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Calculus of Variations

Partial regularity results for subelliptic systems in the Heisenberg group

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Abstract We consider subelliptic systems in the Heisenberg group. We give a new proof for the smoothness of solutions of inhomogeneous systems with constant coefficients. With this result, we prove partial Hölder continuity of the horizontal gradient for non-linear systems with p-growth for $p \geq 2$ via the $\mathcal{A}$-harmonic approximation technique.

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1 Introduction

In this paper we study the regularity of weak solutions of certain non-linear subelliptic systems of second order in the Heisenberg group $\mathbb{H}^n$ in divergence form; more precisely, we study systems of the form

$$-\sum_{i=1}^{2n} X_i A_i(q, \mathfrak{X}u) = 0 \quad \text{in } \Omega \tag{1.1}$$

for Ω a bounded domain in $\mathbb{H}^n$, u taking values in $\mathbb{R}^N$. Here $A(.,.)$ is a mapping $\Omega \times \mathbb{R}^{2n\times N} \to \mathbb{R}^{2n\times N}$, continuously differentiable with respect to the second variable, Hölder continuous in the first variable with exponent $\beta > 0$, i.e.

$$|A(q,z) - A(\tilde{q}, z)| \leq L\,|q^{-1}\tilde{q}|^{\beta}(1 + |z|^{p-1}) \quad \forall q, \tilde{q} \in \Omega, \quad z \in \mathbb{R}^{2n\times N}, \tag{1.2}$$

and satisfying the following ellipticity and boundedness conditions:

$$\frac{\partial A}{\partial z}(q,z)\xi \cdot \xi \geq \lambda|\xi|^2\left(1 + |z|^{p-2}\right), \tag{1.3}$$

$$|A(q,z)| + \left|\frac{\partial A}{\partial z}(q,z)\right|(1 + |z|) \leq L\left(1 + |z|^{p-1}\right) \tag{1.4}$$

for all $q \in \Omega$, $z \in \mathbb{R}^{2n\times N}$.

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Our main result is the proof of partial regularity for weak solutions u of (1.1) in the sense that we show Hölder continuity with exponent β for the horizontal gradient $\mathfrak{X}u$ almost everywhere in Ω. Towards this goal, we first establish smoothness and a priori estimates for the solutions of systems with constant coefficients, and then apply the technique of $\mathcal{A}$-harmonic approximation to the non-linear system.

While similar results have been proved for more general A in the Euclidean setting (see e.g. [7,11,13]), the Heisenberg group poses some new difficulties. Already in the first step, trying to apply the standard difference quotient method, the main difference between Euclidean $\mathbb{R}^n$ and subelliptic $\mathbb{H}^n$ becomes clear: Any time we use horizontal difference quotients (i.e. in the directions X_i), extra terms with derivatives in the T direction arise due non-commutativity, and these cannot be absorbed or controlled using the initial assumptions on the weak solution.

The simplest linear case of (1.1), when $A(q, z) = z$, is called the Kohn-Laplacian. The smoothness of its solutions follows from the central theorem of Hörmander's initial paper [14] on subelliptic operators. Further regularity results on diagonal systems were given by Kohn [16], Folland and Stein [9,10], Xu and Zuily [26].

More general cases, including quasilinear and nonlinear elliptic equations and systems, have since been studied by Capogna [1], Capogna and Garofalo [3], Domokos and Manfredi [5,4,6] and Marchi [21,22]. Results from [1,3] were generalized to Carnot groups of higher step in [24] by Shores and in [2] by Capogna. Recently, first Manfredi and Mingione [20] and, in a far-reaching extension of this last paper, Mingione, Zatorska-Goldstein and Zhong [23] considered subelliptic scalar equations modelled on the subelliptic p-Laplacian, and employed a sophisticated new line of argument to show Hölder regularity for the full Euclidean gradient of solutions and further C^∞ regularity if the coefficients are assumed to be smooth.

The papers [1–3,24] use a technique combining fractional difference quotients and fractional derivatives defined by Fourier transform to ensure the non-horizontal differentiability of solutions to the constant coefficient system, allowing to employ the difference quotient method to obtain $W^{2,2}$ estimates and finally prove interior C^∞ regularity. With this linear theory, the regularity of solutions to quasi-linear systems with p-growth is shown by blow-up.

The works of Marchi, Domokos and Manfredi are mainly concerned with the subelliptic p-Laplacian. Marchi proved $Tu \in L^p_{loc}$ and $\mathfrak{X}^2 u \in L^2_{loc}$ for values of p within certain bounds near 2. Domokos and Manfredi extended these results, and introduced a more direct way of dealing with first-order fractional difference quotients: [4, Theorem 1.1] shows how they can be bounded with the help of second-order difference quotients. This method shall be applied in this paper to give a simpler proof for Tu and $\mathfrak{X}Tu \in L^2$ for solutions of constant coefficient systems, eliminating the need for calculations with implicitly defined fractional differential operators.

The first result of this paper is thus the new proof for the following theorem, originally stated in [1] for equations and in [3] for systems:

Theorem 1.1 *Let $u \in HW^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ be a weak solution of the inhomogeneous constant coefficient system*

$$-X_i(A^{\alpha\beta}_{ij} X_j u^\beta + f^\alpha_i) = F^\alpha \quad \textit{for all } \alpha = 1, \ldots, N,$$

with the coefficients $A^{\alpha\beta}_{ij}$ satisfying the ellipticity and boundedness conditions (3.2) *and* (3.3), *and $f^\alpha_i, Tf^\alpha_i, F^\alpha, TF^\alpha \in L^2_{loc}(\Omega)$ for all $i = 1, \ldots, 2n$, $\alpha = 1, \ldots, 2n$. Then the following hold*:

1. $Tu \in HW^{1,2}_{loc}(\Omega, \mathbb{R}^N)$.

2. *There exists $C = C(n, \lambda, L)$, such that on every $B_{2R}(q_0) \Subset \Omega$ there holds*

$$\int\limits_{B_R(q_0)} R^4|Tu|^2 + R^6|\mathfrak{X}Tu|^2\,dq \le C \int\limits_{B_{2R}(q_0)} |u|^2 + R^4|F|^2 + R^8|TF|^2 + R^2|f|^2 + R^6|Tf|^2\,dq\,.$$

3. *$w = Tu$ is a weak solution to the T-differentiated system*

$$-X_i\left(A_{ij}^{\alpha\beta}X_j w^\beta + Tf_i^\alpha\right) = TF^\alpha.$$

This theorem is the first step on the way to the following a priori estimates needed for the treatment of nonlinear systems with growth $p \ge 2$:

Theorem 1.2 *Let $u \in HW^{1,2}(\Omega, \mathbb{R}^N)$ be a solution to the constant coefficient version of (1.1) and $p \ge 2$. Then there exists $C_a = C_a(n, p, \lambda, L) > 0$ such that for all $0 < r < R$ with $B_R(q_0) \subset \Omega$ one has:*

$$\fint\limits_{B_r(q_0)} |\mathfrak{X}u - (\mathfrak{X}u)_{q_0,r}|^p\,dq \le C_a\left(\frac{r}{R}\right)^p \fint\limits_{B_R(q_0)} |\mathfrak{X}u - (\mathfrak{X}u)_{q_0,R}|^p\,dq \tag{1.5}$$

$$\fint\limits_{B_r(q_0)} |\mathfrak{X}u|^p\,dq \le C_a \fint\limits_{B_R(q_0)} |\mathfrak{X}u|^p\,dq \tag{1.6}$$

(Here $f_{q_0,\rho}$ denotes the average of the function f on the ball $B_\rho(q_0)$).

Once the estimates for solutions of constant coefficient systems are established, we turn to the non-linear system. Systems of this form, in particular the quadratic case $p = 2$, have been intensively studied in the Euclidean setting, and several strategies to prove partial regularity are known. The proofs can be broadly classified into two categories: Direct and indirect. For the direct proofs, as in e.g. [11], a Caccioppoli-type inequality is obtained, and then improved to gain higher integrability of the gradient, making possible an excess decay estimate that implies regularity. This approach involves a great amount of technical detail, but yields the explicit dependency of the constants on the structure parameters. In contrast, in the indirect proof by blow-up (see e.g. [13]) one argues by contradiction: Under the assumption that the desired inequality is false, one constructs a sequence of solutions that, after rescaling ("blow-up"), converges to a solution of a simpler problem, for which the inequality holds. Compactness arguments then give the desired conclusion. A drawback of this method is the lack of control on the sensitivity of the constants to changes in the structure parameters; moreover, it is usually necessary to prove specialized compactness results.

The method of $\mathcal{A}$-harmonic approximation, based on Simon's harmonic approximation technique [25] and developed for nonlinear elliptic systems by Duzaar and Grotowski in [7], unites the advantages of the methods described. The solution u of the non-linear system is rescaled so that it is almost a solution of the system with frozen coefficients. This function is then compared to an L^2-nearby solution of the frozen coefficient system, allowing to derive the excess decay estimate. The only indirect step of this method is the proof of the $\mathcal{A}$-harmonic approximation lemma which is needed to find a suitable solution to the constant coefficient system. Thus, this approach avoids the technical difficulties associated with Gehring's Lemma and having to prove compactness arguments, but still gives good control on the sensitivity of the constants.

In view of these benefits, in Sect. 4, we transfer the $\mathcal{A}$-harmonic Approximation Lemma to the $\mathbb{H}^n$ setting, also generalizing it to the superquadratic case, in order to prove the following regularity theorem:

Theorem 1.3 *Let $u \in HW^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to (1.1), with the coefficients $A : \Omega \times \mathbb{R}^{2n\times N} \to \mathbb{R}^{2n\times N}$ satisfying the structure conditions (1.2)–(1.4). Then there exists a relatively closed set $\Omega_0 \subset \Omega$ such that $u \in \Gamma^{1,\beta}(\Omega \setminus \Omega_0, \mathbb{R}^N)$. Further, $\Omega_0 \subset \Sigma_1 \cup \Sigma_2$ with*

$$\Sigma_1 = \left\{ q_0 \in \Omega : \liminf_{R\searrow 0} \quad ⨍_{B_R(q_0)} |\mathfrak{X}u - (\mathfrak{X}u)_{q_0,R}|^2 + |\mathfrak{X}u - (\mathfrak{X}u)_{q_0,R}|^p \, dq \; > 0 \right\}$$

$$\Sigma_2 = \left\{ q_0 \in \Omega : \limsup_{R\searrow 0} |(\mathfrak{X}u)_{q_0,R}| = \infty \right\}.$$

In particular, $\mu(\Omega_0) = 0$. □

We note that the summation convention will be used throughout the paper. We also note here that more detailed versions of the calculations can be found in [8].

2 Preliminaries

2.1 The Heisenberg group

The Heisenberg Group $\mathbb{H}^n$ can be defined as $\mathbb{R}^{2n+1}$ endowed with the group multiplication

$$\begin{aligned} \cdot \; : \; \mathbb{H}^n \times \mathbb{H}^n &\to \mathbb{H}^n \\ \big((x,t),(y,v)\big) &\mapsto \Big(x + y, t + v + \tfrac{1}{2}\sum_{i=1}^{n}(x_i y_{i+n} - x_{i+n} y_i)\Big), \end{aligned}$$

for all $(x,t), (y,v) \in \mathbb{R}^{2n} \times \mathbb{R}$. Elements of $\mathbb{H}^n$ are usually denoted as $q = (x,t)$ with $x \in \mathbb{R}^{2n}$. This multiplication corresponds to addition in Euclidean $\mathbb{R}^{2n+1}$: Its neutral element is $(0,0)$, and the inverse to (x,t) is given by $(-x,-t)$. In particular, the mapping $(p,q) \mapsto p \cdot q^{-1}$ is smooth, so $(\mathbb{H}^n, \cdot)$ is a Lie group.

