Breaking the Metric Voting Distortion Barrier*

Moses Charikar† Prasanna Ramakrishnan† Kangning Wang† Hongxun Wu‡

Abstract

We consider the following well studied problem of metric distortion in social choice. Suppose we have an election with n voters and m candidates who lie in a shared metric space. We would like to design a voting rule that chooses a candidate whose average distance to the voters is small. However, instead of having direct access to the distances in the metric space, each voter gives us a ranked list of the candidates in order of distance. Can we design a rule that regardless of the election instance and underlying metric space, chooses a candidate whose cost differs from the true optimum by only a small factor (known as the *distortion*)?

A long line of work culminated in finding deterministic voting rules with metric distortion 3, which is the best possible for deterministic rules and many other classes of voting rules. However, without any restrictions, there is still a significant gap in our understanding: Even though the best lower bound is substantially lower at 2.112, the best upper bound is still 3, which is attained even by simple rules such as Random Dictatorship. Finding a rule that guarantees distortion $3-\varepsilon$ for some constant ε has been a major challenge in computational social choice.

In this work, we give a rule that guarantees distortion less than 2.753. To do so we study a handful of voting rules that are new to the problem. One is *Maximal Lotteries*, a rule based on the Nash equilibrium of a natural zero-sum game which dates back to the 60's. The others are novel rules that can be thought of as hybrids of Random Dictatorship and the Copeland rule. Though none of these rules can beat distortion 3 alone, a careful randomization between Maximal Lotteries and any of the novel rules can.

1 Introduction

Elections are a fundamental primitive in societal decision making. Through votes, people express their preferences and make collective decisions for social good. A common example of voting is single-winner elections, where voters select one winner from a pool of candidates. These candidates can be persons that represent the voters, or broader social options such as potential locations to build a public facility. Voting is also applicable to every-day situations, such as choosing one from many lunch options for a group of colleagues, or picking a game to play among a group of friends. A *voting rule* (or *social choice rule*) maps the voters' preferences to a winning candidate.

A standard approach to evaluate the outcomes is to adopt the notion of utilitarian social efficiency. We assume that each voter has a cardinal utility function that maps each possible outcome to a real number quantitatively representing their preference for that outcome. From this utilitarian point of view, the optimal voting rule selects the outcome that optimizes the sum of the utilities.

In what has been classically studied and what has been practically implemented, voting rules are often based on ordinal rankings – these rules make decisions based only on each voter's preference ordering, not the cardinal utilities, on the candidates. There are several considerations behind this. First, the restriction to ordinal rules simplifies the processes and the infrastructures needed for voting. Moreover, even though voters are assumed to have cardinal utilities, they may not be able to articulate them accurately, especially when these numbers represent differences in political stances in an abstract way. Finally, in a sense, this restriction to a common ordinal format gives each voter equal voting power.

Ordinal voting rules cannot always perfectly optimize social efficiency. Aiming to quantify the drawback of this format restriction – or this information loss from the perspective of social optimization – researchers have proposed the powerful notion of *distortion* [42, 10, 11, 3]: It represents the worst-case ratio between the optimal efficiency and the efficiency of a particular ordinal voting rule (or in some contexts, the distortion-optimal ordinal voting rule). The worst-case distortion is generally not bounded by any constant, even after imposing normalization constraints on the cardinal utilities. This naturally calls for structural restrictions for the model.

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†Stanford University.

‡University of California, Berkeley.

In the seminal work of [4, 3], they proposed the influential framework of *metric distortion* – in particular, they imposed the natural assumption that the voters and the candidates lie in a shared (unknown) metric space, and a voter's cardinal cost for a candidate is the distance between them in the metric space. This metric assumption is convincing if we think voters and candidates have positions in a political spectrum in the form of a metric space, or if the candidates are public facilities and voters' costs are their travel costs to the selected facility. More formally, the metric distortion of a social choice rule is defined as the supremum of the ratio between the social cost (i.e., sum of costs of voters) of this rule and the optimal social cost, over all possible metric spaces and all induced ordinal preference profiles. The introduction of this metric constraint reduces the distortion of many social choice rules to constants, and the search for distortion-optimal social choice rules is very intriguing.

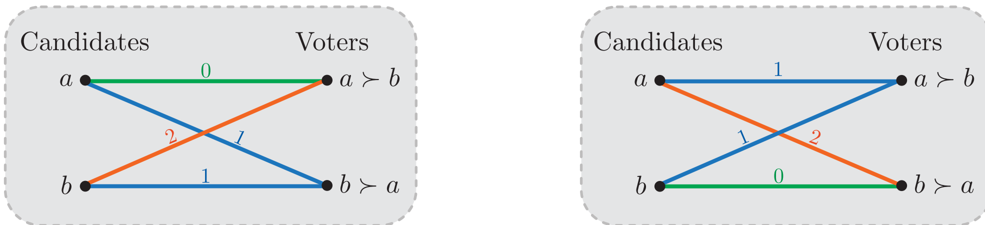

Figure 1: In an election with two candidates and two disagreeing voters, both of the above metric spaces are possible. No deterministic rule can have distortion less than 3, and no randomized rule can have distortion less than 2.

The journey to distortion 3 for deterministic rules. One fruitful line of work on metric distortion, including the original one of [4, 3], focuses on deterministic social choice rules. An immediate lower bound for deterministic rules is 3 (see Figure 1): There are two candidates $\{a, b\}$ and two voters where one voter prefers a to b and the other prefers b to a. Choosing either candidate gives distortion of 3. [4, 3] also showed that any voting rule selecting from the *uncovered set* (which we will introduce later along with its generalizations), such as the Copeland rule, guarantees an upper bound of 5. This gap between 3 and 5 was a tantalizing one and resisted researchers' efforts (e.g. [30]) for a few years. [41] were the first to reduce the gap: They proposed a novel weighted variant of the uncovered set to improve the upper bound to $2 + \sqrt{5} \approx 4.236$, and also showed that selecting from a novel *matching uncovered set* guarantees a distortion upper bound of 3. However, they did not manage to show that the set is always non-empty, and left it as a conjecture. This conjecture had been studied by many researchers (e.g. [32] who identified alternative and additional formulations) since then, until [28], in their breakthrough result, proved it true. [28] identified the crux of the conjecture and proved the existence of a candidate that satisfies their new simplified conditions, hence showing a distortion upper bound of 3 for both their novel rule of Plurality Matching and the one of [41]. This closes the gap for deterministic social choice rules. In the impressive work of [34], they proposed the novel, elegant, and practical voting rule Plurality Veto, which is also by itself a one-paragraph constructive proof of the conjecture of [41]. [35] further identified other related practical voting rules with distortion 3 that connects to the *proportional veto core* [40], a classical notion in social choice.

The need for randomization and the barrier of 3. The above canonical lower bound of 3 only involves two candidates and two disagreeing voters, and one factor that induces this "large" distortion is the limit to *deterministic* social choice rules. This limit is traditionally motivated by people's aversion to randomization for important social issues. However, nowadays, randomization is used in countless democratic (and non-democratic) processes, both those practically implemented and those theoretically studied. In that canonical example, it might even be more natural to randomize over the candidates, than to break the symmetry arbitrarily and pick one candidate deterministically. Moreover, there are situations where fractional solutions are acceptable, such as when allocating funding to the candidates.

In fact, soon after the first work on metric distortion, [22] and [6] independently studied metric distortion without the restriction to deterministic rules. They both showed an upper bound of 3 for the simple rule of Random Dictatorship (which outputs the favorite candidate of a uniformly random voter), and gave a lower bound of 2 using that same two-voter-two-candidate example, leaving open the possibility that much better metric distortion could be achieved by randomized voting rules. This gap between 2 and 3 was a very (if not the most) intriguing question in the field of distortion.

The lack of progress on this question motivated researchers to look at fine-grained distortion analysis within the instance classes of a fixed number of voters or candidates. For example, [6] showed that Random Dictatorship has distortion $3-2/n$ within the instance class of n voters, for any n. Several other works provided upper bounds for the instance class of m candidates, for any m: [21] proposed a rule called Random Oligarchy that can achieve an upper bound slightly worse than $3-2/m$;[1] [33] first showed an upper bound of $3-2/m$ by mixing Random Dictatorship with a rule named Proportional to Squares; [28] proposed Smart Dictatorship, a variant of Random Dictatorship, which gives the same guarantee of $3-2/m$. These improvements over 3 vanish when we consider the supremum over all instances.

