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Settling the Complexity of Arrow-Debreu Equilibria in Markets with Additively Separable Utilities

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Abstract— We prove that the problem of computing an Arrow-Debreu market equilibrium is PPAD-complete even when all traders use additively separable, piecewise-linear and concave utility functions. In fact, our proof shows that this market-equilibrium problem does not have a fully polynomial-time approximation scheme, unless every problem in PPAD is solvable in polynomial time.

1. Introduction

One of the central developments in mathematical economics is the general equilibrium theory, which provides the foundation for competitive pricing [1], [36]. When specialized to exchange economies, it considers an *exchange market* in which there are m traders and n divisible goods, where trader i has an *initial endowment* of $w_{i,j} \geq 0$ of good j, and a *utility function* $u_i : \mathbb{R}_+^n \rightarrow \mathbb{R}$. The *individual goal* of trader i is to obtain a new bundle of goods that maximizes her utility. This new bundle can be specified by a column vector $\mathbf{x}_i \in \mathbb{R}_+^n$ in which the jth entry $x_{i,j}$ is the amount of good j that trader i is able to obtain after the exchange. Naturally, the exchange should satisfy $\sum_i x_{i,j} \leq \sum_i w_{i,j}$, for all good $j \in [m]$.

The pioneering equilibrium theorem of Arrow and Debreu [1] states that if all the utility functions $u_1,\ldots,u_m$ are quasi-concave, then under some mild conditions, the market has an *equilibrium price* $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$: At this price, independently, each trader can sell her initial endowment virtually to the market to obtain a budget and then buys a bundle of goods with this budget from the market — which contains the union of all goods — that maximizes her utility. The equilibrium condition guarantees that the supply equals the demand and hence the market clears: every good is sold and every trader's budget is completely spent.

The existence proof of Arrow and Debreu, based on Kakutani's fixed point theorem [29], is non-constructive in the view of polynomial-time computability. Despite the progress both on algorithms for and on the complexity-theoretic understanding of market equilibria, several fundamental questions, including some seemingly simple ones, remain unsettled. Vijay Vazirani [32] wrote:

> *"Concave utility functions, even if they are additively separable over the goods, are not easy to deal with algorithmically. In fact, obtaining a polynomial time algorithm for such functions is a premier open question today."*

A function $u(\mathbf{x}) : \mathbb{R}_+^n \rightarrow \mathbb{R}$ is said to be *additively separable* and *concave*, if there exist n real-valued concave functions $f_1, \ldots, f_n$ such that $u(x_1, \ldots, x_n) = \sum_{j=1}^n f_j(x_j)$. Noting that every concave function f_j can be approximated by a piecewise linear and concave (PLC) function, Vazirani [32] further asked whether one can find an equilibrium price in a market with additively separable PLC utility functions in polynomial time; or whether the problem is PPAD-hard. This open question has been echoed in several work since then [13], [25], [21], [38].

1.1. Our Contribution

In this paper, we settle the computational complexity of finding an approximate Arrow-Debreu equilibrium in a market with additively separable PLC utilities. We show that this market-equilibrium problem is PPAD-complete.

For a positive integer $t > 0$ we say a real-valued function f is t-segment piecewise linear over $\mathbb{R}_+ = [0, +\infty)$, if f is continuous and $\mathbb{R}_+$ can be divided into t sub-intervals such that f is linear over every sub-interval. If every trader's utility is an additively separable and t-segment PLC function, then we refer to the market as a *t-linear market*. Clearly, a market with linear utilities is a 1-linear market. In contrast to the fact that an Arrow-Debreu equilibrium of a 1-linear market can be computed in polynomial time [18], [31], [12] [14], [26] we show that finding an approximate equilibrium in a 2-linear market is PPAD-complete via a reduction from Sparse Bimatrix: the problem of finding an approximate Nash equilibrium in a sparse two-player game [5] (see 2.1 for the definition).

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Our construction of the PPAD-complete markets has several nice technical elements. First, we introduce a family of simple *linear* markets $\{\mathcal{M}_n\}$ with n goods, which we refer to as the *price-regulating* markets. They have the following nice *price-regulation property*: If $\mathbf{p}$ is a *normalized* [1] approximate equilibrium price vector of $\mathcal{M}_n$, then $p_k \in [1, 2]$ for all $k \in [n]$. This price-regulation property allows us to encode n free variables $x_1, \ldots, x_n$ between 0 and 1 using the n entries of $\mathbf{p}$ by setting $x_k = p_k - 1$, $k \in [n]$.

As a key step in our analysis, we prove that this price-regulation property is *stable* with respect to *small perturbations*: When new traders are added into $\mathcal{M}_n$ without introducing new goods, the price-regulation property remains to hold as long as the total amount of goods they carry with them is *small* compared to those of the traders in $\mathcal{M}_n$. We then show how to set the endowments and utilities of these new traders so that we can control the flows of goods in the new market and set new constraints that every approximate equilibrium price vector $\mathbf{p}$ must satisfy.

Using the stability of the price-regulating markets $\{\mathcal{M}_n\}$ we give a reduction from a two-player game to a 2-linear market $\mathcal{M}$: Given any $n \times n$ two-player game $(\mathbf{A}, \mathbf{B})$, we construct an additively separable PLC market $\mathcal{M}$ by adding new traders — whose endowments are relatively small — to $\mathcal{M}_{2n+2}$, the price-regulating market with $2n + 2$ goods.

We use the first $2n$ entries of $\mathbf{p}$ to encode a pair of probability distributions $(\mathbf{x}, \mathbf{y})$, where

$$x_k = p_k - 1 \quad \text{and} \quad y_k = p_{n+k} - 1, \quad k \in [n]. \quad (1)$$

We then develop a novel way to enforce the Nash equilibrium constraints over $\mathbf{A}$, $\mathbf{B}$, $\mathbf{x}$ and $\mathbf{y}$ by carefully specifying the behaviors of the new traders.

In doing so, we get a market $\mathcal{M}$ with the property that, from any approximate equilibrium $\mathbf{p}$ of $\mathcal{M}$, the pair $(\mathbf{x}, \mathbf{y})$ encoded in $\mathbf{p}$ (after normalization) is an approximate Nash equilibrium of $(\mathbf{A}, \mathbf{B})$. Moreover, if $(\mathbf{A}, \mathbf{B})$ is *sparse*, then the relation of which traders are interested in which goods in $\mathcal{M}$ is also *sparse*.

In the construction, the price-regulation property plays a critical role. It enables us to design the utility functions of the new traders so that we know exactly their preferences over the goods with respect to any approximate equilibrium price $\mathbf{p}$, even though we have no idea in advance about the entries of $\mathbf{p}$ when constructing $\mathcal{M}$.

We anticipate that our reduction techniques will help to resolve more complexity questions concerning other interesting families of markets such as the general CES and the hybrid linear-Leontief markets [6]. Recently, the techniques developed in this paper were further improved to show that even for the much simpler model of Fisher [2] (see Section 1.2.1), finding an approximate market equilibrium remains to be PPAD-complete [7], [39].

[1] We say $\mathbf{p}$ is normalized if its smallest nonzero entry is equal to 1.

1.2. Related Work

The computation of an equilibrium price in an exchange market has been a very challenging problem in mathematical economics [32]. The matter is more complex since some markets only have irrational equilibria, making the computation of exact market equilibria with a finite-precision algorithm impossible.

One approach to handle irrationality is to express market equilibria in some simple algebraic form. However, it turns out that finding an exact equilibrium price in general is not computable [34].

One can also use some notion of *approximate* equilibria. There are various notions of approximate market equilibria: Some require that the approximation solution lies within a small geometric distance from an exact equilibrium point [19], while others only require that the individual optimality condition and/or the supply-demand condition are approximately satisfied. Following Scarf [35] we consider the latter notion of approximate market equilibria in this paper.

1.2.1. Algorithms for market equilibria: Scarf pioneered the algorithmic development of computing general competitive equilibria [35], [36]. His approach combined numerical approximation with combinatorial insights used in Sperner's lemma [37] for fixed points and in Lemke and Howson's algorithm for two-player Nash equilibria. Although his algorithm may not always run in polynomial time, Scarf's work has profound impact to computational economics.

