A Simple and Approximately Optimal Mechanism for an Additive Buyer

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We consider a monopolist seller with n heterogeneous items, facing a single buyer. The buyer has a value for each item drawn independently according to (non-identical) distributions, and her value for a set of items is additive. The seller aims to maximize his revenue.

We suggest using the *a priori better* of two simple pricing methods: *selling the items separately*, each at its optimal price, and *bundling together*, in which the entire set of items is sold as one bundle at its optimal price. We show that for any distribution, this mechanism achieves a constant-factor approximation to the optimal revenue. Beyond its simplicity, this is the first computationally tractable mechanism to obtain a constant-factor approximation for this multi-parameter problem. We additionally discuss extensions to multiple buyers and to valuations that are correlated across items.

CCS Concepts: • **Theory of computation** → **Algorithmic mechanism design**;

Additional Key Words and Phrases: Mechanism design, revenue, approximation, simple vs. optimal, auction design

1 INTRODUCTION

A monopolist seller has a collection of n different items to sell and is facing a single buyer. How should he[1] sell the items to maximize revenue given that the buyer's valuation is private and the buyer acts strategically? When there is only a single item, and a single buyer with private value drawn from a known distribution F, seminal work of Myerson [55] shows that the optimal sale protocol is straightforward: simply post a take-it-or-leave-it price p, chosen to maximize the expected revenue, $p \cdot (1 - F(p))$. This simple deterministic mechanism is optimal among all

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[1]Throughout this article, we shall refer to the seller as "he/him" and the buyer as "she/her."

protocols, including interactive and randomized ones. Myerson's elegant theory extends to single-item auctions with multiple buyers, and further generalizes to all "single-dimensional" domains.[2]

Unfortunately, this rich theory fails to extend even to the simplest multi-item settings, where it was shown that optimal auctions might necessitate randomization [59] and menus of infinite size [31, 60], exhibit non-monotonicity [48], or be computationally intractable [30] (see Section 2 immediately following the introduction for examples illustrating these issues). This is troubling not only from the perspective of analyzing optimal mechanisms but also from the point of view of their usefulness. For an auction to be useful in practice, it should be simple to describe and transparent in its execution. The danger, then, is that mechanisms that obtain exact revenue optimality in theory, regardless of complexity, might share the fate of other mathematically optimal designs and be rarely used in practice [2]. It is therefore crucial to pair the study of revenue optimization with an exploration of the (approximation) power of simple auctions. In other words, *what is the relative strength of simple mechanisms versus complex optimal ones?*

This question was first asked in seminal work of Hartline and Roughgarden [47] for single-item auctions, and by seminal works of Chawla, Hartline, and Kleinberg [21] and Hart and Nisan [46] in multi-item settings. Despite making no direct reference to computation, this paradigm is not unlike those underpinning the theory of approximation algorithms: *optimal* algorithms for NP-hard problems often do not yield much tractable insight, yet approximation algorithms for these same problems often do. The formal notion of "polynomial-time" generally (but not always) separates insightful algorithms from the rest. Similarly, *optimal* auctions for multi-parameter domains do not yield much tractable insight, yet approximately optimal auctions might (indeed, the thesis of this article is that they do). Formal notions of complexity (discussed in Section 2) again generally separate insightful auctions from the rest, but the less formal eyeball-test for simplicity generally does the trick as well.

1.1 A Single Additive Buyer

As discussed above, the goal of this work is to deepen our understanding of the problem of revenue optimization in multi-parameter settings through the lens of approximation. We follow Hart and Nisan [46] in focusing on the simplest such multi-parameter problem: a seller owns n different items and has no value for them (nor cost for production), and a single buyer has non-negative value v_i for each item i, and that value is unknown to the seller. The seller only knows that each v_i is sampled independently from a distribution D_i. The buyer is interested in maximizing her quasi-linear utility (value received minus price paid), while the seller sets the allocation and payment rules with the goal of maximizing expected revenue.[3] Both seller and buyer are risk neutral (the seller aims to maximize the expected payment, and the buyer aims to maximize her expected utility).

Hart and Nisan consider the case of an *additive* buyer, meaning that her value for a set S of items is $\sum_{i \in S} v_i$. As they point out, additivity is a natural starting point for this endeavor, because, at first glance, the additive multi-item problem appears to be just a product of n separate single-item problems. After all, the buyer's value for item i does not depend at all on which other items she receives: it is always v_i, no matter what, by additivity. Moreover, any information the seller possesses about the buyer's values for items other than i tells him absolutely nothing about her value for item i: it is always sampled from D_i, no matter what, by independence. Therefore, there is truly no interaction between the items whatsoever from the buyer's perspective, and it tempting

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[2]Note that results for a single buyer also hold when there are many buyers but there are no supply constraints. In particular, they hold for digital goods (like software and music) that can be duplicated essentially for free.

[3]Note that the seller is constrained to always offer the buyer the option to "stay home"—receive no items and pay nothing.

to think that the optimal mechanism should simply treat this as n separate single-item problems. Myerson's theory tells us exactly how to solve each single-item problem: simply set a price $p_i \in \arg\max\{p \cdot (1 - F_i(p))\}$ on each item, and let the buyer purchase whatever subset of items she likes. Somewhat counter-intuitively, this mechanism is actually *not* optimal, even for extremely simple examples.

Example One: Two i.i.d. Items [46]. Consider the case of $n = 2$, where the buyer's value for each item is drawn i.i.d. from the uniform distribution over the finite set $\{1, 2\}$. Then treating each item as a separate problem, Myerson's theory would say that the seller should post a price of 1 on each item, obtaining expected revenue one per item and two in total.[4] However, there is a better mechanism that offers the buyer only two choices: receive *both* items together for a price of 3, or nothing at all (for free). With probability 3/4 the buyer chooses to purchase the bundle and pay 3, yielding expected revenue 9/4 > 2. So despite the initial intuition, it is indeed suboptimal to solve the problem by stitching together solutions to the separate single-item problems.

To get further intuition behind this example, consider the case of large n, where the buyer's value for each item is still drawn i.i.d. from the uniform distribution over the set $\{1, 2\}$. Then optimally selling each item separately still yields expected revenue of n. Consider instead offering again only two options: receive *all* items together for a price of $(3/2 - \varepsilon)n$, or nothing at all (for free), for an appropriately chosen $\varepsilon > 0$. Now, except with probability exponentially small in $n \cdot \varepsilon^2$, the buyer is indeed willing to purchase the grand bundle, yielding expected revenue approaching $3n/2$ as n grows large (for a multiplicative gap approaching 3/2). Hart and Nisan [46] show how to modify this example to exhibit a gap of $\Omega(\log(n))$ by replacing the uniform distribution with an Equal-Revenue distribution.[5]

What drives this example? Although there is no interaction between the items from the buyer's perspective, this is not the case from the seller's perspective. Indeed, the additional items enriches the strategy space of the seller and enables him to price options for which the buyer's value has much lower variance, allowing the seller to extract more of the buyer's welfare as revenue. At this point, we conclude not only that the multi-item problem is indeed richer than a product of single-item problems, but that treating it as such may come at significant cost: selling the items separately does not guarantee any constant-factor approximation to the optimal attainable revenue.

In this simple example, the seller can gain by combining all the items together into a single grand bundle, to be sold at a take-it-or-leave-it price. However, Hart and Nisan further show that this bundling approach can also be highly suboptimal. Indeed, it guarantees only an $\Omega(n)$ approximation to the optimal revenue in general, considering the case where each value v_i is independently 2^i with probability 2^{-i} (and 0 otherwise). In this example, bundling together achieves revenue $O(1)$, while selling separately achieves revenue n. We must therefore further conclude that **neither selling separately nor bundling together guarantees a constant fraction of the optimal revenue.** The reader should notice, however, that in the above examples, the superior mechanism witnessing that selling separately does not achieve a constant-factor approximation is bundling together, and vice versa.

1.1.1 Further Complexity and the Need for Approximation. So at this point, it is clear that optimal mechanisms are richer than one might naively expect. Still, one might reasonably hope that optimal mechanisms are not too complex. Unfortunately this is not the case, and prior work has

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[4]Note that setting a price of 2 on each item guarantees the same revenue. The existence of multiple optimal prices doesn't drive the example, and the same phenomenon would happen with, e.g., uniform distribution over the set $\{2, 5\}$ instead.

[5]The Equal-Revenue distribution has CDF $F(x) = 0$ for $x \leq 1$, and $F(x) = 1 - 1/x$ for $x \geq 1$. The expected revenue obtained by posting any price $p \geq 1$ is one, thus any two such prices obtain equal revenue.

identified numerous complexities that the optimal mechanism may possess. We briefly highlight one aspect below (menu complexity), and elaborate on the others (computational intractability, non-monotonicity) in Section 2.

Example Two: Randomization [30]. Consider again the case of $n = 2$, where the buyer's value for each item is drawn independently, but not identically. Value v_1 is uniformly drawn from $\{1, 2\}$, and v_2 is uniformly drawn from $\{1, 3\}$. Quick calculations confirm that the revenue achieved by treating the items separately is $1 + 1.5 = 2.5$ (by setting prices of 1 and 3, respectively). The revenue achieved by bundling the items together is $3 \cdot 3/4 = 2.25$ (by setting a price of 3, which sells with probability 3/4). But there is a better mechanism that offers the buyer three choices: (a) receive both items and pay 4; (b) receive item one with probability one and item two with probability 1/2 and pay 2.5; or (c) receive nothing and pay nothing. Then when the buyer has valuations $\langle 1, 3 \rangle$ or $\langle 2, 3 \rangle$, she will choose to pay 4. When her values are $\langle 2, 1 \rangle$, she will choose to pay 2.5. When her values are $\langle 1, 1 \rangle$, she will choose to pay 0, for a total expected revenue of $4 \cdot 1/2 + 2.5 \cdot 1/4 = 2.625 > 2.5$. This randomized mechanism is in fact the *unique* optimal mechanism[6] for this instance, and it beats any deterministic mechanism.

Example Two shows that even for simple instances, optimal mechanisms must sometimes be randomized. Yet, that optimal mechanism uses only one non-trivial lottery, so perhaps the degree of randomization required is not *that* bad. Consider further the following example:

Example Three: Uncountable Menu Complexity [31]. Consider again the case of $n = 2$, but this time the buyer's value for each item is drawn i.i.d. from the distribution supported on $[0, 1]$ with density $f(x) = 2(1 - x)$.[7] Daskalakis et al. prove for this example that the unique (up to differences of measure zero) optimal mechanism has *uncountable menu complexity* [31]. That is, the number of distinct options available for the buyer to purchase is uncountable. They show that the optimal mechanism contains the following four kinds of options: (a) the buyer can receive item one with probability 1, and item two with probability $\frac{2}{(4-5x)^2}$ paying price $\frac{2-3x}{4-5x} + \frac{2x}{(4-5x)^2}$, for *any* $x \in [0, \approx 0.0618)$; (b) the buyer can receive item two with probability 1, and item one with probability $\frac{2}{(4-5x)^2}$ paying price $\frac{2-3x}{4-5x} + \frac{2x}{(4-5x)^2}$, for any $x \in [0, \approx 0.0618)$; (c) the buyer can receive both items and pay ≈ 0.5535; or (d) the buyer can receive neither item and pay nothing.

One can reasonably debate the finer points on the above object's mathematical tractability—on one hand it can at least be succinctly described; on the other hand, it offers an uncountable menu—but what is made clear by this example is that any theory of optimal multi-item mechanisms must include a study of such auctions, which are significantly more complex than their single-item counterparts. At this point, we must further conclude that **the study of truly simple multi-item auctions is unlikely to develop through a theory of *exactly*optimal auctions.**

1.2   Our Results

1.2.1   Main Result. Let us now return to selling separately and bundling together, arguably the two simplest multi-item auctions,[8] neither of which guarantees a constant-fraction of the optimal revenue in all instances. Our main result is that the *a priori maximum* of the revenue generated by these two approaches—selling separately or bundling together—does indeed guarantee a constant-factor approximation to the optimal revenue.

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[6]Note that the mechanism could be needlessly modified to additionally offer, for instance, an option to receive item one for 100. Such an option would of course never be purchased. By "unique," we mean that every optimal mechanism results in the buyer purchasing both items for 4 when $v_2 = 3$, item one w.p. one and item two w.p. 1/2 when $\langle v_1, v_2 \rangle = \langle 2, 1 \rangle$, and purchasing/paying nothing otherwise.

[7]Note that this is the Beta(1,2) distribution.

[8]They are both formally a "black-box reduction" to the single-item case.

Main Result (Informal). *In any market with a single additive buyer with independent item values, either selling separately or bundling together guarantees a 6-approximation to the optimal revenue.*

Prior to our work, it was unknown whether *any* simple mechanism achieves a constant-factor approximation to the optimal revenue, let alone one of these especially simple mechanisms. Moreover, note that our work further implies (Appendix G) that even if the seller does not know the distributions exactly but rather only has some reasonable access to them, a constant-factor approximation can be found in poly-time. Prior to our work, it was unknown whether a constant-factor approximation could be obtained by *any* poly-time mechanism, even without the additional restriction of simplicity. Further, as selling separately and bundling together are both deterministic, our work further shows a constant multiplicative gap between the optimal deterministic and randomized mechanisms. Prior to our work, it was also unknown whether any deterministic mechanism could provide such a guarantee, even without the additional restriction of simplicity. Additionally, both mechanisms have desirable robustness properties (overviewed further in Section 3).

Brief Intuition. To get a sense of why our main result holds, recall the discussion following Example One with n i.i.d. distributions. We noted that if the sum of the buyer's values for all the items tends to concentrate around its expectation, then bundling together (at the right price—slightly below the expected value) will extract a significant fraction of this total value as revenue. The bundling-together mechanism will therefore be approximately optimal when the total value concentrates. However, since the total value is a sum of independent random variables, it *will* concentrate unless the sum is dominated by rare events where one or more items have significantly higher-than-expected value. How should we handle such cases? Intuitively, if these "tail" events are indeed rare, then we are unlikely to see many of them at once and the optimal revenue will be driven by a small number of individual items. Since selling items separately optimizes the revenue from each individual item by itself, the sell-separately mechanism is a good candidate for covering this tail case.

To make this intuition concrete, a surprisingly challenging question is to properly define "concentration" and "tail events." For example, for n i.i.d. Equal Revenue distributions, the expected value for each item is infinite, yet the expected optimal revenue (of selling all items optimally) is finite. In particular, as bundling together (but *not* selling separately) is approximately optimal for n i.i.d. Equal Revenue distributions, we would like our argument to claim that the sum of n i.i.d. Equal Revenue distributions "concentrates," despite the fact that its expected value is infinite. Challenges similarly arise when trying to properly define "tail events."

This led Li and Yao to develop a Core-Tail Decomposition [52], which proposes adequate definitions for tail events, and formally separates the analysis into cases where a tail event occurs ("the tail") and the rest ("the core"). One key difference between our approach and that of Li and Yao is the definition of tail events, and we briefly highlight a top-level distinction here. Consider starting from an arbitrary instance with n items, and adding to that instance n' items whose values are drawn from a point-mass at 0. Clearly, this does not change the underlying instance. Yet, this modification *does* change the definitions for tail events proposed in Reference [52], thereby changing the analysis. Our choice of decomposition has the property that the analysis is invariant under this modification, as discussed in Section 6, and our approach enables a tighter analysis (and thereby improving their approximation guarantees as well—see Section 3). All further distinctions, including formal definitions, are deferred to the technical sections.

1.2.2 Additional Results. Beyond the core single-buyer setting, we consider several extensions using similar techniques. First, we consider the *multi-buyer setting*, where each buyer's value for each item is drawn independently (not necessarily identically) from arbitrary distributions. Here, we show again that selling separately (that is, sell each item using Myerson's optimal single-item

auction) achieves an $O(\log n)$-approximation. Note that this is asymptotically tight, even for the case of a single buyer (via the modification of Example One in Reference [46]). Prior to our work, no non-trivial bounds were known on the approximation guarantee of any class of mechanisms for this setting.

We further show that in contrast to the single-buyer case, the better of selling separately and bundling together (treating the grand bundle of all items as a single item and running Myerson's optimal auction) does *not* guarantee an $o(\log n)$ approximation in the multi-buyer setting. We further extend this lower bound to the class of *partition mechanisms*, which partition the items into disjoint subsets and run Myerson's optimal auction for each subset separately (generalizing both selling separately and bundling together). A preliminary version of this article presented at FOCS 2014 left as its main open problem whether a simple (and/or deterministic, poly-time) mechanism could guarantee a constant-factor approximation in the multi-buyer case, which was resolved by Yao [61] (discussed further in Section 3).

Finally, to better understand this new class of partition mechanisms, we study the performance of selling separately and bundling together against the optimal partition mechanism in a variety of settings. For multiple buyers with independent items, we show that the better of selling separately and bundling together achieves a constant-factor approximation to the optimal partition mechanism when *either* buyers (Theorem 5) *or* items (Theorem 6) are i.i.d., but no better than an $\Omega(\log n)$-approximation in general (Proposition 7). We consider also the single buyer case when values for the items may be *arbitrarily correlated*. While neither class of mechanisms can guarantee any non-zero fraction of the optimal revenue for even a single buyer ([4, 45]), the question remains as to whether simple mechanisms can approximate more complex (though still suboptimal) mechanisms in the presence of correlation. To this end, we prove that selling items separately obtains an $O(\log n)$-approximation to the optimal obtainable revenue by a partition mechanism, and that this is tight. In fact, we show a gap of $\Omega(\log n)$ between the *a priori* better of selling separately and bundling together, versus the optimal partition mechanism. We include several tables in Appendix A displaying the relative power of the various classes of mechanisms studied in this article, noting here that as of our work, all upper and lower bounds are (asymptotically) matching.