The first constant improvement for this gap is on the lower bound. [17] improved the lower bound to 2.1126, while [43] independently showed a lower bound of 2.0631. For the upper bound, there have been many different rules with distortion 3, but also many classes of rules which are known to be *unable* to beat 3: deterministic rules [3], truthful rules [22], rules that only look at the top choices of the voters [31], and rules that only look at pairwise comparisons of candidates (i.e. weighted tournament rules) d[30]. These results rule out a large swath of voting rules that have been studied in the metric distortion literature, which raises the following pressing question.

QUESTION 1. *Is there a voting rule with metric distortion better than* 3*? What might such a voting rule look like?*

In this work, we answer this question by showing that a randomization over simple rules can achieve distortion less than 2.753.

1.1 Our Techniques and Voting Rules

The biased metric framework. Our work uses a new analysis framework that refines the linear programming approach introduced by [17]. Each metric can be viewed as a linear constraint on a potential voting rule, and their insight was to show that a relatively simple class of metrics, called the *biased metrics*, characterizes the most strict constraints. However, they were only able to analyze the biased metrics with some relaxations, and could only prove upper bounds for elections with three candidates. Our approach, on the other hand, allows us to precisely characterize the constraints imposed by the biased metrics. The resulting framework gives us more analysis power while retaining most of the simplicity, and gives us the intuition that leads to the break of the barrier of 3.

In the main body of the paper, we consider three randomized rules and analyze them and their mixtures using this framework. In Appendix C, we also use this framework to revisit a variety of results proved in the metric distortion literature [4, 22, 6, 41, 28, 34] and show that they have short, simple proofs once the biased metric framework has been established. This suggests that the framework may be a helpful primitive for future work in the area.

A note on weighted tournament rules. *Weighted tournament rules* (or *C2 rules* [24]) are a special class of voting rules that only consider pairwise comparisons (for each pair of candidates i, j, the proportion of voters that prefer i over j). Weighted tournament rules are desirable in many settings since they are often simple, interpretable, and efficiently implementable by sampling voters.

The Maximal Lotteries rule discussed in Section 4 is a weighted tournament rule, and is in fact optimal among such rules. Our other rules, including Random Consensus Builder (Section 5.1), Random Dictatorship on the (Weighted) Uncovered Set (Section 5.2), and Random Dictatorship on the Directed Maximal Independent Set (Appendix B), are "almost" weighted tournament rules: They only consider pairwise comparisons of the candidates and the ordering of a uniformly random voter. Note that this modification still allows the rules to be efficiently implemented with sampling.

Maximal Lotteries. The first rule we study is *Maximal Lotteries.* According to [12], it (along with its variants) was first considered by [36], independently rediscovered and studied in detail by [25], and later also independently rediscovered by [37, 26, 23, 44]. This rule has not been studied in the context of distortion.

Maximal Lotteries formulates the following zero-sum game: Two players 1 and 2 each proposes a distribution over the candidates, and then we independently draw a candidate c_1 from Player 1's distribution, a candidate c_2 from Player 2's distribution, and a uniformly random voter v. Player 1 wins if v prefers c_1 to c_2 and vice

[1]Their upper bound is $3-2\cdot\min_{p\in[0,1]}\left(p^2(2-p)+(1-p)^3/(m-1)\right)$, which is $3-2/m+O(1/m^2)$ as $m\to\infty$.

versa, breaking ties uniformly in case $c_1 = c_2$. Each player aims to maximize their winning probability. Maximal Lotteries outputs a Nash equilibrium of this zero-sum game, which can be computed in polynomial time.

We show that Maximal Lotteries has distortion 3. This on its own resolves an interesting question on the optimal distortion of weighted tournament rules. [30] showed that no such rule can have distortion better than 3, and the best previously known upper bound was $2 + \sqrt{5}$ [41]. Additionally, our framework admits a finer characterization on the worst-case instances: Intuitively, if Maximal Lotteries has distortion close to 3 in an instance, then for any candidates c_1 and c_2 where c_1 beats c_2 with a large margin in their pairwise comparison, c_1 cannot be much farther away to the true optimal candidate than c_2. This motivates us to design complementary rules that can deal with these cases where Maximal Lotteries is "bad".

Random Consensus Builder. Motivated by the discussion above, we conceptually build a directed graph where the vertices are the candidates. We draw an edge from c_1 to c_2 if c_1 beats c_2 with a large margin in their pairwise comparison. We are inspired by the following graph theory fact: In any directed graph, there exists an independent set, so that any vertex in the graph can be reached from a vertex in the independent set in at most two steps (see e.g. [9]). When Maximal Lotteries is "bad", a candidate in this independent set must be close to the true optimal candidate. Additionally, candidates in this independent set must be relatively even in their pairwise comparisons due to our construction of the graph. These additional structures make Random Dictatorship on the independent set perform much better than distortion-3 in these cases.

Our Random Consensus Builder rule utilizes this intuition but only implicitly picks this independent set.[2] Random Consensus Builder picks a uniformly random voter and looks at remaining candidates from her least preferred one to her most preferred one. When we encounter a candidate c, we remove all candidates that c can pairwise beat with a large margin. In the end, we output the last candidate that we encounter. Conceptually, Random Consensus Builder naturally balances the opinion of a random voter with the general consensus.

Using our framework, we show that a randomization between Maximal Lotteries and Random Consensus Builder with proper parameters has distortion at most $2\sqrt{2} \approx 2.82843$.

RaDiUS: <u>Ra</u>ndom <u>Di</u>ctatorship on the (Weighted) <u>U</u>ncovered <u>S</u>et. Our analysis of Random Consensus Builder uses properties that are reminiscent of the weighted uncovered set, proposed by [41] who showed that an arbitrary selection from this set (with a proper parameter) gives distortion $2 + \sqrt{5} \approx 4.23607$. This motivates us to propose RaDiUS (<u>Ra</u>ndom <u>Di</u>ctatorship on the (Weighted) <u>U</u>ncovered <u>S</u>et) that outputs a uniformly random voter's favorite candidate within the weighted uncovered set.

It turns out RaDiUS can give better guarantees than Random Consensus Builder. Using our framework, we show that a randomization between Maximal Lotteries and RaDiUS with proper parameters has distortion at most 2.75271.

1.2 Further Related Work

There has been a large body of work on distortion in social choice. We refer the reader to the survey of [5] for a more detailed overview of the field; below we briefly discuss some of them.

The first works on distortion did not impose the metric-space condition, assumed the utilities are non-negative, and defined distortion of a rule as the worst-case ratio between the optimal sum of utilities and the sum of utilities attained by the rule [42]. Many works made the unit-sum utility assumption, where for every voter, her sum of utilities on each candidate equals 1, to avoid uninteresting worst cases. Under this assumption, [15] showed that the Plurality rule has distortion $O(m^2)$ (for the class of instances with m candidates, same below). A matching $\Omega(m^2)$ lower bound for deterministic rules was later given by [14]. Under the same assumption, [10] proposed a randomized rule with distortion $O(\sqrt{m} \log^* m)$ and gave a lower bound of $\Omega(\sqrt{m})$ for any rule. This gap was closed by [20] who proposed a Stable Lottery (and Stable Committee) rule, which was inspired by fair committee selection literature, with distortion $O(\sqrt{m})$.

[29] aimed to provide best-of-both-worlds guarantees for both the metric setting and the non-metric setting. They proposed novel deterministic and randomized social choice rules which guarantee constant metric distortion and almost optimal (for deterministic and randomized rules correspondingly) non-metric distortion.

Researchers have also looked beyond single-winner elections where we select one winner from the candidates. [16] considered a model of multi-winner elections in the metric distortion setting, where they give complete characterizations for the optimal metric distortion. Graph problems such as selecting a perfect bipartite matching

[2]For the interested reader, another voting rule, Random Dictatorship on the Directed Maximal Independent Set, which more directly uses this idea, is discussed in Appendix B.

(e.g. [13, 7, 8, 1, 2]) in both metric and non-metric distortion settings have also received great attention.

Most works in this field aim to optimize the utilitarian aggregation of preferences, i.e., the sum or average of the utilities. Other works consider "fair" ways to aggregate preferences: [30] proposed the *fairness ratio* in the metric setting, which is inspired by the mathematical idea of majorization and replaces the utilitarian aggregation by the worst-case symmetric monotonic norm, and showed a lower bound of 3 and an upper bound of 5. [28] closed this gap by showing their Plurality Matching rule has a fairness ratio of 3. [20] studied proportional fairness, Nash welfare, and the core in the non-metric setting, and gave distortion bounds of $O(\log m)$ for all these objectives.

