pmatrix}
1 & & & & -\frac{1}{K_{x1}} & & & \\
-1 & 1 & & & & -\frac{1}{K_{x2}} & & \\
 & -1 & 1 & & & & -\frac{1}{K_{x3}} & \\
-1 & & & 1 & & & & -\frac{1}{K_{x4}} \\
K_{r1} & & & & 1 & -1 & & -1 \\
 & K_{r2} & & & & 1 & -1 & \\
 & & K_{r3} & & & & 1 & \\
 & & & K_{r4} & & & & 1
\end{pmatrix}}_{C}
*
\underbrace{\begin{pmatrix}
\Psi_{x1} \\ \Psi_{x2} \\ \Psi_{x3} \\ \Psi_{x4} \\ J_{x1} \\ J_{x2} \\ J_{x3} \\ J_{x4}
\end{pmatrix}}_{x}
=
\underbrace{\begin{pmatrix}
\Psi_{collar} \\ 0 \\ 0 \\ 0 \\ \Psi_{sr1}K_{r1} \\ \Psi_{sr2}K_{r2} \\ \Psi_{sr3}K_{r3} \\ \Psi_{sr4}K_{r4}
\end{pmatrix}}_{D}
$$

Finite difference solution

$$
\underbrace{\begin{pmatrix}
1 & & & & -\frac{1}{K_{x1}} & & & \\
-1 & 1 & & & & -\frac{1}{K_{x2}} & & \\
 & -1 & 1 & & & & -\frac{1}{K_{x3}} & \\
-1 & & & 1 & & & & -\frac{1}{K_{x4}} \\
K_{r1} & & & & 1 & -1 & & -1 \\
 & K_{r2} & & & & 1 & -1 & \\
 & & K_{r3} & & & & 1 & \\
 & & & K_{r4} & & & & 1
\end{pmatrix}}_{C}
*
\underbrace{\begin{pmatrix}
\Psi_{x1} \\ \Psi_{x2} \\ \Psi_{x3} \\ \Psi_{x4} \\ J_{x1} \\ J_{x2} \\ J_{x3} \\ J_{x4}
\end{pmatrix}}_{x}
=
\underbrace{\begin{pmatrix}
\Psi_{collar} \\ 0 \\ 0 \\ 0 \\ \Psi_{sr1} K_{r1} \\ \Psi_{sr2} K_{r2} \\ \Psi_{sr3} K_{r3} \\ \Psi_{sr4} K_{r4}
\end{pmatrix}}_{D}
$$

$\rightarrow \quad x = C \backslash D$

Example 1 (Matlab)

Finite difference solution

$$\begin{pmatrix} 1 & & & & -\frac{1}{K_{x1}} & & & \\ -1 & 1 & & & & -\frac{1}{K_{x2}} & & \\ & -1 & 1 & & & & -\frac{1}{K_{x3}} & \\ -1 & & & 1 & & & & -\frac{1}{K_{x4}} \\ K_{r1} & & & & 1 & -1 & & -1 \\ & K_{r2} & & & & 1 & -1 & \\ & & K_{r3} & & & & 1 & \\ & & & K_{r4} & & & & 1 \end{pmatrix} * \begin{pmatrix} \Psi_{x1} \\ \Psi_{x2} \\ \Psi_{x3} \\ \Psi_{x4} \\ J_{x1} \\ J_{x2} \\ J_{x3} \\ J_{x4} \end{pmatrix} = \begin{pmatrix} \Psi_{collar} \\ 0 \\ 0 \\ 0 \\ \Psi_{sr1} K_{r1} \\ \Psi_{sr2} K_{r2} \\ \Psi_{sr3} K_{r3} \\ \Psi_{sr4} K_{r4} \end{pmatrix}$$

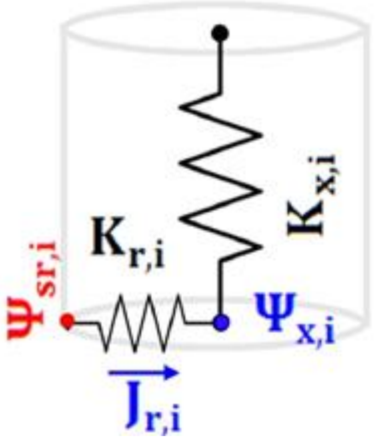
Remarks:

- The actual solution is (N x N), not (2N x 2N)
- Using sparse matrix improves efficiency
- Another solution exists for top boundary condition of flux-type

Examples 2 and 3 (Matlab)

Exact solution

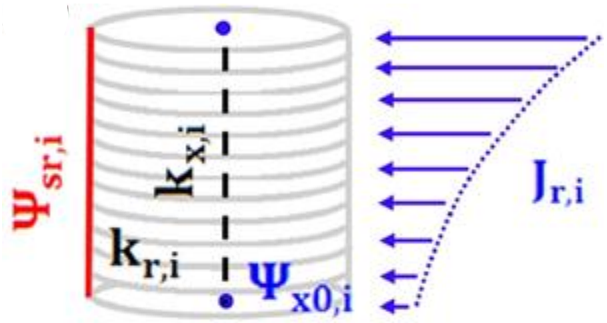
Finite differences



where

- K_{ri} [L^3 P^{-1} T^{-1}] is the segment radial conductance
- K_{xi} [L^3 P^{-1} T^{-1}] is the segment axial conductance

Exact solution

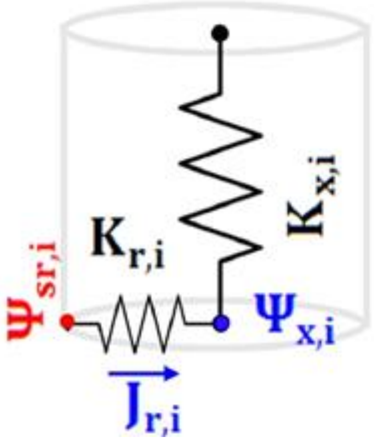


where

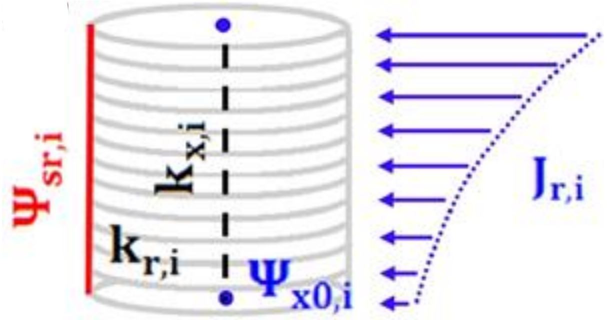
- k_{ri} [L P^{-1} T^{-1}] is the segment radial conductivit
- k_{xi} [L^4 P^{-1} T^{-1}] is the segment axial conductivit
- r [L] is the root radius
- l [L] is the segment length

Exact solution

Finite differences



Exact solution



Equations

$$\sum J_{in} = \sum J_{out}$$

Conservation

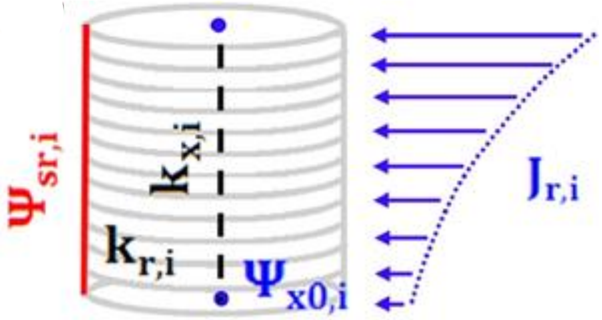
$$q_r = -2\pi r k_r \left(\Psi_x - \Psi_{sr}\right)$$

$$J = K\Delta\Psi$$

Flow

$$J_x(z) = -k_x \frac{d\Psi_x}{dz}$$

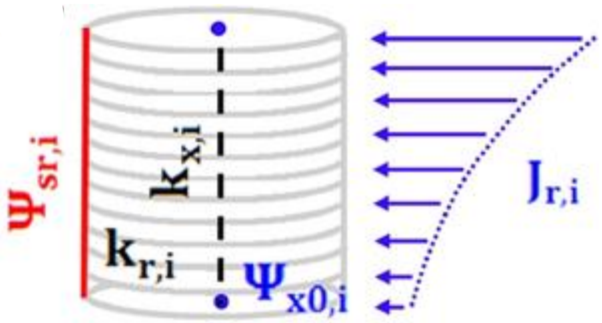
Exact solution



Equations

$$\begin{cases} q_r = -2\pi r k_r \left(\Psi_x - \Psi_{sr}\right) \\ J_x(z) = -k_x \dfrac{d\Psi_x}{dz} \end{cases}$$