2 COMPLEXITY OF OPTIMAL MULTI-ITEM MECHANISMS

Here, we will describe (without proofs) several known examples from the literature that further demonstrate the complexity of optimal multi-item auctions, even with just a single buyer whose value for the items is additive. In each, we will use D_i to denote a distribution and F_i to denote its CDF.

Example Four: Revenue Non-monotonicity [48]. Consider two distributions $D = \times_{i \in [n]} D_i$, and $D^+ = \times_{i \in [n]} D_i^+$, where each D_i^+ *stochastically dominates D_i* (that is, $F_i(x) \geq F_i^+(x)$ for all i, x). This implies that draws (v, v^+) from (D, D^+) can be coupled so that $v(S) \leq v^+(S)$ for all S with probability 1. It seems natural to conjecture that the optimal achievable revenue of D^+ should exceed that of D, as this is true for $n = 1$ (the single item case) and the proof is a trivial corollary of Reference [55].[9] However, Hart and Reny provide explicit distributions D, D^+ *with i.i.d. marginals for two items* such that the optimal achievable revenue for D *strictly exceeds* that for D^+ [48]. That is, the optimal revenue for a "strictly better" distribution is strictly worse. This property has been termed *revenue non-monotonicity*, and provides further evidence of the complexity of optimal multi-item auctions. Indeed, one could imagine D^+ resulting from D after an advertising campaign that increases the values of all consumers in a population for all items. Revenue non-monotonicity

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[9]Specifically, for any price p that might be set, $p \cdot (1 - F^+(p)) \geq p \cdot (1 - F(p))$.

implies that such a campaign may not only harm the revenue of an existing auction but also harm the optimal achievable revenue (and even in cases where D, D^+ are both i.i.d. over two items).

Example Five: Computational Intractability [30]. Consider being given as input a discrete product distribution over valuations for n items, where each value v_i is drawn from D_i and the support of each D_i is of size *two*. That is, each v_i is either a_i or b_i, and is a_i with probability p_i, all of which are rational and of bit complexity POLY(n). Then the entire input can be described by these $3n$ numbers, and has size POLY(n). It is somewhat tricky to formalize exactly what it should mean to "find" the optimal mechanism (since it may, for instance, have exponential menu complexity), but Daskalakis et al. [30] prove that it is #P-hard to find the optimal auction in this setting in the following strong sense: unless ZPP $\supseteq$ P$^{\#P}$, no randomized poly-time procedure can take as input $(\vec{a}, \vec{b}, \vec{p})$, and a further input valuation vector $\vec{v}$, and guess (correctly with probability at least $1/2 + 1/\text{POLY}(n)$) whether the optimal mechanism awards the buyer with valuation $\vec{v}$ item one with probability 0 or 1 (if neither is true, the algorithm is allowed to behave arbitrarily). This rules out any reasonable poly-time solution, as any notion of a "solution" must be able to determine which items are purchased by a given realized $\vec{v}$.

Example Six: Infinite Gaps with Correlation [4, 45]. Finally, consider the case that the buyer's values for the items are *correlated*. That is, there is an arbitrary n-dimensional distribution D over $\mathbb{R}^n_+$, and the buyer's values for all n items are drawn jointly from D. In this case, Hart and Nisan [45] provide an explicit distribution D over $n = 2$ items such that the revenue of the optimal mechanism is *infinite*, yet the revenue of any mechanism of menu complexity C is at most C. It was further observed that this same D is stochastically dominated by some D^+ (that is, couples $(\vec{v}, \vec{v}^+)$ can be drawn from (D, D^+) so that $v(S) \leq v^+(S)$ for all S with probability 1) such that the optimal revenue for D is infinite, yet the optimal revenue for D^+ is at most one [57].[10]

Further related work is described in Section 3, but we have surveyed these examples to emphasize the following themes:

(1) Examples One, Two, and Five highlight that the optimal mechanism may be surprisingly complex, even in extremely simple examples. This suggests that we are unlikely to make progress by simply restricting the allowable input distributions.

(2) Examples Three, Four, and Five highlight the three main complexities commonly associated with optimal auctions that are viewed as impractical: unbounded menu complexity, non-monotonicity, and computational intractability. Note that in contrast, the approximately optimal *a priori* maximum of selling separately and bundling together is deterministic, has polynomial menu complexity,[11] is revenue-monotone,[12] and implementable in poly-time.

(3) In addition to the discussion preceding Example One, Example Six further motivates restricting attention to independent items, as even approximately optimal mechanisms can be arbitrarily complex without this.

3 RELATED WORK

3.1 Prior Work

Seminal work of Hartline and Roughgarden initiated the agenda of "simple versus optimal" mechanisms: the study of simple mechanisms through the lens of approximation [47]. Their work

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[10]In fact, their D^+ simply draws $(v_1, v_2) \leftarrow D$ and sets $(v_1^+, v_2^+) = (\max\{v_1, v_2\}, \max\{v_1, v_2\})$.

[11]Technically, selling separately has exponential menu complexity, but Reference [7] proves that one can get arbitrarily close to the revenue of selling separately with polynomial menu complexity.

[12]To conclude revenue-monotonicity, observe that both selling separately and bundling together are products of single-item auctions, and therefore revenue-monotone. The maximum of revenue-monotone mechanisms is also revenue-monotone.

considers single-dimensional settings and shows that one can often approximate the revenue of Myerson's optimal auction with something even simpler. This agenda has even more bite in multi-dimensional settings, where optimal auctions are *far* more complex (cf. the above examples). On this front, seminal work of Chawla, Hartline, and Kleinberg considers a single unit-demand buyer,[13] and prove that a deterministic item-pricing guarantees a constant-factor approximation to the optimal mechanism [21]. Further follow-up work considers multiple buyers, but faces barriers in moving beyond unit-demand preferences [22, 26, 27, 50].

Hart and Nisan first proposed studying a single additive buyer through the lens of approximation, as this is the simplest possible setting where all previously developed tools remained stuck [46]. Their work provides several simple lemmas, whose composition proves surprisingly strong conclusions: they show that selling separately achieves an $O(\log^2 n)$-approximation, and further that when the items are i.i.d., bundling together achieves an $O(\log n)$-approximation. Follow-up work of Li and Yao introduces the *Core-Tail Decomposition* technique, improving the guarantee of selling separately to $O(\log n)$ (which is tight) and the guarantee of bundling together when all items are i.i.d. to $O(1)$ [52]. Our work makes use of tools developed in both works, and improves the approximation guarantee to 6 without any assumptions.

A related sequence of papers [13, 20] use extreme value theorems to prove that when distributions satisfy the Monotone Hazard Rate (MHR) condition, nearly optimal mechanisms can be found in poly-time and are fairly simple.

3.2   Subsequent Work

An initial presentation of this work at FOCS 2014 posed three open problems, all of which have since been resolved (and then some). The first asked whether a simple mechanism could guarantee a constant-factor approximation for multiple additive buyers (as our lower bounds prove that partition mechanisms cannot achieve this guarantee), which was resolved by Yao [61]. The main conceptual discovery is "the right" extension of bundling together for multiple buyers, which turns out to be an *entry fee*. In the context of our work, bundling together can be interpreted as the mechanism that gives the buyer all items for free, as long as the buyer pays an entry fee to participate.[14] Reference [61] proves that the proper extension of our results to multiple buyers is to replace the "give the buyer all items for free" mechanism with the VCG mechanism (which sells each item separately using a second-price auction), but maintain the entry fee.[15] This idea persists in follow-up works (sometimes further replacing the VCG mechanism with other simple mechanisms) [17, 24, 29].

The second asked whether a simple mechanism could guarantee a constant-factor approximation for a single buyer who was neither unit-demand nor additive (e.g., k-demand, satisfying $v(S) = \max_{T \subseteq S, |T| \leq k} \{\sum_{i \in T} v(\{i\})\}$ for all S). This was resolved in Reference [57], who show again that either selling separately or bundling together achieves a constant-factor approximation even when the buyer has *subadditive* valuations over independent items.[16] Further follow-up work extends this to models of limited complementarity [36].

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[13]A valuation function $v(\cdot)$ is unit-demand if $v(S) = \max_{i \in S} \{v(\{i\})\}$ for all S.
[14]That is, there is an entry fee p. The buyer, with full knowledge of her valuation, decides whether or not to pay p. If she pays, then she participates in the mechanism, which gives her all items for free. If not, then she leaves with no items and pays nothing.
[15]Now, the entry fee p_j for buyer j depends on the valuations of others (yet is still independent of the valuation of buyer j).
[16]Roughly speaking, "independent items" means that the random variables $v(S_1), \ldots, v(S_\ell)$ are independent whenever $S_i \cap S_j = \emptyset$ for all i, j. The formal definition is slightly (but strictly) more restrictive than this.

The third asked whether our results could be extended to models of limited correlation, such as those considered in References [26, 27]. This was resolved in Reference [6], who show essentially that our approach is robust to linear combinations: if there are k independent *features* that a buyer might value, and the value of each item is a fixed linear combination of few features, then again selling separately or bundling together achieves a constant-factor approximation.

However, work has continued far beyond these specific open problems. Most notably, work of Chawla and Miller [24] and Cai and Zhao [29] consider multiple buyers, all of whom are neither unit-demand nor additive, and prove that a posted-price mechanism with entry fee achieves a constant-factor approximation (i.e., posting prices is "the right" extension of selling separately, and adding an entry fee is "the right" extension of bundling together). Reference [29] is the state-of-the-art in this direction, which shows that these mechanisms guarantee a constant-factor approximation when buyers are fractionally subadditive over independent items, and an $O(\log n)$-approximation when buyers are subadditive over independent items. Cai et al. [17] further show how to interpret our work and those following Reference [21] via the same dual solution in a duality framework. This unified presentation of both lines of work is also given in Reference [44].

There are also numerous follow-up works in tangential directions, establishing that the mechanisms studied in these lines of work are quite robust: works of References [1, 9, 10, 14, 53, 58] prove that these mechanisms can be learned using only polynomially many samples from the underlying distributions, and Reference [41] further shows that they can be made *prior-independent* at the cost of additional constant factors. In a similar vein, Reference [23] proves that these results are fairly robust to the Bayesian assumption, and their guarantees hold when the buyers (and not the auctioneer) know the prior. Reference [19] extend our analysis to accommodate a budget-constrained buyer. Reference [56] provides a PTAS for the optimal partition mechanism (and proves that no FPTAS exists unless P = NP). Works of References [12, 42] target max-min guarantees (i.e., the auction that maximizes the minimum achieved revenue over all D in some class $\mathcal{D}$) instead of worst-case approximation guarantees. Our tools have also found use in seemingly unrelated follow-ups studying gains from trade [3, 5], information revelation [18, 33, 37, 40], and "Bulow-Klemperer"-type [8] results [11, 35, 38, 51].

3.3 Recent Work and Open Directions

Nearly simple and Nearly optimal mechanisms. Our work shows that a constant-factor approximation[17] can be found in poly-time, and implies that a constant-factor approximation can be achieved with polynomial menu complexity (the latter claim further requires a result of Reference [7]). Works discussed earlier prove that the optimum cannot be found in poly-time [30], and may have uncountable menu complexity [31], but do not rule out even an FPTAS. Works of References [15, 16, 25] identify similar complexity barriers for related problems. The major open question here is determining whether or not an FPTAS(/PTAS/QPTAS) exists. Extremely recent work now provides a QPTAS for a single unit-demand buyer [49], but the entire spectrum still remains open for additive buyers (along with providing/ruling out a FPTAS/PTAS for unit-demand). The equally significant question for menu complexity also remains open—on this front, only recently was it shown that *some bounded* (as a function of n, ε) menu complexity suffices to guarantee a $(1 - \varepsilon)$-approximation [7], and the first non-trivial lower bounds were proved for the case of $n = 2$ [43].

All the way to Subadditive. Recent work discussed above proves that a posted-price mechanism with entry-fee guarantees a constant-factor approximation when buyers are *fractionally*

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[17]We have originally proved that this constant is at most 7.5, and soon after, Rubinstein has tightened the analysis yielding an upper bound of 6. The current best analysis show that this constant is at most 5.2 [54], yet at least 2 [56].

subadditive with independent items [29], and further shows that posted-price mechanisms alone guarantee an $O(\log n)$-approximation when buyers are subadditive. The key barrier to obtaining a constant-factor approximation for all subadditive valuations is the following: one portion of the analysis in Reference [29] makes use of an analysis in Reference [39] of posted-price mechanisms for *welfare* guarantees. The analysis of Reference [39] guarantees an $O(1)$-approximation to the optimal welfare for fractionally subadditive buyers, and an $O(\log n)$-approximation when buyers are subadditive. It remains an open question whether the [39] analysis can be improved to $O(1)$ for subadditive buyers (explicitly posed in Reference [39]), which would very likely extend [29] to subadditive buyers as well.[18] Independently, it is generally an important open problem to extend [29] all the way to subadditive valuations (explicitly posed in Reference [29]—see therein for a deeper discussion).

4 PRELIMINARIES

The setting we consider is that of a single monopolist seller with n heterogeneous and indivisible items for sale to m additive, risk-neutral, quasi-linear buyers. That is, each buyer j has a non-negative value v_{ij} for item i. While our main results are for the setting of a single buyer, we will define our setting more generally; this will be useful when discussing extensions. If a randomized outcome awards buyer j item i with probability π_{ij} and charges her a price p_j in expectation, then her utility for this outcome is $\sum_i v_{ij}\pi_{ij} - p_j$. Each value v_{ij} is sampled independently from a known distribution D_{ij}, supported on $\mathbb{R}_{\geq 0}$. We make no assumptions on D_{ij} whatsoever. We refer to D as the joint $(m \cdot n)$-dimensional distribution over all buyers' values for all items, D_i as the m-dimensional distribution over all buyers' values for item i, and D^j as the n-dimensional distribution over buyer j's values for all items. Furthermore, we denote by $\vec{v}$ a random sample from D, $\vec{v}_i$ a random sample from D_i, and $\vec{v}^j$ a random sample from D^j. We also denote the maximum value for item i as $v_i^* = \max_j\{v_{ij}\}$. We use the notation D^{-j} to denote the distribution for all buyers but j (and use similar notation for other vectors).

The revelation principle [55] establishes that the optimal revenue of *any* (not necessarily truthful) auction at *any* Bayes-Nash equilibrium is captured by a direct mechanism that is Bayesian Incentive Compatible (and we will therefore restrict attention to optimal mechanisms of this form, as it is w.l.o.g.). That is, this mechanism simply asks each buyer to report a value for each item, and it is in each buyer's interest to report their true value, assuming that all other buyers do so as well. All of the mechanisms we describe will also satisfy the stronger property of Dominant Strategy Incentive Compatibility, where it is in each buyer's interest to report their true values no matter the other buyers' behavior (observe further that these definitions coincide when there is just $m = 1$ buyer). As usual, we also impose the individual rationality constraint, saying that every buyer's utility is non-negative when truthful. Formally:

- For a given direct mechanism, let $x_{ij}(\vec{v})$ denote the probability that item i is awarded to buyer j on bids $\vec{v}$, and $p_j(\vec{v})$ denote the price that buyer j pays. Define also the interim variables $\pi_{ij}(\vec{v}^j) = \mathbb{E}_{\vec{v}^{-j} \leftarrow D^{-j}}[x_{ij}(\vec{v}^{-j}; \vec{v}^j)]$, and $q_j(\vec{v}^j) = \mathbb{E}_{\vec{v}^{-j} \leftarrow D^{-j}}[p_j(\vec{v}^{-j}; \vec{v}^j)]$. When there is just $m = 1$ buyer, observe that $x = \pi$.
- A direct mechanism is *Bayesian Incentive Compatible* (BIC) if for all $j, \vec{v}^j, \vec{w}^j$, it holds that $\sum_i v_{ij} \cdot \pi_{ij}(\vec{v}^j) - q_j(\vec{v}^j) \geq \sum_i v_{ij} \cdot \pi_{ij}(\vec{w}^j) - q_j(\vec{w}^j)$.
- A direct mechanism is *Dominant Strategy Incentive Compatible* (DSIC) if for all $j, \vec{v}^j, \vec{w}^j, \vec{v}^{-j}$, it holds that $\sum_i v_{ij} \cdot x_{ij}(\vec{v}^{-j}; \vec{v}^j) - p_j(\vec{v}^{-j}; \vec{v}^j) \geq \sum_i v_{ij} \cdot x_{ij}(\vec{v}^{-j}; \vec{w}^j) - p_j(\vec{v}^{-j}; \vec{w}^j)$. When there is just $m = 1$ buyer, observe that this is equivalent to BIC.

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[18]Note that Reference [29] does not treat Reference [39] as a black-box, but analysis "in the same spirit" as Reference [39] would suffice.

- A direct mechanism is *Ex-Post Individually Rational* (IR) if for all $j, \vec{v}^j, \vec{v}^{-j}$, it holds that $\sum_i v_{ij} \cdot x_{ij}(\vec{v}^{-j}; \vec{v}^j) - p_j(\vec{v}^{-j}; \vec{v}^j) \geq 0$.