The distortion framework serves as a valuable tool to quantify the efficiency of voting rules, and therefore has been adopted in the study of various aspects of voting, such as the tradeoff between the amount of communication and the efficiency performance of voting rules [31, 21, 38, 33, 39]. The framework is also used to quantify the effect of certain social structures: [18, 19] studied the representativeness of candidates on the population of voters. In particular, they showed that when the candidates are drawn independently from the voter population, the metric distortion of social choice rules becomes much better. [27] studied the effect of public spirit in the non-metric distortion framework. They showed that if every voter altruistically ranks the candidate according to a mixture of her own preference and the average preference of the voters, then the distortion of many social choice rules will drastically improve.

2 Preliminaries and Notation

Elections. An *election instance* is defined by a tuple $\mathcal{E} = (V, C, \succ_V)$, where V is a set of n voters, C is a set of m candidates, and $\succ_V$ is a set of linear orders, one for each voter, where $i \succ_v j$ if voter v prefers candidate i over candidate j. Throughout the paper, we will use i, j, k, a, b, c to refer to candidates and u, v to refer to voters. We will have i^* denote the true best candidate when the metric space is fixed.

For a condition $\mathcal{P}$, we let $S_{\mathcal{P}}$ denote the subset of voters whose preferences satisfy $\mathcal{P}$. We also let $s_{\mathcal{P}} = |S_{\mathcal{P}}|/n$ be the proportion of these voters overall, or equivalently, the probability that a uniformly random voter's preference list satisfies property $\mathcal{P}$. For example, $S_{i\succ j}$ is the set of voters that prefer i over j, $S_{i,j\succ k}$ is the set of voters that prefer i and j over k, and $S_{I\succ j}$ is the set of voters that prefer all the candidates in I over j. Note that if $j \in I$ then $S_{I\succ j} = \varnothing$ and $s_{I\succ j} = 0$. We also let $\text{plu}(i) = s_{i\succ C\setminus\{i\}}$ be the proportion of voters whose first choice is i.

In Section 4, we will also allow the property $\mathcal{P}$ to be randomized, in which case $s_{\mathcal{P}}$ still makes sense but $S_{\mathcal{P}}$ does not. For example, if D is a distribution over candidates, then $s_{D\succ j}$ denotes the probability that a uniformly random voter prefers a candidate $i \sim D$ over j. That is,

$$s_{D\succ j} = \Pr_{i\sim D, v\sim V}[i \succ_v j] = \mathop{\mathbb{E}}_{i\sim D, v\sim V}[\mathbf{1}[i \succ_v j]].$$

Note that in the case that $i = j$, it will be natural to treat $\mathbf{1}[i \succ_v j]$ as $\frac{1}{2}$. To this end, in Section 4 we will let $s_{i\succ i}$ be $\frac{1}{2}$ instead of 0 – this makes it so that the *Condorcet Game* is well defined, and it makes the proof of Theorem 4.1 read much more smoothly.

We use this notation extensively and flexibly in the paper, and we may reiterate what certain instances mean in natural language to be clear.

Metric spaces. A metric space is a pair $(\mathcal{M}, d)$ of a set $\mathcal{M}$ and a distance metric $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{\geq 0}$ with the following three properties:

(1) Positive definiteness: $d(x, y) \geq 0$ with equality if and only if $x = y$,

(2) Symmetry: $d(x, y) = d(y, x)$,

(3) Triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

We will extend the notation of d to operate directly on the voters and candidates rather than the points they occupy in the metric space. Note that for simplicity we allow voters and candidates to be co-located in the space, so their distance may be zero. (That is to say, technically, we consider *pseudometric spaces*. This simplification does not change the distortion of any voting rule.)

When defining the biased metrics in Section 3 and proving lower bounds in Appendix A, we will specify metric spaces where we only explicitly define the distances between candidates and voters, and leave the other distances implicit. We make no use of the implicit distances, but to fully specify the metric space one can use the *graph distance closure* of the explicitly defined distances. i.e., if the distance between two points is not explicitly defined, it should be taken to be the shortest path between those two points using the explicitly defined distances.

Given an election instance $\mathcal{E} = (V, C, \succ_V)$, we say that a distance metric d is *consistent* with $\mathcal{E}$ (denoted $d \triangleright \mathcal{E}$) if for all $v \in V$, we have that $i \succ_v j$ implies $d(i, v) \leq d(j, v)$.

Voting rules, social cost, and distortion. For an election with underlying distance metric d, we denote the social cost of a candidate i to be their average distance to the voters. i.e.,

$$\mathrm{SC}(i, d) := \frac{1}{n} \sum_{v \in V} d(i, v).$$

(In the literature, the social cost of a candidate is usually the *sum*, rather than the average, of distances. Since we are concerned about the ratio between costs, we can equivalently use the average-distance version, which we find easier to work with.) We will often just write $\mathrm{SC}(i)$ when the relevant distance metric has been fixed, or is clear from context.

A *voting rule* (or *social choice rule*) f is a function that maps every election instance $\mathcal{E}$ to a distribution over its candidates. Given this, the *distortion* of f is given by

$$\mathrm{distortion}(f) = \sup_{\mathcal{E}} \sup_{d : d \triangleright \mathcal{E}} \frac{\mathbb{E}_{j \sim f(\mathcal{E})}[\mathrm{SC}(j, d)]}{\min_{i \in C} \mathrm{SC}(i, d)}.$$

We will also often refer to the distortion of f on a particular metric d, which is just the operand of the suprema above. When the election instance and voting rule are fixed, we will use p_j to denote the probability that the rule chooses candidate j on the instance.

The aforementioned *weighted tournament rules* are the class of voting rules that map $\langle s_{i \succ j} \rangle_{i,j \in C}$ to a distribution of candidates.

3 The Biased Metrics

The key tool that we use to understand the metric distortion of the social choice rules in Section 5 is a refinement of the linear programming framework introduced by [17].

Suppose that we have an election instance. If we can design a rule such that for any metric consistent with the instance we have

$$\sum_{j \in C} (\mathrm{SC}(j) - \mathrm{SC}(i^*)) p_j \leq \lambda \cdot 2\, \mathrm{SC}(i^*), \tag{3.1}$$

then the rule has distortion at most $1 + 2\lambda$ on this instance (the reason for the unnecessary factor of 2 will be clear later). In this view, for a rule to have low distortion it has to satisfy a set of linear constraints, with one constraint imposed by each metric. As one might expect, some constraints may be redundant, so it is helpful to try and find a small set of metrics whose constraints imply those for all of the metrics. Then, one can show that a rule has low distortion just by showing that it has low distortion on the small set of metrics.

[17] defined the set of *biased metrics*, and showed that they satisfied this property.

Definition 3.1. *Let $(x_1, \ldots, x_m)$ be a vector of nonnegative real numbers such that $x_{i^*} = 0$ for some i^*. Given an election instance, the* biased metric *for the vector $(x_1, \ldots, x_m)$ is defined as follows. For a voter v and candidate i, let*

$$d(i^*, v) = \frac{1}{2} \max_{i,j : i \succeq_v j} (x_i - x_j),$$
$$d(j, v) - d(i^*, v) = \min_{k : j \succeq_v k} x_k.$$

The rough idea of the biased metrics is the following. Suppose we were given some fixed metric such that the distance from candidate j to the optimal candidate is x_j (so $x_{i^*} = 0$). Then we could imagine throwing out all of the other distances, and remaking them so that the distances from i^* to the voters is as small as possible, and the distances from the other candidates is as large as possible (compared to the distances from i^*). The former is to make the right side of (3.1) smaller and the latter is to make the left side larger, which will tighten the constraint. It turns out that to do this while respecting the triangle inequality and the preferences, one ends up with the above definition.

[17] gave proofs that the biased metrics are indeed valid distance metrics, and that they tighten the constraints in (3.1). For completeness, these proofs are included in Appendix D.