Exact solution

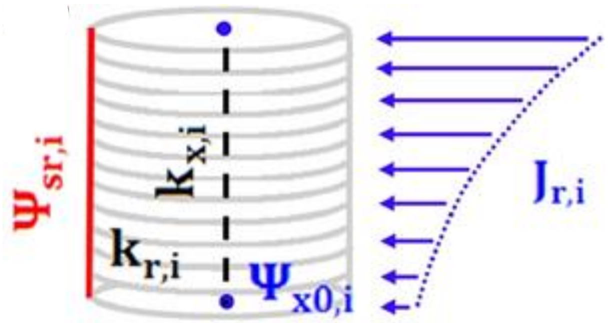


Equations

$$\begin{cases} q_r = -2\pi r k_r \left(\Psi_x - \Psi_{sr}\right) \\ J_x(z) = -k_x \dfrac{d\Psi_x}{dz} \end{cases}$$

→ $$k_x \frac{d^2\Psi_x}{dz^2} = 2\pi r k_r \left(\Psi_x - \Psi_{sr}\right)$$

Exact solution



Equations

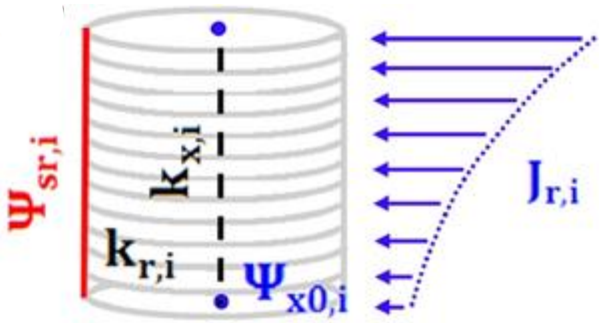
$$\begin{cases} q_r = -2\pi r k_r \left(\Psi_x - \Psi_{sr}\right) \\ J_x(z) = -k_x \dfrac{d\Psi_x}{dz} \end{cases}$$

$$\rightarrow \quad k_x \frac{d^2\Psi_x}{dz^2} = 2\pi r k_r \left(\Psi_x - \Psi_{sr}\right)$$

Generic solution

$$\Psi_x(z) = A + c_1 f_1 + c2 f_2$$

Exact solution



Equations

$$\begin{cases} q_r = -2\pi r k_r \left(\Psi_x - \Psi_{sr}\right) \\ J_x(z) = -k_x \dfrac{d\Psi_x}{dz} \end{cases}$$

$\rightarrow$ $$k_x \frac{d^2\Psi_x}{dz^2} = 2\pi r k_r \left(\Psi_x - \Psi_{sr}\right)$$

Generic solution

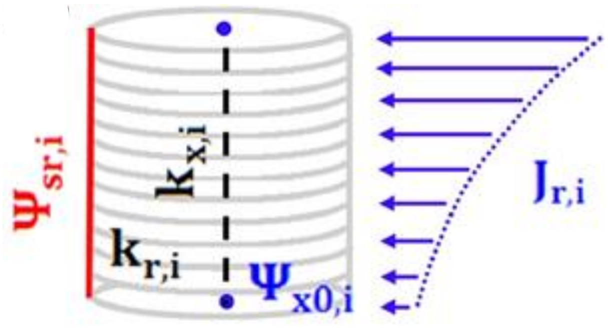
$$\Psi_x(z) = A + c_1 f_1 + c2 f_2$$

General solution (k_r and k_x constant along root segment)

$$f_1 = cosh(\tau z)$$
$$f_2 = sinh(\tau z)$$
$$A = \Psi_{sr}$$

with $\tau = \sqrt{\dfrac{2\pi r k_r}{k_x}}$

Exact solution



Equations

$$\begin{cases} q_r = -2\pi r k_r \left(\Psi_x - \Psi_{sr}\right) \\ J_x(z) = -k_x \dfrac{d\Psi_x}{dz} \end{cases}$$

$$\rightarrow \quad k_x \frac{d^2\Psi_x}{dz^2} = 2\pi r k_r \left(\Psi_x - \Psi_{sr}\right)$$

Generic solution

$$\Psi_x(z) = A + c_1 f_1 + c2 f_2$$

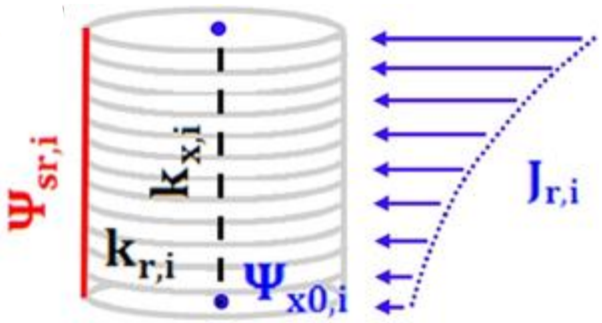
General solution (k_r and k_x constant along root segment)

$$\Psi_x(z) = \Psi_{sr} + c_1 cosh(\tau z) + c_2 sinh(\tau z)$$

Boundary conditions (I)

$$J_x(z=0) = 0 \qquad \Psi_x(z=l) = \Psi_{collar}$$

Exact solution



Equations

$$\begin{cases} q_r = -2\pi r k_r\left(\Psi_x - \Psi_{sr}\right) \\ J_x(z) = -k_x \dfrac{d\Psi_x}{dz} \end{cases}$$

→ $$k_x \frac{d^2\Psi_x}{dz^2} = 2\pi r k_r\left(\Psi_x - \Psi_{sr}\right)$$

Generic solution

$$\Psi_x(z) = A + c_1 f_1 + c2 f_2$$

General solution (k_r and k_x constant along root segment)

$$\Psi_x(z) = \Psi_{sr} + c_1 cosh(\tau z) + c_2 sinh(\tau z)$$

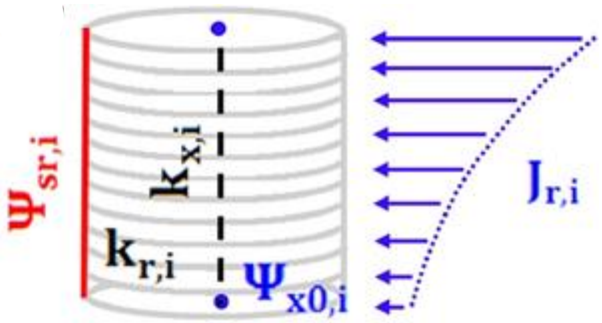
Boundary conditions (I)

$$J_x(z=0) = 0 \qquad \Psi_x(z=l) = \Psi_{collar}$$

Solution (I)

→ $$\Psi_x(z) = (\Psi_{collar} - \Psi_{sr})\frac{cosh(\tau z)}{cosh(\tau l)}$$

Exact solution



Equations

$$\begin{cases} q_r = -2\pi r k_r \left(\Psi_x - \Psi_{sr}\right) \\ J_x(z) = -k_x \dfrac{d\Psi_x}{dz} \end{cases}$$

$$\rightarrow \quad k_x \frac{d^2\Psi_x}{dz^2} = 2\pi r k_r \left(\Psi_x - \Psi_{sr}\right)$$

Generic solution

$$\Psi_x(z) = A + c_1 f_1 + c2 f_2$$

General solution (k_r and k_x constant along root segment)