Building on the success of convex programming [18], polynomial-time algorithms have been developed for some special markets whose sets of market equilibria enjoy some degree of convexity. For Arrow-Debreu markets with linear utility functions, Nenakov and Primak gave a polynomial-time algorithm [31], and there are now several polynomial-time algorithms for computing/approximating market equilibria with linear utility functions [12], [14], [26], [20], [27], [15], [41]. Other polynomial-time algorithms for special markets include Eaves's algorithm for Cobb-Douglas markets [17]; and Devanur and Vazirani's FPTAS for markets with spending constraint utilities [16], building on the algorithm for Fisher's model by Vazirani [38]. The convex programming based approach has also been extended to all markets whose utility functions satisfy weak gross substitutability (WGS) by Codenotti, Pemmaraju, and Varadarajan [10]. In [9], Codenotti, McCune, and Varadarajan showed that for markets satisfying WGS, there is a price-adjustment mechanism called *tâonnement* that converges to an approximate equilibrium efficiently.

A closely related market model is Fisher's model [2]. In this model, there are two types of traders: *producers* and *consumers*. Each consumer comes to the market with a budget and a utility function. Each producer comes to the market with an endowment of goods, and will sell them to the consumers for money. A market equilibrium is then a price vector for goods so that if every consumer spends all her budget to maximize her utility, then the market clears. An (approximate) market equilibrium in a Fisher's market with CES utility functions [18], [41], [40], [14], [28] or with some special class of piecewise linear utility functions [40] can be computed in polynomial time.

However, progress on Arrow-Debreu markets whose sets of equilibria do not enjoy convexity has been relatively slow. There are only a few algorithms in this category. Devanur and Kannan [13] gave a polynomial-time algorithm for PLC markets with a constant number of goods. Codenotti et. al. [8] gave a polynomial-time algorithm for markets with CES functions when the elasticity of substitution $s \geq 1/2$.

1.2.2. Complexity of equilibria: In [33], Papadimitriou introduced the complexity class PPAD. He also proved that the problem of finding a Nash equilibrium in a two-player game and the problem of computing an approximate fixed point are both members of PPAD.

Recently, there has been a sequence of developments that characterized the computational complexity of several equilibrium problems in game theory. Daskalakis, Goldberg and Papadimitriou [23] proved that the problem of computing an exponentially-precise Nash equilibrium of a four-player game is PPAD-complete. Chen and Deng [3] then showed that finding a two-player Nash equilibrium is also PPAD-complete. Chen and Deng's result, together with an earlier reduction of [11], implies that finding an equilibrium in an Arrow-Debreu market with Leontief utilities is PPAD-hard. On the approximation front, Chen, Deng, and Teng proved that it is PPAD-complete to find a polynomially-precise approximate equilibrium in two-player or multi-player games [4]. Huang and Teng then extended it to Leontief market equilibria [25]. Their result implies that the market equilibrium problem with CES utility functions is PPAD-hard, if the elasticity of substitution s is sufficiently small.

2. Preliminaries

2.1. Complexity of Nash Equilibria in Sparse Games

A two-player game is defined by the payoff matrices $\mathbf{A}$ and $\mathbf{B}$ of its two players. In this paper, we assume that both players have n choices of actions and thus, $\mathbf{A}$ and $\mathbf{B}$ are square matrices with n rows and columns. We use Δ^n to denote the set of probability distributions of n dimensions.

Definition 1 (Well-Supported Nash Equilibria). *For $\epsilon > 0$, $(\mathbf{x}, \mathbf{y})$ is an* ϵ-well-supported Nash equilibrium *of $(\mathbf{A}, \mathbf{B})$, if $\mathbf{x}, \mathbf{y} \in \Delta^n$ and for all $i \neq j \in [n]$, we have*

$$\mathbf{A}_i \mathbf{y}^T + \epsilon < \mathbf{A}_j \mathbf{y}^T \implies x_i = 0, \quad \textit{and} \tag{2}$$

$$\mathbf{x}\mathbf{B}_i + \epsilon < \mathbf{x}\mathbf{B}_j \implies y_i = 0, \tag{3}$$

where we use $\mathbf{A}_i$ and $\mathbf{B}_i$ to denote the ith row vector of $\mathbf{A}$ and the ith column vector of $\mathbf{B}$, respectively.

Definition 2. *We say a two-player game $(\mathbf{A}, \mathbf{B})$ is* normalized *if every entry of $\mathbf{A}$ and $\mathbf{B}$ is between -1 and 1. We say a two-player game $(\mathbf{A}, \mathbf{B})$ is* sparse *if every row and every column of $\mathbf{A}$ and $\mathbf{B}$ have at most 10 nonzero entries.*

We let Sparse Bimatrix denote the problem of finding an n^{-6}-well-supported Nash equilibrium in an $n \times n$ sparse normalized two-player game. By [5], it is PPAD-complete.

2.2. Markets with Additively Separable PLC Utilities

Let $\mathcal{G} = \{G_1, \ldots, G_n\}$ denote a set of n divisible goods and $\mathcal{T} = \{T_1, \ldots, T_m\}$ denote a set of m traders. For each T_i, we use $\mathbf{w}_i \in \mathbb{R}^n_+$ to denote her endowment and $u_i(\cdot)$ to denote her utility function. We will focus on markets with additively separable, piecewise linear and concave utilities.

A continuous function $r(\cdot)$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ is said to be *t-segment* piecewise linear and concave (PLC), if $r(0) = 0$ and there exists a tuple $[\theta_0 > \theta_1 > \ldots > \theta_t \geq 0; 0 < a_1 < a_2 < \ldots < a_t]$ of length $2t + 1$, such that

1. For any $i \in [0 : t-1]$, the restriction of f over $[a_i, a_{i+1}]$ $(a_0 = 0)$ is a segment of slope θ_i;
2. The restriction of f over $[a_t, +\infty)$ is a ray of slope θ_t.

The tuple $[\theta_0, \theta_1, \ldots, \theta_t; a_1, a_2, \ldots, a_t]$ is called the *representation* of $r(\cdot)$. We say $r(\cdot)$ is *strictly monotone* if $\theta_t > 0$ and it is *α-bounded* for some $\alpha \geq 1$ if $\alpha \geq \theta_0$ and $\theta_t \geq 1$.

Definition 3. *A function $u(\cdot) : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$ is said to be an* additively separable PLC *function if there exist a set of n* PLC *functions $r_1(\cdot), \ldots, r_n(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

$$u(\mathbf{x}) = \textstyle\sum_{j \in [n]} r_j(x_j), \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n_+.$$

In such a market, we use $r_{i,j}(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ to denote the PLC function of T_i with respect to G_j and thus, we have

$$u_i(\mathbf{x}) = \textstyle\sum_{j \in [n]} r_{i,j}(x_j), \quad \text{for all } \mathbf{x} \in \mathbb{R}^n_+.$$

We let $\mathbf{p} \in \mathbb{R}^n_+$ denote a price vector, where $\mathbf{p} \neq \mathbf{0}$ and p_j is the price of G_j. We always assume that $\mathbf{p}$ is *normalized*. Given $\mathbf{p}$, we use $\text{OPT}(i, \mathbf{p})$ to denote the set of allocations that maximize $u_i(\cdot)$:

$$\text{OPT}(i, \mathbf{p}) = \text{argmax}_{\,\mathbf{x} \in \mathbb{R}^n_+,\ \mathbf{x} \cdot \mathbf{p} \leq \mathbf{w}_i \cdot \mathbf{p}}\ u_i(\mathbf{x}).$$

We use $\mathcal{X} = \{\mathbf{x}_i \in \mathbb{R}^n_+ : i \in [m]\}$ to denote an allocation of the market. For every $T_i \in \mathcal{T}$, $\mathbf{x}_i \in \mathbb{R}^n_+$ is the amount of goods that T_i receives.