For the Lie algebra $\mathfrak{h}^n \cong \mathbb{R}^{2n+1}$, we choose the basis $X_1, \ldots, X_{2n}, T$ which is the left-invariant extension of the standard partial derivatives $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}$ onto $\mathbb{H}^n$. Thus the exponential map $\exp : \mathfrak{h}^n \to \mathbb{H}^n$ is the identity on $\mathbb{R}^{2n+1}$, and we can explicitly calculate the basis vector fields of $\mathfrak{h}^n$ by the formula

$$Zf(q) = \frac{d}{ds}\bigg|_{s=0} f(q \cdot \exp(sZ)) = \frac{d}{ds}\bigg|_{s=0} f(qe^{sZ}). \tag{2.1}$$

We obtain

$$X_i = \frac{\partial}{\partial x_i} - \frac{x_{i+n}}{2}\frac{\partial}{\partial t}, \quad X_{i+n} = \frac{\partial}{\partial x_{i+n}} + \frac{x_i}{2}\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}$$

for $i = 1, \ldots, n$, and notice the special structure of the commutators:

$$[X_i, X_{i+n}] = -[X_{i+n}, X_i] = T, \quad \text{else} \quad [X_i, X_j] = 0, \quad \text{and} \quad [T,T] = [T, X_i] = 0,$$

i.e. $\mathbb{H}^n$ is nilpotent of step 1. We call $\mathfrak{X} = (X_1, \ldots, X_{2n})$ the horizontal gradient, and T the vertical derivative.

Due to the vanishing higher commutators, and also directly by the group multiplication law, the Baker–Campbell–Hausdorff formula takes the simple form

$$\exp(Z)\exp(Y) = \exp\left(Z + Y + \tfrac{1}{2}[Z, Y]\right) \quad \forall Z, Y \in \mathfrak{h}^n.$$

As a substitute for the Euclidean scalar multiplication with $\lambda \in \mathbb{R}$, which is not homogeneous in the last coordinate with respect to group multiplication, the dilation δ_λ is defined as follows:

$$\delta_\lambda\big((x, t)\big) = (\lambda x, \lambda^2 t).$$

With a view to obtaining a suitable distance function, we define the δ_λ-homogeneous pseudo-norm

$$\|(x, t)\| = \left(|x|^4 + t^2\right)^{\frac{1}{4}}.$$

The metric induced by this pseudo-norm,

$$d(q', q) = \|q^{-1} \cdot q'\|,$$

is δ_λ-homogeneous and left-invariant. This metric has been shown to be equivalent to the Carnot-Carathéodory metric on $\mathbb{H}^n$, which is defined as the infimum of the length of the curves connecting q' and q with horizontal tangent in every point [17]. For completeness, we remark that d is not equivalent to any Riemannian metric.

The measure used on $\mathbb{H}^n$ is Haar measure, i.e. the Lebesgue measure on $\mathfrak{h}^n$ pushed forward via the exponential map:

$$\mu(\Omega) = \mathcal{L}^{2n+1}\left(\exp^{-1}(\Omega)\right), \quad \Omega \subset \mathbb{H}^n.$$

By direct calculation of the partial derivatives, we can check that the Jacobians of left- and right-multiplication on $\mathbb{H}^n$ are both 1, while the Jacobian of the dilation δ_λ is λ^{2n+2}. From this follow the translation invariance and δ_λ-homogeneity of μ:

$$\mu\big(B_R(q_0)\big) = |J(l_{q_0})|\,|J(\delta_R)|\,\mu\big(B_1(0)\big) = R^{2n+2}\mu\big(B_1(0)\big).$$

(Here $B_R(q_0)$ denotes the pseudo-ball $\{q \in \mathbb{H}^n : d(q_0, q) < R\}$). The number $Q := 2n + 2$ is called the homogeneous dimension of $\mathbb{H}^n$.

From now on, let Ω be a bounded open set in $\mathbb{H}^n$, $k \in \mathbb{N}_0$ and $1 \le p \le \infty$; let q and q' denote points in Ω. The spaces $C^k(\Omega)$ and $L^p(\Omega)$ are given by the differentiable manifold structure of $\mathbb{H}^n$. The analogs to the usual Hölder- and Sobolev spaces of the Euclidean case are defined as follows:

The Folland–Stein-classes $\Gamma^{k,\alpha}$ correspond to $C^{k,\alpha}$ for $0 < \alpha < 1$:

$$\Gamma^\alpha(\Omega) = \left\{u \in L^\infty(\Omega) : \sup_{q \ne q'} \frac{|u(q') - u(q)|}{d(q', q)^\alpha} < \infty\right\},$$

$$\Gamma^{k,\alpha}(\Omega) = \left\{u \in L^\infty(\Omega) : X_{i_h} \cdots X_{i_1} u \in \Gamma^\alpha \;\; \forall\, (i_h, \ldots, i_1) \in \{1, \ldots, 2n\}^h,\; h \le k\right\}.$$

Endowed with the norm

$$\|u\|_{\Gamma^\alpha} = \sup_{q \ne q'} \frac{|u(q) - u(q')|}{d(q, q')^\alpha} + \|u\|_{L^\infty},$$

Γ^α is a Banach space (see [9]).

The *horizontal Sobolev spaces* $HW^{k,p}(\Omega)$ are defined as

$$\begin{aligned} &HW^{k,p}(\Omega) \\ &\quad = \left\{u \in L^p(\Omega) : X_{i_h} \cdots X_{i_1} u \in L^p(\Omega) \;\; \forall\, (i_h, \ldots, i_1) \in \{1, \ldots, 2n\}^h,\; h \le k\right\}. \end{aligned}$$

Here $X_i u$ denotes the weak derivative of u with respect to X_i, i.e. for $u \in L^1_{loc}(\Omega)$, $g = X_i u \in L^1_{loc}(\Omega)$ if it satisfies

$$\int_\Omega g\, \varphi\, dq = \int_\Omega u\, X_i^* \varphi\, dq \qquad \forall \varphi \in C_c^\infty(\Omega).$$

One can easily show by direct calculation that the formal adjoint Z^* of any left invariant vector field $Z \in \mathfrak{h}^n$ is $-Z$.

While the obstacle of integrating only along curves with horizontal derivatives makes a direct transfer of some basic inequalities and embedding theorems from the Euclidean setting impossible, analogs have been proved on "nice" sets such as pseudo-balls: The fact that the embedding

$$HW^{1,p}\left(B_R(q_0)\right) \hookrightarrow L^p\left(B_R(q_0)\right) \tag{2.2}$$

is compact corresponds to the *Rellich* compactness theorem (cf. [18,19]). The *Poincaré inequality*

$$\fint_{B_R(q_0)} |u - u_{q_0,R}|^p\, dq \le C_P R^p \fint_{B_R(q_0)} |\mathfrak{X} u|^p\, dq\ , \tag{2.3}$$

for $p \ge 1$ with C_P depending only on p and n has been shown in [15]. As an analog to the *Sobolev Embedding Theorem* for $k \in \mathbb{N}$, $1 \le p < \infty$, we have the inclusion (proved in [9])

$$HW^{k,p}\left(B_R(q_0)\right) \subset \Gamma^\alpha\left(B_R(q_0)\right)$$

with $\alpha = k - Q/p > 0$ if $\alpha < 1$ and any $\alpha \in (0,1)$ if $k - Q/p = 1$, as well as the estimate

$$\|u\|_{\Gamma^\alpha(B_R(q_0))} \le C_F \|u\|_{HW^{k,p}(B_R(q_0))}$$

with C_F depending on n, k, p, and α.

For a function $f : \Omega \to \mathbb{R}$, the difference quotient of order $\gamma \in (0,1]$ in the direction of the vector field $Z \in \mathfrak{h}^n$ is defined as

$$\Delta^s_{(Z,\gamma)} f(q) = \frac{f(qe^{sZ}) - f(q)}{s^\gamma}.$$

We note that in contrast to Euclidean difference quotients, the s in the denominator is in general different from the length of the curve described by qe^{sZ}; we only have equality if the vector field Z is horizontal.

2.2 Some useful lemmas

In order to apply the the standard techniques for systems in the Heisenberg group, we need to look at some properties of difference quotients that arise from the group's structure.

Let $f \in C^\infty(\mathbb{H}^n)$, $\gamma \in (0,1]$ and $Y, Z \in \mathfrak{h}^n$. Then we can switch the difference quotient in Z-direction and the derivative Y as follows, generating a commutator term:

$$\Delta^s_{(Z,\gamma)}(Yf)(q) = Y\big(\Delta^s_{(Z,\gamma)} f\big)(q) + s^{1-\gamma}[Z, Y] f(qe^{sZ}). \tag{2.4}$$

We also note that the difference quotient of the product of two functions can be written as

$$\Delta^s_{(Z,\gamma)}(fg)(q) = \Delta^s_{(Z,\gamma)} f(q) g\big(qe^{sZ}\big) + f(q) \Delta^s_{(Z,\gamma)} g(q),$$

and that integration by parts only differs from the Euclidean case in that the direction of the vector field Z can be inverted instead of changing the sign of the integral:

$$\int_{\Omega} \Delta^s_{(Z,\gamma)} f \varphi \, dq \ = \int_{\Omega} f \Delta^s_{(-Z,\gamma)} \varphi \, dq \ .$$

The proofs for all these formulae follow by direct calculation (see [8] for details).

As in the Euclidean case, weak differentiability can be characterised by means of difference quotients. The following lemma is proved in e.g. [1, Proposition 2.3].

Lemma 2.1 *Let $u \in L^p(\Omega)$, $K \Subset \Omega$ and $Z \in \mathfrak{h}^n$.*

1. *If there exist $\varepsilon, C > 0$ such that*

$$\sup_{0<s\leq\varepsilon} \int_K \left| \Delta^s_{(Z,1)} u \right|^p \, dq \ \leq C^p, \tag{2.5}$$

then $Zu \in L^p(K)$ and $\|Zu\|_{L^p(K)} \leq C$.

2. *If $Zu \in L^p(\Omega)$, then*

$$\sup_{0<s\leq\varepsilon} \int_K \left| \Delta^s_{(Z,1)} u \right|^p \, dq \ \leq \|Zu\|^p_{L^p(\Omega)} \tag{2.6}$$

holds for any $\varepsilon > 0$ that satisfies $qe^{\varepsilon Z} \in \Omega$ for all $q \in K$.

The next lemma will be one of the tools in estimating difference quotients in the T direction. The proof is given in [1], and exploits the fact that $e^{t^2 T}$ can be decomposed into a product of four e^{tX_i} type terms.

Lemma 2.2 *Let $w \in HW^{1,2}(\Omega)$, $K \Subset \Omega$ and $\varepsilon = \frac{1}{4} d(K, \partial\Omega)^2$. Then the following inequality holds*:

$$\sup_{0<s<\varepsilon} \int_K |\Delta^s_{(T,1/2)} w|^2 \, dq \ \leq 8 \|\mathfrak{X} w\|^2_{L^2(\Omega)}.$$

The second stepping stone for the proof of T-differentiability of solutions is the following lemma, proved in [5].

Lemma 2.3 *Let $u \in L^2(\Omega)$, $B \Subset \Omega$, $\alpha > 0$, $0 < \varepsilon < \frac{1}{2} d(B, \partial\Omega)^2$ and $M \geq 0$. Suppose that*

$$\sup_{0<s\leq\varepsilon} \frac{\left\| u(qe^{2sT}) + u(q) - 2u(qe^{sT}) \right\|_{L^2(B)}}{s^\alpha} \leq M. \tag{2.7}$$

Then there exists $C = C(\alpha, \beta, \varepsilon)$ such that

$$\sup_{0<s\leq\varepsilon'} \left\| \Delta^s_{(T,\beta)} u \right\|_{L^2(B)} \leq C \left(\|u\|_{L^2(\Omega)} + M \right), \tag{2.8}$$

with $\beta = \min\{\alpha, 1\}$ and $\varepsilon' = \frac{\varepsilon}{4}$ for $\alpha \neq 1$, and β any number in $(0, 1)$ and $\varepsilon' = \frac{\varepsilon}{2} e^{-\frac{1}{1-\beta}}$ for $\alpha = 1$.

Finally, for reference in rescaling arguments, we state how the horizontal and vertical derivatives act on dilations; the proof is by direct calculation.