$$\Psi_x(z) = \Psi_{sr} + c_1 cosh(\tau z) + c_2 sinh(\tau z)$$

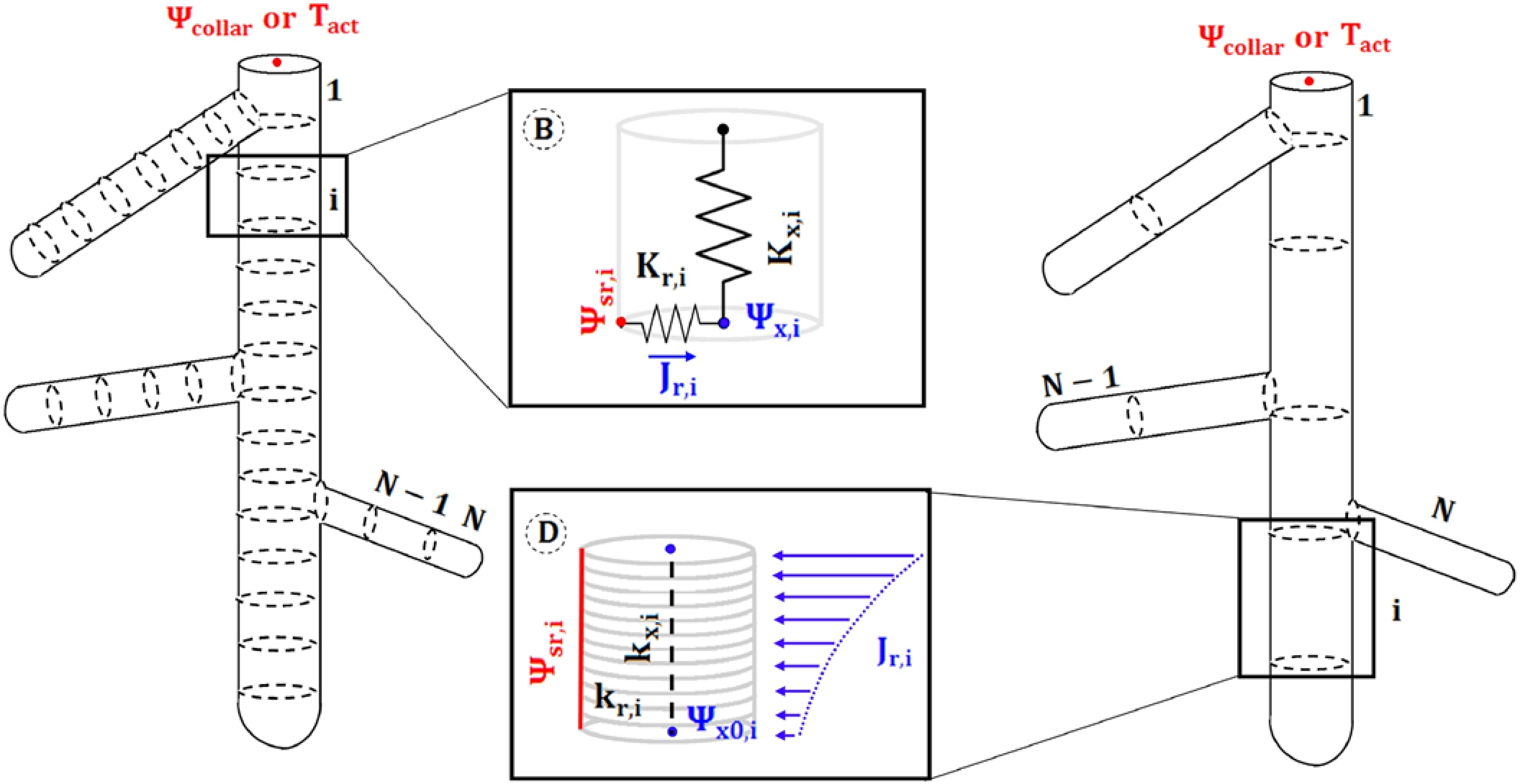
Boundary conditions (II)

$$J_x(z=0) = J_0 \qquad \Psi_x(z=l) = \Psi_{collar}$$

Solution (II)

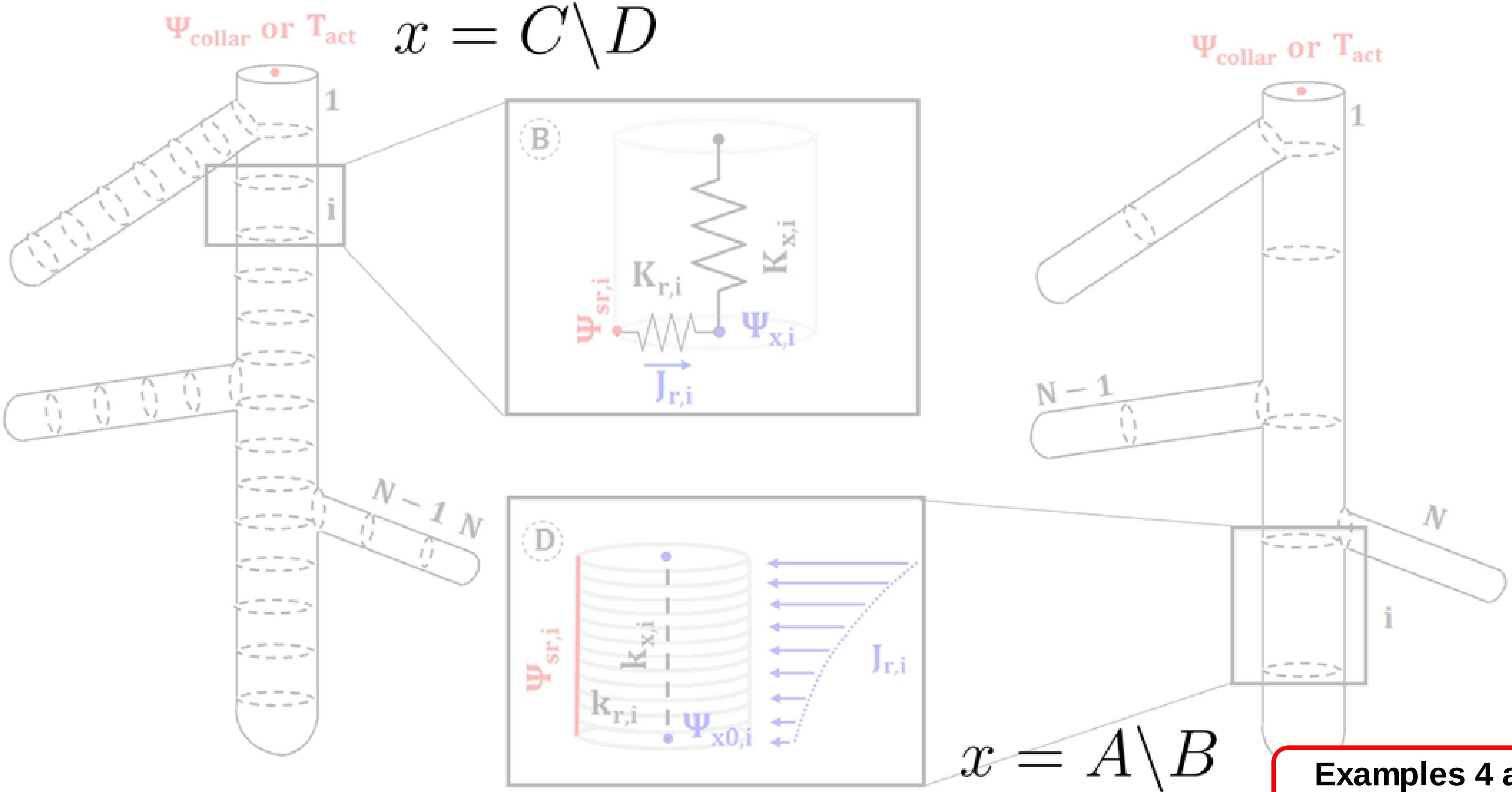
$$\rightarrow \quad \Psi_x(z) = \Psi_{sr} + \frac{cosh(\tau z)}{cosh(\tau l)}\left(\Psi_{collar} - \Psi_{sr} + \frac{J_0}{\kappa} sinh(\tau l)\right) - \frac{J_0}{\kappa} sinh(\tau z)$$

Exact solution



Exact solution

$$x = C \backslash D$$

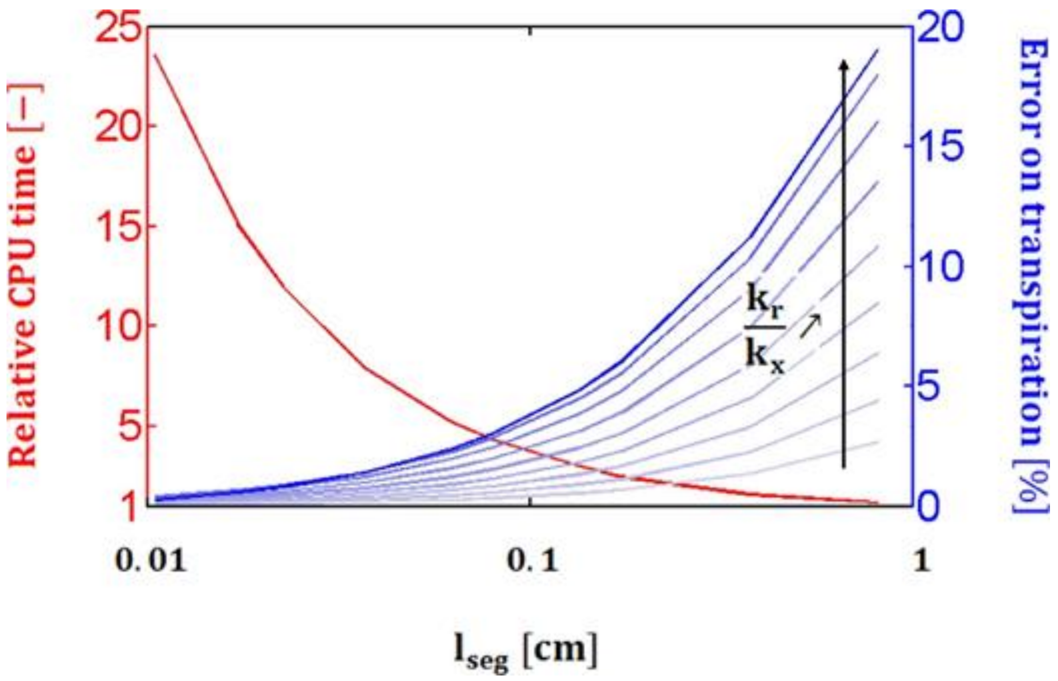


$$x = A \backslash B$$

Examples 4 and 5 (Matlab)