Definition 4 (Arrow-Debreu [1]). *A* market equilibrium *is a non-zero price vector $\mathbf{p} \in \mathbb{R}^n_+$ such that there exists an allocation $\mathcal{X}$ which has the following properties:* 1). *Every trader gets an optimal bundle: For every $T_i \in \mathcal{T}$, we have $\mathbf{x}_i \in \text{OPT}(i, \mathbf{p})$; and* 2). *The market clears: For any G_j,*

$$\textstyle\sum_{i \in [m]} x_{i,j} \leq \sum_{i \in [m]} w_{i,j};$$

If $p_j > 0$, then we have $\sum_{i \in [m]} x_{i,j} = \sum_{i \in [m]} w_{i,j}$.

In general, not every market has an equilibrium, mainly due to some technical issues with traders with zero income. For the additively separable PLC markets considered in this paper, the following condition guarantees the existence of a market equilibrium. It is a corollary of Maxfield [30], and the proof can be found in the appendix.

Definition 5 (Economy Graphs [30])**.** *Given an additively separable* PLC *market $\mathcal{M}$, we define a directed graph $G = (\mathcal{T}, E)$ as follows. The vertex set of G is exactly $\mathcal{T}$, the set of traders in the market. For every two traders $T_i \neq T_j \in \mathcal{T}$ we have an edge from T_i to T_j if there exists an integer $k \in [n]$ such that $w_{i,k} > 0$ and $r_{j,k}(\cdot)$ is strictly monotone. In another word, trader T_i possesses a good that T_j wants. G is called the* economy graph *of the market* [30], [8]. *We say the market is* strongly connected *if G is* strongly connected.

Theorem 1. *Let $\mathcal{M}$ be an Arrow-Debreu market with additively separable* PLC *utility functions. If it is strongly connected, then a market equilibrium $\mathbf{p}$ exists.*

Moreover, if all the parameters of $\mathcal{M}$ are rational numbers, then it has a rational market equilibrium. The number of bits needed to describe it is polynomial in the input size of $\mathcal{M}$ (i.e., the number of bits needed to describe $\mathcal{M}$).

Besides, every quasi-equilibrium of $\mathcal{M}$ (see the appendix for definition) must also be a market equilibrium.

2.3. Definition of the Sparse Market Equilibrium Problem

By Theorem 1, the following problem MARKET is well-defined: The input of the problem is an additively separable PLC market $\mathcal{M}$ that is both rational and strongly connected; and the output is a rational market equilibrium of $\mathcal{M}$.

In the rest of this section, we define SPARSE MARKET, a very restricted version of MARKET. The main result of the paper is that SPARSE MARKET is PPAD-complete.

First the input of SPARSE MARKET is an additively separable PLC market which not only is strongly connected but also satisfies the following three conditions.

We say an additively separable PLC market is *α-bounded* for some $\alpha \geq 1$, if for all T_i and G_j, $r_{i,j}(\cdot)$ is either the zero function ($r_{i,j}(x) = 0$ for all x) or α-bounded. We call an additively separable PLC market a *2-linear* market if for every T_i and G_j, $r_{i,j}(\cdot)$ has at most two segments. Finally, we say an additively separable PLC market is *t-sparse*, for some integer $t > 0$, if *1)* For every $T_i \in \mathcal{T}$, $|supp(\mathbf{w}_i)| \leq t$ and *2)* For every $T_i \in \mathcal{T}$, the number of $j \in [n]$ such that $r_{i,j}(\cdot)$ is not the zero function is at most t. In other words, every trader owns at most t goods at the beginning and is interested in at most t goods.

We use the following notion of approximate equilibria:

Definition 6 (Approximate Market Equilibrium)**.** *Given an additively separable* PLC *market $\mathcal{M}$, we say $\mathbf{p} \in \mathbb{R}^n_+$ is an ϵ-approximate market equilibrium of $\mathcal{M}$ for some $\epsilon \geq 0$, if there exists an allocation $\mathcal{X} = \{\mathbf{x}_i \in \mathbb{R}^n_+ : i \in [m]\}$ such that $\mathbf{x}_i \in \mathrm{OPT}(i, \mathbf{p})$ for all $i \in [m]$; and the market clears approximately: For every $G_j \in \mathcal{G}$,*

$$\Big| \sum_{i\in[m]} x_{i,j} - \sum_{i\in[m]} w_{i,j} \Big| \leq \epsilon \cdot \sum_{i\in[m]} w_{i,j}.$$

We remark that there are various notions of approximate market equilibria. The reason we adopted the one above is to simplify the analysis. The construction in section 4 works for some other notions of approximate equilibria, e.g., the one that allows the allocation to be approximately optimal for each trader.

We let SPARSE MARKET denote the following problem: The input is a 2-linear market $\mathcal{M}$ that is strongly connected 27-bounded and 23-sparse; and the output is an n^{-13}-approximate equilibrium of $\mathcal{M}$, where n is the number of goods. It is tedious but not hard to show that SPARSE MARKET is in PPAD[2]. Besides, one can replace the constant 27 here by any constant larger than 1 and our main result, Theorem 2, below still holds. The constant 23, however, is related to the constant 10 in Definition 2.

Theorem 2 (Main)**.** SPARSE MARKET *is* PPAD*-complete.*

Theorem 2 also implies that the problem of computing an approximate quasi-equilibrium (see the appendix for definition) in an Arrow-Debreu market with additively separable PLC utilities is PPAD-complete.

3. A PRICE-REGULATING MARKET

We now construct the price-regulating markets $\{\mathcal{M}_n\}$ for every positive integer $n \geq 2$. $\mathcal{M}_n$ has n goods and satisfies the following strong price regulation property.

Property 1 (Price Regulation)**.** *A price vector $\mathbf{p} \in \mathbb{R}^n_+$ is a normalized n^{-1}-approximate market equilibrium of $\mathcal{M}_n$ if and only if $1 \leq p_k \leq 2$, for all $k \in [n]$.*

We start with some notation. The goods in $\mathcal{M}_n$ are $\mathcal{G} = \{G_1, \ldots, G_n\}$, and the traders are $\mathcal{T} = \big\{T_{\mathbf{s}} : \mathbf{s} \in S\big\}$,

$$\text{where} \quad S = \big\{\mathbf{s} = (i,j) : 1 \leq i \neq j \leq n\big\}.$$

For each $T_{\mathbf{s}}$, we use $\mathbf{w}_{\mathbf{s}} \in \mathbb{R}^n_+$ to denote her endowment, $u_{\mathbf{s}}(\cdot) : \mathbb{R}^n_+ \to \mathbb{R}_+$ to denote her utility, $r_{\mathbf{s},k}(\cdot)$ to denote her PLC function with respect to G_k, $k \in [n]$, and $\mathrm{OPT}(\mathbf{s}, \mathbf{p})$ to denote the set of bundles that maximize her utility with respect to $\mathbf{p}$.

[2] In [22], the author showed how to construct a continuous map from any market with quasi-concave utilities such that the set of fixed points of the map is precisely the set of equilibria of the market. When the market is additively separable PLC, one can show that the continuous map is indeed Lipschitz continuous. As a result, one can reduce the problem of finding an approximate market equilibrium to the problem of finding an approximate fixed point in a Lipschitz continuous map. This implies a reduction from SPARSE MARKET to the discrete fixed point problem studied in [24] (also see [4] for the high-dimensional version) which is in PPAD, and thus, the former is also in PPAD.

$\mathcal{M}_n$ is *linear*, in which for all $\mathbf{s} \in S$ and $k \in [n]$, $r_{\mathbf{s},k}(\cdot)$ is a ray starting at $(0,0)$. In the construction below, we use $r_{\mathbf{s},k}(\cdot) \Leftarrow [\theta]$ to denote the action of setting $r_{\mathbf{s},k}(\cdot)$ to be the linear function of slope $\theta \geq 0$.

Construction of $\mathcal{M}_n$: First, we set $\mathbf{w_s}$. For every pair $\mathbf{s} = (i,j) \in S$, we have $w_{\mathbf{s},i} = 1/n$; and $w_{\mathbf{s},k} = 0$ for all other $k \in [n]$.

Second, we set the PLC functions $r_{\mathbf{s},k}(\cdot)$. For every $\mathbf{s} = (i,j) \in S$ and $k \in [n]$, we have $r_{\mathbf{s},k}(\cdot) \Leftarrow [\theta]$ where $\theta = 0$ if $k \neq i,j$; $\theta = 1$ if $k = j$; and $\theta = 2$ if $k = i$.