We further use the following terminology to discuss the revenue obtainable by various types of mechanisms, where the first three are taken from Reference [46]. Below, we reference Myerson's optimal single-item auction, which for $m = 1$ buyer simply sets the price $p^* = \arg\max_p \{p \cdot \Pr[v \geq p]\}$ [55]. The precise format of Myerson's auction for $m > 1$ buyers is immaterial for our results, as we use no properties of this auction in our proofs other than its optimality.

- REV(D): The optimal revenue (more precisely, the supremum of revenues) obtained by any (possibly randomized) BIC/IR mechanism when the buyer profile is drawn from D.
- SREV(D): The optimal revenue (more precisely, the supremum of revenues) obtained by selling items separately when the buyer profile is drawn from D. That is, the revenue obtained by running Myerson's optimal auction separately for each item. Recall that when there is just $m = 1$ buyer, SREV(D) is achieved by setting a price p_i on each item i, and letting the buyer pick any subset of items to purchase.
- BREV(D): The optimal revenue (more precisely, the supremum of revenues) obtained by selling the grand bundle when the buyer profile is drawn from D. That is, the revenue obtained by running Myerson's optimal auction when treating the grand bundle as a single item. Recall that when there is just $m = 1$ buyer, BREV(D) is achieved by setting a price p on the grand bundle, and letting the buyer purchase the grand bundle for price p.
- PREV(D): The optimal revenue (more precisely, the supremum of revenues) obtained by any *partition* mechanism when the buyer profile is drawn from D. That is, the maximal revenue obtained by first partitioning the items into disjoint bundles, and then running Myerson's optimal auction separately for each bundle, treating each bundle as a single item.

Observe that some of the terms above may not be well-defined if REV(D) is unbounded. In this case, our proofs establish that SREV(D), BREV(D), PREV(D) are all unbounded as well (although we will not explicitly state these conclusions). Also, for ease of exposition, we will only consider the case when the supremum of revenues is actually achieved by some mechanism, which we will refer to as the "optimal mechanism." While we do not explicitly discuss cases where the supremum is not realized but is instead the limit of a sequence of mechanisms, we note that all of our proofs carry over by standard limiting arguments.

Observe also that selling separately and bundling together are both partition mechanisms, and that all partition mechanisms are in fact DSIC. Given a distribution D over profiles, we will often consider the welfare $\sum_i v_i^* = \sum_i \max_j\{v_{ij}\}$ of a buyer profile $\vec{v}$ drawn from D. We will write VAL(D) for the expected optimal welfare, so that VAL(D) = $\mathbb{E}_{\vec{v} \sim D}[\sum_i v_i^*]$. We will also write var(D) = var$_{\vec{v} \sim D}(\sum_i v_i^*)$ for the variance of the welfare. Observe that, immediately from Individual Rationality, REV(D) $\leq$ VAL(D) for all D.

We will make use of some results from Reference [46] that provide useful bounds on REV(D). We include proofs in Appendix B for completeness. Lemma 1 is stated and proved directly in Reference [46] and also in Reference [20]. Lemma 2 is not directly stated nor proved, but is similar to an implicit result from Reference [46].

In Lemma 1 below, we think of D and D' as being distributions over values for disjoint sets of items, for the same set of m buyers. The distribution $D \times D'$ then draws values for those two sets of items, independently, from D and D', respectively.

LEMMA 1 [20, 46]. REV($D \times D'$) $\leq$ VAL(D) + REV(D').

The next result establishes a weak bound on REV(D) with respect to SREV(D).

Lemma 2. $Rev(D) \leq n \cdot m \cdot SRev(D)$.

5 THE CORE-TAIL DECOMPOSITION

We make use of an idea developed by Li and Yao [52] called the Core-Tail Decomposition of a value distribution for a single buyer. To obtain our stronger results for a single buyer and also extend to many buyers, we define the core differently but in the same spirit. The idea is to separate each m-dimensional value distribution for each item into the core and the tail, the tail being the part where some buyer has an unusually high value for the item. Then the core of the entire nm-dimensional distribution is the product of all the cores, and the tail is everything else.

5.1 Defining the Core and Prior Results

Below, we formalize the notion of the core. We introduce some notation that will be used throughout the article. Many definitions below define distributions conditioned on events. Sometimes, these events will have probability 0 of occurring. For simplicity of notation, we will define a "null" distribution that deterministically outputs 0, and replace any distribution conditioned on a zero probability event with the null distribution.

- r_i: The optimal revenue obtainable by selling just item i, using Myerson's optimal auction.
- r: $\sum_i r_i$. The total revenue from optimally selling the items separately. Note that $r = SRev(D)$, but we introduce this redundant notation for convenience.
- t_i: A parameter for item i, used to define the separation between the core and tail of distribution D_i. We will think of t_i as a multiplier applied to r_i. The core for item i will be supported on the interval $[0, t_i r_i]$, and the tail for item i will be supported on $(t_i r_i, \infty)$. Different results throughout the article will specify different choices for t_i. We will often abuse notation and say that "item i is in the tail" when $v_i^* > t_i r_i$ and "item i is in the core" when $v_i^* \leq t_i r_i$.
- p_i: $Pr[v_i^* > t_i r_i]$, the probability that the highest value on item i lies in the tail. Note that this may be 0, and also that this depends both on the distribution D_i, as well as the choice of t_i.
- D_i^C: The core of D_i, the conditional distribution of $\vec{v}_i$ conditioned on $v_i^* \leq t_i r_i$. Note that this may be the null distribution if $p_i = 1$.
- D_i^T: The tail of D_i, the conditional distribution of $\vec{v}_i$ conditioned on $v_i^* > t_i r_i$. Note that this may be the null distribution if $p_i = 0$.
- A: Throughout our notation, we will use A to represent a subset of items. We often think of A as the items whose values lie in the tail of their respective distributions.
- D_A^T: A is a subset of items, and D_A^T is a product distribution equal to $\times_{i \in A} D_i^T$.
- D_A^C: A is a subset of items, and D_A^C is a product distribution equal to $\times_{i \notin A} D_i^C$. Notice that the product is over items *not* in A. We think of D_A^C as representing the distribution of values in the core, conditional on A being the set of items whose values lie in the tail.
- D_A: $D_A^C \times D_A^T$. Note that this product is taken over the tail of items in A and the core of items not in A. In other words, D_A is the distribution D, conditioned on $v_i^* > t_i r_i$ for all $i \in A$ and conditioned on $v_i^* \leq t_i r_i$ for all $i \notin A$.
- p_A: $(\prod_{i \in A} p_i)(\prod_{i \notin A}(1 - p_i))$. When D_A is not null, this equals $Pr[\vec{v} \in support(D_A)]$.

Before stating our core-tail decomposition lemma, we present some known results about the core. The lemmas below were either stated explicitly in Reference [52] or Reference [46] for a single buyer, or use ideas from one of those papers. We put a citation in the statement of such lemmas, but include all proofs in Appendix C for completeness.

Lemma 3 [52]. $p_i \leq 1/t_i$ for all i.

LEMMA 4 [52]. $\text{REV}(D_i^C) \le r_i$ and $\text{REV}(D_i^T) \le r_i/p_i$.

LEMMA 5 [46]. $\text{REV}(D) \le \sum_A p_A \text{REV}(D_A)$.

5.2 The Core-Tail Decomposition Lemma

In this section, we state our Core-Tail Decomposition Lemma, which relates the optimal revenue from a distribution D to the revenue and welfare that can be extracted from the tail and core of D, respectively. This result is similar in spirit to the main lemma of Reference [52].

Our first result, Lemma 6, is our main decomposition lemma. The lemma states that the optimal revenue from distribution D can be split into a contribution from the core of D and a contribution from the tail of D. One might hope for a bound of the form "the optimal revenue from D is at most the optimal revenue from the tail plus the optimal revenue from the core." Indeed, such a bound is attainable for a single buyer [52] but is problematic for many buyers. We will therefore settle for a weaker bound: the optimal revenue from the tail plus the *expected welfare* from the core. We also note that the approach of Li and Yao eventually upper bounds the optimal revenue of the core with the expected welfare anyway.

LEMMA 6 (CORE-TAIL DECOMPOSITION). $\text{REV}(D) \le \text{VAL}(D_\emptyset^C) + \sum_A p_A \text{REV}(D_A^T)$

PROOF. By Lemma 1,

$$\text{REV}(D_A) \le \text{VAL}(D_A^C) + \text{REV}(D_A^T)$$

for all A. Also, since $\text{VAL}(D_A^C)$ is the expected sum of values for items *not* in A, we have

$$\text{VAL}(D_A^C) \le \text{VAL}(D_\emptyset^C).$$

By Lemma 5,

$$\begin{aligned}
\text{REV}(D) &\le \sum_A p_A \text{REV}(D_A) \\
&\le \sum_A p_A \left(\text{VAL}(D_A^C) + \text{REV}(D_A^T) \right) \\
&\le \left(\sum_A p_A \right) \text{VAL}(D_\emptyset^C) + \sum_A p_A \text{REV}(D_A^T).
\end{aligned}$$

As $\sum_A p_A = 1$ the desired result follows. □

6 MAIN RESULT: REVENUE BOUNDS FOR A SINGLE BUYER

In this section, we focus on the case of a single buyer, $m = 1$. We will work toward proving our main result, which is that $\max\{\text{SREV}(D), \text{BREV}(D)\}$ is a constant-factor approximation to $\text{REV}(D)$ in this setting. Our argument will make use of the Core-Tail decomposition, described in the previous section. We will begin with a simpler result that illustrates our techniques: that $\text{REV}(D)$ is at most $(\ln n + 3)$ times $\text{SREV}(D)$. A logarithmic approximation was already established in Reference [52]; we obtain a slightly tighter bound, but the primary purpose of presenting this result is as a warm-up to introduce our techniques and those of Reference [52]. We will then show how this bound can be improved to a constant by considering the maximum of $\text{SREV}(D)$ and $\text{BREV}(D)$.

6.1 Warm-up: $(\ln n + 3)\text{SREV} \ge \text{REV}$

We first give a simple application of our approach to provide a bound on SREV vs. REV, which is slightly improved relative to the bound obtained in Reference [52].

THEOREM 1. *For a single buyer, and any $c \geq 1/n$, $(2 + 1/c + \ln c + \ln n)SREV(D) \geq REV(D)$. This is minimized at $c = 1$, yielding $(\ln n + 3)SREV(D) \geq REV(D)$.*

The idea of the proof is to consider the Core-Tail decomposition of D, choosing $t_i = cn$ for each item i. By the Core-Tail Decomposition Lemma (Lemma 6), Theorem 1 follows if we can bound the optimal revenue from the tail and the expected welfare from the core, given this choice of $\{t_i\}_{i \in [n]}$.

We begin with Proposition 1, which effectively shows that when c is a constant, the revenue from the tail (when $t_i = cn$ for each item i) is at most a constant times SREV(D). The intuition behind this result is that each item i lies in the tail with probability $p_i \leq 1/t_i = 1/cn$, and hence there will often be at most a single item whose value lies in the tail. In this case, the revenue from the values in the tail is certainly no more than SREV(D), since the optimal mechanism can do no better than setting the optimal price for the single item present. To bound the revenue contribution when many values lie in the tail, the relatively weak bound in Lemma 2 will suffice.

PROPOSITION 1. *For a single buyer, and any $c > 0$, if $t_i = cn$ for all i, then $\sum_A p_A REV(D_A^T) \leq (1 + 1/c)SREV(D)$.*

PROOF. By Lemmas 2 and 4, $REV(D_A^T) \leq |A|SREV(D_A^T) \leq \sum_{i \in A} |A|r_i/p_i$. Therefore, we may rewrite the sum by first summing over item i, and then summing over every set A containing i, obtaining:

$$\sum_A p_A REV(D_A^T) \leq \sum_A p_A \sum_{i \in A} |A|r_i/p_i = \sum_i r_i \sum_{A \ni i} |A| \cdot p_A/p_i.$$

We now interpret the term $\sum_{A \ni i} |A| \cdot p_A/p_i$. Observe that p_A/p_i is exactly the probability that the set A of items are in the tail and all other items are not, conditioned on i being in the tail, and $|A|$ is just the size of A. Summing over all $A \ni i$ therefore yields the expected size of the set of items in the tail, conditioned on i being in the tail.[19] Clearly, as i is in A, this expectation is just $1 + \sum_{j \neq i} p_j$, which is at most $1 + 1/c$ by Lemma 3. As we have just observed that $\sum_{A \ni i} |A|p_A/p_i \leq 1 + 1/c$. Thus, we have now shown that $\sum_A p_A REV(D_A^T) \leq \sum_i (1 + 1/c)r_i$, which is exactly $(1 + 1/c)SREV(D)$. □

Having established a bound on the revenue of the tail, we turn to the welfare of the core. For this, we use the definition of $r_i = SREV(D_i)$ to directly bound $Pr_{v_i \leftarrow D_i}[v_i > x]$ for all x, and then we take an expectation over the range of the core.

PROPOSITION 2. *For a single buyer, and any $c \geq 1/n$, if $t_i = cn$ for all i, then $(1 + \ln c + \ln n)SREV(D) \geq VAL(D_\emptyset^C)$.*

PROOF. Note that $VAL(D_\emptyset^C) = \sum_i VAL(D_i^C) \leq \sum_i \int_0^{cnr_i} Pr_{v_i \leftarrow D_i}[v_i > x]dx$. The last inequality would be equality if we replaced v_i with a random variable drawn from D_i^C, but since v_i stochastically dominates such a random variable, we get an inequality instead. As the optimal revenue of D_i is r_i, this means that $Pr_{v_i \leftarrow D_i}[v_i > x] \leq \min\{1, r_i/x\}$. So, we have

$$VAL(D_i^C) \leq \int_0^{r_i} dx + \int_{r_i}^{cnr_i} (r_i/x)dx = r_i + r_i(\ln(cnr_i) - \ln(r_i)) = r_i(1 + \ln n + \ln c).$$

As $c \geq 1/n$, the above breakdown of the integral is valid, and summing this guarantee over all i yields the proposition. □

Combining Propositions 1 and 2 with Lemma 6 yields Theorem 1.

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[19]This observation is due to Aviad Rubinstein, and we thank him for allowing us to include it. An earlier version of this article presented a $(\ln n + 5)$-approximation in Theorem 1 and a 7.5-approximation in Theorem 2. This observation improved those factors to $(\ln n + 3)$ and 6, respectively.

6.2 Main Result: $6 \cdot \max\{\text{SRev}, \text{BRev}\} \geq \text{Rev}$

In this section, we prove our main result, showing that the best of selling items separately and bundling all of them together is a constant-factor approximation to the optimal mechanism. The proof will follow a similar outline to that of Section 6.1, proving propositions similar to Propositions 1 and 2. The notable difference is that we will need to be more careful in defining the core.

When all D_i are identical, the approach in Section 6.1 (setting each $t_i = cn$) can be leveraged to yield the bound $O(1) \cdot \text{BRev} \geq \text{Rev}$ ([52]), but fails in the case that a small number k of items contributes the majority of the optimal revenue. To see the problem, note that the previous definition of the cutoffs t_i depends on the number of items n, but the number of items can be made arbitrarily large while keeping the problem essentially the same (by adding extra items whose values are deterministically 0). The effect is that analysis of the same underlying instance changes as a result of these dummy items, so one should not expect approximations independent of n. In particular, the cutoffs t_i will be larger than necessary when value distributions are asymmetric, and a few items contribute most of the revenue.

Ideally, our analysis would be invariant under addition of dummy items. To accomplish this, we let t_i scale inverse proportionally to r_i, so that high-revenue items are more likely to occur in the tail. This allows us to capture scenarios in which revenue comes primarily from one heavy item (by analyzing the tail), as well as instances driven by the combined contribution of many light items (by analyzing the core). Indeed, note that if we set $t_i = cr/r_i$, then the boundary between core and tail becomes $t_i r_i = cr = c\text{SRev}(D)$ for each item (so while the choice of t_i is non-uniform, the absolute cutoffs $t_i r_i$ are uniform). This turns out to be precisely the threshold that we need to attain constant-factor approximation bounds for both the core and the tail, simultaneously. We are now ready to state and prove our main result.

THEOREM 2. *For $m = 1$ buyer and n items, $\text{Rev}(D) \leq 6 \max\{\text{SRev}(D), \text{BRev}(D)\}$.*

As in Theorem 1, our approach will be to apply the Core-Tail Decomposition Lemma (Lemma 6) with an appropriate choice of values t_i, then bound separately the revenue from the tail and the welfare from the core.

PROPOSITION 3. *For a single buyer, when $t_i = r/r_i$ for each i, $\sum_A p_A \text{Rev}(D_A^T) \leq 2\text{SRev}(D)$.*

PROOF. We begin similarly to the proof of Proposition 1, using Lemma 2 and Lemma 4 to write $\text{Rev}(D_A^T) \leq |A|\text{SRev}(D_A^T) \leq \sum_{i \in A} |A| r_i/p_i$. Again, summing this over all A yields

$$\sum_A p_A \text{Rev}(D_A^T) \leq \sum_i \sum_{A \ni i} |A| p_A r_i/p_i.$$

Just like in Proposition 1, $\sum_{A \ni i} |A| p_A/p_i$ is exactly the expected number of items in the tail, conditioned on i being in the tail. It's again clear that this sum is exactly $1 + \sum_{j \neq i} p_j$. By Lemma 3, this is at most $1 + \sum_{j \neq i} 1/t_j$. By our choice of t_i, the second term is upper bounded by 1, as $t_j = r/r_j$ and $\sum_j r_j = r$. Therefore, $\sum_{A \ni i} |A| p_A/p_i \leq 2$, and $\sum_A p_A \text{Rev}(D_A^T) \leq 2\text{SRev}(D)$. □

We now turn to bounding the welfare from the core. We will use the small range of the core to derive an upper bound on the variance of its welfare. This will allow us to conclude that the welfare is highly concentrated whenever it is sufficiently large relative to $\text{SRev}(D)$. Thus, if the welfare is "small" compared to $\text{SRev}(D)$, then selling separately extracts most of the welfare (within the core); otherwise, the welfare concentrates and so bundling extracts most of the welfare (within the core). The following lemma of Reference [52] will be helpful for this approach; its proof appears in Appendix D for completeness.