Lemma 2.4 *Let $f \in C^\infty(\mathbb{H}^n)$ and $\lambda \in \mathbb{R}$. Then for all $i = 1, \ldots, 2n$ we have*

$$X_i(f \circ \delta_\lambda)(q) = \lambda X_i f\,(\delta_\lambda(q))\,, \quad \text{and} \quad T(f \circ \delta_\lambda)(q) = \lambda^2 T f\,(\delta_\lambda(q))\,.$$

3 Systems with Constant Coefficients

In this section, we consider weak solutions $u \in HW^{1,2}(\Omega, \mathbb{R}^n)$ of the inhomogeneous constant coefficient system

$$-X_i(A_{ij}^{\alpha\beta} X_j u^\beta + f_i^\alpha) = F^\alpha \quad \forall \alpha = 1, \ldots, N \tag{3.1}$$

with $A_{ij}^{\alpha\beta} \in \mathbb{R}$ satisfying

$$A_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \geq \lambda |\xi|^2, \tag{3.2}$$

$$A_{ij}^{\alpha\beta} \xi_i^\alpha \eta_j^\beta \leq L|\xi||\eta| \tag{3.3}$$

for some fixed $0 < \lambda \leq L$ and all $\xi, \eta \in \mathbb{R}^{2n \times N}$. We presume the functions f_i^α, F^α to be in $L^2(\Omega)$ for all $i, j = 1, \ldots, 2n$, $\alpha, \beta = 1, \ldots, N$.

Our aim is to prove the smoothness of weak solutions to this system, as well as some a priori estimates that will be necessary for the treatment of nonlinear systems. First, we show a Caccioppoli type inequality. Next, we proceed to the $W^{2,2}$ estimates. As differentiating the system in the $\mathfrak{X}$ direction generates T-derivatives of u and $\mathfrak{X}u$, we first have to establish that Tu and $T\mathfrak{X}u$ are L^2. Then we can show a $HW^{2,2}$ estimate for u. After that, we iterate these steps, beginning with the homogeneous system, and prove the smoothness of its solutions via the Sobolev embedding theorem. Finally, we derive the a priori estimates of Theorem 1.2.

In contrast to the Euclidean case, it is not sufficient to prove $W^{2,2}$ estimates only for the homogeneous system: During the horizontal differentiation of the system, extra terms appear, forcing us to deal with inhomogeneities already in the first iteration.

Lemma 3.1 *Let $u \in HW_{loc}^{1,2}(\Omega, R^N)$ be a weak solution of (3.1) with all $F^\alpha \in L^2(\Omega)$, $f_i^\alpha \in HW^{1,2}(\Omega)$, and $B_{2R}(q_0) \subset \Omega$. Then there exists C_c depending on L and λ such that there holds*:

$$\int_{B_R(q_0)} |\mathfrak{X}u|^2 \, dq \leq C_c \left(\int_{B_{2R}(q_0)} R^{-2}|u|^2 \, dq + |f|^2 \, dq + R^2 |F|^2 \, dq \right). \tag{3.4}$$

Proof The proof is exactly the same as in the classical Euclidean setting, except that the full derivative D is replaced with the horizontal gradient $\mathfrak{X}$. □

Our strategy for the proof of Theorem 1.1 is the following: First we estimate the fractional T-direction difference quotient of $\mathfrak{X}u$ of order γ from above by the T-direction difference quotient of u of the same order γ, and from below by the symmetric T-direction difference quotient of order $\gamma + 1/2$. Lemma 2.3 will then allow us to follow the argument of Domokos in [5, Sect. 2.3]: We iteratively increase γ from 1/2 to 2/3 to 1, in each step bounding the higher-order difference quotient by the lower-order one.

Proof of Theorem 1.1 We suppose $B_4(0) \Subset \Omega$, prove (1) and (2) on $B_4(0)$ and then obtain the general result by left translation and dilation to arbitrary balls.

Let $\gamma \in (0, 1]$ and $1/32 < r < 1$. In order to satisfy smallness conditions required by the theorems which will be used, the difference quotients in the following will have increment $0 < s < 1/1024$.

We test (3.1) with

$$\varphi = \Delta^s_{(-T,\gamma)}\big(\eta^2 \Delta^s_{(T,\gamma)} u\big),$$

where η is a cut-off function in $C_c^\infty(B_r(0))$, $\mathbb{1}_{B_{r/2}} \le \eta \le \mathbb{1}_{B_r}$, and $|\mathfrak{X}\eta| \le C_\eta(r)$. We begin by transferring the outer difference quotient to $A_{ij}^{\alpha\beta}$ with the help of (2.4) and a change of variable $q \to qe^{sZ}$ in the second summand:

$$\begin{aligned}
&\int_\Omega \Big(A_{ij}^{\alpha\beta}\Delta^s_{(T,\gamma)}(X_j u^\beta)(q) + \Delta^s_{(T,\gamma)} f_i^\alpha(q)\Big) X_i\big(\eta^2 \Delta^s_{(T,\gamma)} u^\alpha\big)(q)\,dq \\
&\quad + \int_\Omega \Big(A_{ij}^{\alpha\beta} X_j u^\beta(qe^{sT}) + f_i^\alpha(qe^{sT})\Big) s^{1-\gamma}[T, X_i]\big(\eta^2 \Delta^s_{(T,\gamma)} u^\alpha\big)(q)\,dq \\
&= \int_\Omega F^\alpha(q)\Delta^s_{(-T,\gamma)}\big(\eta^2 \Delta^s_{(T,\gamma)} u^\alpha\big)(q)\,dq
\end{aligned} \tag{3.5}$$

Using that $[X_i, T] = 0$ for all $i = 1, \ldots, 2n$, this reduces to

$$\begin{aligned}
&\int_\Omega \big(A_{ij}^{\alpha\beta}\Delta^s_{(T,\gamma)}(X_j u^\beta) + \Delta^s_{(T,\gamma)} f_i^\alpha\big) X_i\big(\eta^2 \Delta^s_{(T,\gamma)} u^\alpha\big)\,dq \\
&\quad = \int_\Omega F^\alpha \Delta^s_{(-T,\gamma)}\big(\eta^2 \Delta^s_{(T,\gamma)} u^\alpha\big)\,dq\ .
\end{aligned} \tag{3.6}$$

Carrying out the X_i differentiation, we expand this expression to

$$\begin{aligned}
&\int_\Omega \eta^2 A_{ij}^{\alpha\beta} X_i \Delta^s_{(T,\gamma)} u^\alpha X_j \Delta^s_{(T,\gamma)} u^\beta + 2\eta A_{ij}^{\alpha\beta} X_i\eta\, \Delta^s_{(T,\gamma)} u^\alpha X_j \Delta^s_{(T,\gamma)} u^\beta\,dq \\
&= \int_\Omega \Delta^s_{(T,\gamma)} F^\alpha \varphi^\alpha\,dq - \int_\Omega \Delta^s_{(T,\gamma)} f_i^\alpha\, X_i\big(\eta^2 \Delta^s_{(T,\gamma)} u^\alpha\big)\,dq\ ,
\end{aligned}$$

where we have used again the commutativity of the T difference quotient with all of the X_i. Now we use the ellipticity condition (3.2), the boundedness condition (3.3) and Young's inequality in the usual way to obtain

$$\begin{aligned}
\int_\Omega |\mathfrak{X}\Delta^s_{(T,\gamma)} u|^2\eta^2\,dq \ &\le C\bigg(\int_\Omega |\Delta^s_{(T,\gamma)} u|^2(\eta^2 + |\mathfrak{X}\eta|^2)\,dq \\
&\qquad + \int_\Omega |\Delta^s_{(T,\gamma)} f|^2\eta^2 + |\Delta^s_{(T,\gamma)} F|^2\eta^2\,dq\bigg)
\end{aligned}$$

with $C = C(n, \lambda, L)$. Exploiting the properties of η, and applying Lemma 2.1 to estimate $\Delta^s_{(T,\gamma)} F$ and $\Delta^s_{(T,\gamma)} f$ by TF and $Tf_i \in L^2_{loc}(\Omega, \mathbb{R}^N)$ on the larger ball $B_{2r}(0)$, we obtain:

$$\int_{B_{r/2}} |\mathfrak{X}\Delta^s_{(T,\gamma)} u|^2\,dq \ \le \frac{C}{r^2}\bigg(\int_{B_r(0)} |\Delta^s_{(T,\gamma)} u|^2\,dq \ + \int_{B_{2r}(0)} |Tf|^2 + |TF|^2\,dq\bigg). \tag{3.7}$$

At this point, we use Lemma 2.2 to bound $\mathfrak{X}\Delta^s_{(T,\gamma)}u$ from below by $\Delta^s_{(T,1/2)}(\Delta^s_{(T,\gamma)}u)$:

$$\int\limits_{B_{r/4}(0)} \frac{|u(qe^{2sT}) - 2u(qe^{sT}) + u(q)|^2}{s^{2(\gamma+\frac{1}{2})}}\,dq \;=\; \int\limits_{B_{r/4}(0)} \left|\Delta^s_{(T,1/2)}(\Delta^s_{(T,\gamma)}u)\right|^2\,dq$$
$$\leq 8 \int\limits_{B_{r/2}(0)} |\mathfrak{X}\Delta^s_{(T,\gamma)}u|^2\,dq\ .$$

Together with (3.7), this gives an estimate for a higher-order symmetric difference quotient by a lower-order simple difference quotient:

$$\int\limits_{B_{r/4}(0)} \frac{|u(qe^{2sT}) - 2u(qe^{sT}) + u(q)|^2}{s^{2(\gamma+\frac{1}{2})}}\,dq$$
$$\leq C(r)\Bigg(\int\limits_{B_r(0)} |\Delta^s_{(T,\gamma)}u|^2\,dq \;+\; \int\limits_{B_{2r}(0)} |Tf|^2 + |TF|^2\,dq \Bigg). \tag{3.8}$$

Let us first set $\gamma = 1/2$ and $r = 1$. Lemma 2.2 allows us to control the term $\Delta^s_{(T,1/2)}u$ on the R.H.S. by $\mathfrak{X}u$, giving an upper bound on the symmetric difference quotient of order 1:

$$\int\limits_{B_{1/4}(0)} \frac{|u(qe^{2sT}) - 2u(qe^{sT}) + u(q)|^2}{s^2}\,dq \;\leq C \int\limits_{B_2(0)} |\mathfrak{X}u|^2 + |Tf|^2 + |TF|^2\,dq\ .$$

Lemma 2.3 now yields a bound on the L^2-norm of $\Delta^s_{(T,\beta)}u$ for any $\beta \in (0,1)$, in particular for $\beta = 2/3$:

$$\|\Delta^s_{(T,2/3)}u\|^2_{L^2(B_{1/4}(0))} \leq C \int\limits_{B_2(0)} |u|^2 + |\mathfrak{X}u|^2 + |Tf|^2 + |TF|^2\,dq\ .$$

Having gained control over $\Delta^s_{(T,2/3)}u$, we set $\gamma = 2/3$ and $r = 1/4$ in (3.8), and repeat the last step. This gives

$$\int\limits_{B_{1/16}(0)} \frac{|u(qe^{2sT}) - 2u(qe^{sT}) + u(q)|^2}{s^{2\cdot\frac{7}{6}}}\,dq$$
$$\leq C \int\limits_{B_2(0)} |u|^2 + |\mathfrak{X}u|^2 + |Tf|^2 + |TF|^2\,dq\ ,$$

so that with a repeated application of Lemma 2.3, we can estimate the full difference quotient in the T direction:

$$\|\Delta^s_{(T,1)}u\|^2_{L^2(B_{1/16}(0))} \leq C \int\limits_{B_2(0)} |u|^2 + |\mathfrak{X}u|^2 + |Tf|^2 + |TF|^2\,dq\ .$$

In particular, this means that u is weakly differentiable in the T direction with $Tu \in L^2(B_{1/16}(0))$ and

$$\|Tu\|^2_{L^2(B_{1/16}(0))} \leq C \int\limits_{B_2(0)} |u|^2 + |\mathfrak{X}u|^2 + |Tf|^2 + |TF|^2\,dq\ .$$

This in turn enables us to use $\gamma = 1$ and $r = 1/16$ in (3.7) to establish $\mathfrak{X}Tu \in L^2(B_{1/32}(0))$:

$$\|\mathfrak{X}Tu\|^2_{L^2(B_{1/32}(0))} \le \sup_{0<s<1024} \int_{B_{1/32}(0)} |\mathfrak{X}\Delta^s_{(T,1)}u|^2\, dq$$
$$\le C \int_{B_2(0)} |u|^2 + |\mathfrak{X}u|^2 + |Tf|^2 + |TF|^2\, dq\ .$$

Finally, we sum up the estimates for Tu and $\mathfrak{X}Tu$ and eliminate $\mathfrak{X}u$ from the R.H.S. via the Caccioppoli inequality (3.4):

$$\|Tu\|^2_{L^2(B_{1/32}(0)} + \|\mathfrak{X}Tu\|^2_{L^2(B_{1/32}(0)} \le C \int_{B_4(0)} |u|^2 + |f|^2 + |Tf|^2 + |F|^2 + |TF|^2\, dq\ .$$

From this, a standard rescaling argument yields (ii), keeping in mind Lemma 2.4.