It is easy to check that $\mathcal{M}_n$ constructed above is strongly connected, 2-bounded and 2-sparse.

Proof of Property 1: One of the directions is trivial. If $1 \leq p_k \leq 2$ for all $k \in [n]$, then $\mathcal{X} = \{\mathbf{x_s} = \mathbf{w_s} : \mathbf{s} \in S\}$ is an optimal market clearing allocation at price $\mathbf{p}$.

The other direction is less trivial. Let $\mathbf{p}$ be a normalized n^{-1}-approximate equilibrium of $\mathcal{M}_n$, and $\mathcal{X}$ be an optimal allocation that clears the market approximately. First, it is easy to check that p_k must be positive for all $k \in [n]$ since otherwise, we have $x_{\mathbf{s},k} = +\infty$ for all $\mathbf{s} = (i,j)$ such that $k = i$ or j, contradicting the market clearing condition.

Since $\mathbf{p}$ is normalized, we have $p_k \geq 1$, for all $k \in [n]$. Assume for contradiction that Property 1 is not true, then without loss of generality, we assume that $p_1 = \max_k p_k > 2$ and $p_2 = \min_k p_k = 1$. To reach a contradiction, we focus on the amount of G_1 each trader gets in the allocation $\mathcal{X}$.

First, if $1 \notin \{i,j\}$ where $\mathbf{s} = (i,j)$, then we have $x_{\mathbf{s},1} = 0$; Second, if $i = 1$ and $j = 2$, then $x_{\mathbf{s},1} = 0$ since $2/p_1 < 1/p_2$ and trader $T_\mathbf{s}$ likes G_2 better than G_1 with respect to the price vector $\mathbf{p}$; Third, if $j = 1$, then we have $x_{\mathbf{s},1} = 0$ since $1/p_1 < 2/p_i$ and $T_\mathbf{s}$ likes G_i better than G_1; Finally, for every $\mathbf{s} = (i,j)$ with $i = 1$ and $j \neq 2$, we always have $x_{\mathbf{s},1} \leq 1/n$ since the budget of $T_\mathbf{s}$ is exactly $(1/n) \cdot p_1$. As a result, we have

$$\textstyle\sum_{\mathbf{s}\in S} x_{\mathbf{s},1} \leq (n-2)/n \quad \text{and} \quad \sum_{\mathbf{s}\in S} w_{\mathbf{s},1} = (n-1)/n,$$

which contradicts the assumption that $\mathbf{p}$ is an n^{-1}-approximate market equilibrium. ■

4. From Sparse Bimatrix to Sparse Market

In this section, we give a reduction from Sparse Bimatrix to Sparse Market.

Given an $n \times n$ sparse and normalized two-player game $(\mathbf{A}, \mathbf{B})$, we build an additively separable PLC market $\mathcal{M}$ by adding more traders to the price-regulating market $\mathcal{M}_{2n+2}$. There are $2n+2$ goods in $\mathcal{M}$: $\mathcal{G} = \{G_1, \ldots, G_{2n+2}\}$, and the traders $\mathcal{T}$ in $\mathcal{M}$ are

$$\mathcal{T} = \{T_\mathbf{s}, T_\mathbf{u}, T_\mathbf{v}, T_i : \mathbf{s} \in S, \mathbf{u} \in U, \mathbf{v} \in V, i \in [2n]\},$$

where $S = \{(i,j) : 1 \leq i \neq j \leq 2n+2\}$, $U = \{(i,j,1) : 1 \leq i \neq j \leq n\}$ and $V = \{(i,j,2) : 1 \leq i \neq j \leq n\}$.

The traders $T_\mathbf{s}$, $\mathbf{s} \in S$, have almost the same initial endowments $\mathbf{w_s}$ and PLC functions $r_{\mathbf{s},k}(\cdot)$ as in $\mathcal{M}_{2n+2}$. We only slightly modify the parameters to ease the analysis.

For each $T \in \mathcal{T}$, we will set her PLC function $r(\cdot)$ with respect to $G_k \in \mathcal{G}$ to one of the following functions:

1) $r(\cdot)$ is the zero function: $r(x) = 0$ for all $x \geq 0$ (denoted by $r(\cdot) \Leftarrow [0]$);
2) $r(\cdot)$ is a ray: $r(x) = \theta \cdot x$ for all $x \geq 0$ (denoted by $r(\cdot) \Leftarrow [\theta]$); or
3) $r(\cdot)$ is a 2-segment PLC function with representation $[\theta_0, \theta_1; a_1]$ (denoted by $r(\cdot) \Leftarrow [\theta_0, \theta_1; a_1]$).

4.1. Setting up the Market

4.1.1. Traders $T_\mathbf{s}$, where $\mathbf{s} \in S$: For $\mathbf{s} = (i,j) \in S$, we set the vector $\mathbf{w_s}$ as follows: $w_{\mathbf{s},i} = 1/n$; and $w_{\mathbf{s},k} = 0$ for all other $k \in [2n+2]$. We set the PLC functions $r_{\mathbf{s},k}(\cdot)$ as follows: $r_{\mathbf{s},k}(\cdot) \Leftarrow [\theta]$ and $\theta = 1$ if $k = j$; $\theta = 2$ if $k = i$; and $\theta = 0$ for all other $k \in [2n+2]$.

4.1.2. Traders $T_\mathbf{u}$, where $\mathbf{u} \in U$: We let $\mathbf{u} = (i,j,1)$ be a triple in U with $1 \leq i \neq j \leq n$. We let $\mathbf{A}_i$ and $\mathbf{A}_j$ denote the ith and jth row vectors of $\mathbf{A}$. We define $\mathbf{C}$ and $\mathbf{D}$ to be the following two n-dimensional vectors: For $k \in [n]$,

$$(C_k, D_k) = (A_{i,k} - A_{j,k}, 0) \text{ if } A_{i,k} - A_{j,k} \geq 0; \quad \text{and}$$
$$(C_k, D_k) = (0, A_{j,k} - A_{i,k}) \text{ otherwise.}$$

By definition, we have $\mathbf{C} - \mathbf{D} = \mathbf{A}_i - \mathbf{A}_j$ while both vectors $\mathbf{C}$ and $\mathbf{D}$ are non-negative. Moreover, because $\mathbf{A}$ is sparse, the number of non-zero entries in either $\mathbf{C}$ or $\mathbf{D}$ is at most 20, and each entry is between 0 and 2. We also let E, F be the following non-negative numbers: Let $C = \sum_{k\in[n]} C_k$ and $D = \sum_{k\in[n]} D_k$, then

$$(E,F) = (D - C, 0) \text{ if } D \geq C; \quad \text{and} \tag{4}$$
$$(E,F) = (0, C - D) \text{ otherwise.}$$

It is clear that $E, F \geq 0$, and $E + C = F + D$. Moreover, since $\mathbf{C}$ and $\mathbf{D}$ are sparse, we have $E, F \leq 20 \cdot 2$.

Using $\mathbf{C}$ and E, we set the vector $\mathbf{w_u}$ of $T_\mathbf{u}$ as follows:

1) $w_{\mathbf{u},i} = 1/n^4$; $w_{\mathbf{u},k} = w_{\mathbf{u},2n+2} = 0$, $\forall\, k \neq i \in [n]$;
2) $w_{\mathbf{u},n+k} = C_k/n^5$, $\forall\, k \in [n]$; and $w_{\mathbf{u},2n+1} = E/n^5$.

The number of nonzero entries in $\mathbf{w_u}$ is at most 22.

Using $\mathbf{D}$ and F, we set $r_{\mathbf{u},k}(\cdot)$, $k \in [2n+2]$, as follows:

1) $r_{\mathbf{u},i}(\cdot) \Leftarrow [9, 1; 1/n^4]$; $r_{\mathbf{u},k}(\cdot) \Leftarrow [0]$, $\forall\, k \neq i \in [n]$;
2) $r_{\mathbf{u},2n+2}(\cdot) \Leftarrow [3]$;
3) $r_{\mathbf{u},n+k}(\cdot) \Leftarrow [0]$, $\forall\, k \in [n]$ with $D_k = 0$;
4) $r_{\mathbf{u},n+k}(\cdot) \Leftarrow [27, 1; D_k/n^5]$, $\forall\, k \in [n]$ with $D_k > 0$;
5) $r_{\mathbf{u},2n+1}(\cdot) \Leftarrow [0]$ if $F = 0$; and $r_{\mathbf{u},2n+1}(\cdot) \Leftarrow [27, 1; F/n^5]$ if $F > 0$.