Now that it has been established that we only need to consider the constraints imposed by the biased metrics, let us see how to express these constraints. Suppose that we have a fixed biased metric given by a vector $(x_1, \ldots, x_m)$. Let $I_t = \{k \in C : x_k \leq t\}$. Notice then that $d(j,v) - d(i^*,v) > t$ if and only if $v \in S_{I_t \succ j}$. To be clear, note that if $j \in I_t$ then $S_{I_t \succ j} = \varnothing$ and $s_{I_t \succ j} = 0$. It follows that

$$\mathrm{SC}(j) - \mathrm{SC}(i^*) = \mathop{\mathbb{E}}_{v \sim V}[d(j,v) - d(i^*,v)] = \int_0^\infty \Pr_{v \sim V}[d(j,v) - d(i^*,v) > t] \, \mathrm{d}t = \int_0^\infty s_{I_t \succ j} \, \mathrm{d}t$$

and so

$$\sum_{j \in C} (\mathrm{SC}(j) - \mathrm{SC}(i^*)) p_j = \int_0^\infty \sum_{j \notin I_t} s_{I_t \succ j} p_j \, \mathrm{d}t.$$

We can use a similar approach to express $2\,\mathrm{SC}(i^*)$. We have that $2d(i^*,v) \leq t$ if and only if $v \in S_{\forall i \succ j, x_i - x_j \leq t}$. This is the set of voters v such that whenever $i \succ_v j$, we have $x_i - x_j \leq t$. It follows that

$$2\,\mathrm{SC}(i^*) = \mathop{\mathbb{E}}_{v \sim V}[2d(i^*,v)] = \int_0^\infty (1 - \Pr_{v \sim V}[2d(i^*,v) \leq t]) \, \mathrm{d}t = \int_0^\infty (1 - s_{\forall i \succ j, x_i - x_j \leq t}) \, \mathrm{d}t.$$

Therefore, the constraint imposed by the biased metric is

$$\int_0^\infty \sum_{j \notin I_t} s_{I_t \succ j} p_j \, \mathrm{d}t \leq \lambda \int_0^\infty (1 - s_{\forall i \succ j, x_i - x_j \leq t}) \, \mathrm{d}t. \tag{3.2}$$

To get a sense of how the right side of this expression behaves, note that $s_{i^* \succ I_t^c} \geq s_{\forall i \succ j, x_i - x_j \leq t}$. If a voter v satisfies the condition that $i \succ_v j$, we have that $x_i - x_j \leq t$, then for all k such that $x_k > t$ (the candidates in I_t^c), we must have $i^* \succ_v k$. In a lot of situations, using $s_{i^* \succ I_t^c}$ in place of the more complicated expression is sufficient. To this end, note that the following constraint implies (3.2).

$$\int_0^\infty \sum_{j \notin I_t} s_{I_t \succ j} p_j \, \mathrm{d}t \leq \lambda \int_0^\infty (1 - s_{i^* \succ I_t^c}) \, \mathrm{d}t. \tag{3.3}$$

[17] derived (3.3) (though written in a more discrete form), and then noted that one could consider an even stricter collection of constraints, which we will use to analyze Maximal Lotteries in Section 4.

$$\sum_{j \notin I} s_{I \succ j} p_j \leq \lambda (1 - s_{i^* \succ I^c}). \tag{3.4}$$

In particular, to show (3.2) for all possible metrics, it suffices to show the above inequality for all sets $I \neq \varnothing, C$, and all $i^* \in I$. The convenience of this is that there are finitely many possible constraints rather than the infinitely many constraints we would have if we used (3.2) or (3.3) (since each choice of the vector $(x_1, \ldots, x_m)$ may give a different constraint). Moreover, the metric space has been completely abstracted away – these constraints only involve terms that come from the election instance alone. However, it should be noted that (3.2) loses no generality (a rule has distortion $1 + 2\lambda$ if and only if it satisfies (3.2) for all biased metrics), but (3.3) and (3.4) may lose generality (the implication only goes one way).

To conclude this section, we introduce some notation that makes (3.2) easier to discuss.

Once an election instance is fixed, we let

$$r(t) = 1 - s_{\forall i \succ j, x_i - x_j \leq t} \qquad \text{and} \qquad R = \int_0^\infty r(t) \, \mathrm{d}t.$$

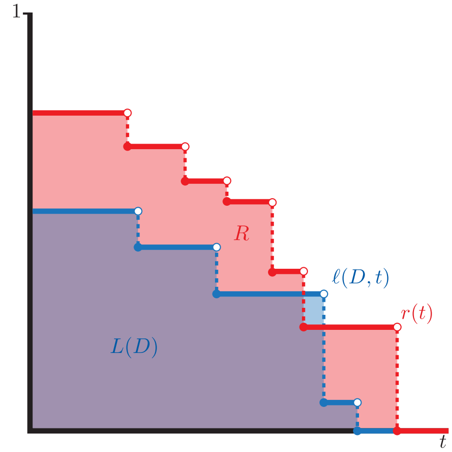

Figure 2: An example of the functions $r(t)$, $\ell(D,t)$, and the areas R and $L(D)$.

Given a distribution D over the candidates which chooses candidate j with probability p_j, we let

$$\ell(D,t) = \sum_{j \notin I_t} s_{I_t \succ j} p_j \qquad \text{and} \qquad L(D) = \int_0^\infty \ell(D,t)\, \mathrm{d}t.$$

With this notation, we would like to design a rule which outputs a distribution D such that for all biased metrics, $L(D)/R \leq \lambda$ for a small fixed λ (to get distortion less than 3 we need $\lambda < 1$). Though the expression for $r(t)$ may seem complicated and difficult to work with, ultimately we only use it in two simple ways. As mentioned, in Section 4 we only use $r(t) \geq 1 - s_{i^* \succ I_t^c}$. In section 5 it is only needed for Proposition 5.1, which roughly says that if $r(t)$ is small, then the metric admits a nice structure that can be leveraged to get better distortion bounds.

4 Maximal Lotteries

In this section, we will study the distortion of Maximal Lotteries [36]. The voting rule is based on a zero-sum game (in our formulation, a constant-sum game) between two players Alice and Bob. The details of the game and the voting rule are below.

The Condorcet Game

- Simultaneously, Alice picks a distribution D_A and Bob picks a distribution D_B over the candidates.
- We sample $a \sim D_A$ and $b \sim D_B$.
- Alice and Bob's payoffs are $s_{a \succ b}$ and $s_{b \succ a}$ respectively. If $a = b$ then each player gets $\frac{1}{2}$.

Maximal Lotteries (ML)

- Choose a candidate from any Nash equilibrium distribution of the Condorcet game.

For the Condorcet game, note that under the notation we introduced in Section 2, once D_A and D_B are picked the payoffs for Alice and Bob are $s_{D_A \succ D_B}$ and $s_{D_B \succ D_A}$ respectively. We remind the reader that in this section we treat $s_{i \succ i}$ as $\frac{1}{2}$ so that if both players choose the same candidate, their payoffs are equal.

Suppose that we fix an election instance. Let D be the distribution output by ML, which chooses candidate i with probability p_i. Let $P(I) = \sum_{i \in I} p_i$ be the probability that a candidate in I is chosen. Let $D(I)$ be the distribution conditioned on the chosen candidate coming from I. i.e., a candidate $i \in I$ is chosen with probability $p_i/P(I)$. Note that $D(I)$ is not well-defined if $P(I) = 0$, so we will deal with cases where this comes up separately.

We will prove the following theorem.

THEOREM 4.1. *For any fixed biased metric and any $t \geq 0$, we have*

$$\ell(D,t) \leq \frac{P(I_t^c)}{2} \leq r(t).$$

In particular, this implies that ML has distortion at most 3.

By a theorem due to [30], no randomized or deterministic weighted tournament rule can have distortion better than 3, so ML is optimal among weighted tournament rules.

Proof of Theorem 4.1. For ease of notation, let us fix t and set $I = I_t$. Let us first prove the theorem in the cases where $P(I)$ or $P(I^c)$ are zero, so that afterwards we can assume that $D(I)$ and $D(I^c)$ are well-defined.

If $P(I^c) = 0$, then we simply have that $\ell(D,t) = \frac{P(I^c)}{2} = 0$, and the theorem easily follows. If $P(I) = 0$, we need to show that $\ell(D,t) \leq \frac{1}{2} \leq r(t)$. Then note that

$$\ell(D,t) = \sum_{j \notin I} s_{I \succ j} p_j \leq \sum_{j \notin I} s_{i^* \succ j} p_j = s_{i^* \succ D}.$$

On the other hand, we have

$$r(t) \geq 1 - s_{i^* \succ I^c} \geq 1 - \min_{j \notin I} s_{i^* \succ j} \geq 1 - \sum_{j \in I^c} s_{i^* \succ j} p_j = 1 - s_{i^* \succ D}.$$

We have that $s_{i^* \succ D} \leq \frac{1}{2}$ since D weakly beats the strategy of deterministically picking i^*, so $\ell(D,t) \leq \frac{1}{2} \leq r(t)$. As desired.