To prove (3), we test the weak formulation of (3.1) with $T\varphi$. Then integration by parts and using $X_iT = TX_i$ gives the desired result. □

With $Tu \in HW^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ thus established, we can deal with the commutator term in (3.5) which does not vanish when testing (3.1) with an X_{i_0}-direction difference quotient instead of a vertical one. Applying the standard difference quotient method while keeping track of the additional commutators, we obtain $X_{i_0}u \in HW^{1,2}_{loc}(\Omega, \mathbb{R}^N)$, and hence the desired estimate. For details of the proof, we refer the reader to [1,2].

Lemma 3.2 *Let $u \in HW^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ be a weak solution of* (3.1) *in Ω, and f_i^α, ∇f_i^α, F^α, $TF^\alpha \in L^2_{loc}(\Omega)$ for all $i = 1, \ldots, 2n$, $\alpha = 1, \ldots, 2n$. Then for all $i_0 = 1, \ldots, 2n$ the following hold*:

1. $X_{i_0}u \in HW^{1,2}_{loc}(\Omega, \mathbb{R}^N)$.
2. *There exists $C = C(n, \lambda, L)$ such that*

$$\int_{B_R(q_0)} R^4|\mathfrak{X}X_{i_0}u|^2\, dq$$
$$\le C \int_{B_{2R}(q_0)} |u|^2 + R^4|F|^2 + R^8|TF|^2 + \sum_{i=1}^{2n} \left(R^2|f_i|^2 + R^4|\mathfrak{X}f_i|^2 + R^6|Tf_i|^2\right)\, dq$$

   holds for every $B_{2R}(q_0) \subset \Omega$.
3. *Suppose $F \in HW^{1,2}_{loc}(\Omega)$. Then the function $w = X_{i_0}u$ is a weak solution to the X_{i_0}-differentiated system*

$$X_i\left(A^{\alpha\beta}_{ij}X_jw^\beta + A^{\alpha\beta}_{ij}[X_{i_0}, X_j]u^\beta + X_{i_0}f_i^\alpha\right) + [X_{i_0}, X_i]\left(A^{\alpha\beta}_{ij}X_ju^\beta + f_i\right) = -X_{i_0}F^\alpha.$$

Now that we have the full gradient $\nabla u \in HW^{1,2}$, and know how to differentiate system (3.1), we iterate the results of the last two theorems to achieve higher differentiablity.

Lemma 3.3 *Let $u \in HW^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ be a weak solution of*

$$-X_i(A^{\alpha\beta}_{ij}X_ju^\beta) = F^\alpha \quad \alpha = 1, \ldots, N.$$

For a multiindex $I_h = (i_1, \ldots, i_h) \in \{1, \ldots, 2n\}^k$, $h \in \mathbb{N}_0$ and $\ell \in N_0$ define

$$v_{(\ell,h)} = T^\ell \mathfrak{X}^{I_h} u^\beta = T \cdots T X_{i_h} \cdots X_{i_1} u^\beta.$$

Suppose $F \in C^\infty(\Omega, \mathbb{R}^N)$*. Then* $v_{(\ell,h)}$ *is in* $HW^{1,2}_{loc}(\Omega, \mathbb{R}^n)$*, and satisfies*

$$-X_i\left(A^{\alpha\beta}_{ij} X_j v_{(\ell,h)} + f^\alpha_{i(\ell,h)}\right) = F^\alpha_{(\ell,h)} \quad \alpha = 1, \dots, N \tag{3.9}$$

with:

$$\begin{aligned} F^\alpha_{(0,0)} &= F^\alpha \\ F^\alpha_{(\ell,h+1)} &= X_{i_{h+1}} F^\alpha_{(\ell,h)} + [X_{i_{h+1}}, X_i](A^{\alpha\beta}_{ij} X_j v^\beta_{(\ell,h)} + f^\alpha_{i(\ell,h)}) \\ F^\alpha_{(\ell+1,h)} &= T F^\alpha_{(\ell,h)} \end{aligned}$$

and

$$\begin{aligned} f^\alpha_{i(0,0)} &= 0 \\ f^\alpha_{i(\ell,h+1)} &= A^{\alpha\beta}_{ij}[X_{i_{h+1}}, X_j] v^\beta_{(\ell,h)} + X_{i_{h+1}} f^\alpha_{i(\ell,h)} \\ f^\alpha_{i(\ell+1,h)} &= T f^\alpha_{i(\ell,h)}. \end{aligned}$$

Proof By induction: To increase the index ℓ, we use Theorem 1.1 (3) with $v_{(\ell+1,h)} = Tv_{(\ell,h)}$; for $h \to h+1$, we apply Lemma 3.2 (3) with $v_{(\ell,h+1)} = X_{i_{h+1}} v_{(\ell,h)}$. □

Remark We may construct the following explicit formulae from the recursion in Lemma 3.3:

$$\begin{aligned} f^\alpha_{i(\ell,h)} &= \sum_{m=1}^h A^{\alpha\beta}_{ij}[X_{i_m}, X_j] T^\ell X_{i_h} \cdots \hat{X}_{i_m} \cdots X_{i_1} u^\beta \\ F^\alpha_{(\ell,h)} &= T^\ell \mathfrak{X}^{I_h} F^\alpha + \sum_{m=1}^h A^{\alpha\beta}_{ij}[X_{i_m}, X_i] T^\ell X_{i_h} \cdots X_{i_{m+1}} X_j X_{i_{m-1}} \cdots X_{i_1} u^\beta \\ &\quad + \sum_{1 \le k < m \le h} A^{\alpha\beta}_{ij}[X_{i_m}, X_i]\,[X_{i_k}, X_j] T^\ell X_{i_h} \cdots \hat{X}_{i_m} \cdots \hat{X}_{i_k} \cdots X_{i_1} u^\beta \end{aligned}$$

for all $\ell \in \mathbb{N}_0$ and $h \in \mathbb{N}$ with $h \ge 2$ for F. As usual, the notation $\hat{X}_{i_m}$ stands for omitting the vector field X_{i_m}. □

We need an analog to the $W^{k,2}$ norm that keeps track of the horizontal and vertical derivatives separately:

$$\|u\|^2_{\ell,h,R} = \sum_{\substack{0 \le j \le \ell \\ 0 \le |I| \le h}} \|T^j \mathfrak{X}^I u\|^2_{L^2(B_R(q_0))}$$

With this norm, we can estimate higher derivatives of u: The formulae from the previous remark are used to convert $f^\alpha_{i(\ell,h)}$ and $F^\alpha_{(\ell,h)}$ into derivatives of u. Then, applying Lemma 3.2 to (3.9), the horizontal derivatives are reduced in steps of 2, until only T-derivatives are left. These are then successively eliminated by repeated appication of Theorem 1.1. (For technical details, see [2,8].) Thus, we arrive at the following theorem:

Theorem 3.4 *Let u be as in Lemma 3.3, and* $B_{2R}(q_0) \Subset \Omega$*. Then*

$$R^{4\ell+4h+4} \|u\|^2_{\ell,2h+2,R} \le C\left(\|u\|_{0,0,2R} + R^4 \|F\|^2_{\ell+3h+1,2h,2R}\right)$$

holds for any $h, \ell \in \mathbb{N}_0$ *and* $R \le 1$*, with* $C = C(L, \lambda, n, \ell, h)$*.* □

The L^2-boundedness of all derivatives of u now allows us to derive the smoothness of u and the a priori estimates in Theorem 1.2 for solutions of homogeneous systems, i.e. when $F \equiv 0$, in a similar manner to the Euclidean setting: In view of the last theorem, we first apply the Sobolev embedding theorem to derivatives of $v_{(\ell,h)}$ with ℓ, h suitably large, to obtain (Hölder) continuity and thereby local boundedness of the derivatives of u, as well as the estimate

$$\sup_{B_R(q_0)} \left(|u|^2 + R^2|\mathfrak{X}u|^2 + R^4|\mathfrak{X}^2u|^2\right) \leq C \fint_{B_{2R}(q_0)} |u|^2\, dq\ . \tag{3.10}$$

We can then show the a priori estimates by arguments similar to those in [2], but adapting them for exponents $p \geq 2$ as in [12].

Proof of Theorem 1.2 We first assume $r \leq R/2$.

As in the Euclidean setting, we define $w = u - u_{q_0,R}$, noting that w solves (3.1) and $\mathfrak{X}u = \mathfrak{X}w$. We use the supremum estimate (3.10) and the Poincaré inequality (2.3) to show the desired estimate (1.6):

$$\begin{aligned}\fint_{B_r(q_0)} |\mathfrak{X}u|^p\, dq &= \fint_{B_r(q_0)} |\mathfrak{X}w|^p\, dq \leq \Big(\sup_{B_r(q_0)} |\mathfrak{X}w|^2\Big)^{p/2} \leq CR^{-p} \fint_{B_R(q_0)} |w|^p\, dq \\ &= CR^{-p} \fint_{B_R(q_0)} |u - u_{q_0,R}|^p\, dq \leq C \fint_{B_R(q_0)} |\mathfrak{X}u|^p\, dq\ .\end{aligned}$$

For the proof of (1.5), we first note that the function

$$v(q) = u(q) - u_{q_0,R} - \sum_{i=1}^{2n} (x_i(q) - x_i(q_0))(X_iu)_{q_0,R}$$

is a solution to (3.1). We have $v_{q_0,R} = 0$ and $\mathfrak{X}v = \mathfrak{X}u - (\mathfrak{X}u)_{q_0,R}$, and further $\mathfrak{X}^2u = \mathfrak{X}^2v$. We prove (1.5) by using the Poincaré inequality twice and an application of (3.10):

$$\begin{aligned}&\fint_{B_r(q_0)} |\mathfrak{X}u - (\mathfrak{X}u)_{q_0,r}|^p\, dq \\ &\leq Cr^p \fint_{B_r(q_0)} |\mathfrak{X}^2u|^p\, dq = Cr^p \fint_{B_r(q_0)} |\mathfrak{X}^2v|^p\, dq \\ &\leq Cr^p\Big(\sup_{B_{R/2}(q_0)} |\mathfrak{X}^2v|^2\Big)^{p/2} \leq Cr^p(R^{-4})^{p/2} \fint_{B_R(q_0)} |v|^p\, dq \\ &= CR^{-2p} \fint_{B_R(q_0)} |v - v_{q_0,R}|^p\, dq \leq Cr^pR^{-p} \fint_{B_R(q_0)} |\mathfrak{X}v|^p\, dq \\ &= C\left(\frac{r}{R}\right)^p \fint_{B_R(q_0)} |\mathfrak{X}u - (\mathfrak{X}u)_{q_0,r}|^p\, dq\ .\end{aligned}$$

In the case $r > R/2$, we simply rescale the ball $B_r(q_0)$ to $B_R(q_0)$ multiplying the radius by $\frac{R}{r} \in (1/2, 1]$. ◻

Remark From the proof of the previous theorem, we also obtain the estimate

$$\fint_{B_r(q_0)} |\mathfrak{X}^2 u|^p \, dq \ \leq C \fint_{B_R(q_0)} |\mathfrak{X}^2 u|^p \, dq$$

by applying the Poincaré inequalty once more.