Exact solution

Comparison I (Single root)

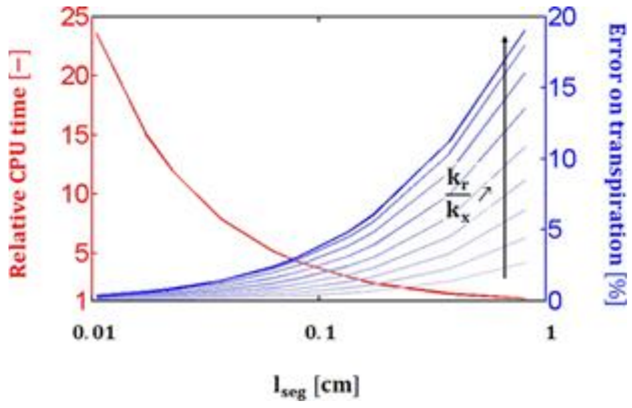


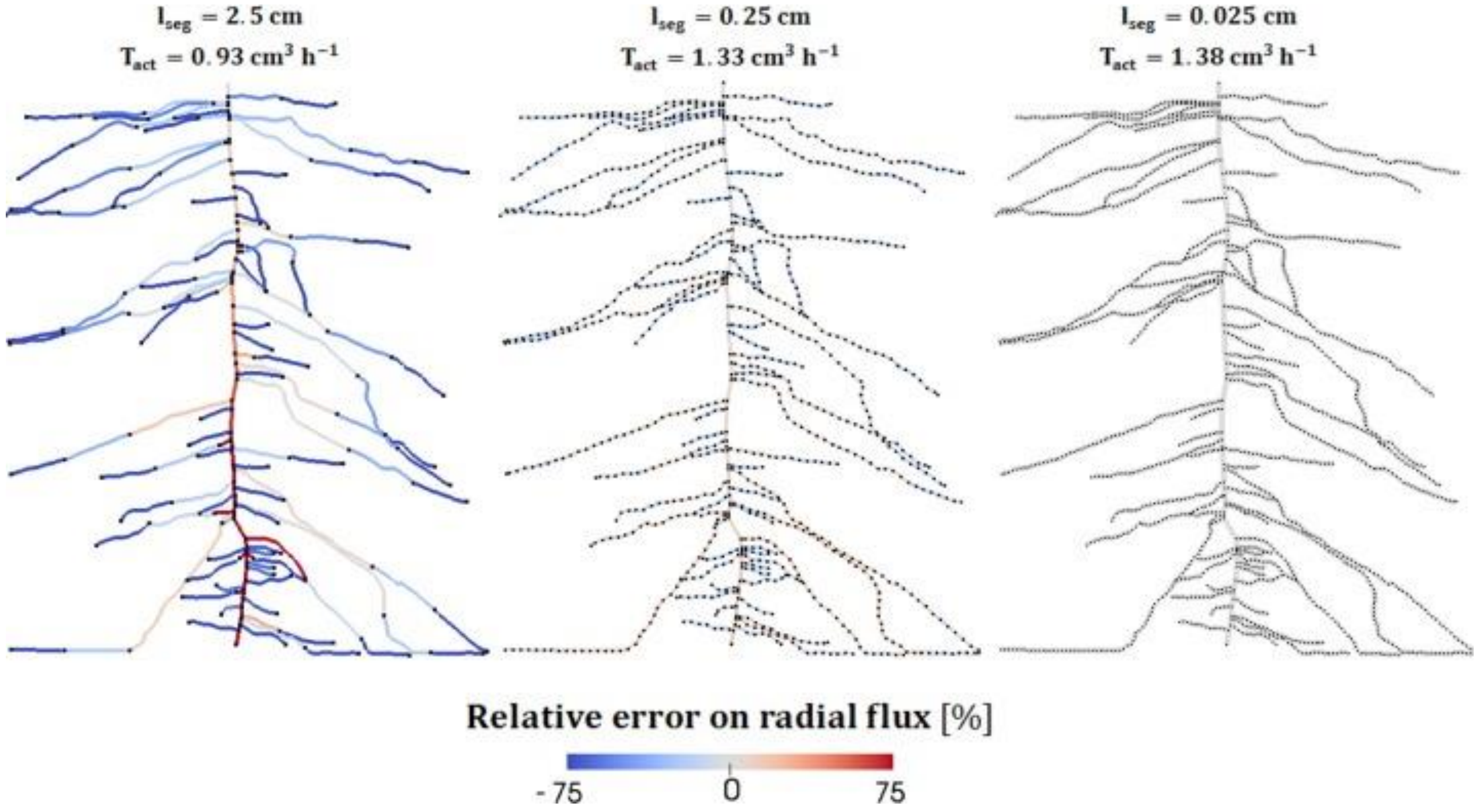
Example 5 (Matlab)

Exact solution

Comparison II (Root system)