Also note that the number of $k \in [2n+2]$ such that $r_{\mathbf{u},k}(\cdot)$ is not the zero function is at most 23.

We give some intuition for the construction above. First the use of the constants $1, 3, 9, 27$ is to enforce $T_{\mathbf{u}}$, where $\mathbf{u} = (i, j, 1)$, to have a fixed preference over the goods (see the proof of Lemma 4 for the preference of $T_{\mathbf{u}}$) even if we do not know the prices of the goods in $\mathbf{p}$. This is possible because the same price-regulation property: $1 \le p_k \le 2$ for all k, still holds for any approximate market equilibrium $\mathbf{p}$ of $\mathcal{M}$ (Lemma 1). The price-regulation property also allows us to encode a pair of vectors $(\mathbf{x}, \mathbf{y})$ using (1).

Second, the role of $T_{\mathbf{u}}$ is to implement the Nash equilibrium constraint (2). The idea is that, if $\mathbf{A}_i\mathbf{y}^T < \mathbf{A}_j\mathbf{y}^T$, then the preference of $T_{\mathbf{u}}$ guarantees that the amount of G_i she buys is less than the amount of G_i she possesses in her endowment. Then using the market clearing condition, we show that for traders $T_{\mathbf{s}}$, $\mathbf{s} \in S$, the amount of G_i they buy must be more than the amount of G_i they possess in their endowments. Intuitively this implies that the price p_i of G_i in the approximate equilibrium is low and indeed, we show that p_i must be 1 and thus, $x_i = p_i - 1 = 0$.

4.1.3. Traders $T_{\mathbf{v}}$, where $\mathbf{v} \in V$: The behavior of $T_{\mathbf{v}}$ is similar except that it works on $\mathbf{B}$. Let $\mathbf{v} = (i, j, 2) \in V$. We let $\mathbf{B}_i$ and $\mathbf{B}_j$ denote the ith and jth column vectors of $\mathbf{B}$. Similarly, we define vectors $\mathbf{C}$ and $\mathbf{D}$: For $k \in [n]$,

$$(C_k, D_k) = (B_{k,i} - B_{k,j}, 0) \text{ if } B_{k,i} - B_{k,j} \ge 0; \quad \text{and}$$
$$(C_k, D_k) = (0, B_{k,j} - B_{k,i}) \text{ otherwise.}$$

As a result, we have $\mathbf{C} - \mathbf{D} = \mathbf{B}_i - \mathbf{B}_j$ while $\mathbf{C}, \mathbf{D}$ are nonnegative and sparse. We also define $E, F \ge 0$ in a similar way so that $0 \le E, F \le 40$ and

$$E + \textstyle\sum_{k \in [n]} C_k = F + \sum_{k \in [n]} D_k.$$

Using $\mathbf{C}$ and E, we set the vector $\mathbf{w}_{\mathbf{v}}$ of $T_{\mathbf{v}}$ to be

1) $w_{\mathbf{v},n+i} = 1/n^4$; and $w_{\mathbf{v},n+k} = w_{\mathbf{v},2n+2} = 0$ for all other $k \in [n]$;
2) $w_{\mathbf{v},k} = C_k/n^5$ for all $k \in [n]$; and $w_{\mathbf{v},2n+1} = E/n^5$.

Using $\mathbf{D}$ and F, we set $r_{\mathbf{v},k}(\cdot)$, $k \in [2n+2]$, as follows:

1) $r_{\mathbf{v},n+i}(\cdot) \Leftarrow [9, 1; 1/n^4]$; and $r_{\mathbf{v},n+k}(\cdot) \Leftarrow [0]$ for all other $k \in [n]$;
2) $r_{\mathbf{v},2n+2}(\cdot) \Leftarrow [3]$;
3) $r_{\mathbf{v},k}(\cdot) \Leftarrow [0]$, $\forall\, k \in [n]$ with $D_k = 0$;
4) $r_{\mathbf{v},k}(\cdot) \Leftarrow [27, 1; D_k/n^5]$, $\forall\, k \in [n]$ with $D_k > 0$;
5) $r_{\mathbf{v},2n+1}(\cdot) \Leftarrow [0]$ if $F = 0$; and $r_{\mathbf{v},2n+1}(\cdot) \Leftarrow [27, 1; F/n^5]$ if $F > 0$.

Again we have $|\mathrm{supp}(\mathbf{w}_{\mathbf{v}})| \le 22$ and the number of indices k such that $r_{\mathbf{v},k}(\cdot)$ is not the zero function is at most 23.

4.1.4. Traders T_i, where $i \in [2n]$: For each $i \in [2n]$, we set the endowment $\mathbf{w}_i$ of T_i as follows: $w_{i,2n+1} = 1/n^{12}$; and $w_{i,k} = 0$, for all other $k \in [2n+2]$. We set her PLC functions $r_{i,k}(\cdot)$, $k \in [2n+2]$, as follows: $r_{i,i}(\cdot) \Leftarrow [1]$; and $r_{i,k}(\cdot) \Leftarrow [0]$ for all other $k \in [2n+2]$.

4.2. From Approximate Market Equilibria to Approximate Nash Equilibria

By definition, it is easy to verify that $\mathcal{M}$ is a 2-linear additively separable PLC market which is strongly connected, 27-bounded and 23-sparse. Let $N = 2n+2$ be the number of goods. To prove Theorem 2, we only need to show that given any N^{-13}-approximate equilibrium $\mathbf{p}$ of $\mathcal{M}$, one can construct an n^{-6}-well-supported Nash equilibrium $(\mathbf{x}, \mathbf{y})$ of $(\mathbf{A}, \mathbf{B})$ in polynomial time.

To this end, we use $\mathbf{x}'$ and $\mathbf{y}'$ to denote the following n-dimensional vectors: $x'_k = p_k - 1$ and $y'_k = p_{n+k} - 1$ for all $k \in [n]$. Then, we normalize $(\mathbf{x}', \mathbf{y}')$ to get a pair $(\mathbf{x}, \mathbf{y})$ of probability distributions (we will show that $\mathbf{x}', \mathbf{y}' \ne \mathbf{0}$):

$$x_k = x'_k \Big/ \textstyle\sum_{i \in [n]} x'_i \quad \text{and} \quad y_k = y'_k \Big/ \sum_{i \in [n]} y'_i, \qquad (5)$$

for every $k \in [n]$.

Theorem 2 follows from Theorem 3 which we will prove in Section 5 (since every N^{-13}-approximate equilibrium is also an n^{-13}-approximate equilibrium by definition.)

Theorem 3. *If $\mathbf{p}$ is an n^{-13}-approximate market equilibrium of $\mathcal{M}$, then $(\mathbf{x}, \mathbf{y})$ constructed above must be an n^{-6}-well-supported Nash equilibrium of $(\mathbf{A}, \mathbf{B})$.*

5. Correctness of the Reduction

In this section, we prove Theorem 3. Let $\mathbf{p}$ be a normalized n^{-13}-approximate market equilibrium of $\mathcal{M}$. By the same argument used earlier, one can show that p_k must be positive for all $k \in [2n+2]$. As a result, we have $p_k \ge 1$ for all k and $\min_k p_k = 1$. Let $\mathcal{X}$ be an optimal allocation with respect to $\mathbf{p}$ that clears the market approximately: $\mathcal{X} = \{\mathbf{a}_{\mathbf{s}}, \mathbf{a}_{\mathbf{u}}, \mathbf{a}_{\mathbf{v}}, \mathbf{a}_i : \mathbf{s} \in S, \mathbf{u} \in U, \mathbf{v} \in V, i \in [2n]\}$.

We start with the following notation. Let $\mathcal{T}' \subseteq \mathcal{T}$ be a subset of traders and $k \in [2n+2]$. Then we use $w_k[\mathcal{T}']$ to denote the amount of good G_k that traders in $\mathcal{T}'$ possess at the beginning; and $a_k[\mathcal{T}']$ to denote the amount of good G_k that $\mathcal{T}'$ receives in the final allocation $\mathcal{X}$.