4 Nonlinear systems with superquadratic growth

In this section, we consider the case of a homogeneous system of second-order subelliptic equations in divergence form with superquadratic growth, i.e. weak solutions to (1.1) under the structure conditions (1.2), (1.3) and (1.4).

A weak solution of (1.1) is $u \in HW^{1,p}(\Omega, \mathbb{R}^N)$ such that

$$\int_{\Omega} A(q, \mathfrak{X}u) \cdot \mathfrak{X}\varphi \, dq \ = \int_{\Omega} A_i^{\alpha}(q, \mathfrak{X}u) \, X_i \varphi^{\alpha} \, dq \ = 0 \quad \forall \varphi \in C_c^{\infty}(\Omega, \mathbb{R}^N). \tag{4.1}$$

The continuity of $\frac{\partial A}{\partial z}$ combined with the growth condition allows us to deduce for each $M > 0$ the existence of a concave and non-decreasing modulus of continuity $\omega_M : [0, \infty) \to [0, \infty)$ with $\omega_M(0) = 0$, and also concave square ω_M^2, such that we have

$$\left| \frac{\partial A}{\partial z}(\tilde{q}, \tilde{z}) - \frac{\partial A}{\partial z}(q, z) \right| \leq \omega_M \left(\|q^{-1}\tilde{q}\| + |\tilde{z} - z| \right) \tag{4.2}$$

for all $q, \tilde{q} \in \Omega$, $z, \tilde{z} \in \mathbb{R}^{2n \times N}$ with $|z|, |\tilde{z}| \leq M$. The M-dependency is necessary due to the growth of $\frac{\partial A}{\partial z}$ in $|z|$, which means that ω_M increases with M.

Our approach follows the line of argument in [7, Sect. 3]. We first show the $\mathcal{A}$-harmonic approximation lemma. Next, we derive a Caccioppoli inequality. The main step then consists of proving an excess decay estimate for u provided some smallness conditions are satisfied. We achieve this by comparing an appropriately rescaled version of u to a solution h of (1.1) frozen at some point. It is here that the $\mathcal{A}$-harmonic approximation lemma is used to ensure the closeness of this solution to the rescaled version of u. The estimate thus obtained will finally enable us to show Hölder continuity of $\mathfrak{X}u$, i.e. that u is in the Folland-Stein-Class $\Gamma^{1,\beta}$.

Roughly speaking, the $\mathcal{A}$-harmonic approximation lemma states that if a g is an "almost-solution" to a constant coefficient system with coefficient matrix $\mathcal{A}$, then we can find an actual solution h that is arbitrarily close in an appropriate sense. The precise statement, adapted to the $p \geq 2$ situation, is as follows:

Lemma 4.1 *Consider fixed positive $\lambda, L > 0$, $\gamma \in (0, 1]$, and $n, N \in \mathbb{N}$ with $n \geq 2$. Then for any given $\varepsilon > 0$ there exists $\delta = \delta(n, N, p, \lambda, L, \gamma, \varepsilon) \in (0, 1]$ with the following property: For any $\mathcal{A} \in Bil(\mathbb{R}^{2n \times N})$ satisfying*

$$\mathcal{A}(\nu, \nu) \geq \lambda |\nu|^2 \quad and \quad \mathcal{A}(\nu, \tilde{\nu}) \leq L |\nu| |\tilde{\nu}| \quad \forall \nu, \tilde{\nu} \in \mathbb{R}^{2n \times N} \tag{4.3}$$

and any $g \in HW^{1,p}(B_r(q_0), \mathbb{R}^N)$ *satisfying*

$$\fint_{B_r(q_0)} |\mathfrak{X}g|^2 + \gamma^{p-2}|\mathfrak{X}g|^p \, dq \ \le 1, \tag{4.4}$$

$$\left| \fint_{B_r(q_0)} \mathcal{A}(\mathfrak{X}g, \mathfrak{X}\varphi) \, dq \right| \le \delta \sup_{B_r(q_0)} |\mathfrak{X}\varphi| \quad \forall \varphi \in C_c^1\big(B_r(q_0), \mathbb{R}^{2n\times N}\big), \tag{4.5}$$

there exists a function $h \in HW^{1,p}\big(B_{r/2}(q_0), \mathbb{R}^{2n\times N}\big)$ *that is* $\mathcal{A}$*-harmonic on* $B_{r/2}(q_0)$*, with the properties*

$$\fint_{B_{r/2}(q_0)} |\mathfrak{X}h|^2 + \gamma^{p-2}|\mathfrak{X}h|^p \, dq \ \le 2^{Q+1} \tag{4.6}$$

and

$$\fint_{B_{r/2}(q_0)} \Big(\frac{|h-g|}{r/2}\Big)^2 + \gamma^{p-2}\Big(\frac{|h-g|}{r/2}\Big)^p \, dq \ \le \varepsilon. \tag{4.7}$$

Proof We argue as in the proof of [7, Lemma 2.1], with some modifications. Without loss of generality, we can set $B_r(q_0) = B_1(0) =: B$; the general case follows by rescaling.

We assume the conclusion to be false, and thus for some $\varepsilon > 0$ we find a sequence of triples $(\mathcal{A}_k, \gamma_k, g_k)$ of coefficients, parameters and functions which fulfill the conditions (4.3)–(4.5) with $\delta_k = \frac{1}{k}$, but fail to satisfy (4.7) for any $\mathcal{A}_k$-harmonic h_k obeying condition (4.6). Without loss of generality we may assume $(g_k)_{B_1} = 0$ (otherwise consider $g_k - (g_k)_{B_1}$).

We now define the auxiliary functions

$$w_k = \gamma_k^{\frac{p-2}{p}} g_k,$$

and with the Poincaré inequality and the L^2-boundedness of $\mathfrak{X}g_k$ see that $\{w_k\}_{k\in\mathbb{N}}$ is bounded in $HW^{1,p}(B, \mathbb{R}^{2n\times N})$.

By the compact embedding of $HW^{1,p}$ into L^p, we find subsequences of g_k and w_k (also denoted by g_k and w_k) as well as g, w, γ and $\mathcal{A}$ such that

$$\begin{cases} g_k \to g & \text{strongly in } L^2, \quad w_k \to w \quad \text{strongly in } L^p, \\ g_k \rightharpoonup g & \text{weakly in } HW^{1,2}, \quad w_k \rightharpoonup w \quad \text{weakly in } HW^{1,p}, \\ \mathcal{A}_k \to \mathcal{A}, & \qquad\qquad\qquad \gamma_k \to \gamma \in [0,1], \end{cases}$$

with $\mathcal{A}$ also satisfying (4.3).

Almost everywhere, we have $w = \lim_{k\to\infty} \gamma_k^{\frac{p-2}{p}} g_k = \gamma^{\frac{p-2}{p}} g$; in particular, we find by lower semicontinuity of the norm that g also fulfills (4.4), and has mean value 0 on B_1. We further find that g is $\mathcal{A}$-harmonic by using the weak $HW^{1,2}$ convergence and $\mathcal{A}_k$-harmonicity of the g_k and the convergence of the $\mathcal{A}_k$,

$$\begin{aligned} \fint_B \mathcal{A}(\mathfrak{X}g, \mathfrak{X}\varphi) \, dq \ &= \fint_B \mathcal{A}(\mathfrak{X}(g-g_k), \mathfrak{X}\varphi) \\ &\quad + \fint_B (\mathcal{A} - \mathcal{A}_k)(\mathfrak{X}g, \mathfrak{X}\varphi) \, dq \ + \fint_B \mathcal{A}_k(\mathfrak{X}g, \mathfrak{X}\varphi) \, dq \ \stackrel{k\to\infty}{\longrightarrow} 0, \end{aligned}$$

so in particular g is smooth inside of B due the results in Sect. 3.

We now consider for the Dirichlet problems

$$\fint_{B_{3/4}} \mathcal{A}_k(\mathfrak{X}v, \mathfrak{X}\varphi)\, dq \;= 0 \qquad \forall \varphi \in C_c^1(B_{3/4}, \mathbb{R}^N), \qquad v - g \in HW_0^{1,2}(B_{3/4}, \mathbb{R}^N),$$

and with the Lax–Milgram theorem find the respective solutions h_k, which are smooth in $B_{3/4}$ and in particular on $\overline{B}_{1/2}$. Using the ellipticity of $\mathcal{A}_k$, exploiting the properties of h_k and g, and applying Hölder's inequality, we see that

$$\begin{aligned}
\lambda \fint_{B_{3/4}} |\mathfrak{X}h_k - \mathfrak{X}g|^2\, dq \;&\le \fint_{B_{3/4}} \mathcal{A}_k(\mathfrak{X}h_k - \mathfrak{X}g, \mathfrak{X}h_k - \mathfrak{X}g)\, dq \\
&= - \fint_{B_{3/4}} (\mathcal{A} - \mathcal{A}_k)(\mathfrak{X}g, \mathfrak{X}h_k - \mathfrak{X}g)\, dq \\
&\le \left(\tfrac{3}{4}\right)^{-\frac{Q}{2}} |\mathcal{A} - \mathcal{A}_k| \bigg(\fint_{B_{3/4}} |\mathfrak{X}h_k - \mathfrak{X}g|^2\, dq \bigg)^{1/2},
\end{aligned}$$

i.e. since $h_k - g \in HW_0^{1,2}(B_{3/4})$, h_k converges to g strongly in $HW^{1,2}(B_{3/4}, \mathbb{R}^N)$. By means of the supremum estimate (3.10), the Poincaré inequality, and the boundedness of $\mathfrak{X}g$, we see that $|h_k|$ and $|\mathfrak{X}h_k|$ are uniformly bounded on $B_{1/2}$:

$$\sup_{B_{1/2}} \left(|h_k|^2 + |\mathfrak{X}h_k|^2\right) \le C \fint_{B_{3/4}} |h_k|^2\, dq \;\le C\bigg(\fint_{B_{3/4}} |h_k - g|^2\, dq \;+ \fint_{B_1} |\mathfrak{X}g|^2\, dq \bigg) \overset{k\to\infty}{\to} C$$

Hence, the $HW^{1,2}(B_{3/4})$-convergence of h_k to g and the L^p-bound on $\gamma^{\frac{p-2}{p}} \mathfrak{X}g$ imply that $\gamma_k^{\frac{p-2}{p}} h_k$ converges to $\gamma^{\frac{p-2}{p}} g$ strongly in $W^{1,p}(B_{1/2})$. In particular, there holds

$$\lim_{k\to\infty} \fint_{B_{1/2}} |\mathfrak{X}h_k|^2 + \gamma_k^{p-2} |\mathfrak{X}h_k|^p\, dq \;\le 2^Q \fint_{B_1} |\mathfrak{X}g|^2 + \gamma^{p-2}|\mathfrak{X}g|^p\, dq \;\le 2^Q,$$

so the h_k satisfy (4.6) for k greater than some k_0. Combining the convergence of h_k and g_k to g in the appropriate spaces, we conclude

$$\lim_{k\to\infty} \fint_{B_{1/2}} \left(\frac{|h_k - g_k|}{1/2}\right)^2 + \gamma_k^{p-2} \left(\frac{|h_k - g_k|}{1/2}\right)^p dq \;= 0.$$

Since the set of $\mathcal{A}_k$-harmonic functions satisfying (4.6) is non-empty, this yields the desired contradiction. □

Next, we deduce a Caccioppoli-type inequality.

Lemma 4.2 *Let* $\xi \in \mathbb{R}^N$ *and* $P \in \mathbb{R}^{2n\times N}$ *with* $|P| \le M$ *for some* $M > 0$*; and let* $u \in HW^{1,p}(\Omega, \mathbb{R}^N)$ *be a weak solution to* (1.1) *with the coefficient function* A *satisfying the* *structure conditions* (1.2)–(1.4). *Then on every* $B_R(q_0) \subset \Omega$ *we have*

$$\fint_{B_{R/2}(q_0)} |\mathfrak{X}u - P|^2 + |\mathfrak{X}u - P|^p \, dq$$

$$\leq C_c \left(\fint_{B_R(q_0)} \left| \frac{u - \xi - P(x - x_0)}{R} \right|^2 + \left| \frac{u - \xi - P(x - x_0)}{R} \right|^p \, dq \; + R^{2\beta} \right), \tag{4.8}$$

where the constant C_c *depends only on* L/λ, M *and* p, *and* x *denotes the horizontal component of* $q = (x, t)$.