LEMMA 7 [52]. *Let F be a one-dimensional distribution with optimal revenue at most y supported on $[0, ty]$. Then $\text{var}(F) \leq (2t - 1)y^2$.*

COROLLARY 1. *For a single buyer, and any choice of t_i, $var(D_i^C) \leq 2t_i r_i^2$.*

PROOF. $\text{REV}(D_i^C) \leq r_i$, and the distribution D_i^C is supported on $[0, t_i r_i]$. Therefore, plugging into Lemma 7 (and relaxing) yields the desired bound. □

PROPOSITION 4. *For a single buyer, when $t_i = r/r_i$ for every i, it holds that*
$\max\{S\text{REV}(D), B\text{REV}(D)\} \geq \frac{1}{4} \text{VAL}(D_\emptyset^C).$

PROOF. There are two cases to consider. If $\text{VAL}(D_\emptyset^C) \leq 4r$, then we trivially have that $\text{SREV}(D) = r \geq \frac{1}{4}\text{VAL}(D_\emptyset^C)$ as required.

However, if $\text{VAL}(D_\emptyset^C) \geq 4r$, then Corollary 1 tells us that $var(D_i^C) \leq 2t_i r_i^2$. Summing over all i and recalling that $t_i = r/r_i$, we get

$$\text{var}(D_\emptyset^C) = \sum_i \text{var}(D_i^C) \leq 2 \sum_i t_i r_i^2 = 2r^2.$$

So, $\text{var}(D_\emptyset^C) \leq 2r^2$ and $\text{VAL}(D_\emptyset^C) \geq 4r$. By Chebyshev's inequality, we get

$$\Pr_{\vec{v} \leftarrow D}\left[\sum_i v_i \leq \frac{2}{5} \cdot \text{VAL}(D_\emptyset^C)\right] \leq \frac{2r^2}{\left(1 - \frac{2}{5}\right)^2 \cdot \text{VAL}(D_\emptyset^C)^2}$$

$$\leq \frac{25r^2}{72r^2} = \frac{25}{72}.$$

Since $\text{BREV}(D)$ is at least the revenue obtained by setting price $\frac{2}{5} \cdot \text{VAL}(D_\emptyset^C)$ on the grand bundle, $\text{BREV}(D) \geq (\frac{2}{5} \cdot \text{VAL}(D_\emptyset^C)) \cdot \frac{47}{72} = \frac{47}{180} \cdot \text{VAL}(D_\emptyset^C)$. As $\frac{47}{180} > \frac{1}{4}$, $\text{BREV}(D) > \frac{1}{4}\text{VAL}(D_\emptyset^C)$ as required. □

Combining Propositions 3 and 4 with Lemma 6 yields Theorem 2. Our analysis was improved to provide a bound of 5.2 in Reference [54]. Reference [56] provides a construction D (with $m = 1$) such that $\max\{S\text{REV}(D), B\text{REV}(D)\} = (1/2 + o(1)) \cdot \text{REV}(D)$ (proving the analysis cannot be improved beyond a bound of 2). It is an interesting open question to further narrow the gap between 2 and 5.2.

7 REVENUE BOUNDS FOR MULTIPLE BUYERS

Here, we extend our results to multiple buyers with all valuations sampled independently (again, not necessarily identically). We first show in Theorem 3 that selling items separately achieves a logarithmic (in the number of items n) approximation to the optimal revenue. In Section 7.4 (Theorem 7), we explore the limits of our techniques in the multi-buyer case. Specifically, we establish that, like in the single buyer case, the only case in which selling items separately fails to achieve a good approximation is if welfare is highly concentrated. Unfortunately, such concentration is no longer sufficient to achieve a constant approximation by bundling all items together. This is so, because even though the welfare is concentrated, the allocation of items to buyers that provides such welfare can change dramatically between realizations. Indeed, in Proposition 8, we show not only that $\text{BREV}(D)$ fails to provide a constant approximation to the optimal mechanism, but even $\text{PREV}(D)$ fails, and this is so even when item values are sampled i.i.d. for all items and buyers.

Finally, we explore the connection between $\text{SREV}(D)$, $\text{BREV}(D)$, and $\text{PREV}(D)$ for multiple buyers. In Section 7.2, we establish that when *either buyers or items* (not necessarily both) are i.i.d., then $\max\{S\text{REV}(D), B\text{REV}(D)\}$ achieves a constant-factor approximation to $\text{PREV}(D)$ (Theorems 5 and 6). We also establish that this approximation guarantee fails when neither buyers nor items are i.i.d. (Proposition 7)—$\max\{S\text{REV}(D), B\text{REV}(D)\}$ can guarantee at best an $\Omega(\ln(n))$-approximation to $\text{PREV}(D)$ in general (which is already achieved by $\text{SREV}(D)$ itself, even when compared to $\text{REV}(D)$).

Together, these provide a complete picture of the gaps between these three quantities (see Table 2 in Appendix A for a summary of these results).

7.1 Extension: $(\ln n + 6)$SRev $\geq$ Rev

We first show that selling items separately achieves a logarithmic (in the number of items, n) approximation to the optimal revenue.

THEOREM 3. *For any number m of buyers and n items, $(2 + 2e^{1/4} + \ln 4 + \ln n)$SRev$(D) \geq$ Rev(D). (Note that $2 + 2e^{1/4} + \ln 4 < 6$.)*

Our proof will proceed via amplification. We will begin with the (weak) bound on SRev vs. Rev from Lemma 2, then show in Theorem 4 how to amplify any such bound into an improved bound. We will then iterate this amplification process over and over, until we reach the desired logarithmic approximation (which will be a fixed point of the amplification process). To prove the amplification theorem, we use an approach similar to the single-buyer analysis from Section 6.1. That is, we will apply the Core-Tail Decomposition Lemma (Lemma 6), then bound the revenue of the tail and the welfare of the core with respect to SRev(D). The first step in the proof of Theorem 3 is the following amplification theorem, where any current bound (in terms of a) is improved.

THEOREM 4 (AMPLIFICATION). *Assume that for some $a, m > 0$, it holds that for any number n of items and all D on m buyers and n items that $a \cdot n \cdot$ SRev$(D) \geq$ Rev(D). Then, for any $c \geq 1/a$, and all D on m buyers and n items, $(2 + 2e^{1/ca}/c + \ln c + \ln a + \ln n)$SRev$(D) \geq$ Rev(D) as well. In particular, when $a \geq 1$, setting $c = 1$ yields $(2 + 2e^{1/a} + \ln a + \ln n)$SRev$(D) \geq$ Rev(D).*

To prove Theorem 4, we will apply the Core-Tail Decomposition Lemma (Lemma 6), using $t_i = c \cdot a \cdot n$ for each i. Theorem 4 will then follow from bounds on the revenue from the tail and the expected welfare from the core, which we establish in the two following propositions.

PROPOSITION 5. *Assume that for some $a, m > 0$, it holds that for any number n of items and all D on m buyers and n items that $a \cdot n \cdot$ SRev$(D) \geq$ Rev(D). Then for all D on m buyers and n items and $c > 0$, setting $t_i = c \cdot a \cdot n$ for all i implies $\sum_A p_A$Rev$(D_A^T) \leq (1 + 2e^{1/ca}/c)$SRev$(D)$.*

PROOF. The following proof is similar to that of Proposition 1, with two differences. First, we start with the bound Rev$(D_A^T) \leq a|A|$SRev(D_A^T) (since that is our starting hypothesis, instead of Rev$(D_A^T) \leq |A|$SRev(D_A^T) as in the single-buyer case). Second, we have to make use of the fact that when there is only one item, SRev$(D_A^T) =$ Rev(D_A^T) and use this tighter bound whenever $|A| = 1$. We continue now with the proof.

By hypothesis and Lemma 4, for A with $|A| > 1$ it holds that Rev$(D_A^T) \leq a|A|$SRev$(D_A^T) \leq \sum_{i \in A} a|A|r_i/p_i$. Combining with SRev$(D_A^T) =$ Rev(D_A^T) for the case that $|A| = 1$, we can rewrite

$$\sum_A p_A\text{Rev}(D_A^T) \leq \sum_i \left(r_i + \sum_{j=2}^n aj \sum_{A \ni i, |A|=j} p_A r_i/p_i \right).$$

Observe that $p_A = (\prod_{i \in A} p_i)(\prod_{i \notin A}(1 - p_i)) \leq \prod_{i \in A} p_i$ and thus $p_A r_i/p_i \leq \prod_{k \in A - \{i\}} p_k r_i$. We then have that

$$\sum_{A \ni i, |A|=j} p_A r_i/p_i \leq r_i \sum_{A \ni i, |A|=j} \prod_{k \in A - \{i\}} p_k.$$

Furthermore, by Lemma 3, we have that each $p_k \leq 1/(c \cdot a \cdot n)$, so we have

$$\sum_{j=2}^{n} \sum_{A \ni i, |A|=j} aj \prod_{k \in A-\{i\}} p_k \leq \sum_{j=2}^{n} aj \binom{n-1}{j-1}/(c \cdot a \cdot n)^{j-1}$$

$$\leq \sum_{j=2}^{n} \frac{j(n-1)^{j-1}}{(j-1)!c^{j-1}a^{j-2}n^{j-1}}$$

$$\leq \sum_{j=2}^{n} \frac{2(j-1)}{(j-1)!c^{j-1}a^{j-2}}$$

$$\leq \sum_{j=2}^{n} \frac{2}{(j-2)!c^{j-1}a^{j-2}}$$

$$\leq \frac{2e^{1/ca}}{c}.$$

The last inequality makes use of the fact that $\sum_{j=0}^{\infty} \frac{1}{j!c^j a^j}$ is the Taylor expansion for $e^{x/ca}$ evaluated at $x = 1$. Adding back the $j = 1$ term that we handled outside the sum (making use of the fact that SREV = REV on single-item distributions) and summing over all i of r_i times the above inequality yields the proposition. □

The following bound on the welfare from the core follows similarly to Proposition 2.

PROPOSITION 6. *For any number of buyers and any positive a and c with $a \cdot c \geq 1/n$, if $t_i = c \cdot a \cdot n$ for all i, then $(1 + \ln c + \ln a + \ln n)SRev(D) \geq Val(D_\emptyset^C)$.*

PROOF. Note that $\text{VAL}(D_\emptyset^C) = \sum_i \text{VAL}(D_i^C) \leq \sum_i \int_0^{canr_i} Pr[v_i^* > x]dx$ (recall that $v_i^* := \max_j\{v_{ij}\}$). The last inequality would be equality if we replaced v_i^* with a random variable that is the maximum value in a sample drawn from D_i^C, but since v_i^* stochastically dominates such a random variable, we get an inequality instead. As the optimal revenue of D_i is r_i, this means that $Pr[v_i^* > x] \leq \min\{1, r_i/x\}$. So, we have

$$\text{VAL}(D_i^C) \leq \int_0^{r_i} dx + \int_{r_i}^{canr_i} (r_i/x)dx$$

$$= r_i + r_i(\ln(c \cdot a \cdot n \cdot r_i) - \ln(r_i))$$

$$= r_i(1 + \ln n + \ln c + \ln a).$$

Summing this bound over all i yields the proposition. □

Theorem 4 then follows from Propositions 5 and 6, together with Lemma 6. We now show how to prove Theorem 3 using Theorem 4.

PROOF OF THEOREM 3. The goal is to iteratively apply Theorem 4 starting with $a = m$ (which is a valid hypothesis, by Lemma 2), until we can apply it once with $a \leq 4$.

So, let us start with an application of Theorem 4 from $a = m$. This yields a bound of the form $a'n\text{SREV}(D) \geq \text{REV}(D)$ for some new a'. First, perhaps already $a' \leq 4$. If not, then we can then apply Theorem 4 again, taking a to be this new value a'. We can iteratively apply Theorem 4 until we reach $a \leq 4$. One can verify that, for all $n \geq 2$, $a \geq 4$, the function $f(a) = (2 + 2e^{1/a} + \ln a + \ln n)/n$ satisfies $f(a) < a - 1$. Therefore, $mn - 4$ iterations suffice to get $a \leq 4$. Once the hypothesis holds with some $a \leq 4$, we can apply Theorem 4 one final time to conclude Theorem 3. □

7.2 Comparing SREV(D), BREV(D), PREV(D) for Multiple Buyers

In this section, we investigate the relationship between $\max\{\text{SREV}(D), \text{BREV}(D)\}$ and $\text{PREV}(D)$ for multiple buyer with independent items. The main results of this section establish that when *either* buyers *or* items are i.i.d. that the gap is at most a constant factor. Proposition 7 establishes that this gap can be $\Omega(\ln n)$ when neither buyers nor items are i.i.d. (which is tight, as $\max\{\text{SREV}(D), \text{BREV}(D)\} \geq \text{SREV}(D) \geq \Omega(1/\ln(n)) \cdot \text{REV}(D) \geq \text{PREV}(D)$ by Theorem 3).

THEOREM 5. *Let D have any number of items and any number of i.i.d. buyers (that is, $D^j = D^{j'}$ for all buyers j, j', but perhaps $D_i \neq D_{i'}$ for some i, i'). Then $\text{PREV}(D) \leq O(\max\{\text{SREV}(D), \text{BREV}(D)\})$.*

THEOREM 6. *Let D have any number of i.i.d. items and any number of buyers (that is, $D_i = D_{i'}$ for all items i, i', but perhaps $D^j \neq D^{j'}$ for some j, j'). Then $\text{PREV}(D) \leq O(\max\{\text{SREV}(D), \text{BREV}(D)\})$.*

The intuition for Theorems 5 and 6 is the following. Consider any partition mechanism that partitions the items into $S_1, \ldots, S_k$, and refer by D_{S_i} to the distribution D restricted to only items in S_i (and therefore the partition mechanism achieves revenue $\sum_i \text{BREV}(D_{S_i})$). We say that S_i is *separable* if $\text{SREV}(D_{S_i}) = \Omega(\text{BREV}(D_{S_i}))$. If $\mathcal{S}$ denotes the set of all indices i for which S_i is separable, then it immediately follows that $\text{SREV}(D) = \Omega(\sum_{i \in \mathcal{S}} \text{BREV}(D_{S_i}))$ (Lemma 12). Similarly, we say that S_i is *bundlable* for j if selling S_i as a bundle *only to buyer j*, ignoring all other buyers, generates revenue $\Omega(\text{BREV}(D_{S_i}))$. We then argue that if $\mathcal{B}_j$ denotes the set of all indexes i for which S_i is bundlable for buyer j, that $\text{BREV}(D) = \Omega(\sum_{i \in \mathcal{B}_j} \text{BREV}(D_{S_i}))$ (Lemma 13). This step is not quite as trivial as Lemma 12, but still fairly simple. The most interesting step of the proof for both theorems is showing that *when either buyers or items are i.i.d.* there is an approximately optimal partition mechanism and buyer j such that *every set S_i that is not separable, is bundlable for that buyer j*. The key step is stated formally in Propositions 11 and 12, and full details for all proofs are in Appendix E.

Proposition 7 below establishes, however, that at least one of the i.i.d. assumptions used in Theorems 5 and 6 is necessary.

Definition 1. We denote by ER_k the Equal Revenue curve truncated at k: the single-dimensional distribution with $F(x) = 1 - 1/x$ for all $x \in [1, k]$, $F(x) = 0$ for all $x < 1$, and $F(x) = 1$ for all $x \geq k$ (i.e., it is an equal revenue curve with all mass above k moved to a point mass at k).

PROPOSITION 7. *Let D have n items and $m = \sqrt{n}$ buyers. Partition the items into $\sqrt{n}$ disjoint sets of size $\sqrt{n}$, $S_1, \ldots, S_m$. Let buyer j have value 0 for every item not in S_j, and value independently drawn from $\text{ER}_{n^{1/8}}$ for each item in S_j. Then $\max\{\text{SREV}(D), \text{BREV}(D)\} \leq \text{PREV}(D)/\Omega(\log n)$.*

A full proof of Proposition 7 appears in Appendix E. The high-level idea is that each buyer j is only interested in items in S_j, which are disjoint. So, we should partition the items and run the optimal single-buyer auction within each (which would generate revenue $\Omega(n \ln n)$). Selling instead each partition separately to the intended buyer generates revenue only $O(n)$, and selling the entire bundle together causes $n - \sqrt{n}$ items to be wasted, and generates revenue only $O(\sqrt{n} \ln(n))$. To help process this example in the language of our proof outline of Theorems 5 and 6, observe that no S_i is separable (because $\text{SREV}(D_{S_i}) = \sqrt{n}$, while $\text{BREV}(D_{S_i}) = \Theta(\sqrt{n} \cdot \ln(n))$). Also, observe that *only* S_j is bundlable for j (because j has value 0 for all items not in S_j). The key step (Proposition 12) towards Theorems 5 and 6 states that this phenomenon cannot occur in an example with either i.i.d. items or i.i.d. buyers.