Comparison I





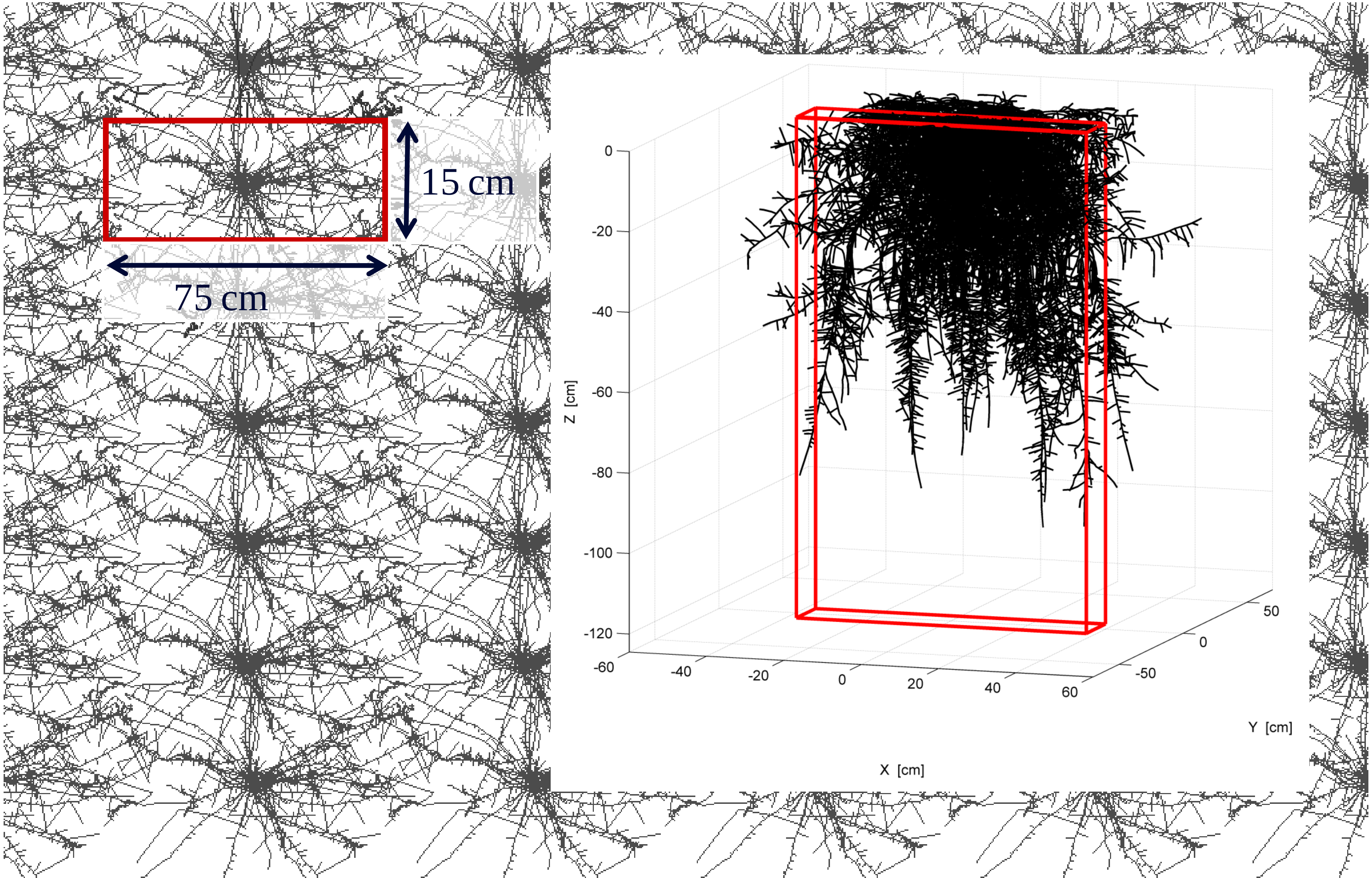
UCLouvain

Macroscopic approximation

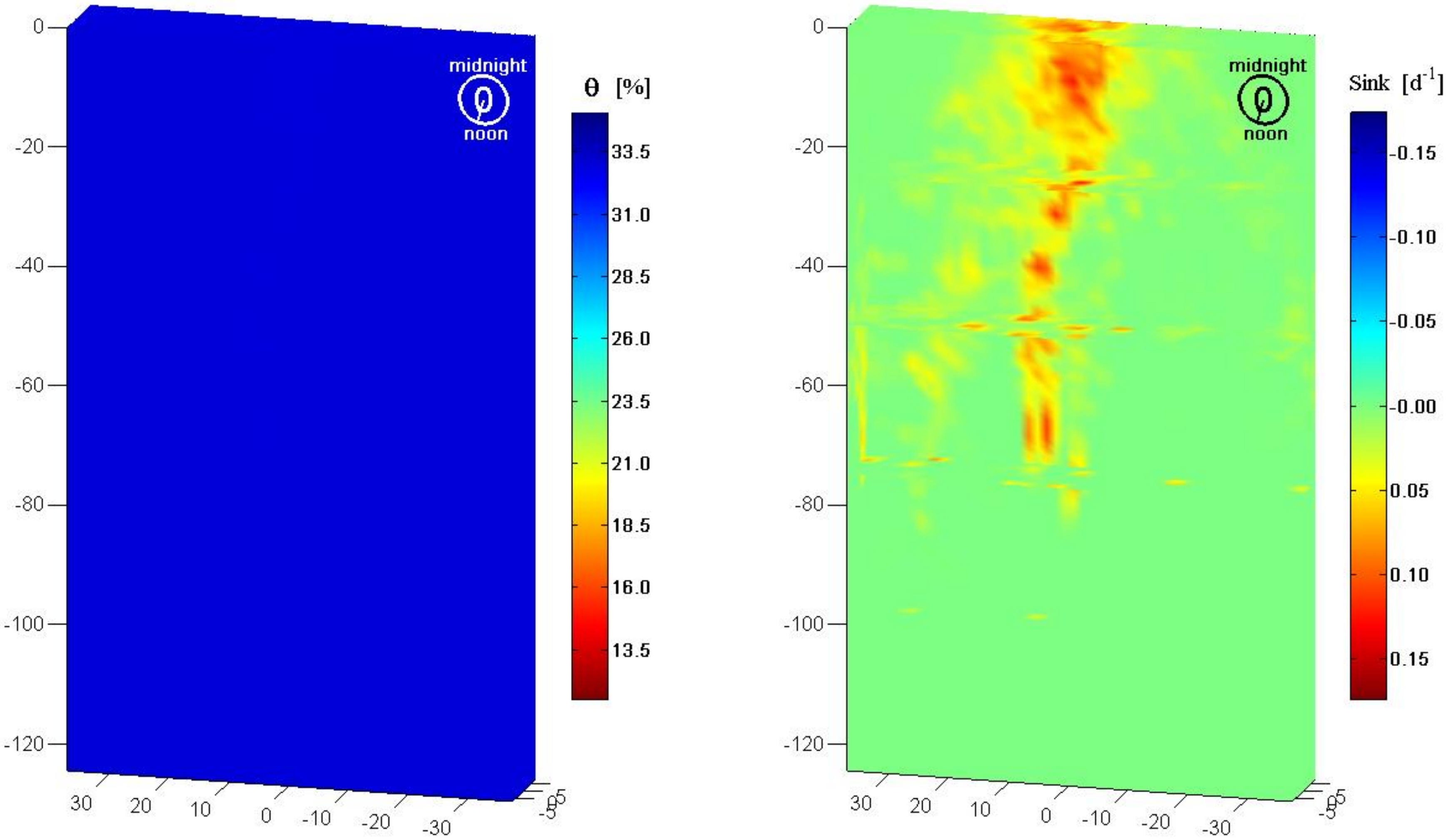
How it started …

and where we ended up …

Macroscopic approximation



Where do plants take up water ?



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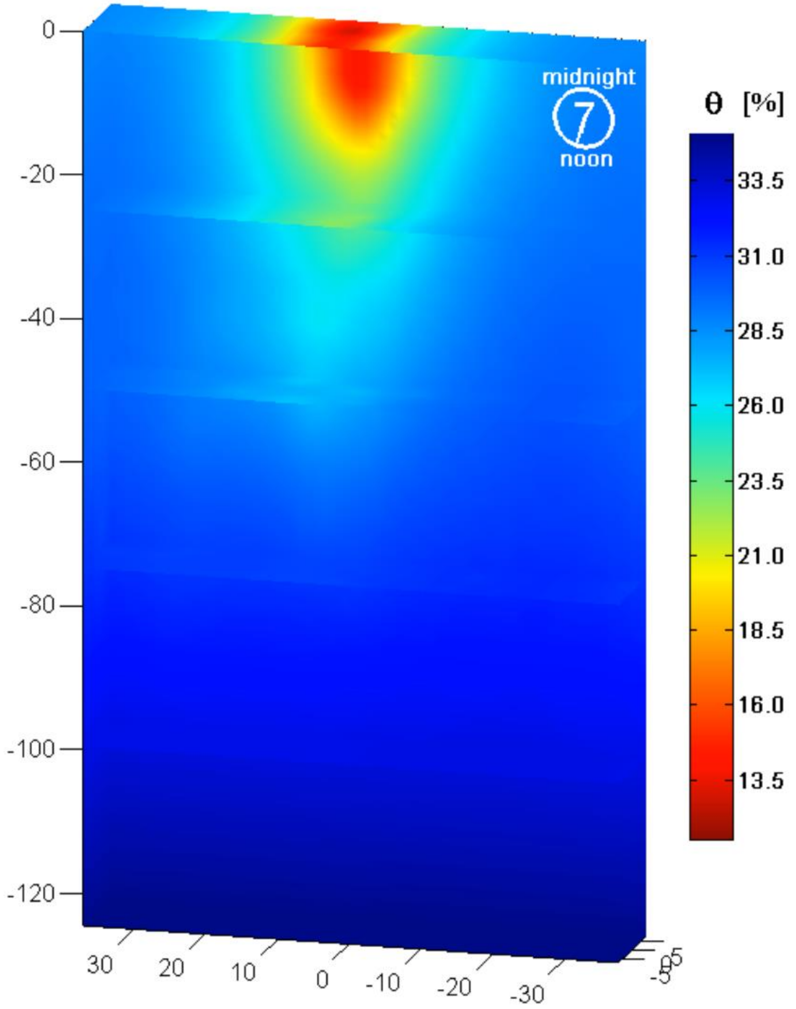
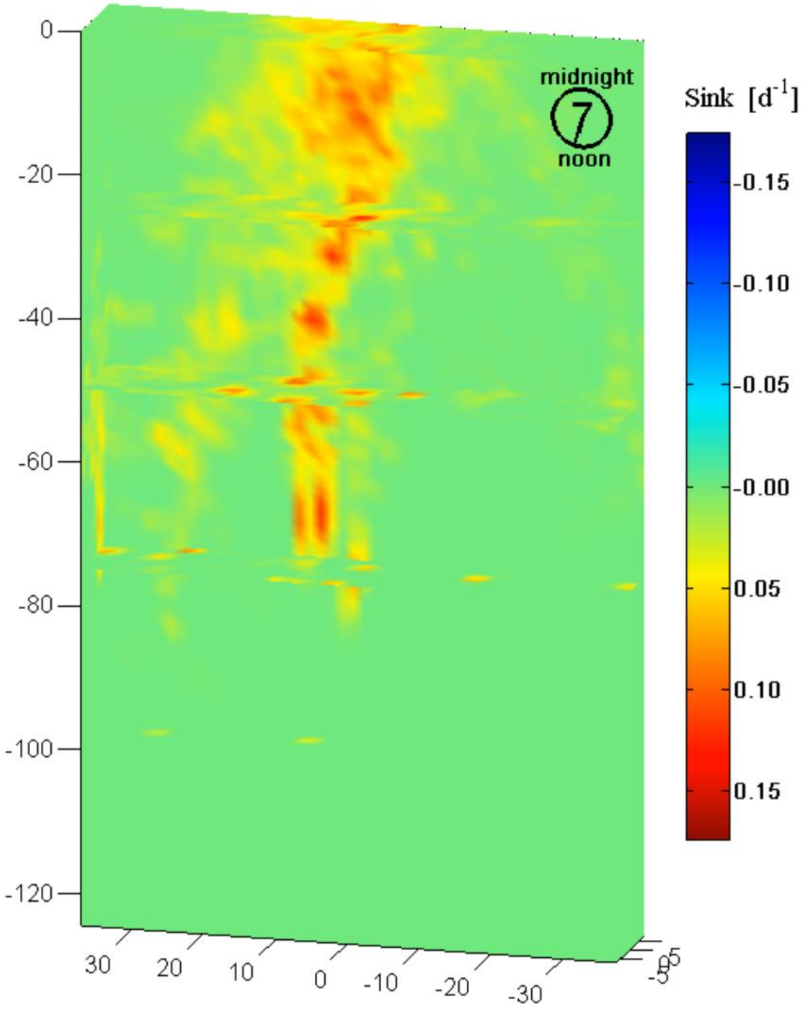
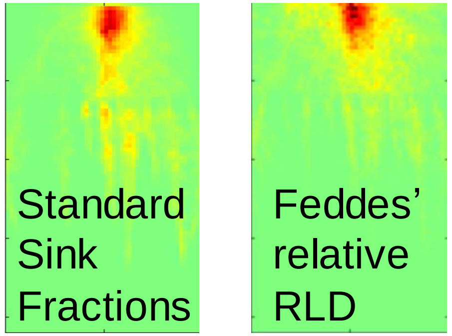
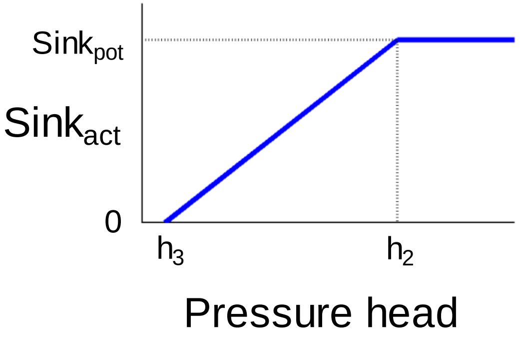