By the construction of $\mathcal{M}$, we have $2 \le w_k[\mathcal{T}] \le 3$ for all $k \in [2n+2]$. We further divide the traders $\mathcal{T}$ into two groups: $\mathcal{T}_1 = \{T_{\mathbf{s}} : \mathbf{s} \in S\}$ and $\mathcal{T}_2 = \mathcal{T} - \mathcal{T}_1$. Then by the definition of approximate equilibria, we have for all k,

$$\big| w_k[\mathcal{T}_1] - a_k[\mathcal{T}_1] + w_k[\mathcal{T}_2] - a_k[\mathcal{T}_2] \big| \le 3/n^{13}. \qquad (6)$$

First, we prove that, the price vector $\mathbf{p}$ must still satisfy the price-regulation property as in the price-regulating market $\mathcal{M}_{2n+2}$. The proof mainly uses the fact that traders in $\mathcal{T}_1$ possess almost all the goods in $\mathcal{M}$.

Lemma 1 (Price Regulation)**.** $\forall\, k \in [2n+2]$, $1 \le p_k \le 2$.

Proof: Assume for contradiction that $\mathbf{p}$ does not satisfies the price-regulation property. Without loss of generality, we assume that $p_1 = \max_k p_k > 2$ and $p_2 = 1$. By the same argument used in the proof of Property 1, we have

$$w_1[\mathcal{T}_1] = (2n+1) \cdot \frac{1}{n} \quad \text{and} \quad a_1[\mathcal{T}_1] \le 2n \cdot \frac{1}{n}$$

and thus, $w_1[\mathcal{T}_1] - a_1[\mathcal{T}_1] \ge 1/n$. By (6), we have

$$w_1[\mathcal{T}_2] - a_1[\mathcal{T}_2] \le -\frac{1}{n} + \frac{3}{n^{13}}.$$

As a result, we have

$$a_1[\mathcal{T}_2] \ge w_1[\mathcal{T}_2] + \frac{1}{n} - \frac{3}{n^{13}} \ge \frac{1}{n} - \frac{3}{n^{13}} \tag{7}$$

because $w_1[\mathcal{T}_2] \ge 0$. However, this cannot be true since the amount of goods the traders in $\mathcal{T}_2$ possess at the beginning is much smaller compared to $1/n$. Even if they spend all the money on G_1, $a_1[\mathcal{T}_2]$ is at most

$$\frac{\sum_{k\in[2n+2]} p_k \cdot w_k[\mathcal{T}_2]}{p_1} \le \sum_{k\in[2n+2]} w_k[\mathcal{T}_2] = O(n^{-2}) \ll \frac{1}{n},$$

since $p_1 = \max_k p_k$. This contradicts with (7). ■

Next, we prove two very useful relations between p_k and $w_k[\mathcal{T}_2] - a_k[\mathcal{T}_2]$, $k \in [2n+2]$.

Lemma 2. *Let $\mathbf{p}$ be a normalized n^{-13}-approximate market equilibrium and $\mathcal{X}$ be an optimal allocation that clears the market approximately. If $w_k[\mathcal{T}_2] - a_k[\mathcal{T}_2] > 3/n^{13}$ for some $k \in [2n+2]$, then $p_k = 1$.*

Proof: Without loss of generality, we prove the lemma for the case when $k = 1$. By (6) we have $w_1[\mathcal{T}_1] - a_1[\mathcal{T}_1] < 0$. This means that in the market participated by traders $T_{\mathbf{s}}$, the amount of G_1 which they would like to buy is strictly more than the amount of G_1 they possess at the beginning. Intuitively, this implies that the price p_1 of G_1 is lower than what it should be, and indeed we show below that $p_1 = 1$.

First, by the construction, only the following traders $T_{\mathbf{s}}$, $\mathbf{s} \in S$, are interested in G_1:

$$S_1 = \{\mathbf{s} = (1, j) : j \ne 1\} \quad \text{and} \quad S_2 = \{\mathbf{s} = (i, 1) : i \ne 1\}.$$

However, we have

$$a_1\big[T_{\mathbf{s}}, \mathbf{s} \in S_1\big] \le w_1\big[T_{\mathbf{s}}, \mathbf{s} \in S_1\big] = w_1[\mathcal{T}_1]$$

due to the budget limitation. As a result, there must exist an $\mathbf{s} = (i, 1) \in S_2$ such that $a_{\mathbf{s},1} > 0$. Since $\mathbf{a_s}$ is an optimal bundle for $T_{\mathbf{s}}$ with respect to $\mathbf{p}$, we have

$$1/p_1 \ge 2/p_i \implies p_1 \le p_i/2.$$

By Lemma 1 the price-regulation property, we conclude that $p_1 = 1$ and the lemma is proved. ■

Lemma 3. *Let $\mathbf{p}$ be a normalized n^{-13}-approximate market equilibrium and $\mathcal{X}$ be an optimal allocation that clears the market approximately. If $w_k[\mathcal{T}_2] - a_k[\mathcal{T}_2] < -3/n^{13}$ for some $k \in [2n+2]$, then $p_k = 2$.*

Proof: Without loss of generality, we prove the lemma for the case when $k = 1$. By (6) we have $w_1[\mathcal{T}_1] - a_1[\mathcal{T}_1] > 0$. This means that in the market participated by traders $T_{\mathbf{s}}$, the amount of G_1 which they would like to buy is strictly less than the amount of G_1 they possess at the beginning. Intuitively this implies that the price p_1 of G_1 is higher than what it should be, and indeed we show below that $p_1 = 2$.

Because $a_1[\mathcal{T}_1] < w_1[\mathcal{T}_1]$, there must exist a $j \in [2n+2]$ with $j \ne 1$ such that $\mathbf{s} = (1, j)$ and $a_{\mathbf{s},1} < w_{\mathbf{s},1}$. (Otherwise $a_1[\mathcal{T}_1] \ge w_1[\mathcal{T}_1]$.) This implies that trader $T_{\mathbf{s}}$ spends some of its money to buy G_j and thus,

$$1/p_j \ge 2/p_1 \implies p_1 \ge 2 \cdot p_j.$$

By Lemma 1 the price-regulation property, we conclude that $p_1 = 2$ and the lemma is proved. ■

We also need the following lemma.

Lemma 4. *Let $\mathbf{u} = (i, j, 1) \in U$ and $\mathbf{u}' = (j, i, 1) \in U$, then $w_{\mathbf{u},k} + w_{\mathbf{u}',k} \ge a_{\mathbf{u},k} + a_{\mathbf{u}',k}$, for all $k \in [2n+1]$.*

Let $\mathbf{v} = (i, j, 2) \in V$ and $\mathbf{v}' = (j, i, 2) \in V$, then $w_{\mathbf{v},k} + w_{\mathbf{v}',k} \ge a_{\mathbf{v},k} + a_{\mathbf{v}',k}$, for all $k \in [2n+1]$.

Proof: Without loss of generality, we only prove the first part of Lemma 4 for the case when $\mathbf{u} = (1, 2, 1)$ and $\mathbf{u}' = (2, 1, 1)$. Let $\mathbf{C}$ and $\mathbf{D}$ denote the following two n-dimensional vectors: For $k \in [n]$,

$$\begin{gathered}(C_k, D_k) = (A_{1,k} - A_{2,k}, 0) \text{ if } A_{1,k} - A_{2,k} \ge 0; \\ (C_k, D_k) = (0, A_{2,k} - A_{1,k}) \text{ otherwise.}\end{gathered} \tag{8}$$

We also define E and F as in (4). By the construction,

$$\begin{gathered}w_{\mathbf{u},n+k} = C_k/n^5, \quad w_{\mathbf{u}',n+k} = D_k/n^5, \quad \text{for all } k \in [n], \\ w_{\mathbf{u},1} = w_{\mathbf{u}',2} = 1/n^4, \quad w_{\mathbf{u},2n+1} = E/n^5, \quad w_{\mathbf{u}',2n+1} = F/n^5,\end{gathered}$$

and all other entries of $\mathbf{w_u}$ and $\mathbf{w_{u'}}$ are 0.