Proof We set $v(q) = u(q) - \xi - P(x - x_0)$, and take the standard cut-off function $\varphi = \eta^p v$, where $\eta \in C_c^\infty(B_R(q_0))$ with $\mathbb{1}_{B_{R/2}(q_0)} \leq \eta \leq \mathbb{1}_{B_R(q_0)}$ and $|\mathfrak{X}\eta| \leq \frac{12}{R}$. By simple calculation, we check that

$$\mathfrak{X}v(q) = \mathfrak{X}u(q) - \mathfrak{X}\big(P(x - x_0)\big) = \mathfrak{X}u(q) - P.$$

Testing (4.1) with φ, then repeating this with the second coefficient of A fixed as P, then with both coefficients fixed, and adding up the equations, we obtain:

$$\begin{aligned}
&\fint_{B_R(q_0)} \big(A(q, \mathfrak{X}u) - A(q, P)\big) \cdot (\mathfrak{X}u - P)\eta^p \, dq \\
&= -p \fint_{B_R(q_0)} \big(A(q, \mathfrak{X}u) - A(q, P)\big) \cdot \eta^{p-1} v \otimes \mathfrak{X}\eta \, dq \\
&\quad - \fint_{B_R(q_0)} \big(A(q, P) - A(q_0, P)\big) \cdot \mathfrak{X}\varphi \, dq \quad =: I + II.
\end{aligned} \tag{4.9}$$

We estimate the first term using the properties of η, growth condition (1.4) and Young's inequality with parameter $\varepsilon > 0$:

$$\begin{aligned}
|I| &\leq \frac{12p}{R} \fint_{B_R(q_0)} \int_0^1 \left| \frac{\partial A}{\partial z}\big(q, P + t(\mathfrak{X}u - P)\big) \right| dt \, |v||Xu - P|\eta^{p-1} \, dq \\
&\leq \frac{12Lp}{R} \fint_{B_R(q_0)} \int_0^1 \big(1 + |P + t(\mathfrak{X}u - P)|^{p-2}\big) \, dt \, |v||Xu - P|\eta^{p-1} \, dq \\
&\leq \frac{12 \cdot \max\{1, 2^{p-3}\}Lp}{R} \fint_{B_R(q_0)} \big(1 + |P|^{p-2} + |(\mathfrak{X}u - P)|^{p-2}\big) \, |v||\mathfrak{X}u - P|\eta^{p-1} \, dq \\
&\leq \frac{c(p)L}{R} \fint_{B_R(q_0)} \big(1 + |P|^{p-2}\big) \, |v|\eta^{p/2-1}|\mathfrak{X}u - P|\eta^{p/2} + |\mathfrak{X}u - P|^{p-1}|v|\eta^{p-1} \, dq \\
&\leq \fint_{B_R(q_0)} \varepsilon|\mathfrak{X}u - P|^2\eta^p + \frac{c(p)L^2}{\varepsilon R^2} \big(1 + |P|^{p-2}\big)^2 |v|^2\eta^{p-2} + \varepsilon|\mathfrak{X}u - P|^p\eta^p \\
&\quad + \left(\frac{\varepsilon^{1-p}c(p)L}{R} \right)^p |v|^p \, dq
\end{aligned}$$

$$\leq \varepsilon \fint_{B_R(q_0)} \left(|\mathfrak{X}u - P|^2 + |\mathfrak{X}u - P|^p\right) \eta^p \, dq \; + \frac{c(p)L^2}{\varepsilon} \fint_{B_R(q_0)} \left(1+|P|^{p-2}\right)^2 \left|\frac{v}{R}\right|^2 dq$$
$$+\varepsilon^{1-p} c(p) L^p \fint_{B_R(q_0)} \left|\frac{v}{R}\right|^p dq \, .$$

For the second integral, we use structure condition (1.2), the properties of φ and Young's inequality to obtain

$$\begin{aligned}
|II| \leq L & \fint_{B_R(q_0)} |q_0^{-1}q|^\beta \left(1 + |P|^{p-1}\right) |\mathfrak{X}\varphi| \, dq \\
&\leq L R^\beta \left(1 + |P|^{p-1}\right) \fint_{B_R(q_0)} \eta^p |\mathfrak{X}u - P| + p\eta^{p-1}|v||\mathfrak{X}\eta| \, dq \\
&\leq \fint_{B_R(q_0)} \varepsilon |\mathfrak{X}u - P|^2 \eta^p + \frac{L^2}{4\varepsilon} R^{2\beta} \left(1 + |P|^{p-1}\right)^2 \eta^p \\
&\quad + \frac{c(p)\varepsilon}{R^2} |v|^2 \eta^p + \frac{L^2}{4\varepsilon} R^{2\beta} \left(1 + |P|^{p-1}\right)^2 \eta^{p-2} \, dq \\
&\leq \fint_{B_R(q_0)} \varepsilon |\mathfrak{X}u - P|^2 \eta^p \, dq \; + c(p)\varepsilon \fint_{B_R(q_0)} \left|\frac{v}{R}\right|^2 dq \; + \frac{L^2}{2\varepsilon} R^{2\beta} \left(1 + |P|^{p-1}\right)^2
\end{aligned}$$

Finally, we recall the technical fact that for vectors $a, b \in \mathbb{R}^{2n \times N}$ there holds

$$\int_0^1 |a + tb|^{p-2} \, dt \geq c(p) |b|^{p-2}$$

with $c(p) \in (0, 1]$ (for a proof, see e.g. [8]), and apply it on the L.H.S. to estimate it from below:

$$\begin{aligned}
& \fint_{B_R(q_0)} \left(A(q, \mathfrak{X}u) - A(q, P)\right) \cdot (\mathfrak{X}u - P) \eta^p \, dq \\
&= \fint_{B_R(q_0)} \int_0^1 \frac{\partial}{\partial t} A\left(q, P + t(\mathfrak{X}u - P)\right) dt \cdot (\mathfrak{X}u - P) \eta^p \, dq \\
&= \fint_{B_R(q_0)} \int_0^1 \frac{\partial A}{\partial z}\left(q, P + t(\mathfrak{X}u - P)\right) dt (\mathfrak{X}u - P) \cdot (\mathfrak{X}u - P) \eta^p \, dq \\
&\geq \lambda \fint_{B_R(q_0)} \int_0^1 \left(1 + |P + t(\mathfrak{X}u - P)|^{p-2}\right) dt |\mathfrak{X}u - P|^2 \eta^p \, dq \\
&\geq c(p)\lambda \fint_{B_R(q_0)} \left(1 + |\mathfrak{X}u - P|^{p-2}\right) |\mathfrak{X}u - P|^2 \eta^p \, dq
\end{aligned}$$

$$= c(p)\lambda \fint_{B_R(q_0)} \left(|\mathfrak{X}u - P|^2 + |\mathfrak{X}u - P|^p\right) \eta^p \, dq \, .$$

Combining these estimates, choosing ε appropriately and using the properties of η completes the proof. □

For the main step, the proof of the excess decay estimate, we adapt the calculations from [7] to the situation of p-growth.

Lemma 4.3 *Fix* $\alpha \in (\beta, 1)$. *For given* $M > 0$, *there exist constants* $c_6, c_7 \in \mathbb{R}$, $\tau \in (0, \frac{1}{4}]$, $\delta \in (0, 1]$ *with the following properties*: *If the smallness conditions*

$$\omega^2\big(\Phi(q_0, R)\big) + \Phi(q_0, R) \le \frac{\delta^2}{2}, \qquad c_6 R^{2\beta} \le \delta^2$$

and

$$|(\mathfrak{X}u)_{q_0,R}| \le M$$

are fulfilled, we have

$$\Phi_p(q_0, \tau R) \le \tau^{2\alpha} \Phi(q_0, R) + c_7 R^{2\beta},$$

where $\Phi_p(q_0, R)$ *denotes the* p*-excess*

$$\Phi_p(q_0, R) = \fint_{B_r(q_0)} |\mathfrak{X}u - (\mathfrak{X}u)_{q_0,r}|^2 + |\mathfrak{X}u - (\mathfrak{X}u)_{q_0,r}|^p \, dq \, .$$

The constants have the following dependencies: $\tau = \tau(n, L, \lambda, p, M, \alpha)$, $\delta = \delta(n, N, L, \lambda, p, M, \alpha)$, $c_6 = c_6(n, L, \lambda, p, M, \alpha)$, $c_7 = c_7(n, L, \lambda, p, M)$. *In particular*, c_6 *and* c_7 *are non-decreasing in* M.

Proof We want to compare the weak solution u of (1.1) to a solution h of the system with frozen coefficients in some point. The **first step** will be to find a h sufficiently close to u via the $\mathcal{A}$-harmonic approximation lemma, so we have to ensure that the prerequisites (4.3)–(4.5) are satisfied. For this, we have to rescale u suitably.

Let $0 < R \le 1$, take a test function $\varphi \in C_c^\infty(B_R(q_0), \mathbb{R}^N)$, and consider fixed $|z| < M$. We further define

$$\Phi = \fint_{B_R(q_0)} |\mathfrak{X}u - z|^2 + |\mathfrak{X}u - z|^p \, dq \, .$$

Using the fact that

$$\int_{B_R(q_0)} A(q, \mathfrak{X}u)\mathfrak{X}\varphi \, dq = 0 = \int_{B_R(q_0)} \frac{\partial A}{\partial z}(q_0, z)\mathfrak{X}\varphi \, dq \, ,$$

we first re-write the integral appearing (4.5):

$$
\begin{aligned}
&\fint_{B_R(q_0)} \frac{\partial A}{\partial z}(q_0, z)(\mathfrak{X}u - z) \cdot \mathfrak{X}\varphi \, dq \\
&= \int_{B_R(q_0)} \int_0^1 \frac{\partial A}{\partial z}(q_0, z) \, dt \; (\mathfrak{X}u - z) \cdot \mathfrak{X}\varphi \, dq \\
&= \fint_{B_R(q_0)} \int_0^1 \left(\frac{\partial A}{\partial z}(q_0, z) - \frac{\partial A}{\partial z}\big(q_0, z + t(\mathfrak{X}u - z)\big) \right) dt \; (\mathfrak{X}u - z) \cdot \mathfrak{X}\varphi \, dq \\
&\qquad + \fint_{B_R(q_0)} \big[A(q_0, \mathfrak{X}u) - A(q_0, z) \big] \cdot \mathfrak{X}\varphi \, dq \\
&= \fint_{B_R(q_0)} \int_0^1 \left(\frac{\partial A}{\partial z}(q_0, z) - \frac{\partial A}{\partial z}\big(q_0, z + t(\mathfrak{X}u - z)\big) \right) dt \; (\mathfrak{X}u - z) \cdot \mathfrak{X}\varphi \, dq \\
&\quad + \fint_{B_R(q_0)} \big[A(q_0, \mathfrak{X}u) - A(q, \mathfrak{X}u) \big] \cdot \mathfrak{X}\varphi \, dq \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}
$$

We split $B_R(q_0)$ into two parts:

$$
S_1 = \big\{ q \in B_R(q_0) : |\mathfrak{X}u(q) - z| \le 1 \big\} \quad \text{and} \quad S_2 = \big\{ q \in B_R(q_0) : |\mathfrak{X}u - z| > 1 \big\}.
$$