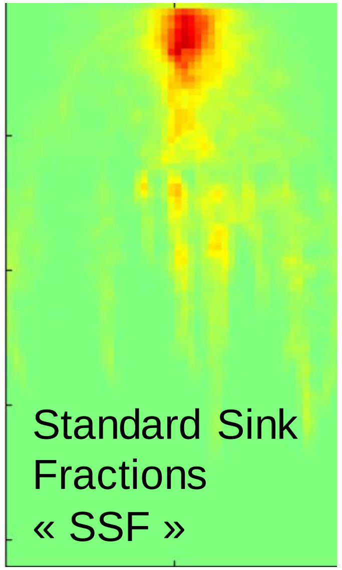
Where do plants take up water when equally available?

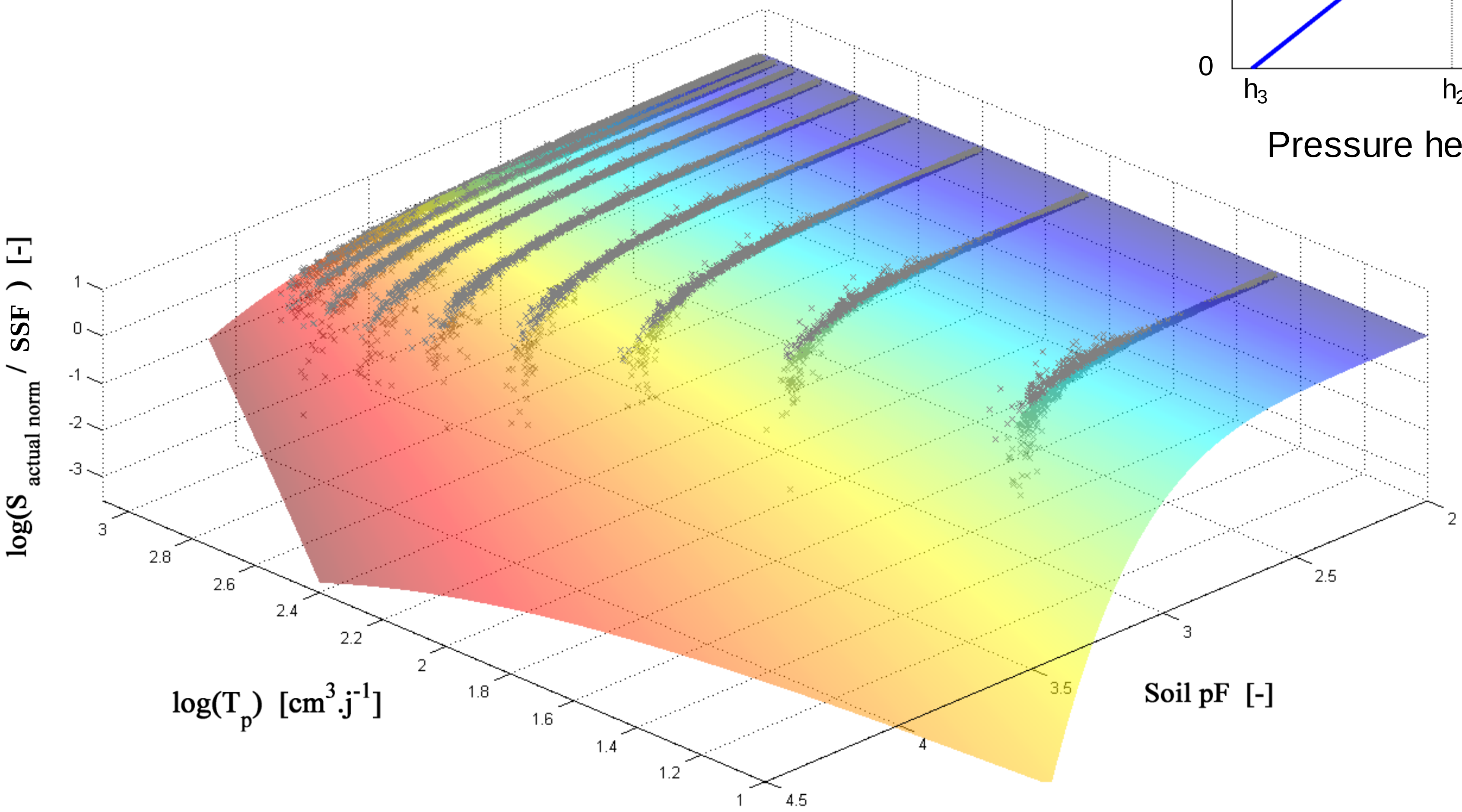


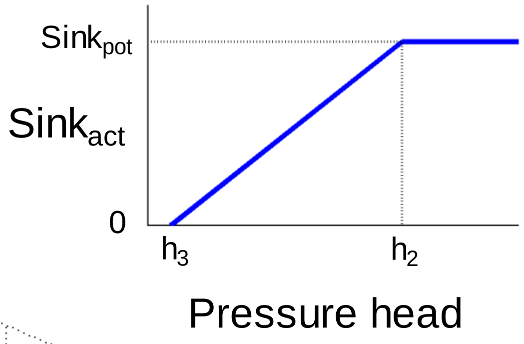
Can we find back the Feddes water stress model?



Can we find back the Feddes water stress model?



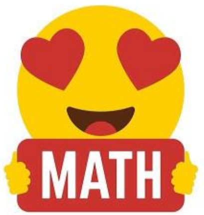


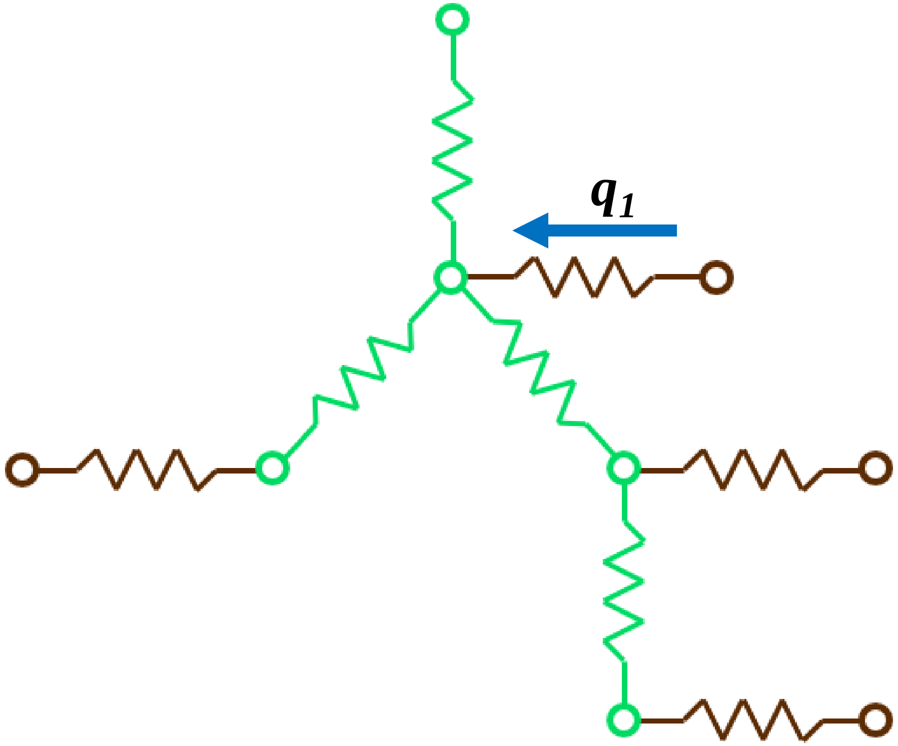


Jan: « Hey let's just start over and do things completely differently »