We focus on the preference of $T_{\mathbf{u}}$. After selling its initial endowment, the budget of $T_{\mathbf{u}}$ is $\Theta(1/n^4)$ by Lemma 1 since the total amount of goods she possesses is $\Theta(1/n^4)$. The PLC utilities $r_{\mathbf{u},k}(\cdot)$ of $T_{\mathbf{u}}$ are designed carefully, so that even though we do not know what exactly $\mathbf{p}$ is, we know the behavior of $T_{\mathbf{u}}$ due to the price-regulation property: $T_{\mathbf{u}}$ first buys the following bundle of goods from the market

$$\big\{D_k/n^5 \text{ of } G_{n+k} \text{ and } F/n^5 \text{ of } G_{2n+1} : k \in [n]\big\}. \tag{9}$$

Because $\mathbf{D}$ has at most 20 nonzero entries and every entry is between 0 and 2, the cost of this bundle is $O(1/n^5)$. $T_{\mathbf{u}}$ then buys as much G_1 as she can up to $1/n^4$, and spends all the money left, if any, on G_{2n+2}.

The behavior of $T_{\mathbf{u}'}$ is similar.

$T_{\mathbf{u}'}$ first buys the following bundle from the market:

$$\big\{C_k/n^5 \text{ of } G_{n+k} \text{ and } E/n^5 \text{ of } G_{2n+1} : k \in [n]\big\}. \quad (10)$$

$T_{\mathbf{u}'}$ then buys as much G_2 as she can up to $1/n^4$ and spends all the money left, if any, on G_{2n+2}.

Now we are ready to prove the lemma. The case when $k \in [n]$ but $k \neq 1, 2$ is trivial because

$$w_{\mathbf{u},k} = w_{\mathbf{u}',k} = a_{\mathbf{u},k} = a_{\mathbf{u}',k} = 0.$$

When $k = 1$, we have $w_{\mathbf{u},1} + w_{\mathbf{u}',1} = 1/n^4$, $a_{\mathbf{u}',1} = 0$ and $a_{\mathbf{u},1} \leq 1/n^4$. Lemma 4 then follows. The case when $k = 2$ can be proved similarly. For the case of $n + k$, $k \in [n]$, and for the case of $2n + 1$, we have

$$w_{\mathbf{u},n+k} = C_k/n^5, \quad w_{\mathbf{u}',n+k} = D_k/n^5, \quad a_{\mathbf{u},n+k} = D_k/n^5,$$
$$a_{\mathbf{u}',n+k} = C_k/n^5, \quad w_{\mathbf{u},2n+1} = E/n^5, \quad w_{\mathbf{u}',2n+1} = F/n^5,$$
$$a_{\mathbf{u},2n+1} = F/n^5, \quad \text{and} \quad a_{\mathbf{u}',2n+1} = E/n^5,$$

and Lemma 4 follows. This finishes the proof. ∎

Combining Lemma 4 and Lemma 2, we immediately get the following important corollary concerning p_{2n+1}.

Corollary 1. $p_{2n+1} = 1$.

Proof: By Lemma 4, we have

$$w_{2n+1}\big[T_{\mathbf{u}} : \mathbf{u} \in U \cup V\big] \geq a_{2n+1}\big[T_{\mathbf{u}} : \mathbf{u} \in U \cup V\big].$$

However, by the construction, we also have

$$w_{2n+1}\big[T_i : i \in [2n]\big] = 2n \cdot (1/n^{12}) = 2/n^{11}$$

and $a_{2n+1}[T_i : i \in [2n]] = 0$. As a result, we have

$$w_{2n+1}[\mathcal{T}_2] - a_{2n+1}[\mathcal{T}_2] > 3/n^{13}.$$

It then follows from Lemma 2 that $p_{2n+1} = 1$. ∎

Let $\mathbf{x}'$ and $\mathbf{y}'$ denote the vectors where $x'_k = p_k - 1$ and $y'_k = p_{n+k} - 1$. By Lemma 1, we have $0 \leq x'_k, y'_k \leq 1$ for all $k \in [n]$. We state the following two properties of $\mathbf{x}'$ and $\mathbf{y}'$ and use them to prove Theorem 3.

Property 2. *For all $1 \leq i \neq j \leq n$, we have*

$$(\mathbf{A}_i - \mathbf{A}_j)\mathbf{y}'^T < -\epsilon \implies x'_i = 0; \quad \textit{and} \quad (11)$$
$$\mathbf{x}'(\mathbf{B}_i - \mathbf{B}_j) < -\epsilon \implies y'_i = 0, \quad (12)$$

where $\epsilon = n^{-6}$, $\mathbf{A}_i$ denotes the ith *row vector of* $\mathbf{A}$, *and* $\mathbf{B}_i$ *denotes the* ith *column vector of* $\mathbf{B}$.

Property 3. $\exists\, i, j \in [n]$ *such that $x'_i = 1$ and $y'_j = 1$.*

Now we assume that $\mathbf{x}'$ and $\mathbf{y}'$ satisfy both properties. In particular, Property 3 implies that $\mathbf{x}', \mathbf{y}' \neq \mathbf{0}$. As a result, we can normalize them to get two probability distribution $\mathbf{x}$ and $\mathbf{y}$ using (5). Theorem 3 then follows.

Proof of Theorem 3: As both $\mathbf{x}$ and $\mathbf{y}$ are probability distributions, we only need to prove that $(\mathbf{x}, \mathbf{y})$ satisfies (2) and (3) for all $i \neq j \in [n]$. We only prove (2) here. Assume $\mathbf{A}_i\mathbf{y}^T + \epsilon < \mathbf{A}_j\mathbf{y}^T$, then by Property 3

$$(\mathbf{A}_i - \mathbf{A}_j)\mathbf{y}'^T = (\mathbf{A}_i - \mathbf{A}_j)\mathbf{y}^T \cdot \textstyle\sum_{k\in[n]} y'_k < -\epsilon.$$

By Property 2, we have $x'_i = 0$ and thus, $x_i = 0$. ∎

Proof of Property 2: We prove (11) for $i = 1$ and $j = 2$. The other part can be proved similarly.

Let $\mathbf{u} = (1, 2, 1)$, and $\mathbf{u}' = (2, 1, 1)$. Let $\mathbf{C}$ and $\mathbf{D}$ be the two nonnegative vectors defined in (8) and E and F be the two nonnegative numbers defined in (4).

Assume $(\mathbf{A}_1 - \mathbf{A}_2)\mathbf{y}'^T < -\epsilon$, then the money of $T_{\mathbf{u}}$ left after purchasing the bundle in (9) is

$$p_1 \cdot \frac{1}{n^4} + \sum_{k\in[n]} p_{n+k}\left(\frac{C_k}{n^5} - \frac{D_k}{n^5}\right) + p_{2n+1} \cdot \left(\frac{E}{n^5} - \frac{F}{n^5}\right).$$

By Corollary 1, $p_{2n+1} = 1$. Using $\mathbf{C} - \mathbf{D} = \mathbf{A}_1 - \mathbf{A}_2$ and

$$E + \textstyle\sum_k C_k = F + \sum_k D_k,$$

the money of $T_{\mathbf{u}}$ left is

$$p_1 \cdot \frac{1}{n^4} + \frac{1}{n^5}\sum_{k\in[n]} y'_k \cdot (C_k - D_k) < p_1 \cdot \frac{1}{n^4} - \frac{\epsilon}{n^5}.$$

This implies that the amount $a_{\mathbf{u},1}$ of G_1 that $T_{\mathbf{u}}$ buys is

$$a_{\mathbf{u},1} < 1/n^4 - \epsilon/(p_1 n^5) \leq 1/n^4 - 1/(2n^{11}).$$

Since $w_{\mathbf{u},1} = 1/n^4$, we have $w_{\mathbf{u},1} - a_{\mathbf{u},1} > 1/(2n^{11})$.