On S_1, we have $|z| + |\mathfrak{X}u - z| < M + 1$, so the first term can be estimated using (4.2), Hölder's and Jensen's inequality as follows:

$$
\begin{aligned}
|I_1| &= \frac{1}{|B_R(q_0)|} \Bigg| \int_{B_R(q_0) \cap S_1} \int_0^1 \left(\frac{\partial A}{\partial z}(q_0, z) - \frac{\partial A}{\partial z}(q_0, z + t(\mathfrak{X}u - z)) \right) dt \\
&\qquad\qquad \cdot (\mathfrak{X}u - z) \cdot \mathfrak{X}\varphi \, dq \Bigg| \\
&\le \frac{1}{|B_R(q_0)|} \int_{B_R(q_0) \cap S_1} \int_0^1 \omega_{M+1}\big(t^2 |\mathfrak{X}u - z|^2\big) \, dt \; |\mathfrak{X}u - z| \, |\mathfrak{X}\varphi| \, dq \\
&\le \omega_{M+1} \left(\fint_{B_R(q_0)} |\mathfrak{X}u - z|^2 \, dq \right) \left(\fint_{B_R(q_0)} |\mathfrak{X}u - z|^2 \, dq \right)^{1/2} \sup_{B_R(q_0)} |\mathfrak{X}\varphi| \\
&= \sqrt{\Phi} \; \omega_{M+1}(\Phi) \sup_{B_R(q_0)} |\mathfrak{X}\varphi|,
\end{aligned}
$$

and for the second term we use the Hölder continuity condition (1.2):

$$
\begin{aligned}
|II_1| &= \frac{1}{|B_R(q_0)|}\left| \int\limits_{B_R(q_0)\cap S_1} \big[A(q_0,\mathfrak{X}u) - A(q,\mathfrak{X}u)\big]\mathfrak{X}\varphi\, dq \right| \\
&\le \frac{1}{|B_R(q_0)|} \int\limits_{B_R(q_0)\cap S_1} L|q^{-1}q_0|^{\beta}(1+|\mathfrak{X}u|^{p-1})|\mathfrak{X}\varphi|\, dq \\
&\le \frac{LR^{\beta}}{|B_R(q_0)|}|B_R(q_0)\cap S_1|\cdot\big(1+(M+1)^{p-1}\big) \sup_{B_R(q_0)} |\mathfrak{X}\varphi| \\
&\le 2^{p-1}L(1+M^{p-1})R^{\beta} \sup_{B_R(q_0)} |\mathfrak{X}\varphi|.
\end{aligned}
$$

The definition of the set S_2 implies $|\mathfrak{X}u - z|^{p-1} \le |\mathfrak{X}u - z|^p$ on it. Therefore, with growth condition (1.4) and $|z| < M$ we obtain

$$
\begin{aligned}
|I_2| &= \frac{1}{|B_R(q_0)|}\Bigg| \int\limits_{B_R(q_0)\cap S_2} \int\limits_0^1 \left(\frac{\partial A}{\partial z}(q_0,z) - \frac{\partial A}{\partial z}(q_0, z+t(\mathfrak{X}u - z))\right) dt \\
&\qquad\qquad \cdot (\mathfrak{X}u - z)\cdot \mathfrak{X}\varphi\, dq \Bigg| \\
&\le \frac{2L}{|B_R(q_0)|} \int\limits_{B_R(q_0)\cap S_2} \big(1+(|z|+|\mathfrak{X}u-z|)^{p-2}\big)\, |\mathfrak{X}u - z|\,|\mathfrak{X}\varphi|\, dq \\
&\le 2^{p-1}L(1+M^{p-2}) \fint\limits_{B_R(q_0)} |\mathfrak{X}u - z| + |\mathfrak{X}u - z|^{p-1}\, dq \sup_{B_R(q_0)} |\mathfrak{X}\varphi| \\
&\le 2^{p-1}L(1+M^{p-2})\Phi \sup_{B_R(q_0)} |\mathfrak{X}\varphi|,
\end{aligned}
$$

and

$$
\begin{aligned}
|II_2| &= \frac{1}{|B_R(q_0)|} \int\limits_{B_R(q_0)\cap S_2} \big[A(q_0,\mathfrak{X}u) - A(q,\mathfrak{X}u)\big]\cdot\mathfrak{X}\varphi\, dq \\
&\le \frac{2L}{|B_R(q_0)|} \int\limits_{B_R(q_0)\cap S_2} (1+|\mathfrak{X}u|^{p-1})\, dq \sup_{B_R(q_0)} |\mathfrak{X}\varphi| \\
&\le 2^{p}L(1+M^{p-1})\Phi \sup_{B_R(q_0)} |\mathfrak{X}\varphi|.
\end{aligned}
$$

Combining these four estimates yields

$$
\fint\limits_{B_R(q_0)} \frac{\partial A}{\partial z}(q_0,z)(\mathfrak{X}u - z)\cdot\mathfrak{X}\varphi\, dq \ \le c_3\left[\omega(\Phi)\sqrt{\Phi} + \Phi + \frac{R^{\beta}}{\sqrt{2}}\right] \sup_{B_R(q_0)} |\mathfrak{X}\varphi|,
$$

with $c_3 = 2^{p+1}L(1+M^{p-2}+M^{p-1}) > 1$. For a parameter $\delta > 0$ to be fixed later, we define $\gamma = 2c_3\sqrt{\Phi + \delta^{-2}R^{2\beta}}$ and consider the rescaled function

$$
w(q) = \gamma^{-1}(u(q) - z(x - x_0))
$$

on $B_R(q_0)$, so we have

$$\mathfrak{X}w(q) = \gamma^{-1}(\mathfrak{X}u(q) - z).$$

We check by direct calculation that

$$\left| \fint_{B_R(q_0)} \frac{\partial A}{\partial z}(q_0, z)\mathfrak{X}w \cdot \mathfrak{X}\varphi \, dq \right| \leq \left[\omega^2(\Phi) + \Phi + \tfrac{\delta^2}{2}\right]^{1/2} \sup_{B_R(q_0)} |\mathfrak{X}\varphi| \tag{4.10}$$

and

$$\begin{aligned} &\fint_{B_R(q_0)} |\mathfrak{X}w|^2 + \gamma^{p-2}|\mathfrak{X}w|^p \, dq \\ &\quad \leq \frac{1}{4c_3^2\Phi} \fint_{B_R(q_0)} |\mathfrak{X}u - z|^2 + |\mathfrak{X}u - z|^p \, dq = \frac{\Phi}{4c_3^2\Phi} \leq 1. \end{aligned}$$

Now we set $\mathcal{A} = \frac{\partial A}{\partial z}(q_0, z)$; from (1.3) and (1.4) we see that $\mathcal{A}$ is elliptic with constant λ and upper bound $L(1+M^{p-2})$. For $\varepsilon > 0$ to be determined later, we fix $\delta = \delta(n, N, p, \lambda, L, M, \varepsilon) \in (0, 1]$ as required in the $\mathcal{A}$-harmonic approximation lemma. Then, supposing that the smallness condition

$$\omega^2(\Phi) + \Phi \leq \frac{\delta^2}{2} \tag{4.11}$$

is satisfied, the term in front of $\sup_{B_R(q_0)} |\mathfrak{X}\varphi|$ in (4.10) becomes $\leq \delta$. This means that w and $\mathcal{A}$ satisfy the prerequisites of Lemma 4.1. Additionally requiring that $\gamma \leq 1$, we can thus find a $\frac{\partial A}{\partial z}(q_0, z)$-harmonic function $h \in HW^{1,p}(B_{R/2}(q_0), \mathbb{R}^N)$ with

$$\begin{aligned} \fint_{B_{R/2}(q_0)} |\mathfrak{X}h|^2 + \gamma^{p-2}|\mathfrak{X}h|^p \, dq &\leq 2^{Q+1}, \\ \fint_{B_{R/2}(q_0)} \left(\frac{|h - w|}{r/2}\right)^2 + \gamma^{p-2}\left(\frac{|h - w|}{r/2}\right)^p dq &\leq \varepsilon. \end{aligned}$$

In the second step, we compare w to h in order to gain control over $\Phi(q_0, \tau R)$ via the Caccioppoli inequality. From the estimates for systems with constant coefficients in Theorem 1.2, we have the following inequalities for h (with $s \in \{2, p\}$, $C_a = C_a(n, L, \lambda, s)$):

$$\begin{aligned} \fint_{B_r(q_0)} |\mathfrak{X}h|^s \, dq &\leq C_a \fint_{B_R(q_0)} |\mathfrak{X}h|^s \, dq \, , \\ \fint_{B_r(q_0)} |\mathfrak{X}h - (\mathfrak{X}h)_{q_0,r}|^s \, dq &\leq \left(\frac{r}{R}\right)^s C_a \fint_{B_R(q_0)} |\mathfrak{X}h - (\mathfrak{X}h)_{q_0,R}|^s \, dq \, . \end{aligned}$$

For $\tau \in (0, 1]$ to be fixed later, we consider the p-excess of $\mathfrak{X}u$ on the ball $B_{\tau R/4}(q_0)$, and use the Caccioppoli inequality (4.8) with $P = z + \gamma(\mathfrak{X}h)_{q_0,\tau R/2}$ and $\xi = \gamma h_{q_0,\tau R/2}$:

$$\begin{aligned}
&\fint_{B_{\tau R/4}(q_0)} |\mathfrak{X}u - (\mathfrak{X}u)_{q_0,\tau R/4}|^2 + |\mathfrak{X}u - (\mathfrak{X}u)_{q_0,\tau R/4}|^p \, dq \\
&\le 2^p \fint_{B_{\tau R/4}(q_0)} |\mathfrak{X}u - P|^2 + |\mathfrak{X}u - P|^p \, dq \\
&\le 2^p C_c \Bigg[\fint_{B_{\tau R/2}(q_0)} \left| \frac{u - \gamma h_{q_0,\tau R/2} - P(x - x_0)}{\tau R/2} \right|^2 \\
&\quad + \left| \frac{u - \gamma h_{q_0,\tau R/2} - P(x - x_0)}{\tau R/2} \right|^p \, dq \; + (\tau R/2)^{2\beta} \Bigg] \\
&=: 2^p C_c \left(I_2 + I_p + (\tau R/2)^{2\beta} \right).
\end{aligned} \tag{4.12}$$

Now we estimate I_s for $s = 2, p$ as follows:

$$\begin{aligned}
I_s &= \fint_{B_{\tau R/2}(q_0)} \left| \frac{u - \gamma h_{q_0,\tau R/2} - P(x - x_0)}{\tau R/2} \right|^s \, dq \\
&= \gamma^s \fint_{B_{\tau R/2}(q_0)} \left| \frac{w - \left(h_{q_0,\tau R/2} + (\mathfrak{X}h)_{q_0,\tau R/2}(x - x_0)\right)}{\tau R/2} \right|^s \, dq \\
&\le 2^{s-1}\gamma^s \Bigg[\fint_{B_{\tau R/2}(q_0)} \left| \frac{w - h}{\tau R/2} \right|^s \, dq \\
&\quad + (\tau R/2)^{-s} \fint_{B_{\tau R/2}(q_0)} |h - h_{q_0,\tau R/2} - (\mathfrak{X}h)_{q_0,\tau R/2}(x - x_0)|^s \, dq \Bigg] \\
&\le 2^{s-1}\gamma^s \Bigg[\tau^{-Q-s} \fint_{B_{R/2}(q_0)} \left| \frac{w - h}{R/2} \right|^s \, dq \; + C_P(s) \fint_{B_{\tau R/2}(q_0)} |\mathfrak{X}h - (\mathfrak{X}h)_{q_0,\tau R/2}|^s \, dq \Bigg] \\
&\le 2^{s-1}\gamma^s \Bigg[\tau^{-Q-s} \fint_{B_{R/2}(q_0)} \left| \frac{w - h}{R/2} \right|^s \, dq \; + C_a C_P(s)\tau^s \fint_{B_{R/2}(q_0)} |\mathfrak{X}h - (\mathfrak{X}h)_{q_0,R/2}|^s \, dq \Bigg] \\
&\le 2^{s-1}\tau^{-Q-s} \fint_{B_{R/2}(q_0)} \left| \frac{w - h}{R/2} \right|^s \gamma^s \, dq \; + 2^{2s-1} C_a C_P(s)\tau^s\gamma^s \fint_{B_{R/2}(q_0)} |\mathfrak{X}h|^s \, dq \, .
\end{aligned}$$

(Here we have used the definition of P, the definition of w, the Poincaré inequality and the properties of h from linear theory.) Addition of I_2 and I_p allows us to exploit the properties of h from the $\mathcal{A}$-harmonic approximation (note $0 < \tau < 1$):

$$I_2 + I_p \le 2^{p-1}\tau^{-Q-p}\gamma^2 \fint_{B_{R/2}(q_0)} \left| \frac{w - h}{R/2} \right|^2 + \gamma^{p-2} \left| \frac{w - h}{R/2} \right|^2 \, dq$$

$$+2^{2p-1}C_aC_P\tau^2\gamma^2 \fint_{B_{R/2}(q_0)} |\mathfrak{X}h|^2+\gamma^{p-2}|\mathfrak{X}h|^p\,dq$$
$$\leq\ 2^{p-1}\tau^{-Q-p}\gamma^2\varepsilon+2^{2p+Q}C_aC_P\tau^2\gamma^2$$