Let's solve water flow equations in a simple hydraulic architecture

& check if we find back Feddes' water stress function





Homogeneous soil water potential

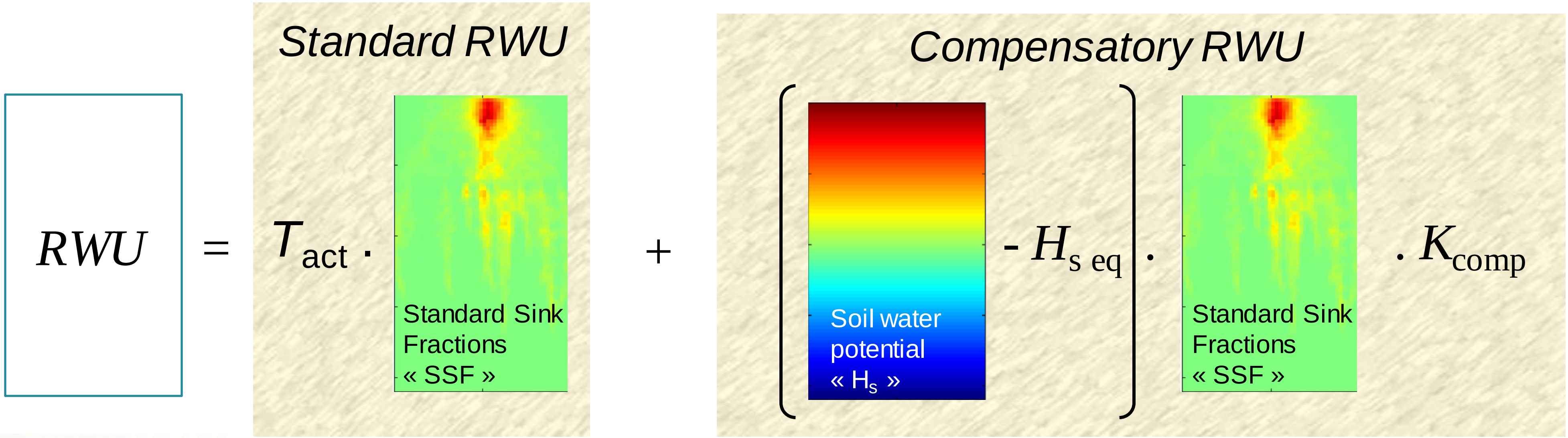
$$q_1 = \left(T_{act} + \cancel{(H_{s,3} - H_{s,4}) \cdot \frac{1}{(r_4 + R_4) \cdot R_3 \cdot \left(\frac{1}{R_3} + \frac{1}{r_4 + R_4} + \frac{1}{r_3}\right)} - (H_{s,3} - H_{s,1}) \cdot \frac{1}{R_3 + \frac{1}{\frac{1}{r_3} + \frac{1}{r_4 + R_4}}} + (H_{s,1} - H_{s,2}) \cdot \frac{1}{r_2 + R_2}} \right) \cdot \frac{\rho}{r_1}$$

$$\rho = \frac{1}{\frac{1}{r_1} + \frac{1}{r_2 + R_2} + \frac{1}{R_3 + \frac{1}{\frac{1}{r_3} + \frac{1}{r_4 + R_4}}}}$$

Standard Uptake Fraction
at soil-root interface 1

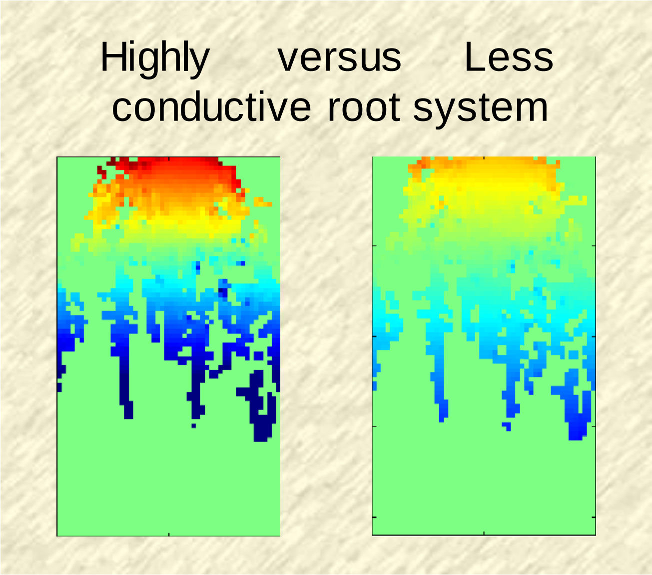
What the analytical solution tells us too

- Root water uptake is the **sum** of a term independent from H_s and another that controls "compensation"
- Compensation reduces (increases) water uptake where H_s is lower (higher) than a threshold "$H_{s,eq}$", an **"equivalent" soil water potential**
- $H_{s,eq}$ is the **SSF-weighted-average H_s**
- When root axial conductances are much larger than radial conductances, the last multiplicative term of the compensation tends to a **unique value "K_{comp}"**



Reality check in a complex maize hydraulic architecture

Is compensation proportional to H_s-$H_{s,eq}$ and to a unique value of K_{comp} ?



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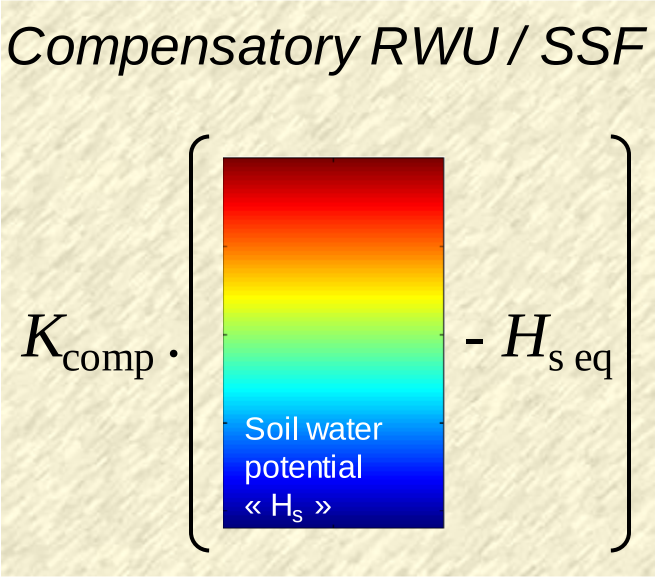
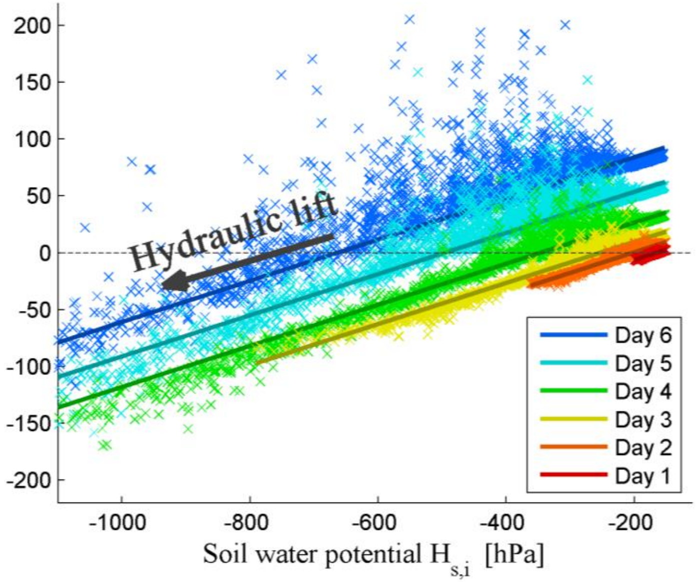




Rather so !