On the other hand, it is easy to check that $w_{\mathbf{u}',1} = 0$ and $a_{\mathbf{u}',1} = 0$. By Lemma 4, we have

$$w_1\big[T_{\mathbf{u}} : \mathbf{u} \in U \cup V\big] - a_1\big[T_{\mathbf{u}} : \mathbf{u} \in U \cup V\big] > 1/(2n^{11}). \quad (13)$$

Next we bound $w_1[T_i : i \in [2n]] - a_1[T_i : i \in [2n]]$. By the construction, we have

$$a_1\big[T_i : i \in [2n]\big] = a_{1,1} = 1 \cdot w_{1,2n+1}/p_1 \leq 1/n^{12}$$

and thus,

$$w_1\big[T_i : i \in [2n]\big] - a_1\big[T_i : i \in [2n]\big] \geq -1/n^{12}.$$

Combining (13) we have $w_1[\mathcal{T}_2] - a_1[\mathcal{T}_2] \gg 3/n^{13}$. It then follows from Lemma 2 that $x'_1 = 0$. ∎

Proof of Property 3: Let $\ell \in [n]$ be one of the indices that maximize $\mathbf{A}_\ell\mathbf{y}'^T$. We prove Property 3 by showing that $x'_\ell = 1$. Without loss of generality, we assume $\ell = 1$.

First, we consider a pair $\mathbf{v} = (i, j, 2)$, $\mathbf{v}' = (j, i, 2)$ in V. In the proof of Lemma 4, we actually showed that

$$w_{\mathbf{u},n+k} + w_{\mathbf{u}',n+k} = a_{\mathbf{u},n+k} + a_{\mathbf{u}',n+k},$$

for all $\mathbf{u} = (i, j, 1)$ and $\mathbf{u}' = (j, i, 1)$ in U, and all $k \in [n]$. Similarly, we have $w_{\mathbf{v},1} + w_{\mathbf{v}',1} = a_{\mathbf{v},1} + a_{\mathbf{v}',1}$.

Second, for any $\mathbf{u} = (i, j, 1) \in U$, we have $w_{\mathbf{u},1} = a_{\mathbf{u},1}$. This is because: If $i \neq 1$, then $w_{\mathbf{u},1} = a_{\mathbf{u},1} = 0$; if $i = 1$, then the money of $T_{\mathbf{u}}$ left after buying the bundle of goods in (9) is at least p_1/n^4, so $w_{\mathbf{u},1} = a_{\mathbf{u},1}$. As a result,

$$w_1\big[T_{\mathbf{u}} : \mathbf{u} \in U \cup V\big] = a_1\big[T_{\mathbf{u}} : \mathbf{u} \in U \cup V\big].$$

However, the amount of G_1 that T_1 buys is

$$p_{2n+1} \cdot w_{1,2n+1}/p_1 \geq 1/(2n^{12}).$$

As a result, we have

$$w_1\big[T_i, i \in [2n]\big] - a_1\big[T_i, i \in [2n]\big] \leq -1/(2n^{12}).$$

Finally, we have $w_1[\mathcal{T}_2] - a_1[\mathcal{T}_2] \ll -3/n^{13}$. By Lemma 3 we conclude that $p_1 = 2$ and thus, $x_1' = 1$. ■

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APPENDIX

In the appendix, we prove Theorem 1.

First, we prove that under the conditions of Theorem 1, $\mathcal{M}$ has at least one *quasi-equilibrium*. Then we show that any quasi-equilibrium of $\mathcal{M}$ is indeed a market equilibrium.

Definition 7. *A quasi-equilibrium of $\mathcal{M}$ is a price vector $\mathbf{p} \in \mathbb{R}_+^n$ such that there exists an allocation $\mathcal{X} = \{\mathbf{x}_i \in \mathbb{R}_+^n : i \in [m]\}$ which has the following properties:*

1) *The market clears: For every good $G_j \in \mathcal{G}$,*

$$\textstyle\sum_{i\in[m]} x_{i,j} \leq \sum_{i\in[m]} w_{i,j};$$

If $p_j > 0$, then $\sum_{i\in[m]} x_{i,j} = \sum_{i\in[m]} w_{i,j}$;

2) *For every T_i, at least one of the following is true:*
   a) $\mathbf{x}_i \in \mathrm{OPT}(i, \mathbf{p})$*; or*
   b) $\mathbf{p} \cdot \mathbf{x}_i = \mathbf{p} \cdot \mathbf{w}_i = 0$ (*zero income*).

One can also define ϵ-approximate quasi-equilibria similarly as in Definition 6.

The only difference between market equilibria and quasi-equilibria is that in the latter, we *do not* require the optimality of allocations for traders who have a zero income: If a trader has a zero income, then we can assign her any bundle of zero cost. However, if $\mathbf{p}$ is a quasi-equilibrium and the income of every trader is positive with respect to $\mathbf{p}$, then by definition $\mathbf{p}$ must be also a market equilibrium.

In [30], Maxfield gave a set of conditions that are sufficient for the existence of a quasi-equilibrium in an Arrow-Debreu market. We use the following simplified version:

Theorem 4 ([30])**.** *$\mathcal{M}$ has a quasi-equilibrium $\mathbf{p}$ if*

1) *For each trader $T_i \in \mathcal{T}$, its utility function $u_i : \mathbb{R}_+^n \to \mathbb{R}$ is both continuous and quasi-concave; and*
2) *For each trader $T_i \in \mathcal{T}$, $u_i(\cdot)$ is non-satiable, i.e., for any $\mathbf{x} \in \mathbb{R}_+^n$, there exists a vector $\mathbf{y} \in \mathbb{R}_+^n$ such that $u_i(\mathbf{y}) > u_i(\mathbf{x})$.*

Proof of Theorem 1: First, it is not hard to check that if $\mathcal{M}$ is an additively separable PLC market that is strongly connected, then it satisfies all the conditions in Theorem 4. As a result, $\mathcal{M}$ has a quasi-equilibrium $\mathbf{p}$.

We use $\mathcal{X} = \{\mathbf{x}_i \in \mathbb{R}_+^n : i \in [m]\}$ to denote an allocation that clears the market. Because $\mathbf{p} \neq \mathbf{0}$, there is at least one trader in $\mathcal{T}$, say T_1, has a positive income.

Second, we show that for every trader, its income is positive and thus $\mathbf{p}$ is indeed an equilibrium of $\mathcal{M}$. Suppose this is not true, then there is at least one trader T_2 whose income is zero. Since the economy graph is strongly connected, there is a directed path from T_2 to T_1. As a result, there must be a directed edge T_3T_4 on the path such that the income of T_3 is zero and the income of T_4 is positive. By definition, there exists a $j \in [n]$ such that the amount of G_j that T_3 owns at the beginning is positive and the PLC utility function of T_4 with respect to G_j is strictly monotone. However, since the income of T_3 is zero, we have $p_j = 0$. Therefore, the amount of G_j that T_4 wants to buy is $+\infty$, contradicting the assumption that $\mathbf{p}$ is a quasi-equilibrium of $\mathcal{M}$ (because the income of T_4 is positive but the bundle she receives is not optimal).

Now we have proved the existence of a market equilibrium. The second part of Theorem 1 follows from the work of Devanur and Kannan [13] (also see [39]). In [13] the authors proposed an algorithm for computing an equilibrium in an additively separable PLC market. (When the number of goods is constant, its running time is polynomial.) They divide the search space $\mathbb{R}_+^n$ of $\mathbf{p}$ into "cells" $C \subset \mathbb{R}_+^n$ using hyperplanes. Then for each cell C, there is a rational linear program LP_C which characterizes the set of equilibria in C: $\mathbf{p} \in C$ is an equilibrium iff it is a feasible solution to LP_C. Moreover, the size of LP_C for any cell C, is polynomial in the input size of $\mathcal{M}$.

Now let $\mathbf{p}$ be a market equilibrium of $\mathcal{M}$, which is not necessarily rational. We let C^* denote the cell that $\mathbf{p}$ lies in then $\mathbf{p}$ must be a feasible solution to LP_{C^*}. Since LP_{C^*} is rational, it must have a rational solution $\mathbf{p}^*$ and the number of bits one needs to describe $\mathbf{p}^*$ is polynomial in the size of LP_{C^*} and thus, is polynomial in the input size of $\mathcal{M}$. Theorem 1 follows since $\mathbf{p}^*$ is also an equilibrium. ■