Lastly, C_c depends on the upper bound on $|P|$. Using the definition of P, Hölder's inequality and the a priori estimate for $\mathfrak{X}h$, we have

$$|P|\leq M+\gamma\left(C_a\tau^2 \fint_{B_{R/2}(q_0)} |\mathfrak{X}h|^2\,dq\right)^{1/2}\leq M+\gamma\big(C_a\cdot 2^{Q+1}\big)^{1/2}=:M+c_4\gamma,$$

so we can ensure that $|P|\leq 1+M$ by imposing the smallness condition

$$c_4\gamma\leq 1, \tag{4.13}$$

which also guarantees that the requirement $\gamma\leq 1$ from $\mathcal{A}$-harmonic approximation is met. Plugging all this into (4.12), and re-substituting γ, we obtain

$$\begin{aligned}&\fint_{B_{\tau R/4}(q_0)} |\mathfrak{X}u-(\mathfrak{X}u)_{q_0,B_{\tau R/2}(q_0)}|^2\,dq\\&\leq 2^pC_c\left[\gamma^2\big(2^{p-1}\tau^{-Q-p}\varepsilon+2^{2p+Q}C_aC_P(p)\tau^2\big)+(\tau R/2)^{2\beta}\right]\\&\leq\tilde{c}_5\left[(\tau^{-Q-p}\varepsilon+\tau^2)(\Phi+\delta^{-2}R^{2\beta})+(\tau R)^{2\beta}\right],\end{aligned}$$

with $\tilde{c}_5=\tilde{c}_5(n,p,L,\lambda,M)$. Now fixing $z=(\mathfrak{X}u)_{q_0,R}$ and substituting $\tau\in(0,1]$ by 4τ with $\tau\in(0,\frac{1}{4}]$, we arrive at

$$\Phi(q_0,\tau R)\leq c_5\left[(\tau^{-Q-p}\varepsilon+\tau^2)\left(\Phi(q_0,R)+\delta^{-2}R^{2\beta}\right)+(\tau R)^{2\beta}\right].$$

For any $\alpha\in(\beta,1]$, we find $\tau\in(0,\frac{1}{4}]$, $\tau=\tau(n,p,L,\lambda,M,\alpha)$ such that

$$2c_5\tau^2\leq\tau^{2\alpha}\quad\Leftrightarrow\quad\tau\leq(2c_5)^{1/(2\alpha-2)}.$$

We then set $\varepsilon=\tau^{Q+p+2}$ and fix the corresponding $\delta=\delta(n,N,p,L,\lambda,p,M,\alpha)$ from the $\mathcal{A}$-harmonic approximation lemma in such a way that $\delta\leq(2c_3c_4)^{-1}$ is also satisfied. Further, we set $c_6=8c_3^2c_4^2$.

With this choice of constants and recalling (4.11), we can derive the smallness condition (4.13) from

$$c_6R^{2\beta}\leq\delta^2. \tag{4.14}$$

Therefore, supposing (4.11) and (4.14) are satisfied, the estimate for $\Phi(q_0,\tau R)$ can be completed as follows:

$$\begin{aligned}\Phi(q_0,\tau R)&\leq c_5\left[(\tau^{-Q-p}\tau^{Q+p+2}+\tau^2)\left(\Phi(q_0,R)+\delta^{-2}R^{2\beta}\right)+(\tau R)^{2\beta}\right]\\&\leq\tau^{2\alpha}\left(\Phi(q_0,R)+\delta^{-2}R^{2\beta}\right)+c_5(\tau R)^{2\beta}\\&\leq\tau^{2\alpha}\Phi(q_0,R)+(\delta^{-2}+c_5)R^{2\beta}=:\tau^{2\alpha}\Phi(q_0,R)+c_7R^{2\beta}.\end{aligned}$$

□

The estimate in Lemma 4.3 for a single τ can now be iterated to obtain a similar estimate on each scale $\tau^j R$, and then extended to a continuous excess decay estimate. First we note that for any $M > 0$ there exists $\Phi_0(M) > 0$, depending also on n, N, λ, L and α, such that

$$\omega^2_{M+1}\big(2\Phi_0(M)\big) + 2\Phi_0(M) \le \frac{\delta^2}{2} \quad \text{and} \quad \Phi_0(M) \le \tfrac{1}{16} M^2 \tau^Q (1-\tau^\alpha)^2.$$

For this $\Phi_0(M)$ we can find a radius $R_0(M) \in (0, 1]$ satisfying

$$\frac{c_6(M) + c_7(M)}{\tau^{2\beta} - \tau^{2\alpha}} R_0(M)^{2\beta} \le \min\left\{\delta^2, \Phi_0(M), \tfrac{1}{16} M^2 \tau^Q (1-\tau^\beta)^2\right\}.$$

With these definitions, we can state the following iteration lemma.

Lemma 4.4 *Consider $M_0 > 0$ such that the following smallness conditions are satisfied in $B_R(q_0) \Subset \Omega$:*

$$|(\mathfrak{X}u)_{q_0,R}| \le M_0, \qquad R \le R_0(M_0) \quad \textit{and} \quad \Phi(q_0, R) \le \Phi_0(M_0),$$

with Φ_0 and R_0 defined as above. Then the prerequisites of Lemma 4.3 are fulfilled on every $B_{\tau^j R}(q_0)$, i.e., for all $j \in \mathbb{N}_0$ there holds:

$$\omega^2_{M_0+1}(\Phi(q_0, \tau^j R)) + \Phi(q_0, \tau^j R) \le \frac{\delta^2}{2}$$

$$c_7(M_0)(\tau^j R)^{2\beta} \le \delta^2$$

Further, there exists

$$\Xi_{q_0} := \lim_{j\to\infty} (\mathfrak{X}u)_{q_0,\tau^j R}$$

and for all $0 < r < R$ we have

$$\fint_{B_r(q_0)} |\mathfrak{X}u - \Xi_{q_0}|^2 + |\mathfrak{X}u - \Xi_{q_0}|^p \, dq \le c_8 \left(\left(\frac{r}{R}\right)^{2\alpha} \Phi(q_0, R) + r^{2\beta}\right)$$

with a constant $c_8(M_0, n, N, p, \lambda, L, \alpha, \beta)$. □

Proof The proof for this iteration lemma is standard (see e.g. [7, p. 284] or [8, Lemma 5.5]). We only have to check for every ball $B_{\tau^j R}(q_0)$, $j \in \mathbb{N}$, that the conditions (4.11) and (4.14) are satisfied. □

The last step is to put all the pieces together for the proof of interior partial regularity of u. In the Euclidean setting, Hölder continuity would follow from the continuous excess-decay estimate by a well-known result of Campanato. We use a similar line of argument to the proof of this result given in [25, Sect. 1] to establish the Hölder continuity of $\mathfrak{X}u$ in the Heisenberg group setting.

Proof of Theorem 1.3 We fix $q_0 \in \Omega \setminus (\Sigma_1 \cup \Sigma_2)$. As $q_0 \notin \Sigma_2$, there exists some M_0 and corresponding $0 < R_1 < 1$ such that:

$$|(\mathfrak{X}u)_{q_0,R}| < M_0 \quad \forall 0 < R \le R_1.$$

Since $q_0 \notin \Sigma_1$, we can find $R \le \min\{R_0(M_0), R_1\}$ with $B_{4R}(q_0) \Subset \Omega$ such that

$$\Phi(q_0, R) < \Phi_0(M_0).$$

Due to the continuity of the mappings $y \mapsto (\mathfrak{X}u)_{y,R}$ and $y \mapsto \Phi(y, R)$, we can then find a ball $B_\rho(q_0)$ with $0 < \rho \leq \frac{R}{2}$ such that we have

$$B_R(y) \subset B_{2R}(q_0), \quad |(\mathfrak{X}u)_{y,R}| < M_0, \quad \Phi(y, R) < \Phi_0(M_0) \quad \text{for all } y \in B_\rho(q_0).$$

Thus we can apply Lemma 4.4 on every ball $B_R(y)$ with centre $y \in B_\rho(q_0)$. This yields the existence of $\Xi_y \in \mathbb{R}^{2n \times N}$ with the property

$$\fint_{B_r(y)} |\mathfrak{X}u - \Xi_y|^2 + |\mathfrak{X}u - \Xi_y|^p \, dq \ \leq c_8 \left(\left(\frac{r}{R} \right)^{2\alpha} \Phi(y, R) + r^{2\beta} \right) \quad \text{for all } r \in (0, R]$$

Now choose two points $y \neq \tilde{y} \in B_\rho(q_0)$ with $r := d(y, \tilde{y}) \leq 2\rho$; for $e^Z := y^{-1}\tilde{y}$, let $a := y \cdot e^{\frac{1}{2}Z}$. For this a we have

$$\frac{r}{2} \leq d(a, y) = d(a, \tilde{y}) = \|e^{\frac{1}{2}Z}\| \leq \frac{r}{\sqrt{2}} \leq R,$$

implying $B_{r/8}(a) \subset B_r(y) \cap B_r(\tilde{y})$ by the triangle inequality. Thus we have

$$\begin{aligned} |\Xi_y - \Xi_{\tilde{y}}|^2 &= \fint_{B_{r/8}(a)} |\Xi_y - \Xi_{\tilde{y}}|^2 \, dq \ \leq \frac{2^{3Q}}{\alpha_n r^Q} \int_{B_r(y) \cap B_r(\tilde{y})} |\Xi_y - \Xi_{\tilde{y}}|^2 \, dq \\ &\leq 2^{3Q+1} \Bigg[\fint_{B_r(y)} |\mathfrak{X}u - \Xi_y|^2 \, dq \ + \fint_{B_r(\tilde{y})} |\mathfrak{X}u - \Xi_{\tilde{y}}|^2 \, dq \ \Bigg] \\ &\leq 2^{3Q+1} c_8 \Bigg[\left(\frac{r}{R} \right)^{2\alpha} (\Phi(y, R) + \Phi(\tilde{y}, R)) + 2r^{2\beta} \Bigg] \\ &\leq 2^{4Q+2} c_8 \Bigg[\left(\frac{d(y, \tilde{y})}{R} \right)^{2\alpha} \Phi(q_0, 2R) + d(y, \tilde{y})^{2\beta} \Bigg], \end{aligned} \tag{4.15}$$

where we have used that $\Phi(y, R) + \Phi(\tilde{y}, R) \leq 2^{Q+1} \Phi(q_0, 2R)$. Taking the square root on both sides, and dividing through by $d(y, \tilde{y})^\beta$ in the last inequality, we obtain

$$\frac{|\Xi_y - \Xi_{\tilde{y}}|}{d(y, \tilde{y})^\beta} \leq 2^{2Q+1} \sqrt{c_8} \left[R^{-\beta} \Phi(\tilde{y}, 2R) + 1 \right].$$

Since $y \mapsto \Xi_y$ is the Lebesgue representative of $\mathfrak{X}u$, this last estimate implies the Hölder continuity of $\mathfrak{X}u$ in all points $q_0 \in \Omega \setminus (\Sigma_1 \cup \Sigma_2)$.

Moreover, as $u \in HW^{1,p}(\Omega, \mathbb{R}^N)$, the sets Σ_1 and Σ_2 are of measure 0, yielding $\mu(\Omega_0) = 0$. □

Remark With some additional technical effort, the results of this chapter could be extended to more general classes of subelliptic systems, i.e. systems of the type

$$-\sum_{i=1}^{2n} X_i A_i(q, u, \mathfrak{X}u) = b(x, u, \mathfrak{X}u) \quad \text{in } \Omega$$

where one has to impose a mild continuity assumption on the coefficients with respect to the variable u, and the R.H.S. has to satisfy a so-called critical resp. controllable growth condition (cf. [7]).

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