Mathieu: « Could it be that $H_{s,eq}$ allows H_{stem} calculation ? »

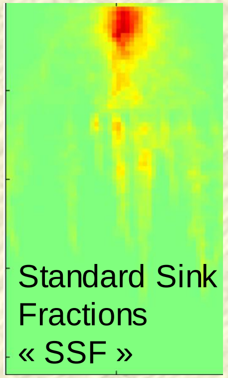
- Stem water potential H_{stem} **exactly** corresponds to the sum of Hs,eq and of the frictions across the whole root system (whose conductance is K_{rs})

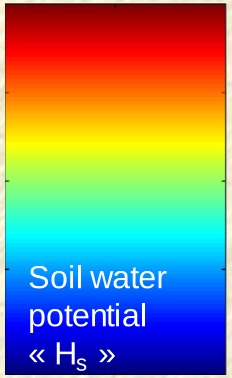
- The approximate value of $\mathbf{K_{comp}}$ **tends to the exact value of** $\mathbf{K_{rs}}$

Frictions across the root system

Equivalent soil water potential $H_{s,eq}$

$$H_{stem} = -\frac{T_{act}}{K_{rs}} + \text{SSF} \cdot H_s$$





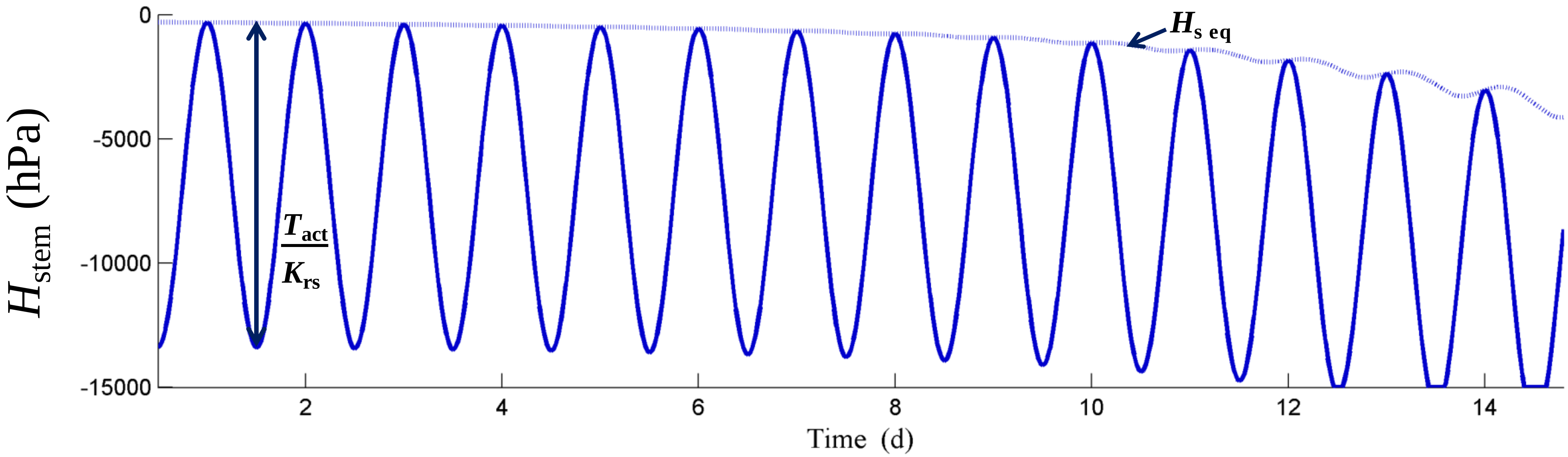
Different ways to evaluate the model parameters

- Super-informed (root architecture, root types and hydraulic prop.)

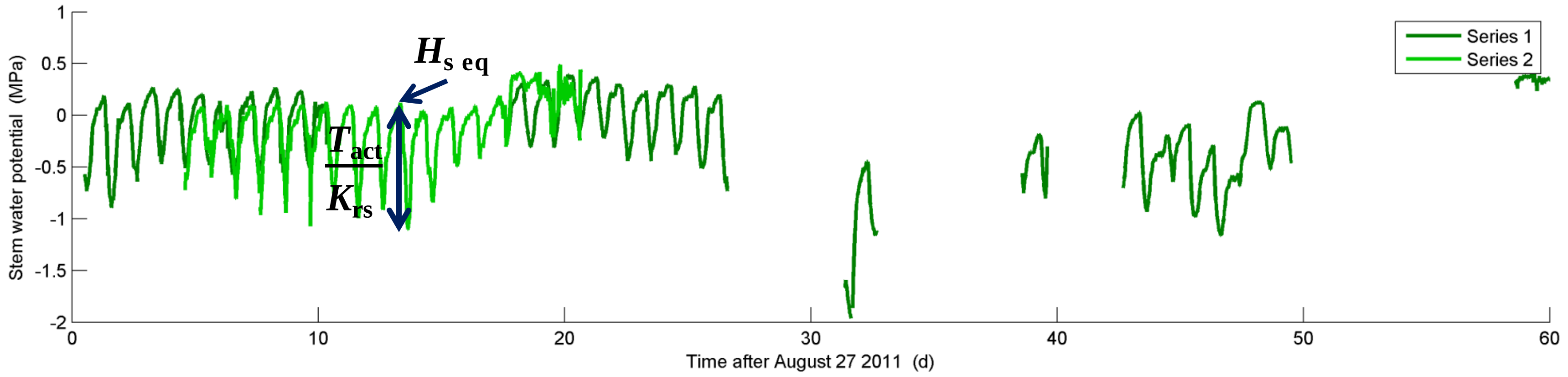
  Mathematical routine ⟶ ***SSF***, K_{rs}, K_{comp}

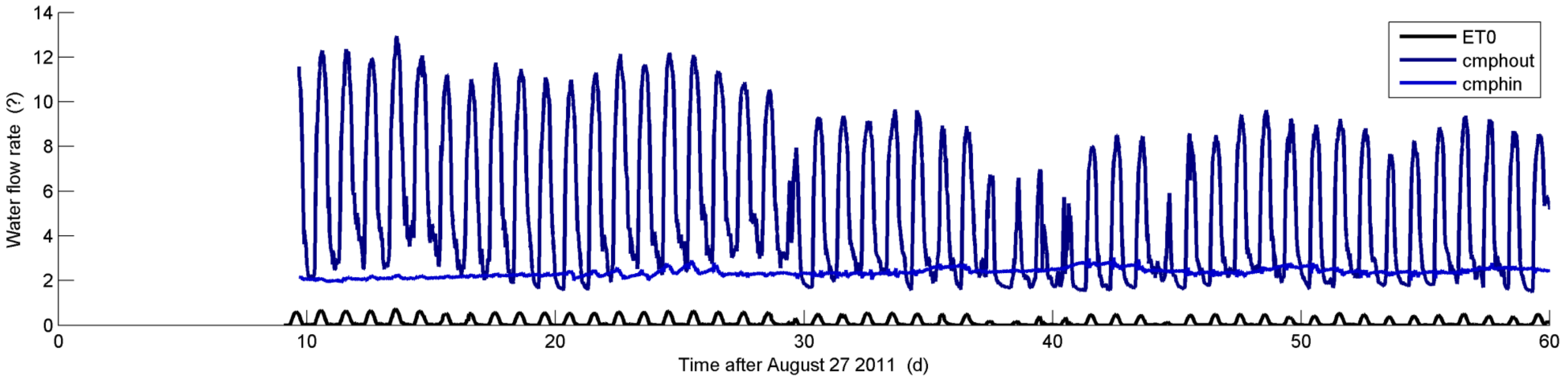
- Partly-informed

  Stem water potential and stem flow measurements



How it would look in a real dataset



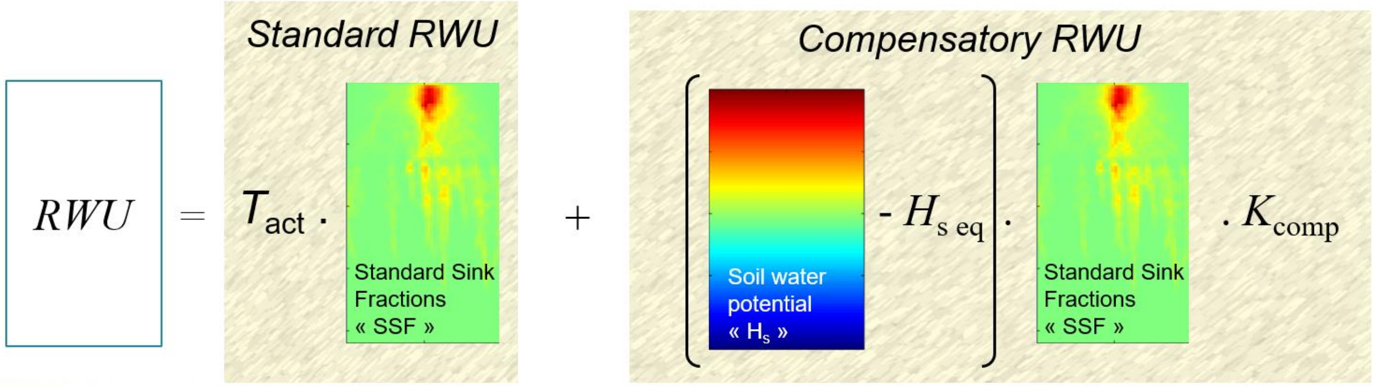


Macroscopic approximation conclusion

Can we identify plant scale **emergent hydraulic laws and properties**, which explain plant root water uptake in simple terms?

Yes, we can!

- Standard RWU, compensatory RWU, $H_{s\,eq}$, ***SSF***, K_{comp} and K_{rs} are **macroscopic processes, variable and parameters** emerging from the system elemental laws and properties.



- Macroscopic parameters have a **physical and intuitive** meaning.
- They may be directly measured or simply estimated.