7.3 A Lower Bound: PREV ≤ REV/$\Omega(\log n)$ Even for i.i.d. Item Values

We next show that there is a setting with many buyers with item valuations that are sampled i.i.d from the same distribution, for which $\text{PREV}(D)$ (and thus also $\max\{\text{SREV}(D), \text{BREV}(D)\}$) provides

a poor approximation to Rᴇᴠ(D). Intuitively, the key feature our example possesses is that for any *fixed* set S of $\Theta(\sqrt{n})$ items, it is extremely unlikely that any buyer values S particularly highly. Yet, for all buyers, it is extremely *likely* that they value *some* set of $\Theta(\sqrt{n})$ items highly. The former property allows us to claim that all partition mechanisms perform poorly, while the latter property allows us to design a posted-price mechanism that performs well. The construction will make use of the following distribution:

Pʀᴏᴘᴏsɪᴛɪᴏɴ 8. *When D is such that each of $m = \sqrt{n}$ buyers have i.i.d. values for each of n items drawn from a distribution that is a point-mass at 0 with probability $1 - 1/\sqrt{n}$ and drawn from $ER_{n^{1/8}}$ with the remaining probability: $PRev(D) \leq Rev(D)/\Omega(\log n)$.*

A full proof of Proposition 8 appears in Appendix E. The high-level idea is that only one buyer in expectation has non-zero value for each item, but each buyer has non-zero value for $\sqrt{n}$ items in expectation. As a result, selling the entire grand bundle at once achieves poor revenue ($O(\sqrt{n}\ln(n))$), since $n - \sqrt{n}$ are likely to be unvalued by the winner. Similarly, selling separately is suboptimal ($O(n)$), because each buyer's value for the $\sqrt{n}$ items they like concentrates around its expectation, and selling separately doesn't exploit this. Instead, a posted-price mechanism that allows each buyer to pick any subset of $\Theta(\sqrt{n})$ remaining items for $\Theta(\sqrt{n}\ln(n))$ has the property that with high probability, each buyer wishes to purchase a set of remaining items, generating revenue $\Omega(n\ln(n))$. Interestingly, note that, because SRᴇᴠ(D) is a $(\ln(n) + 6)$-approximation to Rᴇᴠ(D), and we have just claimed that this posted-price mechanism achieves a $\Theta(\ln(n))$-factor more revenue than PRᴇᴠ(D) (and therefore SRᴇᴠ(D)), this posted-price mechanism must be a constant-factor approximation.

7.4 A Concentration Result

Finally, we explore the limits of our single-buyer approach for multiple buyers. Specifically, we establish sufficient conditions for SRᴇᴠ(D) to be a constant-factor approximation to Rᴇᴠ(D) with multiple buyers. We will show (Theorem 7) that this occurs unless the welfare of D is sufficiently well concentrated around its expectation. Proposition 8 establishes that this concentration *does not suffice* even for PRᴇᴠ(D) to guarantee a constant-factor approximation, so this result identifies that the main challenge in extending our work to multiple buyers is leveraging concentration of welfare to get a constant-factor approximation (and Reference [61] accomplishes this via an entry fee).

We begin with a corollary of Theorem 3, which will be useful for our analysis.

Cᴏʀᴏʟʟᴀʀʏ 2. *For any number of buyers and n items, $4 \cdot n \cdot SRev(D) \geq Rev(D)$.*

Pʀᴏᴏf. This is a direct application of Theorem 3 and noting that $6 + \ln n \leq 4n$ for all $n \geq 2$. □

We next prove an alternative bound on the revenue from the tail of the distribution D, using a familiar choice of t_i.

Pʀᴏᴘᴏsɪᴛɪᴏɴ 9. *For any number of buyers, if $t_i = 4r/r_i$ for all i, then $\sum_A p_A Rev(D_A^T) \leq 5SRev(D)$.*

Pʀᴏᴏf. Again, we begin by rewriting $\sum_A p_A$Rᴇᴠ(D_A^T) using Corollary 2, and reordering:

$$\sum_A p_A\text{Rᴇᴠ}(D_A^T) \leq \sum_A 4p_A|A|\sum_{i \in A} r_i/p_i = \sum_i r_i \sum_{A \ni i} 4|A|p_A/p_i = 4\sum_i r_i \sum_{A \ni i} |A|p_A/p_i.$$

Again, the value $\sum_{A \ni i} |A|p_A/p_i$ is exactly the expected number of items in the tail, conditioned on i being in the tail. Therefore, $\sum_{A \ni i} |A|p_A/p_i \leq 1 + \sum_{j \neq i} p_j \leq 1 + 1/4$ by Lemma 3. Therefore, $\sum_A p_A$Rᴇᴠ(D_A^T) $\leq 5\sum_i r_i = 5$SRᴇᴠ. □

We are now ready to establish the claimed bound between SREV and REV, subject to the welfare of D not being too concentrated around its expectation.

Definition 2. We say that a one-dimensional distribution F is d-concentrated if there exists a value C such that $Pr_{x \sim F}[|x - C| \leq C/2] \geq d$.

THEOREM 7. *For any number of buyers, and any $c \geq 4\sqrt{2}$, either $(c + 5)SREV(D) \geq REV(D)$ or the welfare of D (the random variable with expectation $VAL(D)$) is $(3/4 - \frac{24}{c^2})$-concentrated.*

PROOF. Let all $t_i = 4r/r_i$. Then combining Proposition 9 and Lemma 6 yields

$$5 \cdot SREV(D) + VAL(D_\emptyset^C) \geq REV(D).$$

There are two cases to consider. First, perhaps $c \cdot SREV(D) \geq VAL(D_\emptyset^C)$. In this case, we have $(c + 5)SREV(D) \geq REV(D)$ and the claim follows.

In the other case, $VAL(D_\emptyset^C) \geq c \cdot SREV(D)$. In this case, Corollary 1 tells us that $var(D_i^C) \leq 2t_i r_i^2$. Summing over all i and recalling that $t_i = 4r/r_i$, we get

$$var(D_\emptyset^C) \leq 2 \sum_i t_i r_i^2 = 2 \sum_i (4r) r_i = 8r^2.$$

So, $var(D_\emptyset^C) \leq 8r^2$ and $VAL(D_\emptyset^C) \geq cr$. By Chebyshev's inequality, we get

$$Pr\left[\left|\sum_i v_i^* - VAL(D_\emptyset^C)\right| \geq VAL(D_\emptyset^C)/2\right] \leq \frac{8r^2}{VAL(D_\emptyset^C)^2/4} \leq \frac{32r^2}{c^2 r^2} = \frac{32}{c^2},$$

meaning that the welfare of $D_\emptyset^C$ is $(1 - \frac{32}{c^2})$-concentrated. The last step is observing that $\vec{v}$ is sampled in the support of $D_\emptyset^C$ with probability exactly $\prod_i(1 - p_i)$. As $\sum_i p_i \leq \sum_i 1/t_i \leq \sum_i \frac{r_i}{4r} = 1/4$ and each $p_i \geq 0$, this is minimized when exactly one p_i is $1/4$ and the rest are 0, yielding $\prod_i(1 - p_i) = 3/4$. So, with probability at least $3/4$ $\vec{v}$ is in the support of $D_\emptyset^C$. When this happens, the welfare is $(1 - \frac{32}{c^2})$ concentrated. So, the welfare of D is $(3/4 - \frac{24}{c^2})$-concentrated. □

8 ONE BUYER WITH CORRELATED VALUES

In this section, we study the relationship between SREV(D), Max{SREV(D), BREV(D)}, and PREV(D) for a single buyer with correlated values. Prior work of References [4, 45] already shows that there is no hope of obtaining a non-zero bound between any of these quantities and REV(D) (because partition mechanisms are deterministic, and References [4, 45] show that no deterministic mechanism achieves any non-zero approximation), even when there are only two items. But it is still important to understand the relationship between these mechanisms of varying complexity even if their revenue cannot compare to that of the optimal mechanism. We show in Theorem 8 that for any correlated distribution D for a single buyer and n items, SREV(D) is a $O(\ln n)$ approximation to BREV(D), and thus also to Max{SREV(D), BREV(D)} and PREV(D).[20] We then show in Proposition 10 that this bound is tight, there exists a distribution D witnessing max{SREV(D), BREV(D)} ≤ PREV(D)/$\Omega(\ln n)$. In other words, SREV(D) provides a logarithmic

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[20]As SREV approximates BREV for any set of items, it can do so for any part in the partition in PREV separately and thus also approximates PREV.

approximation to PREV(D), but taking max{SREV(D), BREV(D)} can't guarantee anything better (even for $m = 1$ buyer). Both proofs appear in Appendix F.

THEOREM 8. *For any distribution D for a single buyer and n items (arbitrarily correlated),* $BREV(D) \leq 5\ln(n)SREV(D)$. *Therefore,* $PREV(D) \leq 5\ln(n)SREV(D)$ *as well.*

PROPOSITION 10. *There exists a (correlated) distribution D of the valuation of a single buyer over* n *items for which* max{$SREV(D), BREV(D)$} $\leq PREV(D)/\Omega(\ln n)$.

APPENDICES

A SUMMARY OF KNOWN RESULTS

Tables 1 and 2 present the best results known for one additive buyer with item values sampled independently or arbitrarily, respectively. Additionally, Table 3 presents the best results known for many additive buyers in the independent setting. In each cell there is the known upper and lower bounds of the ratio between the corresponding column quantity and row quantity, and the source of the result. For example, in Table 1, the table entry that corresponds to the row marked by max{SREV, BREV} and column marked by REV there is the upper bound of 5.2 that slightly improves the 6-approximation from Theorem 2 for the ratio REV/ max{SREV, BREV}, which holds for every distribution D. Results that are implied from other results, point to the results that imply them.

Regarding Table 3, recall that the middle box (max{SREV, BREV} vs. PREV) becomes $O(1)$ and $\Omega(1)$ when D has either i.i.d. buyers (Theorem 5) or i.i.d. items (Theorem 6).

Table 1. One Buyer, Independent Item Values

	max{SREV,BREV}	REV
SREV	$O(\log n)$ [→] $\Omega(\log n)$ [46]	$O(\log n)$ [52] $\Omega(\log n)$ [←]
max{SREV, BREV}	1 1	5.2 [54] 2 [56]

When the top number in a box is x, it means that x times the row quantity exceeds the column quantity for all distributions. When the bottom number is x, it means there exists a distribution such that the row quantity times x does not exceed the column quantity.

Table 2. One Buyer, Correlated Item Values

	max{ SREV,BREV}	PREV	REV
SREV	$O(\log n)$ [→] $\Omega(\log n)$ [46]	**$O(\log n)$ [Theorem 8]** $\Omega(\log n)$ [↓]	∞ ∞ [↓]
max{SREV, BREV}	1 1	$O(\log n)$ [↑] **$\Omega(\log n)$ [Proposition 10]**	∞ ∞ [↓]
PREV	1 1	1 1	∞ ∞ [4, 45]

When the top value in a row is ∞, it simply means that ∞ times the row quantity exceeds the column quantity (which is trivial). When the bottom value in a row is ∞, it means that there exists a distribution such that for all finite x, x times the row quantity does not exceed the column quantity.

Table 3. Many Buyers, Independent Item Values

	max{ SREV,BREV}	PREV	REV
SREV	$O(\log n)$ [→] $\Omega(\log n)$ [46]	$O(\log n)$ [→] $\Omega(\log n)$ [←]	**$O(\log n)$ [Theorem 3]** $\Omega(\log n)$ [←]
max{SREV, BREV}	1 1	$O(\log n)$ [↑] $\Omega(\log n)$ [Propostion 7]	$O(\log n)$ [↑] $\Omega(\log n)$ [↓]
PREV	1 1	1 1	$O(\log n)$ [↑] $\Omega(\log n)$ [Proposition 8]

B OMITTED PROOFS FROM SECTION 4

The proofs of Lemmas 1 and 2 require some technical lemmas from Reference [46]. We include them below with proofs for completeness. In Lemma 8 below, D and D' are distributions over values for disjoint sets of items for the same buyers, and D and D' may be dependent. By REV(D, D'), we mean the optimal revenue obtainable by selling to buyers whose values for items are sampled from the joint distribution according to D and D'. The Marginal Mechanism lemma below essentially states that while selling disjoint sets of items jointly may achieve significantly greater revenue than selling them separately (i.e., REV(D, D') can be significantly larger than REV(D) + REV(D')), it cannot exceed the *welfare* of one set plus the revenue generated by the other.

Below, and throughout this section, we use the notation $D|E$ to denote the distribution D conditioned on event E. In particular, the distribution $D'|\vec{v} = \vec{w}$ draws a valuation $(\vec{v}, \vec{v}') \leftarrow (D, D')$ conditioned on $\vec{v} = \vec{w}$ and outputs $\vec{v}'$. We also use the notation REV$_M(D)$ to denote the revenue that a particular mechanism M guarantees on distribution D.

LEMMA 8. *("Marginal Mechanism" [20, 46]) REV$(D, D') \leq$ VAL$(D) + \mathbb{E}_{\vec{w} \leftarrow D}[REV(D'|\vec{v} = \vec{w})]$.*

PROOF. We will establish a lower bound on REV$(D'|\vec{v} = \vec{w})$ by constructing a truthful mechanism for selling items in the support of D', based on one for those in the support of (D, D'). To sell items in the support of D', first make "imaginary items" for each item in the support of D. Then, announce that whenever buyer j receives an imaginary item i in the support of D, she will instead receive money equal to w_{ij}. Note that, due to this announcement, each buyer j now has a value w_{ij} for each imaginary item i in the support of D.

Next, take any optimal mechanism M for selling items in the support of (D, D') and run exactly this mechanism for buyers with values drawn from $D'|\vec{v} = \vec{w}$ (with make-believe values $\vec{w}$ for items in the support of D). Observe that a buyer with value $\vec{v}'^j$ has exactly the same incentives in M as a buyer with values $(\vec{w}^j, \vec{v}'^j)$ (because we have explicitly given them value $\vec{w}$ for all items in the support of D). So, this mechanism is truthful and generates revenue REV$_M((D, D')|\vec{v} = \vec{w})$, minus the money awarded for the imaginary items, on distribution $D'|\vec{v} = \vec{w}$. Observe that the total money awarded is at most $\sum_i \max_j\{w_{ij}\}$ (if the buyer with highest value for each imaginary item purchases it). So, we get

$$\text{REV}(D'|\vec{v} = \vec{w}) \geq \text{REV}_M((D, D')|\vec{v} = \vec{w}) - \sum_i \max_j\{w_{ij}\}.$$

Now, let us take an expectation over $\vec{w}$ of both sides:

$$\mathbb{E}_{\vec{w} \leftarrow D}\left[\text{REV}(D'|\vec{v} = \vec{w})\right] \geq \mathbb{E}_{\vec{w} \leftarrow D}\left[\text{REV}_M(D, D')|\vec{v} = \vec{w}) - \sum_i \max_j\{w_{ij}\}\right]$$

$$= \text{REV}_M(D, D') - \text{VAL}(D) = \text{REV}(D, D') - \text{VAL}(D). \qquad \square$$

Proof of Lemma 1. This is an immediate corollary of Lemma 8. As D and D' are independent, $\text{Rev}(D'|\vec{v} = \vec{w}) = \text{Rev}(D')$, for all $\vec{w}$.   □

Lemma 9. *("Sub-Domain Stitching" [46]) Let $S_1, \ldots, S_k$ form a partition of $\mathbb{R}^{nm}_+$ and let $s_i = \text{Pr}_{\vec{v} \leftarrow D}[\vec{v} \in S_i]$. Then $\sum_i s_i \cdot \text{REV}(D|\vec{v} \in S_i) \geq \text{REV}(D)$.*

Proof. Let M be the optimal mechanism for D, and $\text{REV}_M(D)$ denote the revenue of M when valuations are sampled from D. Then, we have $\text{REV}_M(D) = \sum_i s_i \cdot \text{REV}_M(D|\vec{v} \in S_i)$. We also clearly have $\text{REV}(D) = \text{REV}_M(D)$, and $\text{REV}_M(D|\vec{v} \in S_i) \leq \text{REV}(D|\vec{v} \in S_i)$ for all i, proving the lemma.   □

In the lemma below, again think of D and D' as independent distributions for the same buyers over disjoint sets of items.