Henceforth, let's assume that $D(I)$ and $D(I^c)$ are well-defined. First, using the fact that $s_{I \succ j} \leq \min_{i \in I} s_{i \succ j} \leq s_{D(I) \succ j}$, we have,

$$\ell(D,t) = \sum_{j \notin I} s_{I \succ j} p_j \leq \sum_{j \notin I} s_{D(I) \succ j} p_j = s_{D(I) \succ D(I^c)} \cdot P(I^c).$$

Similarly, using $s_{i^* \succ I^c} \leq \min_{j \notin I} s_{i^* \succ j} \leq s_{i^* \succ D(I^c)}$, we have

$$r(t) \geq 1 - s_{i^* \succ I^c} \geq 1 - s_{i^* \succ D(I^c)}.$$

Therefore, it suffices to show that

$$s_{D(I) \succ D(I^c)} \cdot P(I^c) \leq \frac{P(I^c)}{2} \leq 1 - s_{i^* \succ D(I^c)}. \tag{4.5}$$

To prove this, we will rely on two somewhat general properties of any equilibrium. The first claim, below, is equivalent to the first inequality in (4.5).

CLAIM 4.1. $s_{D(I) \succ D(I^c)} \leq \frac{1}{2}$.

Proof. Intuitively, this is true because if it were the case that $s_{D(I) \succ D(I^c)} > \frac{1}{2}$ then $D(I)$ could strictly beat D, contradicting the fact that it is an equilibrium.

The proof of the claim relies on two facts. For any distribution X over the candidates, we have

1. $s_{X \succ D} \leq \frac{1}{2}$, and
2. $s_{X \succ X} = \frac{1}{2}$.

The first follows by definition of D being an equilibrium, and the second follows by symmetry. Then we have that by the law of conditional expectation,

$$\begin{aligned}\frac{1}{2} \geq s_{D(I) \succ D} &= P(I^c) s_{D(I) \succ D(I^c)} + P(I) s_{D(I) \succ D(I)} \\ &= P(I^c) s_{D(I) \succ D(I^c)} + (1 - P(I^c)) \cdot \frac{1}{2}\end{aligned}$$

which means that $s_{D(I) \succ D(I^c)} \leq \frac{1}{2}$ as claimed. $\square$

Note that, applying the claim with I replaced with I^c we can in fact conclude that $s_{D(I)\succ D(I^c)} = s_{D(I^c)\succ D(I)} = \frac{1}{2}$.

Now it remains to show that

$$\frac{P(I^c)}{2} \leq 1 - s_{i^*\succ D(I^c)}.$$

For brevity, let $p = P(I^c)$ and $q = s_{i^*\succ D(I^c)}$. We want to show that $\frac{p}{2} \leq 1-q$. We will also introduce $r = s_{D(I)\succ i^*}$. Consider three different strategies for the game: $D(I)$, $D(I^c)$, and just deterministically choosing i^*. Then by

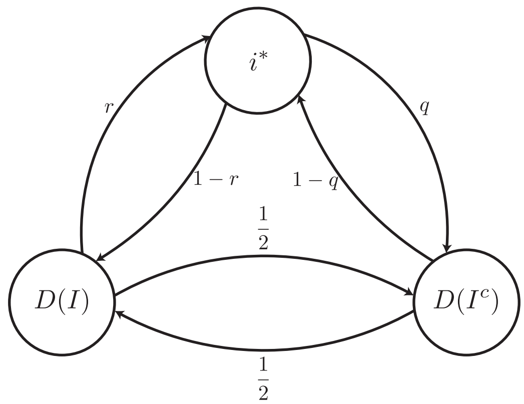

Figure 3: Three strategies $D(I), D(I^c), i^*$. The edge (A, B) is labeled with $s_{A\succ B}$.

conditional expectation and the assumption that D is an equilibrium, we have that

$$\frac{1}{2} \leq s_{D\succ i^*} = P(I^c)s_{D(I^c)\succ i^*} + P(I)s_{D(I)\succ i^*} = p(1-q) + (1-p)r.$$

On the other hand, the payoffs for these strategies satisfy a kind of triangle inequality, by the following claim.

Claim 4.2. *For three strategies A, B, C, we have $s_{A\succ B} \leq s_{A\succ C} + s_{C\succ B}$.*

Proof. We have that $s_{A\succ B} = \mathop{\mathbb{E}}_{i\sim A, j\sim B, v\sim V}[\mathbf{1}[i \succ_v j]]$, so

$$-s_{A\succ B} + s_{A\succ C} + s_{C\succ B} = \mathop{\mathbb{E}}_{i\sim A, j\sim B, k\sim C, v\sim V}[-\mathbf{1}[i \succ_v j] + \mathbf{1}[i \succ_v k] + \mathbf{1}[k \succ_v j]].$$

We claim that the term in the expectation is always nonnegative. If i, j, k are all different then this is clear because it is not possible that $i \succ_v j$ but then $k \succ_v i$ and $j \succ_v k$. If $j = k$ or $i = k$ then the first term cancels with the second or third terms. If $i = j$ then this term is

$$-\mathbf{1}[i \succ_v i] + \mathbf{1}[i \succ_v k] + \mathbf{1}[k \succ_v i]] = \tfrac{1}{2}.$$

This means that the term in the expectation is always nonnegative, which proves the claim. ∎

Now, applying the claim to our three strategies, we have that

$$\begin{aligned} s_{D(I)\succ i^*} &\leq s_{D(I)\succ D(I^c)} + s_{D(I^c)\succ i^*} \\ \implies r &\leq \tfrac{1}{2} + (1-q). \end{aligned}$$

Combining this with our previous inequality relating p, q, r we have

$$\frac{1}{2} \leq p(1-q) + (1-p)r \leq p(1-q) + (1-p)(\tfrac{3}{2} - q) = (1-q) + \frac{1-p}{2}$$

which implies that $\frac{p}{2} \leq 1-q$ as desired. This completes the proof of Theorem 4.1. □

5 Two Rules that Complement Maximal Lotteries

In this section, we introduce two novel social choice rules, each of which can be mixed with Maximal Lotteries to get distortion better than 3: Random Consensus Builder (RCB) and Random Dictatorship on the Uncovered Set (RaDiUS). The two rules are very much cousins of one another, and their analyses have many commonalities. RCB has a slightly worse distortion guarantee, but we include it for several reasons: First, its analysis mirrors that of RaDiUS but is simpler, making it a natural warm-up for the tighter bound given by RaDiUS. Moreover, the final specification of the mixed rule using RCB is easier to state, and the distortion bound ends up being a clean algebraic number, $2\sqrt{2}$. These nice properties, along with the fact that RCB itself has a clean interpretation, make us believe that RCB, as a voting rule, can be of independent interest.

We will analyze the rules using our biased metric framework. Intuitively, the constraints will be harder to satisfy when $r(t)$ is often small, which makes R small. However, when $r(t)$ is small, we can show that the election instance and the metric admit a certain structure which can be leveraged in the analysis. This structure is characterized by the following definition, and it interplays with the function $r(t)$ via the subsequent proposition. This is actually all we need from $r(t)$ in this section.

Definition 5.1. *Given an election instance, a biased metric is* (α, β)-consistent *if whenever $s_{k\succ i^*} \geq \beta$, we have $x_k \leq \alpha R$.*

Proposition 5.1. *If $r(\alpha R) < \beta$ then the metric is (α, β)-consistent.*

Proof. $r(\alpha R) < \beta$ means $s_{\forall i\succ j, x_i - x_j \leq \alpha R} > 1-\beta$. Then, whenever $s_{k\succ i^*} \geq \beta$, we have $s_{k\succ i^*} + s_{\forall i\succ j, x_i - x_j \leq \alpha R} > 1$, which means that there exists a voter v such that $k \succ_v i^*$ and whenever $i \succ_v j$, we have $x_i - x_j \leq \alpha R$. This exactly implies that $x_k \leq \alpha R$. ∎

Once the election instance and metric space are fixed, we will analyze both rules under the assumption that the metric is (α, β)-consistent. This assumption will crucially come into play in Section 6 where we want these rules to perform particularly well when α is close to 0 and β is close to $\frac{1}{2}$. The results in this section will still extend to the general case by the following proposition.

Proposition 5.2. *All biased metrics are $(\frac{1}{\beta}, \beta)$-consistent for all $\beta \in (0, 1)$.*

Proof. If $s_{k\succ i^*} \geq \beta$ we have

$$R = 2\,\mathrm{SC}(i^*) \geq (x_k - x_{i^*})s_{k\succ i^*} \geq x_k \cdot \beta.$$

This means that $x_k \leq \frac{1}{\beta}R$, and the proposition follows. ∎

Both of our rules, RCB and RaDiUS, are parameterized by a tunable value $\beta \in (\frac{1}{2}, 1)$. Very roughly speaking, they both construct the graph on candidates where (i, j) is an edge whenever $s_{i\succ j} \geq \beta$, with the goal of choosing candidates that can always reach the low-distortion candidates in few hops. Note that the rules have natural interpretations when $\beta = \frac{1}{2}$ and $\beta = 1$ which we will briefly discuss, but to avoid dealing with these (ultimately irrelevant) edge cases in the proofs, we will restrict β to the open interval.