Lemma 10. *("Marginal Mechanism on Sub-Domain" [46]) Let there be $m = 1$ buyer, and let S be any subset of $\mathbb{R}^n_+$, and $s = \text{Pr}_{(\vec{v}, \vec{v}') \leftarrow D \times D'}[(\vec{v}, \vec{v}') \in S]$. Then $s \cdot \text{REV}(D \times D'|(\vec{v}, \vec{v}') \in S) \leq s \cdot \text{VAL}(D|(\vec{v}, \vec{v}') \in S) + \text{REV}(D')$.*

Proof. Let T be the set of items in the support of D and let T' be the set of items in the support of D'. We will use the same approach as in the proof of Lemma 8: design a mechanism for selling items in T' given the optimal mechanism for selling items in $T \times T'$ with distribution $D \times D' \mid (\vec{v}, \vec{v}') \in S$, and show that this mechanism has revenue at least $s \cdot \text{REV}(D \times D'|(\vec{v}, \vec{v}') \in S) - s \cdot \text{VAL}(D|(\vec{v}, \vec{v}') \in S)$. Let M denote this optimal (truthful direct) mechanism. To sell items in T', first draw $\vec{v} \leftarrow D$ and announce it to the buyer. Then solicit the buyer's type $\vec{v}'$ and consider the allocation x and price p returned by M for buyer type $(\vec{v}, \vec{v}')$. If $(\vec{v}, \vec{v}') \notin S$, then the buyer receives nothing and pays nothing. If $(\vec{v}, \vec{v}') \in S$, then we copy the allocation and price of M, using rebates on the payment to simulate the buyer's expected value for the allocated items in T (so long as the resulting total price is non-negative). More precisely, if $\sum_{i \in T} x_i v_i \leq p$, allocate the buyer each item $i \in T'$ independently with probability x_i and charge the buyer $p - \sum_{i \in T} x_i v_i$. Otherwise (if $\sum_{i \in T} x_i v_i > p$), the buyer receives nothing and pays nothing.

If the buyer has a value $\vec{v}'$ such that $(\vec{v}, \vec{v}') \notin S$, then we can make no guarantees about what the buyer will report. Indeed, they may lie about $\vec{v}'$, because it will cause them to get items and rebates that they otherwise would not. But certainly the payment made by such $\vec{v}'$ is non-negative (because we removed any options where the rebates exceed the original payment).

Similarly, if the buyer has a value $\vec{v}'$ such that $(\vec{v}, \vec{v}') \in S$, but the rebates exceed the price, then the buyer may misreport. However, the payment made by these buyers is non-negative and so at least as large as the payment in M minus the rebates.

Finally, if the buyer has a value $\vec{v}'$ such that $(\vec{v}, \vec{v}') \in S$ and the rebates do not exceed the price then, because M is truthful, then we can guarantee that the buyer prefers to tell the truth. Indeed, because the buyer will get the rebates by telling the truth, their utility is exactly the same as in M. Their utility for any lie is at most the utility by reporting that lie to M, since they may additionally lose the rebates. So, the buyer's payment is equal to their payment in M minus the rebate.

To summarize, we have argued that we get non-negative revenue from all $(\vec{v}, \vec{v}') \notin S$, and that from all $(\vec{v}, \vec{v}') \in S$, we get revenue at least their payment in M, minus their rebate.

Putting this together, this means that the revenue of our mechanism is at least as large as the revenue obtained by M, *only counting revenue when $(\vec{v}, \vec{v}') \in S$* (which is $s \cdot \text{REV}_M(D \times D'|(\vec{v}, \vec{v}') \in S) = s \cdot \text{REV}(D \times D'|(\vec{v}, \vec{v}') \in S)$), minus the rebates given to $(\vec{v}, \vec{v}') \in S$. Observe that the maximum given back in rebates to such values is $s \cdot \text{VAL}(D|(\vec{v}, \vec{v}') \in S)$. So, we have a mechanism

for selling items in T' with distribution D' guaranteeing revenue at least $s \cdot \text{Rev}(D \times D'|(\vec{v}, \vec{v}') \in S) - s \cdot \text{Val}(D|(\vec{v}, \vec{v}') \in S)$, completing the proof.[21]   □

PROOF OF LEMMA 2. We first prove the lemma in the case of $m = 1$. For each item i, let S_i be the set of types where item i is the buyer's favorite item, tie-breaking lexicographically (i.e., $S_i = \{\vec{v} \mid \forall j \neq i, v_i > v_j\}$, where ">" tie-breaks lexicographically), let $D^{(i)}$ be the conditional distribution of D given the event that $\vec{v} \in S_i$, let s_i be the probability of this event, and let M be the optimal mechanism for distribution D. Then $\text{Rev}_M(D) \leq \sum_i s_i \text{Rev}_M(D^{(i)}) \leq \sum_i s_i \text{Rev}(D^{(i)})$. By Lemma 10,

$$s_i \cdot \text{Rev}(D^{(i)}) \leq s_i \cdot \text{Val}(D_{-i}^{(i)}) + \text{Rev}(D_i).$$

Furthermore, we claim that $s_i \cdot \text{Val}(D_{-i}^{(i)}) \leq (n-1) \cdot \text{Rev}(D_i)$. To see this, observe that one truthful mechanism for selling just item i first samples $\vec{v}_{-i} \leftarrow D_{-i}$, and then sets a price of $\max_{j \neq i}\{v_j\}$. Conditioned on $\vec{v} \in S_i$, the item will always sell (by the definition of S_i), and will generate revenue $\max_{j \neq i}\{v_j\} \geq \frac{1}{n-1} \sum_{j \neq i} v_j$. So, the item sells with probability s_i, and makes expected revenue at least $\frac{1}{n-1} \text{Val}(D_{-i}^{(i)})$ when this occurs, implying $\text{Rev}(D_i) \geq \frac{s_i}{n-1} \text{Val}(D_{-i}^{(i)})$, as claimed. Plugging this into the bound from Lemma 10 above, we have now shown that $s_i \cdot \text{Rev}(D^{(i)}) \leq n \cdot \text{Rev}(D_i)$. Summing over all i, we get the desired bound $\text{Rev}(D) \leq n \cdot \text{SRev}(D)$ for $m = 1$.

We conclude by proving the $m > 1$ case. To extend to $m > 1$ buyers, observe that any truthful m-buyer mechanism M induces m truthful single-buyer mechanisms $M_1, \ldots, M_m$ such that $\text{Rev}_M(D) = \sum_j \text{Rev}_{M_j}(D^j)$ (i.e., for each M_j, just sample $m - 1$ make-believe buyers and have them play M). As $\text{Rev}_{M_j}(D^j) \leq n \cdot \text{SRev}(D^j) \leq n \cdot \text{SRev}(D)$ for every j, the mechanism M cannot have revenue more than $n \cdot m \cdot \text{SRev}(D)$.   □

C OMITTED PROOFS FROM SECTION 5

PROOF OF LEMMA 3. One could sell item i using a second price auction with reserve $t_i r_i$ to guarantee revenue at least $p_i t_i r_i$. If $p_i > 1/t_i$, then this contradicts the fact that the optimal revenue is r_i.   □

The proof of Lemma 4 (and future proofs) will rely on the following well-known fact about single-item auctions. Note that the proof for $m = 1$ buyer is straightforward, but the proof for $m > 1$ buyer is not obvious—we refer the reader to Reference [32] (Theorem 2.1) for a proof of an even stronger claim. Notice that the lemma below does *not* hold when there are $n > 1$ items [48].

LEMMA 11. *Let there be $n = 1$ item and m buyers, whose values are drawn independently. Further, consider two instances, D and D^+ such that every marginal of D^+ stochastically dominates the corresponding marginal of D. Then $\text{Rev}(D^+) \geq \text{Rev}(D)$.*

PROOF OF LEMMA 4. Each distribution D_{ij}^C is stochastically dominated by D_{ij}. By Lemma 11, we conclude that each $\text{Rev}(D_i^C) \leq r_i$.

It is possible to obtain revenue $p_i \text{Rev}(D_i^T)$ when selling to buyers from D_i. Simply use whatever mechanism is used to obtain revenue $\text{Rev}(D_i^T)$. With probability p_i, the buyers will be sampled from D_i^T and yield this much revenue. Therefore, we must have $p_i \text{Rev}(D_i^T) \leq r_i$.   □

PROOF OF LEMMA 5. This is a direct application of Lemma 9. Applied here, observe that the supports of D_A form a partition of the support of D when taken over all A.   □

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[21]Briefly observe that the above analysis considers a fixed menu, and allows the buyer to purchase whatever option they like from that menu. By the taxation principle, this is equivalent to a truthful mechanism.

D   OMITTED PROOFS FROM SECTION 6

PROOF OF LEMMA 7. $\text{var}(F) \leq E_{X \sim F}[X^2]$. As the optimal revenue of F is at most y, we know that $Pr_{X \sim F}[X \geq x] \leq y/x$ for all x. Additionally, $Pr_{X \sim F}[X \geq x] \leq 1$, as it is a probability. So,

$$
\begin{aligned}
E_{X \sim F}[X^2] &= \int_0^{t^2 y^2} Pr_{X \sim F}[X^2 \geq x] dx \\
&\leq \int_0^{y^2} dx + \int_{y^2}^{t^2 y^2} (y/\sqrt{x}) dx \\
&= y^2 + 2y\sqrt{x}\big|_{y^2}^{t^2 y^2} \\
&= y^2 + 2ty^2 - 2y^2 = (2t - 1)y^2.
\end{aligned}
$$
□

E   OMITTED PROOFS FROM SECTION 7

Analysis of our examples, as well as the proofs of Theorems 5 and 6, use the following theorem of Reference [22], which describes a simple constant-factor approximation mechanism to REV(D) in the case of a single item ($n = 1$) that will be easier to analyze than REV(D) itself. Below, a *posted-price mechanism* simply sets a price p^j on the item for buyer j, and lets the lexicographically first buyer j whose value exceeds p^j take the item and pay p^j (we refer to a posted-price mechanism as *anonymous* if all p^j are i.i.d.). To properly use the result of Reference [22] for arbitrary distributions, we will also let p^j be a random variable. That is, the mechanism visits buyers one at a time in lexicographical order. When visiting buyer j, it draws the random variable p^j, and offers the item at price p^j to buyer j. Throughout this section, when discussing a random variable X, we will assume there exists an x such that $\Pr[X \geq x] = c$ for any $c \in (0, 1)$. If X is continuous, then clearly such an x exists. If X is discrete, then we will "make X continuous" by additionally drawing a tie-breaker uniformly from $[0, 1]$ and attaching it to each draw of X (and when comparing two draws from X with the same value, we say the one with larger tie-breaker is larger).[22]

We state two versions of their theorem below, which differ only in the choice of prices to set. The distinction is not necessary when all distributions D_j are regular[23] (that is, the prices in Theorem 9 can be taken to be deterministic if all D_j are regular). For arbitrary distributions, however, the prices can either be deterministic, as in Theorem 10, or be randomized but also ensure that the probability of sale is at most half, as in Theorem 9. Theorem 9 will be used to prove Theorems 5 and 6, and Theorem 10 will be used to analyze our examples.

THEOREM 9 ([22], VERSION 1).   *Let there be a single item ($n = 1$) and m buyers. Then there exists a posted-price mechanism that achieves expected revenue at least REV(D)/2. Moreover, if $p^1, \ldots, p^m$ denote the (random variable) prices used, then $\sum_j \Pr[v^j \geq p^j] \leq 1/2$. Moreover, if the buyers are i.i.d., then the random variables $\{p^j\}_{j \in [m]}$ are i.i.d. as well.*

THEOREM 10 ([22], VERSION 2).   *Let there be a single item ($n = 1$) and m buyers. Then there exists a posted-price mechanism that achieves expected revenue at least REV(D)/2. Moreover, the prices $p^1, \ldots, p^m$ used are deterministic. Moreover, if the buyers are i.i.d., then $p^1, \ldots, p^m$ are identical.*

The following corollary simply observes that the revenue of a posted-price mechanism with (random) prices $p^1, \ldots, p^m$ is clearly upper-bounded by $\sum_j \mathbb{E}_{p \leftarrow p^j}[p \cdot \Pr[v^j \geq p]]$.

---

[22]So, for example, if X is a point-mass at 1, then the value 1 with tie-breaker 1/3 is an x satisfying $\Pr[X \geq x = 2/3]$.

[23]A one-dimensional distribution is regular if $x - \frac{1-F(x)}{f(x)}$ is monotone non-decreasing

COROLLARY 3. *Let there be a single item (n = 1) and m buyers. Then there exist random variables* $p^1, \ldots, p^m$ *such that* $\sum_j \mathbb{E}_{p \leftarrow p^j}[p \cdot \Pr[v^j \geq p]] \geq Rev(D)/2$, *and* $\sum_j \Pr[v^j \geq p^j] \leq 1/2$. *Moreover, if the buyers are i.i.d., then the random variables* $\{p^j\}_{j \in [m]}$ *are i.i.d. as well.*

E.1 Proof of Theorems 5 and 6

Recall that our proof outline first defines a set S as *separable* or *bundlable*, based on whether we will target it with $\text{SRev}(D)$ or $\text{BRev}(D)$. We first begin by providing these definitions.

Definition 3 (Separable set). Say that a set S is α-separable for D if $\text{SRev}(D_S) \geq \alpha \cdot \text{BRev}(D_S)$.

Definition 4 (Bundlable set). Say that a set S is β-bundlable for buyer j and distribution D if there exists a price p such that $\Pr[\sum_{i \in S} v_{ij} \geq p] \geq 1/2$ and $p \geq \beta \cdot \text{BRev}(D_S)$.

We now show that $\text{SRev}(D)$ covers revenue from separable sets, and $\text{BRev}(D)$ covers revenue from bundlable sets. Importantly, the latter claim requires that it is the same buyer j, which witnesses that S is β-bundlable *for all S that are not separable*. This aspect is what enables Theorem 5 when buyers are i.i.d. (because if the condition holds for one buyer, it holds for all of them), or when items are i.i.d. (as we will argue that there is a "dominant buyer" to whom we can restrict attention for all S) but not for arbitrary instances (see Proposition 7 for a counterexample).

LEMMA 12. *Let $\mathcal{S}$ be any collection of disjoint subsets of items such that S is α-separable for D for all $S \in \mathcal{S}$. Then $SRev(D) \geq \alpha \cdot \sum_{S \in \mathcal{S}} BRev(D_S)$.*

PROOF. As $\text{SRev}(D)$ sells all items, $\mathcal{S}$ is a collection of disjoint subsets of items, and items have non-negative values, we clearly have $\text{SRev}(D) \geq \sum_{S \in \mathcal{S}} \text{SRev}(D_S)$. By definition of α-separability, we have that $\text{SRev}(D_S) \geq \alpha \cdot \text{BRev}(D_S)$ for all $S \in \mathcal{S}$, completing the inequality. □

LEMMA 13. *Let $\mathcal{B}$ be any collection of disjoint subsets of items such that there exists a buyer j such that S is β-bundlable for j and D for all $S \in \mathcal{B}$. Then $BRev(D) \geq \frac{\beta}{4} \cdot \sum_{S \in \mathcal{B}} BRev(D_S)$.*

PROOF. Let p_S denote the price promised by β-bundlability for set $S \in \mathcal{B}$. Consider setting price $p := \sum_{S \in \mathcal{B}} p_S/2$ on the grand bundle of all items and running a posted-price mechanism with price p (the same p for all buyers). We claim that buyer j will choose to purchase the grand bundle with probability at least $1/2$, and therefore $\text{BRev}(D) \geq \sum_{S \in \mathcal{B}} p_S/4$. As we are promised that $p_S \geq \beta \cdot \text{BRev}(D_S)$ for all $S \in \mathcal{B}$, this would imply that $\text{BRev}(D) \geq \frac{\beta}{4} \cdot \sum_{S \in \mathcal{B}} \text{BRev}(D_S)$.

It remains to establish that buyer j will choose to purchase the bundle with probability at least $1/2$ at price p. Consider a random variable V_S, which is equal to p_S whenever $\sum_{i \in S} v_{ij} \geq p_S$ and 0 otherwise. Observe that $\sum_{i \in S} v_{ij}$ stochastically dominates V_S (immediately from the definition of V_S: We defined it to first sample $\sum_{i \in S} v_{ij}$ and then lower it to either p_S or 0). Consider finally the random variable W_S, which is equal to p_S with probability $1/2$ and 0 otherwise. It is also clear that V_S stochastically dominates W_S for all $S \in \mathcal{B}$ (because β-bundability of S guarantees that $V_S = p_S$ with probability at least $1/2$). Therefore, $\sum_{i \in S} v_{ij}$ stochastically dominates W_S as well. We proceed to analyze the random variable $W = \sum_{S \in \mathcal{B}} W_S$.

Observe that W is *symmetric about its mean*, which is p. That is, for all x, $\Pr[W = p + x] = \Pr[W = p - x]$. To see this, couple draws $\langle W_S \rangle_{S \in \mathcal{B}}$ with $\langle p_S - W_S \rangle_{S \in \mathcal{B}}$. Observe that because $\Pr[W_S = 0] = 1/2 = \Pr[W_S = p_S]$ for all $S \in \mathcal{B}$, the two coupled draws are equally likely. Moreover, observe that for all coupled draws, $\sum_{S \in \mathcal{B}} W_S + \sum_{S \in \mathcal{B}} (p_S - W_S) = \sum_{S \in \mathcal{B}} p_S = 2p$. Therefore, all coupled draws have equal distance from p. As this coupling maps between outcomes with equal probability, and these outcomes are symmetric about p, the entire random variable W is symmetric about p. We therefore conclude that $\Pr[W \geq p] = 1/2$, and as $\sum_i v_{ij}$ stochastically dominates W, we therefore get that $\Pr[\sum_i v_{ij} \geq p] \geq 1/2$. Therefore, selling the entire grand bundle at price p results in revenue at least $p/2$ (even if only sold to buyer j). □

Next, we establish that every set S must be either separable, or bundlable for some buyer j (Proposition 12). We will later use this to prove Theorems 5 and 6, showing that *when either buyers or items are i.i.d.* there is an approximately optimal partition mechanism and buyer j such that *every set in the partition that is not separable, is bundlable for that buyer j.*

The remainder of this section is dedicated to proving this claim and the theorems it implies, and we begin with some setup and technical lemmas. First, recall by Theorem 9 that we may relate $\text{BRev}(D_S)$ to the revenue of a posted-price mechanism (because $\text{BRev}(D_S)$ is just the revenue of the optimal single-item auction for the "item" S). We define the following notation below (which is used for the rest of this section):

- Denote $p_S := \text{BRev}(D_S)/2$, we introduce this redundant notation for convenience.
- $p_S^1, \ldots, p_S^m$ denote the random variables guaranteed to exist by Theorem 9, for distribution D_S.
- Denote $q_S^j := \Pr[\sum_{i \in S} v_{ij} \geq p_S^j]$. Recall that both $\sum_{i \in S} v_{ij}$ and p_S^j are random variables.