5.1 Random Consensus Builder

Below is the description of our first rule.

β-Random Consensus Builder (RCB)

- Choose a uniformly random voter v. Initially the candidate under consideration, i, is v's least favorite candidate.
- Until a candidate is chosen:
  - Eliminate all of the candidates j such that $j \succ_v i$ and $s_{i\succ j} \geq \beta$.
  - If there are no uneliminated candidates that v prefers over i then we choose i. Otherwise, update i to be v's least favorite uneliminated candidate that v prefers over i.

We will say that i *eliminates* j if j is eliminated in the iteration where i is the candidate under consideration.

One interpretation of this rule is that it chooses a random voter v (the consensus builder) whose preferences are the main guide of the rule, but if the voters as a whole have consensus disagreements with v's preferences, then their view overrules. In particular, in each stage of the rule v considers a candidate that is undesirable according to v's preferences, but if a large fraction of voters views this candidate as preferable over some other candidates, then those candidates are removed from contention. The threshold for how strong the consensus needs to be is tuned by this parameter β.

We can find another interesting interpretation of this rule by considering how it would operate at the extremes where $\beta = \frac{1}{2}$ and $\beta = 1$. If $\beta = 1$, the chosen candidate is always v's top choice, and so the rule is exactly Random Dictatorship. On the other hand, if $\beta = \frac{1}{2}$, then it is not hard to see that if the chosen candidate is a, for every other candidate b, either a defeats b (i.e. $s_{a\succ b} \geq \frac{1}{2}$) or a defeats a candidate who defeats b. The candidates that satisfy this property are called the *uncovered set*. Some well known voting rules always output a candidate in the uncovered set, including the well known Copeland rule. In this sense, β also is a measure of interpolation between these kinds of rules and Random Dictatorship.

We will show the following theorem.

THEOREM 5.1. *Suppose that we have an election instance with an (α, β)-consistent underlying metric. Then if D is the distribution output by β-RCB, we have*

$$L(D) \leq (\alpha + \beta)R.$$

COROLLARY 5.1. *β-RCB guarantees distortion at most $1 + 2(\beta + \frac{1}{\beta})$.*

Proof of Theorem 5.1. Suppose v is the consensus builder. Let j_v be the candidate that β-RCB picks. Note that each candidate is either at some point the candidate under consideration (candidate i), or it is eliminated by some other candidate.

If i^* is not eliminated during the rule then let $k_v = i^*$, and otherwise let k_v be the candidate that eliminates i^*. In order to prove the theorem, we will use the following three critical properties of k_v:

(I) $x_{k_v} \leq \alpha R$,

(II) $j_v \succeq_v k_v$,

(III) $s_{k_v \succ j_v} < \beta$.

(I) follows because either $k_v = i^*$ in which case $x_{k_v} = 0$, or k_v eliminates i which means that $s_{k_v\succ i^*} \geq \beta$ and so by the fact that the metric is (α, β)-consistent, we have $x_{k_v} \leq \alpha R$. (II) follows because both k_v and j_v are at some point under consideration, and we consider candidates from lowest to highest on v's preference list. (III) follows because at some point k_v is under consideration, and so either $k_v = j_v$ in which case $s_{k_v\succ j_v} = 0 < \beta$, or k_v did not eliminate j_v which means $s_{k_v\succ j_v} < \beta$.

We will use (I) and (III) to get a good upper bound on $L(D)$, and (I) and (II) to get a good lower bound on R.

We have that

$$\begin{aligned}
\mathrm{SC}(j_v) - \mathrm{SC}(i^*) &\leq s_{j_v\succeq k_v} \min(x_{k_v}, x_{j_v}) + s_{k_v\succ j_v} x_{j_v} \\
&\leq (1-\beta)\min(x_{k_v}, x_{j_v}) + \beta x_{j_v} \\
&\leq (1-\beta)\alpha R + \beta x_{j_v}.
\end{aligned}$$

The second line follows from the first because $x_{j_v} \geq \min(x_{k_v}, x_{j_v})$ and so the expression is maximized when $s_{k_v\succ j_v}$ (which is bounded above by β) is as large as possible. It follows that

$$L(D) = \frac{1}{n}\sum_{v\in V}(\mathrm{SC}(j_v) - \mathrm{SC}(i^*)) \leq \alpha(1-\beta)R + \beta \cdot \frac{1}{n}\sum_{v\in V} x_{j_v}.$$

On the other hand, since $j_v \succeq_v k_v$, we have $2d(v, i^*) \geq x_{j_v} - x_{k_v} \geq x_{j_v} - \alpha R$. It follows that

$$R = 2\,\mathrm{SC}(i^*) \geq -\alpha R + \frac{1}{n}\sum_{v \in V} x_{j_v} \implies \frac{1}{n}\sum_{v \in V} x_{j_v} \leq (1+\alpha)R.$$

Plugging this into our upper bound on $L(D)$, we get

$$L(D) \leq \alpha(1-\beta)R + \beta(1+\alpha)R = (\alpha+\beta)R$$

as desired. □

5.2 Random Dictatorship on the (Weighted) Uncovered Set Next, we consider a rule similar in spirit to RCB, but with better distortion guarantees.

β-Random Dictatorship on the (Weighted) Uncovered Set (RaDiUS)

- Say that *a covers b* if $s_{a \succ b} \geq \beta$ and for any c, if $s_{c \succ a} \geq \beta$ then $s_{c \succ b} \geq \beta$.
- Let U be the set of candidates that are not covered by any other candidate.
- Choose a uniformly random voter and output their favorite candidate in U.

The set U was previously considered by [41] in the context of deterministic rules. They showed that there exists a β such that any candidate from the set U (which they called the *weighted uncovered set*) has distortion at most $2 + \sqrt{5}$.

To see how this rule is similar to β-RCB, consider the following proposition. It also conveniently gives a proof that the set U is always non-empty (which was proved in a different way in [41]).

Proposition 5.3. *Suppose that U is the weighted uncovered set constructed by β-RaDiUS. Then β-RCB always outputs a member from U.*

Proof. Let's suppose towards a contradiction that for some voter v, the candidate j_v chosen by RCB is covered by some other candidate a.

For j_v to be chosen, it must have been the case that a was eliminated by some candidate c such that $j_v \succ_v c$. Otherwise, if $j_v \succ_v a$ then a would have eliminated j_v and if $a \succ_v j_v$ then a would not have been eliminated when j_v is the candidate under consideration (or before by c) and so the rule would not have terminated in that iteration.

But then $s_{c \succ a} \geq \beta$, and so by the definition of a covering j_v, we must have $s_{c \succ j_v} \geq \beta$. But then c would have eliminated j_v, which is a contradiction. □

Before we get into the distortion guarantee for β-RaDiUS, we prove two more facts that will be helpful to us.

Proposition 5.4. *The covering relation is transitive.*

Proof. Suppose a covers b and b covers c. We claim that a covers c. Since b covers c and $s_{a \succ b} \geq \beta$ we have $s_{a \succ c} \geq \beta$. Now suppose that for some d, $s_{d \succ a} \geq \beta$. Then since a covers b we have $s_{d \succ b} \geq \beta$ but then since b covers c we have $s_{d \succ c} \geq \beta$. So indeed, a covers c. □

Proposition 5.5. *If a candidate is not in U then it is covered by a candidate in U.*

Proof. Suppose that we build a graph on the candidates where (a, b) is an edge if a covers b. We cannot have a cycle in this graph, because by transitivity this would imply that some candidate i covers itself, which would imply the impossible $s_{i \succ i} \geq \beta > \frac{1}{2} > 0$.

If a candidate i is not in U, it must have positive in-degree. Since the graph is acyclic, by arbitrarily following edges backwards from i, we must eventually reach a candidate j with in-degree zero. This means that $j \in U$ and there is a path from j to i, which by transitivity means that j covers i. □

Now we prove the following distortion guarantee.

THEOREM 5.2. *Suppose that we have an election instance with an (α, β)-consistent underlying metric. Then if D is the distribution output by β-RaDiUS, we have*

$$L(D) \leq (\alpha(1-\beta^2)+\beta)R.$$

COROLLARY 5.2. *β-RaDiUS guarantees distortion at most $1+2/\beta$.*

The proof is similar in structure to the proof of Theorem 5.1. The key difference is that rather than using the same candidate k_v which satisfies the properties (I), (II), (III), we will have one candidate k_v which satisfies properties (I) and (III) and another candidate k^* that satisfies properties (I) and (II). The advantage is that in the later case, we have the same candidate k^* for *all* voters v, which will allow us to get a stronger lower bound on R. However, having different candidates for the two cases makes the argument a little more complicated.

Proof of Theorem 5.2. Once again, let j_v be the candidate that is output when v is the randomly chosen voter. Let's assume that $j_v \neq i^*$, otherwise the rule picks the best candidate and all of the bounds will only be improved. Then since $j_v \in U$ it must be the case that i^* *does not* cover j_v. Unpacking the definition, this means that either

(a) $s_{i^* \succ j_v} < \beta$, or

(b) there exists some k such that $s_{k \succ i^*} \geq \beta$ but $s_{k \succ j_v} < \beta$.

Define k_v so that $k_v = i^*$ if (a) occurs and $k_v = k$ if (b) occurs. In either case we have once again that $x_{k_v} \leq \alpha R$ and $s_{k_v \succ j_v} < \beta$. These are the properties (I) and (III) used in the proof of Theorem 5.1, and so by an identical argument we can show that

$$L(D) = \frac{1}{n}\sum_{v\in V}(\mathrm{SC}(j_v) - \mathrm{SC}(i^*)) \leq \alpha(1-\beta)R + \beta \cdot \frac{1}{n}\sum_{v\in V} x_{j_v}.$$

Now define k^* so that $k^* = i^*$ if $i^* \in U$ and otherwise, k^* is some member of U which covers i^* (which exists by Proposition 5.5). Once again, we have that either $k^* = i^*$ and so $x_{k^*} = 0$, or $s_{k^* \succ i^*} \geq \beta$ and since the metric is (α, β)-consistent we have $x_{k^*} \leq \alpha R$. In addition, we have that $j_v \succeq_v k^*$, since $k^* \in U$ and j_v is v's favorite candidate in U. Thus, k^* satisfies properties (I) and (II) used in the proof of Theorem 5.1.

It follows that for every voter v,

$$2d(v, i^*) \geq x_{j_v} - x_{k^*}.$$

Moreover, if v satisfies $k^* \succeq_v i^*$, the inequality can be stronger. In this case, $j_v \succeq_v k^* \succeq_v i^*$ and so

$$2d(v, i^*) \geq x_{j_v} = x_{k^*} + (x_{j_v} - x_{k^*}).$$

Since $k^* \succeq_v i^*$ for at least a β fraction of voters v, we have

$$R = 2\,\mathrm{SC}(i^*) \geq \beta x_{k^*} + \frac{1}{n}\sum_{v\in V}(x_{j_v} - x_{k^*}) = -(1-\beta)x_{k^*} + \frac{1}{n}\sum_{v\in V} x_{j_v},$$

where we crucially use the fact that all voters share the same k^*. It follows that

$$\frac{1}{n}\sum_{v\in V} x_{j_v} \leq R + (1-\beta)x_{k^*} \leq (1+(1-\beta)\alpha)R.$$

Plugging this into our upper bound on $L(D)$, we have

$$\begin{aligned} L(D) &\leq \alpha(1-\beta)R + \beta(1+(1-\beta)\alpha)R \\ &= (\alpha(1-\beta^2)+\beta)R \end{aligned}$$

as claimed. □

6 Mixing Rules

Even though none of the three social choice rules we introduced can beat distortion 3 (see Appendix A), it turns out that mixing them in a careful way can.

Let us introduce the general technique for analyzing the mixture of these rules. Suppose that we have a given election instance and a biased metric. We would like to show that a particular rule achieves low distortion on this instance and metric. Consider the graph of $r(t)$ that is fixed by the metric. For each $\beta \in (\frac{1}{2}, 1)$, let $\alpha(\cdot)$ be the function such that $\alpha(\beta) \cdot R = \min\{t : r(t) < \beta\}$. Informally, if we draw the horizontal line $y = \beta$, then this line intersects the graph of $r(t)$ at the point $(\alpha(\beta)R, \beta)$. (If the intersection is a line segment, we take the rightmost point on the segment.)

Unsurprisingly, the function $\alpha(\cdot)$ is directly related to the α we were considering in Section 5: Proposition 5.1 tells us that if we have $\alpha(\cdot)$ corresponding to a biased metric, then the metric is $(\alpha(\beta), \beta)$-consistent for all $\beta \in (\frac{1}{2}, 1)$.

Moreover, we can use this function $\alpha(\beta)$ to get a tighter bound on the distortion of ML. Let D_{ML} be the distribution output by ML. Then Theorem 4.1 tells us that

$$\ell(D_{\mathrm{ML}}, t) \leq \frac{P(I_t^c)}{2} \leq r(t)$$

and since $P(I_t^c) \leq 1$, we have $\ell(D_{\mathrm{ML}}, t) \leq \min(\frac{1}{2}, r(t))$. On the other hand, the area that is below $r(t)$ but above the horizontal line $\frac{1}{2}$ is exactly $R \int_{\frac{1}{2}}^{1} \alpha(\beta)\, \mathrm{d}\beta$, and so it follows that

$$L(D_{\mathrm{ML}}) + R \int_{\frac{1}{2}}^{1} \alpha(\beta)\, \mathrm{d}\beta \leq R \implies L(D_{\mathrm{ML}}) \leq \left(1 - \int_{\frac{1}{2}}^{1} \alpha(\beta)\, \mathrm{d}\beta\right) R. \tag{6.6}$$

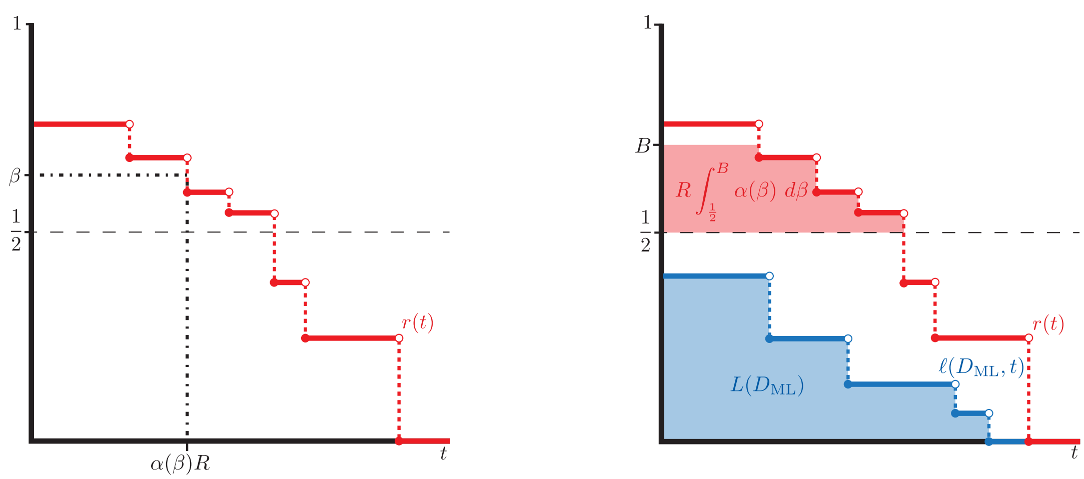

Figure 4: Left: For each β, the horizontal line at β intersects $r(t)$ at $(\alpha(\beta)R, \beta)$. Right: If the area above $\frac{1}{2}$ and below $r(t)$ is large, we can get a better bound on $L(D_{\mathrm{ML}})/R$.

This mixture of rules does well because there is a sense in which ML and β-RCB/β-RaDiUS are complementary. For the analysis of ML not to have much wiggle room, the curve $r(t)$ should be above the line $\frac{1}{2}$ very little. But then it means that $\alpha(\beta)$ is small for values of β that are slightly larger than $\frac{1}{2}$, and with smaller α and β, we get much better guarantees in Theorem 5.1 and Theorem 5.2.

The rules will have three parameters: p, B, and $\rho(\cdot)$. Both rules run ML with probability p, and otherwise run β-RCB or β-RaDiUS where $\beta \in (\frac{1}{2}, B)$ is drawn from a distribution with probability density function ρ. It will turn out that in the analysis, we will fix p and ρ as a function of B, and then to get the best distortion we just need to optimize a single variable function. In the warm-up, the right ρ is just uniform, so to simplify we will omit that parameter.

6.1 Warm-up: ML and RCB Get Distortion $2\sqrt{2}$

We note that the rule and proof might read more smoothly with the right values for p and B baked in, but we will keep these as variables to illustrate the mechanics of the proof technique.