Recall that by Corollary 3 , $\text{BRev}(D_S) \leq 2 \cdot \sum_{j \in [m]} \mathbb{E}_{p \leftarrow p_S^j}[p \cdot \Pr[\sum_{i \in S} v_{ij} \geq p]]$, and $\sum_j q_S^j \leq 1/2$.

We will now fix attention to a single set S, and wish to understand, for a single buyer j, whether the events in which $\sum_{i \in S} v_{ij}$ is large are driven mostly by a few *large*-value items, or several items with *tiny* value (below L stands for large, and T for tiny).

- For each item i, let t_i be such that $\Pr[v_i^* \geq t_i] = 1/2$ (i.e., t_i is the median of the random variable v_i^*. Recall that $v_i^* := \max_{j \in [m]}\{v_{ij}\}$).
- Define $L_{ij} := v_{ij} \cdot \mathbb{I}(v_{ij} \geq t_i)$.
- Define $T_{ij} := v_{ij} \cdot \mathbb{I}(v_{ij} < t_i)$.
- Define $C_{ijp} := \min\{L_{ij}, p\}$ (think of this as L_{ij} "capped at p").
- Let R_i^j denote the optimal achievable revenue selling a single item to a single buyer with value distributed according to L_{ij} (observe that $R_i^j = \max_{x \geq t_i}\{x \cdot \Pr[v_{ij} \geq x]\}$). For ease of notation, also define $R_S^j := \sum_{i \in S} R_i^j$.

Next, we will conclude a few basic properties of the above-defined random variables. Importantly, we will start connecting the expectation and variance of these random variables to $\text{SRev}(D_S)$.

LEMMA 14. *$SRev(D_S) \geq \sum_{i \in S} t_i/2$. Also, $SRev(D_S) \geq \sum_{j \in [m]} R_S^j/2$.*

PROOF. To see the first bound, simply set a posted price of t_i on item i, for all i. By definition of t_i, item i will be purchased with probability exactly $1/2$, giving expected revenue $t_i/2$. Summing over all items gives the bound.

To see the second bound, use a posted-price mechanism that sets a price for buyer j to purchase item i of $z_{ij} = \arg\max_{x \geq t_i}\{x \cdot \Pr[v_{ij} \geq x]\}$. Then visit the buyers in arbitrary order and allow them to purchase any remaining items, offering an available item i to buyer j at its personalized price z_{ij}. Because for any buyer j the price z_{ij} of item i is always at least t_i, this means that for any buyer j, item i is available for buyer j to purchase with probability at least $1/2$. When item i is available for buyer j, we get expected revenue exactly R_i^j from selling item i to buyer j. Therefore, the total revenue achieved from this scheme (which sells items separately) is at least $\sum_{i \in S} \sum_{j \in [m]} R_i^j/2 = \sum_{j \in [m]} R_S^j/2$. □

COROLLARY 4. *For any buyer j, $\sum_{i \in S} T_{ij} \leq 2 \cdot SRev(D_S)$ with probability 1.*

PROOF. Each T_{ij} is a random variable supported on $[0, t_i]$, so their sum is at most $\sum_{i \in S} t_i$, with probability 1. Also, $\sum_{i \in S} t_i \leq 2 \cdot \text{SREV}(D_S)$, by Lemma 14. □

COROLLARY 5. *For any buyer j, $\text{var}(\sum_{i \in S} C_{ijp}) \leq 2 \cdot p \cdot R_S^j$.*

PROOF. Each C_{ijp} is a random variable supported on $[0, p]$, whose optimal revenue is at most R_i^j (because C_{ijp} is stochastically dominated by L_{ij}, whose optimal revenue is R_i^j). Therefore, by Lemma 7, $\text{var}(C_{ijp}) \leq 2pR_i^j$. As all C_{ijp} are independent, we have that $\text{var}(\sum_{i \in S} C_{ijp}) = \sum_{i \in S} \text{var}(C_{ijp}) \leq 2p \sum_{i \in S} R_i^j = 2pR_S^j$. □

We now quickly apply Chebyshev's inequality using Corollary 5.

COROLLARY 6. *For any buyer j, set S, and price p: if $\mathbb{E}[\sum_{i \in S} C_{ijp}] \leq p/4$, then $\Pr[\sum_{i \in S} C_{ij} \geq p/2] \leq \frac{32R_S^j}{p}$. If instead $\mathbb{E}[\sum_{i \in S} C_{ijp}] \geq p/4$, then $\Pr[\sum_{i \in S} C_{ijp} \geq p/8] \geq 1 - \frac{128R_S^j}{p}$.*

PROOF. We apply Chebyshev's inequality to the random variable $\sum_{i \in S} C_{ijp}$. Corollary 5 establishes that $\text{var}(\sum_{i \in S} C_{ijp}) \leq 2pR_S^j$, and therefore

$$\Pr\left[\left|\sum_{i \in S} C_{ijp} - \mathbb{E}\left[\sum_{i \in S} C_{ijp}\right]\right| > x\right] \leq \frac{2pR_S^j}{x^2}.$$

For the first case of the corollary statement, the deviation must be at least $x = p/4$, resulting in a bound of $\frac{32R_S^j}{p}$. For the second case to *not* occur, the deviation must be at least $x = p/8$, resulting in a (lower) bound of $1 - \frac{128R_S^j}{p}$. □

Now, we state our two main technical propositions. We first show that a "good" (Definition 5 immediately below) buyer always exists when S is not $(1/1, 024)$-separable. We later establish that S is either separable, or bundlable for any good buyer. To get intuition how we will eventually leverage these when items or buyers are i.i.d., observe that when buyers are i.i.d., either all buyers are good, or none are good (and therefore by Proposition 11, they are all good whenever S is not separable). However, when items are i.i.d., observe that whether a buyer is good depends only on $|S|$ (we will later use this to show that the *same buyer* is good for every non-separable S in the partition).

DEFINITION 5. *A buyer j is good for S if there exists a $p \geq p_S$ such that $\mathbb{E}[\sum_{i \in S} C_{ijp}] \geq p/4$.*

PROPOSITION 11. *If S is not $(1/1, 024)$-separable, then there exists at least one buyer j that is good for S.*

PROOF. We will make use of the fact that $\sum_{j \in [m]} \mathbb{E}_{p \leftarrow p_S^j}[p \cdot \Pr[\sum_{i \in S} v_{ij} \geq p]] \geq \text{BREV}(D_S)/2 = p_S$ (by definition, as promised by Theorem 9). We will derive a contradiction by showing that this sum is too small if no j is good for S, and S is not $(1/1.024)$-separable.

We will now break the sum into two pieces, by expanding $\sum_{j \in [m]} \mathbb{E}_{p \leftarrow p_S^j}[p \cdot \Pr[\sum_{i \in S} v_{ij} \geq p]]$. By simply splitting into contributions from cases where p is smaller than p_S and when it is larger, we can write

$$\sum_{j \in [m]} \mathbb{E}_{p \leftarrow p_S^j}\left[p \cdot \Pr\left[\sum_{i \in S} v_{ij} \geq p\right]\right] = \sum_{j \in [m]} \mathbb{E}_{p \leftarrow p_S^j}\left[p \cdot \Pr\left[\sum_{i \in S} v_{ij} \geq p\right] \cdot \mathbb{I}(p \leq p_S)\right]$$

$$+ \sum_{j \in [m]} \mathbb{E}_{p \leftarrow p_S^j}\left[p \cdot \Pr\left[\sum_{i \in S} v_{ij} \geq p\right] \cdot \mathbb{I}(p > p_S)\right]. \qquad \square$$

We now proceed to analyze these terms separately.

Lemma 15. $\sum_{j\in[m]} \mathbb{E}_{p\leftarrow p_S^j}[p\cdot\Pr[\sum_{i\in S} v_{ij}\geq p]\cdot\mathbb{I}(p\leq p_S)]\leq p_S/2.$

Proof. The proof follows by simply expanding the sum:

$$\sum_{j\in[m]}\mathbb{E}_{p\leftarrow p_S^j}\left[p\cdot\Pr\left[\sum_{i\in S}v_{ij}\geq p\right]\cdot\mathbb{I}(p\leq p_S)\right]\leq\sum_{j\in[m]}\mathbb{E}_{p\leftarrow p_S^j}\left[p_S\cdot\Pr\left[\sum_{i\in S}v_{ij}\geq p\right]\cdot\mathbb{I}(p\leq p_S)\right]$$

$$\leq\sum_{j\in[m]}\mathbb{E}_{p\leftarrow p_S^j}\left[p_S\cdot\Pr\left[\sum_{i\in S}v_{ij}\geq p\right]\right]$$

$$=\sum_{j\in[m]}p_S\cdot q_S^j$$

$$\leq p_S/2.$$

Indeed, the first inequality simply observes that whenever the indicator is non-zero, $p\leq p_S$. The second inequality upper bounds an indicator by 1. The equality observes that p_S is a constant, and that $q_S^j := \Pr[\sum_{i\in S} v_{ij}\geq p_S^j] = \mathbb{E}_{p\leftarrow p_S^j}[\Pr[\sum_{i\in S} v_{ij}\geq p]]$. The final inequality makes use of the promise from Theorem 9 that $\sum_{j\in[m]} q_S^j\leq 1/2$.   □

Next, we transition to the second term in the sum. We will argue that if no j are good for S, then the second term is small as well.

Lemma 16. *Assume that S is not $(1/1,024)$-separable, and assume further that no buyer j is good for S. Then $\sum_{j\in[m]}\mathbb{E}_{p\leftarrow p_S^j}[p\cdot\Pr[\sum_{i\in S} v_{ij}\geq p]\cdot\mathbb{I}(p>p_S)]\leq BRev(D_S)/16$.*

Proof. We begin by observing that $v_{ij} = T_{ij} + L_{ij}$ (with probability 1). Therefore, if we are to possibly have $\sum_{i\in S} v_{ij}\geq p$, then we must also have either $\sum_{i\in S} T_{ij}\geq p/2$, or $\sum_{i\in S} L_{ij}\geq p/2$. By Corollary 4, however, we know that $\sum_{i\in S} T_{ij}\leq 2\cdot\mathrm{SRev}(D_S)$ with probability 1. Therefore, we conclude that for any $p > 4\cdot\mathrm{SRev}(D_S)$, $\sum_{i\in S} v_{ij}\geq p\Rightarrow\sum_{i\in S} L_{ij}\geq p/2$. Observe even further that whenever $\sum_{i\in S} L_{ij}\geq p/2$, we must further have $\sum_{i\in S} C_{ijp}\geq p/2$ (because the only reason we would have $C_{ijp}\neq L_{ij}$ for any i,j is if both are already at least p).

In particular, this allows us to conclude that, *whenever $p > 4SRev(D_S)$*:

$$\Pr\left[\sum_{i\in S} v_{ij}\geq p\right]\leq\Pr\left[\sum_{i\in S} C_{ijp}\geq p/2\right]. \tag{1}$$

We can now use this to complete the following chain of inequalities:

$$\sum_{j\in[m]}\mathbb{E}_{p\leftarrow p_S^j}\left[p\cdot\Pr\left[\sum_{i\in S}v_{ij}\geq p\right]\cdot\mathbb{I}(p>p_S)\right]\leq\sum_{j\in[m]}\mathbb{E}_{p\leftarrow p_S^j}\left[p\cdot\Pr\left[\sum_{i\in S}C_{ijp}\geq p/2\right]\cdot\mathbb{I}(p>p_S)\right]$$

$$\leq\sum_{j\in[m]}\mathbb{E}_{p\leftarrow p_S^j}\left[p\cdot\frac{32R_S^j}{p}\cdot\mathbb{I}(p>p_S)\right]$$

$$\leq\sum_{j\in[m]}32R_S^j$$

$$\leq 64\cdot\mathrm{SRev}(D_S)$$

$$\leq\mathrm{BRev}(D_S)/16.$$

Indeed, the first line follows by Equation (1) and the fact that S is not $(1/1,024)$-separable: because S is not $(1/1,024)$-separable, whenever the indicator is non-zero, we have $p > p_S =$

BREV$(D_S)/2 > 512 \cdot$ SREV$(D_S) > 4 \cdot$ SREV(D_S), so we can use Equation (1). The second line follows directly from Corollary 6, because of our hypothesis that no j is good for S. The third follows by upper-bounding an indicator random variable by 1. The fourth follows directly from Lemma 14. The final inequality follows as S is not $(1/1,024)$-separable. □

Using Lemmas 15 and 16, we can now complete the proof of Proposition 11. Assume for contradiction that S is not $(1/1,024)$-separable and also that no buyer j is good for S. Then Lemmas 15 and 16 immediately imply that

$$\sum_{j \in [m]} \mathbb{E}_{p \leftarrow p_S^j} \left[p \cdot \Pr\left[\sum_{i \in S} v_{ij} \geq p \right] \cdot \mathbb{I}(p \leq p_S) \right] + \sum_{j \in [m]} \mathbb{E}_{p \leftarrow p_S^j} \left[p \cdot \Pr\left[\sum_{i \in S} v_{ij} \geq p \right] \cdot \mathbb{I}(p > p_S) \right]$$
$$\leq \text{BREV}(D_S)/4 + \text{BREV}(D_S)/16$$
$$< \text{BREV}(D_S)/2.$$

But Theorem 9 guarantees that $\sum_{j \in [m]} \mathbb{E}_{p \leftarrow p_S^j}[p \cdot \Pr[\sum_{i \in S} v_{ij} \geq p]] \geq$ BREV$(D_S)/2$, a contradiction.

The only assumptions necessary to reach a contradiction are that S is not $(1/1,024)$-separable, and that no buyer j is good for S. So, one of these assumptions must fail (and in particular, if S is not $(1/1,024)$-separable, it must be the latter one).

We now prove our second key technical proposition, which states that S is either separable, or bundlable for any good j.

PROPOSITION 12 (SEPARABLE OR BUNDLABLE). *Let buyer j be good for S. Then S is either $(1/1,024)$-separable for D, or $(1/16)$-bundlable for j and D.*

PROOF. We assume throughout the proof that S is not $(1/1,024)$-separable (clearly, if this assumption is false, then S is $(1/1,024)$-separable and we are done). Let p be any witness that j is good for S. The proof will follow from the chain of inequalities below:

$$\Pr\left[\sum_{i \in S} v_{ij} \geq p/8 \right] \geq \Pr\left[\sum_{i \in S} C_{ijp} \geq p/8 \right]$$
$$\geq 1 - \frac{128 R_S^j}{p}$$
$$\geq 1 - \frac{256 \cdot \text{SREV}(D_S)}{\text{BREV}(D_S)/2}$$
$$\geq 1/2.$$

The first inequality follows simply as v_{ij} stochastically dominates C_{ijp} for all i, j, p. The second follows from Corollary 6, and the hypothesis that j is good for S (witnessed by p). The third follows as $R_S^j \leq \sum_{\ell \in [m]} R_S^\ell \leq 2 \cdot$ SREV(D_S) by Lemma 14, and $p \geq p_S =$ BREV$(D_S)/2$ by definition. The final inequality follows as S is not $(1/1,024)$-separable. Therefore, we have shown that buyer j will purchase S at price $p/8$ with probability at least $1/2$. As $p \geq p_S =$ BREV$(D_S)/2$ it follows that $p/8 \geq$ BREV$(D_S)/16$, and thus S is $(1/16)$-bundlable for j, D. □

Propositions 11 and 12 are the key ingredients in Theorems 5 and 6, which we now prove below.