ML mixed with RCB
• With probability p, run Maximal Lotteries. • With probability $1-p$, choose a *uniformly* random $\beta \in (\frac{1}{2}, B)$ and run β-Random Consensus Builder.

THEOREM 6.1. *With $p = \frac{1}{\sqrt{2}}$ and $B = \sqrt{2} - \frac{1}{2}$, the rule ML mixed with RCB has distortion at most $2\sqrt{2} \approx 2.828$.*

Proof. Suppose that we have a fixed election instance and metric space. Let D_{ML} be the distribution output by ML, let D_β be the distribution output by β-RCB, and let D be the overall distribution of the rule. Note that β is chosen according to the probability density function $\frac{1}{B-\frac{1}{2}}$, so

$$\begin{aligned}
L(D) &= pL(D_{\mathrm{ML}}) + (1-p)\int_{\frac{1}{2}}^{B} \frac{1}{B-\frac{1}{2}} \cdot L(D_\beta)\,\mathrm{d}\beta \\
&= pL(D_{\mathrm{ML}}) + \frac{1-p}{B-\frac{1}{2}} \int_{\frac{1}{2}}^{B} L(D_\beta)\,\mathrm{d}\beta.
\end{aligned}$$

So then applying (6.6) and Theorem 5.1, we get

$$\begin{aligned}
\frac{L(D)}{R} &\le p\left(1 - \int_{\frac{1}{2}}^{B} \alpha(\beta)\,\mathrm{d}\beta\right) + \frac{1-p}{B-\frac{1}{2}} \int_{\frac{1}{2}}^{B} (\alpha(\beta)+\beta)\,\mathrm{d}\beta \\
&= p + \frac{1-p}{B-\frac{1}{2}} \int_{\frac{1}{2}}^{B} \beta\,\mathrm{d}\beta + \left(-p + \frac{1-p}{B-\frac{1}{2}}\right) \int_{\frac{1}{2}}^{B} \alpha(\beta)\,\mathrm{d}\beta \\
&= p + \tfrac{1}{2}(1-p)(B+\tfrac{1}{2}) + \left(\frac{1-p(B+\frac{1}{2})}{B-\frac{1}{2}}\right) \int_{\frac{1}{2}}^{B} \alpha(\beta)\,\mathrm{d}\beta.
\end{aligned}$$

Now, $\alpha(\beta)$ is a function which depends on the metric, which could be chosen adversarially. However, we can completely eliminate this "dangerous" term by choosing p and B such that its coefficient is 0. This is perhaps where the magic of the proof happens – by carefully balancing the two rules, we can get a kind of destructive interference that eliminates any dangerous terms.

Choosing $p = \frac{1}{B+\frac{1}{2}}$, we get

$$\frac{L(D)}{R} \le \frac{1}{B+\frac{1}{2}} + \tfrac{1}{2}(B - \tfrac{1}{2}).$$

It is not hard to check that choosing $B = \sqrt{2} - \frac{1}{2}$ minimizes the above expression, at which point it is also $\sqrt{2} - \frac{1}{2}$. This gives us distortion $1 + 2(\sqrt{2} - \frac{1}{2}) = 2\sqrt{2}$. ∎

6.2 ML and RaDiUS Get Distortion 2.753

ML mixed with RaDiUS
• With probability p, run Maximal Lotteries. • With probability $1-p$, sample $\beta \in (\frac{1}{2}, B)$ according to the probability density function $\rho(\beta)$ and run β-RaDiUS.

THEOREM 6.2. *With appropriate choices for $p, B,$ and $\rho(\cdot)$ the rule ML mixed with RaDiUS has distortion at most 2.753.*

Proof. Let D_{ML} and D_β be defined as in the proof of Theorem 6.1, but with β-RaDiUS in place of β-RCB. Then we get

$$\begin{aligned}
\frac{L(D)}{R} &\le p\left(1-\int_{\frac12}^{B}\alpha(\beta)\,\mathrm{d}\beta\right)+(1-p)\int_{\frac12}^{B}\rho(\beta)\left(\alpha(\beta)(1-\beta^2)+\beta\right)\mathrm{d}\beta\\
&= p+(1-p)\int_{\frac12}^{B}\rho(\beta)\beta\,\mathrm{d}\beta+\int_{\frac12}^{B}\alpha(\beta)\left(-p+\rho(\beta)(1-p)(1-\beta^2)\right)\mathrm{d}\beta\\
&= 1-(1-p)\int_{\frac12}^{B}\rho(\beta)(1-\beta)\,\mathrm{d}\beta+\int_{\frac12}^{B}\alpha(\beta)\left(-p+\rho(\beta)(1-p)(1-\beta^2)\right)\mathrm{d}\beta.
\end{aligned}$$

The last line uses the fact that $\int_{\frac12}^{B}\rho(\beta)\,\mathrm{d}\beta = 1$. In order to make the coefficient of $\alpha(\beta)$ equal to 0, we set

$$\rho(\beta)=\frac{p}{(1-p)(1-\beta^2)}$$

which means that

$$1=\int_{\frac12}^{B}\rho(\beta)\,\mathrm{d}\beta=\frac{p}{1-p}\int_{\frac12}^{B}\frac{\mathrm{d}\beta}{1-\beta^2}\implies p=\frac{1}{1+\int_{\frac12}^{B}\frac{\mathrm{d}\beta}{1-\beta^2}}.$$

With these choices, we have

$$\begin{aligned}
\frac{L(D)}{R} &\le 1-p\int_{\frac12}^{B}\frac{\mathrm{d}\beta}{1+\beta}\\
&= 1-\frac{\int_{\frac12}^{B}\frac{\mathrm{d}\beta}{1+\beta}}{1+\int_{\frac12}^{B}\frac{\mathrm{d}\beta}{1-\beta^2}}\\
&= 1-\frac{\ln\frac23+\ln(1+B)}{1-\frac12\ln 3+\frac12(\ln(1+B)-\ln(1-B))}.
\end{aligned}$$

Using numerical optimization methods, we find that the best choice is $B\approx 0.876353$, which gives distortion 2.75271. ∎

7 Discussion

In this work, we studied Maximal Lotteries in the distortion setting and proposed novel simple rules of Random Consensus Builder and RaDiUS. Using our biased metric framework, we show that a mix between ML and RCB has metric distortion at most $2\sqrt{2}$, and a mix between ML and RaDiUS has distortion at most 2.753.

An immediate future direction is to further close the gap $(2.112, 2.753)$ of optimal metric distortion. Towards this, we propose the following ideas:

- Our RaDiUS rule is a hybrid of Random Dictatorship and a deterministic weighted tournament rule of [41]. Is there a deterministic weighted tournament rule with better distortion than $2+\sqrt{5}\approx 4.236$? Such a result is very interesting on its own, and can potentially serve as an ingredient for a rule with better distortion than 2.753. (Note that our analysis for ML pinned down the optimal metric distortion for weighted tournament rules at 3. For deterministic weighted tournament rules, the gap is $[3, 2+\sqrt{5}]$.)
- Our RaDiUS rule uses the notion of weighted uncovered set, which was designed to show a deterministic rule with good metric distortion. Would ideas that lead to distortion-optimal deterministic rules be useful, such as the matching uncovered set [41] and related ideas [32], Plurality Matching [28], Plurality Veto [34] and its variants [34, 35]?

- The biased metric framework potentially has more power than we have utilized in our proofs. Theorems 5.1 and 5.2 show that for some function $f(\cdot,\cdot)$, their respective rules have distortion $1+2f(\alpha,\beta)$ under this assumption. If one can show a similar theorem for a new rule, but with a smaller function $f(\cdot,\cdot)$, then this would improve the distortion upper bound. One could also attempt to go beyond our proof structure. For example, (3.3) is a set of simpler and stricter constraints (derived by [17]) than what we use, but we do not know what distortion bounds we can achieve after this simplification. Further understanding the structures of biased metrics can be helpful in improving the metric distortion bounds.

Another intriguing direction is to find "simpler" rules that have good metric distortion:

- We managed to break the barrier of 3 by mixing simple rules. Can we do this using an even simpler rule, e.g., one which does not look like a randomization between simple rules? A similar question can be asked for some non-metric distortion settings, where the Stable Lottery (or Stable Committee) rule, which looks like a randomization between two simple rules, gives optimal $\Theta(\sqrt{m})$ distortion [20].
- Can we break the barrier of 3 by using a minimal amount of randomness – for example, randomizing between at most two candidates, or only using randomness to sample a single voter (as RCB, RaDiUS, and Random Dictatorship do)?

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The appendices can be found in the full version of the paper: `https://arxiv.org/abs/2306.17838`.