PROOF OF THEOREM 5. The theorem now follows by Propositions 11 and 12 and Lemmas 12 and 13. For the optimal partition of the items, let $\mathcal{S}$ denote the $(1/1,024)$-separable sets, and $\mathcal{B}$ denote the $(1/16)$-bundlable sets for buyer 1. Because buyers are i.i.d., for all S it is the case that either every buyer is good for S, or no buyer is good for S. Proposition 11 proves that for

all $S \notin \mathcal{S}$, all buyers are good for S. Propositions 12 then guarantees that all $S \notin \mathcal{S}$ are $(1/16)$-bundlable for all buyers (and in particular, buyer 1). Therefore, $\mathcal{S}$ and $\mathcal{B}$ form a cover of the partition. Lemma 12 guarantees that $\text{SRev}(D) \geq \frac{1}{1,024} \sum_{S \in \mathcal{S}} \text{BRev}(D_S)$, and Lemma 13 guarantees that $\text{BRev}(D) \geq \frac{1}{64} \sum_{S \in \mathcal{B}} \text{BRev}(D_S)$. Therefore, either $\text{SRev}(D)$ or $\text{BRev}(D)$ guarantees a $(1/2048)$-approximation to $\sum_S \text{BRev}(D_S) = \text{PRev}(D)$.   □

PROOF OF THEOREM 6. The theorem again follows from Propositions 11 and 12, and Lemmas 12 and 13, although one extra step is needed. Consider the optimal partition of $[n]$ into $S_1 \sqcup \ldots \sqcup S_k$ and consider $\sum_i \text{BRev}(D_{S_i})$. First, we observe that because the items are i.i.d., $\text{BRev}(D_{S_i})$ depends only on $|S_i|$, and not the precise items in S_i (exactly because the items are i.i.d.). Therefore, we will abuse notation and write $\text{BRev}(x)$ instead of $\text{BRev}(D_S)$ for any S with $|S| = x$. Next, let x be any value maximizing $\text{BRev}(x)/x$ among all $x \in \{1, \ldots, n\}$. That is, bundling x items together generates the most "bang per buck" (revenue per item bundled). Then, we get

$$\sum_i \text{BRev}(D_{S_i}) \leq \sum_i |S_i| \cdot \text{BRev}(x)/x \leq n\text{BRev}(x)/x.$$

Therefore, if we instead partition the items into $\lfloor n/x \rfloor$ sets of size x, and one remaining set, then we would get revenue at least:

$$\lfloor n/x \rfloor \cdot x \cdot \text{BRev}(x)/x \geq (n/2) \cdot \text{BRev}(x)/x \geq \sum_i \text{BRev}(D_{S_i})/2.$$

The first inequality follows, because $\lfloor n/x \rfloor \cdot x \geq n/2$ when $x \leq n$ (i.e., if $\geq n/2$ items are left over, it is because $x \leq n/2$, in which case another bundle can be made). Therefore, up to a factor of 2, we may analyze $\lfloor n/x \rfloor \cdot x \cdot \text{BRev}(x)$ instead. Importantly, observe that because items are i.i.d., and we are only considering bundles S of the same size, each instance S considered is identically distributed, and therefore every buyer is either *good for every such set S, or good for no S*. This observation, together with Proposition 11, implies that if no buyer is good for all S, then all S are $(1/1,024)$-separable. Otherwise, there is a buyer j that is good for all S, and Proposition 12 states that either *all S are* $(1/1,024)$-*separable* or *all S are* $(1/16)$-*bundlable* for j (because all S are identically sized). Whichever the case is, $\max\{\text{SRev}(D), \text{BRev}(D)\}$ guarantees at least a $(1/1,024)$-approximation by Lemmas 12 and 13, for a total of a $(1/2048)$-approximation after losing the factor of 2 by restricting to the case of identically sized S.   □

E.2 Analysis of Multi-buyer Lower Bounds

The following lemma slightly improves concentration bounds for sums of i.i.d. Equal Revenue random variables provided in Reference [46].

LEMMA 17. *Let X be the sum of ℓ i.i.d. random variables all drawn according to ER_k. Then:*

- $\mathbb{E}[X] = \ell + \ell \ln(k)$.
- $\Pr[|X - \ell - \ell \ln(k)|/(\ell + \ell \ln(k)) > \delta] \leq 2e^{-\delta^2 \ell \ln(k)/(3k)}$.

PROOF. The expected value of a single draw from ER_k is $\int_0^\infty 1 - F(x)dx = \int_0^1 1dx + \int_1^k (1/x)dx + \int_k^\infty 0 = 1 + \ln(k)$. So, the expected value of X is just ℓ times this.

To derive the second bullet, we simply apply the multiplicative Chernoff bound (see Appendix H for statement). Observe that each individual draw from ER_k is bounded in $[0, k]$, so after normalizing to lie in $[0, 1]$, we get that the probability that the sum deviates by its expectation by more than a multiplicative factor of δ is at most $2e^{-\delta^2(\ell+\ell \ln(k))/(3k)}$ (in the lemma statement, we relax this by lower bounding $\ell \geq 0$ for cleanliness).   □

PROOF OF PROPOSITION 8. To prove the claim, we show that $\text{REV}(D) \in \Omega(n \log n)$ while $\text{PREV}(D) \in O(n)$ (actually, since $\text{SREV} \in \Omega(n)$ it holds that $\text{PREV}(D) \in \Theta(n)$).

To see that $\text{REV}(D) \in \Omega(n \log n)$ consider the mechanism that sequentially visits the $\sqrt{n}$ buyers, allowing each to pick any set of $\sqrt{n}/2$ items that are still available, and pay $c(\sqrt{n} \log n)$ for some $c > 0$ to be determined later. For each of the first $\sqrt{n}/4$ buyers, at least $7n/8$ items are remaining, and with probability $1 - e^{-\Omega(\sqrt{n})}$, the buyer has non-zero value for at least $\sqrt{n}/2$ of these items. This follows directly from the multiplicative Chernoff bound (Appendix H for statement), as the expected number of items for which the buyer has non-zero value is at least $7\sqrt{n}/8$, and the buyer has non-zero value for each item independently. Conditioned on having non-zero value for each of at least $\sqrt{n}/2$ remaining items, pick an arbitrary set of remaining items of size $\sqrt{n}/2$ for which the buyer has non-zero value and call it S. Then the buyer's value for each item is S is drawn from $\text{ER}_{n^{1/8}}$, and therefore by Lemma 17 her expected value for this set is at least $\sqrt{n} \ln(n)/16$ (plugging in $\ell = \sqrt{n}/2$ items and $k = n^{1/8}$ and dropping the $+\ell$ term). Moreover, the probability that her value for the set falls below $\sqrt{n} \ln(n)/32$ is at most $e^{-\Omega(n^{3/8})}$ (again, just plugging into Lemma 17). Therefore, if we set $c = 1/32$, then no matter what happens for the first $j-1$ buyers, buyer $j \leq \sqrt{n}/4$ will choose to purchase with probability at least $1/2$ (in fact, much closer to 1 than this), just over the randomness in drawing v^j from D^j. This means that our expected revenue from the first $\sqrt{n}/4$ buyers is at least $\sqrt{n}/4 \cdot 1/2 \cdot c\sqrt{n} \ln(n) = n \ln(n)/256$. So, $\text{REV}(D) \in \Omega(n \log n)$.

To see that $\text{PREV}(D) \in O(n)$ consider any partitioning mechanism into sets $S_1, \ldots, S_x$, and denote by $s_k = |S_k|$. Denote by $\text{BREV}(k)$ and $\text{SREV}(k)$ the revenue obtained by bundling together and selling separately the items in S_k, respectively. Note that $\sum_k s_k = n$ and also that $\text{PREV}(D) = \sum_k \text{BREV}(k)$.

We first analyze all k such that $s_k \leq n^{1/4}$. In this case, observe that the probability that a fixed buyer has non-zero value for at least 12 items can be upper bounded by taking a union bound over all $\binom{s_k}{12} \leq n^{(1/4) \cdot 12} = n^3$ subsets of S_k of size 12 of the probability that the buyer has non-zero value for this entire subset S_k (which is just $1/\sqrt{n}^{12} = 1/n^6$ as each event is independent). Taking another union bound over all $\sqrt{n}$ buyers yields that the probability that *any* buyer has non-zero value for more than 12 items is at most $n^{3-6+1/2} = n^{-5/2}$. So, even if our mechanism achieved the maximum possible welfare from S_k ($s_k \cdot n^{1/8} \leq n^{3/8}$) whenever this occurred, the total contribution to the revenue would still be $o(1)$. To finish analyzing k such that $s_k \leq n^{1/4}$, it remains to analyze the case when every buyer has non-zero value for 12 or fewer items in part k.

When every buyer has non-zero value for 12 or fewer items in part k, we show that $24\text{SREV}(k) \geq \text{BREV}(k)$. To see this, let p be the anonymous posted-price promised by Theorem 10 such that $p \cdot \Pr[\text{exists a buyer who values the bundle above } p] \geq \text{BREV}(k)/2$. Consider instead the anonymous price mechanism that sets price $p/12$ for each item separately. Then clearly, whenever some buyer is willing to pay p for the grand bundle, and only 12 of these items provide non-zero value, she is also willing to pay $p/12$ for at least one item separately. So, we get that $24\text{SREV}(k) \geq \text{BREV}(k)$. Finally, a trivial upper bound on $\text{SREV}(k)$ is the revenue obtainable by selling goods separately without any supply constraint (i.e., the seller has infinitely many copies of each good instead of just one). For a single item and single buyer, the optimal price is 1 (or any value between 1 and $n^{1/8}$), and generates revenue $1/\sqrt{n}$. So, the total revenue for selling a single item separately to $\sqrt{n}$ buyers is at most 1, and therefore $\text{SREV}(k) \leq s_k$. So, we have now shown that $\text{BREV}(k) \leq 24s_k + o(1)$ whenever $s_k \leq n^{1/4}$. It remains now to consider parts where $s_k > n^{1/4}$.

Next, we analyze the case where $s_k > n^{1/4}$. In this case, we will argue that with high probability, no buyer has large value for the bundle S_k (implying that $\text{BREV}(k)$ must be small). To see this, observe that the number of items for which buyer i has non-zero value is a sum of s_k independent random variables with expectation $s_k/\sqrt{n}$. The multiplicative Chernoff bound for large deviations

(Appendix H for statement) implies that with probability at most $e^{-\Omega(\ln^2 n)}$, the number of items for which buyer i has non-zero value is at most $s_k \cdot \ln^2(n)/n^{1/4}$.[24] So, even if some buyer had the maximum possible value for every item she valued above 0, her total value would be at most $2s_k \ln^2(n)/n^{1/8} < s_k$. So, the revenue contribution from cases where no buyer values more than $s_k \cdot \ln^2(n)/n^{1/4}$ items above 0 is $\leq s_k$. In the unlikely event that some buyer has non-zero value for many items, the maximum possible revenue is still at most $s_k \cdot n^{1/8}$. But this event occurs with probability $e^{-\Omega(\ln^2 n)}$, and $s_k \cdot n^{1/8} \cdot e^{-\Omega(\ln^2 n)} = o(s_k)$. So, our total revenue is $\leq s_k$.

So, now we have shown that for all k, $\text{BREV}(k) \leq 25 s_k$, and therefore $\sum_k \text{BREV}(k) \leq 25n$, and $\text{PREV}(D) \in O(n)$. □

PROOF OF PROPOSITION 7. First recall the example: D has n items and $m = \sqrt{n}$ buyers. The items are partitioned into $\sqrt{n}$ disjoint sets of size $\sqrt{n}$, $S_1, \ldots, S_m$. Each buyer j has value 0 for every item not in S_j, and value independently drawn from $\text{ER}_{n^{1/8}}$ for each item in S_j. Our goal is to show that $\max\{\text{SREV}(D), \text{BREV}(D)\} \leq \text{PREV}(D)/\Omega(\log n)$.

We note first that $\text{SREV}(D) = n$, as each item has exactly one buyer with non-zero value, and the item can be sold to that buyer at any price $p \in [1, n^{1/8}]$ for expected revenue of 1 (and higher revenue is not achievable). We further claim that $\text{BREV}(D) \in O(\sqrt{n} \ln(n))$. To see this, we'll upper bound the revenue of any anonymous posted-price mechanism, and then apply Theorem 10 to obtain a bound on $\text{BREV}(D)$.

First observe that any posted-price mechanism with price $p > n^{5/8}$ for the grand bundle achieves revenue 0, as the maximum possible value of any buyer for the grand bundle is $n^{5/8}$. Also, any price $p < 3\sqrt{n} \ln(n)$ achieves revenue $O(\sqrt{n} \ln(n))$ (even if it sells with probability 1). The remaining prices to rule out are $p \in [3\sqrt{n} \ln(n), n^{5/8}]$.

To this end, first conclude from Lemma 17 (taking $\ell = \sqrt{n}, k = n^{1/8}$, and taking the bound $\ln(k)/3 \geq 1$) that for any buyer j, their value for the grand bundle exceeds $3\sqrt{n} \ln(n)$ with probability at most $e^{-n^{3/8}}$. Taking a union bound over all $\sqrt{n}$ buyers, we see that the probability that any buyer values the grand bundle above $3\sqrt{n} \ln(n)$ is at most $\sqrt{n} \cdot e^{-n^{3/8}} \leq e^{-n^{1/4}}$.

As the probability that any buyer values the grand bundle above $3\sqrt{n} \ln(n)$ is at most $e^{-n^{1/4}}$ by the work above, we can immediately conclude that any anonymous posted-price mechanism with price $p \in [3\sqrt{n} \ln(n), n^{5/8}]$ achieves revenue at most $n^{5/8} \cdot e^{-n^{1/4}} = o(1)$. Therefore, the revenue of any anonymous posted-price mechanism is $O(\sqrt{n} \ln(n))$. By Theorem 10, this implies that $\text{BREV}(D) \in O(\sqrt{n} \ln(n))$ as well. We conclude that $\max\{\text{SREV}(D), \text{BREV}(D)\} \in O(n)$.

Finally, consider a partition mechanism that bundles each of the sets of size $\sqrt{n}$ separately and sells it to the interested buyer at price $\sqrt{n} \ln(n)/4$ (which sells with high probability by Lemma 17). This mechanism gets a total revenue of $\sqrt{n} \cdot \Omega(\sqrt{n} \log \sqrt{n}) = \Omega(n \log n)$. We conclude that $\text{PREV}(D) = \Omega(n \log n)$, and hence $\max\{\text{SREV}(D), \text{BREV}(D)\} \leq \text{PREV}(D)/\Omega(\log n)$. □

F OMITTED PROOFS FROM SECTION 8

F.1 Proof of Theorem 8

Theorem 8 follows directly from Lemmas 18, 19, and 20 below. We first present two helpful definitions.

Definition 6. We say that an n-dimensional distribution D is a *point-mass in sum* distribution if there exists a p such that when $\vec{v}$ is sampled from D, $\sum_i v_i = p$ with probability 1.

Definition 7. We say that an n-dimensional distribution D is *symmetric* if all marginals D_i are the same.

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[24]To see this, plug in $\varepsilon = \ln^2(n) \cdot n^{1/4}$. The resulting bound is $e^{-\Omega(\ln^2(n) n^{1/4} s_k/\sqrt{n})}$. Because $s_k \geq n^{1/4}$, this is $e^{-\Omega(\ln^2(n))}$.

LEMMA 18. *For any n-dimensional distribution D, there exists a point-mass in sum n-dimensional distribution D' such that BREV(D')/SREV(D') ≥ BREV(D)/SREV(D).*

PROOF. Pick any instance D whose optimal grand bundle price is p, and the grand bundle sells at price p with probability q. We will transform D into a point-mass in sum distribution D' without decreasing the ratio BREV(D)/SREV(D).

If $\vec{v}$ denotes a sample from D, then observe that we may modify D to D'' such that BREV$(D'') \geq$ BREV(D) and SREV$(D'') \leq$ SREV(D) (and therefore BREV(D'')/SREV$(D'') \geq$ BREV(D)/SREV(D)). Whenever $\sum_i v_i > p$, lower some values so that $\sum_i v_i = p$. Whenever $\sum_i v_i < p$, set all $v_i = 0$. It is clear that BREV$(D'') \geq$ BREV(D), as the buyer is still willing to pay p with probability q. It is also clear that SREV$(D'') \leq$ SREV(D), as we have only lowered the buyer's value for each item in a stochastically dominating way.

Next, define D' to be the distribution that is exactly D'' conditioned on $\sum_i v_i = p$. Then D'' samples from D' with probability q, and sets all values to 0 otherwise. It is also clear that SREV$(D'') = q$SREV(D'), because whatever price is set for each item sells with probability exactly q times the probability it sells when the buyer is drawn from D' (because the buyer will never pay anything for the item if instead all values are 0). Therefore, because BREV$(D'') = qp$, and BREV$(D') = p$, the two ratios are equal. That is: BREV(D'')/SREV$(D'') =$ BREV(D')/SREV(D'). It is clear that D' is a point-mass in sum distribution. □

LEMMA 19. *For any n-dimensional distribution D, there exists a symmetric n-dimensional distribution D' such that BREV(D')/SREV(D') ≥ BREV(D)/SREV(D). If D was point-mass in sum, then D' is point-mass in sum as well.*

PROOF. Define D' in the following way: sample $\vec{v}$ from D, then randomly permute the components of $\vec{v}$ to form $\vec{v}'$. It is clear that D' is symmetric. It is also clear that BREV$(D) =$ BREV(D'). We just have to show that SREV$(D') \leq$ SREV(D). Let D_i denote the ith marginal of D, D'_j denote the jth marginal of D', v_i denote a sample from D_i, and v'_j a sample from D'_j. Then D'_j samples from each D_i with probability $1/n$.

Now observe that SREV$(D') = \sum_j \max_p \{p Pr[v'_j \geq p]\}$. As each D'_j samples each D_i with probability $1/n$, we get that $Pr[v'_j \geq p] = \sum_i Pr[v_i > p]/n$. This means that we can rewrite SREV$(D') = \sum_j \max_p \{p \sum_i Pr[v_i > p]/n\} = \max_p \{p \sum_i Pr[v_i > p]\}$. And observe also that SREV$(D) = \sum_i \max_p \{p Pr[v_i \geq p]\}$. In other words, SREV$(D')$ is exactly SREV(D) after swapping the order of the max and sum, which can only decrease SREV(D'). □

LEMMA 20. *Let D be any symmetric point-mass in sum distribution. Then BREV(D) ≤ 5 ln(n)SREV(D).*

PROOF. Without loss of generality, scale D down so that SREV$(D) = n$. We are essentially asking how large VAL(D) can possibly be subject to SREV$(D) = n$ (as VAL$(D) =$ BREV(D) for point-mass in sum distributions), plus the symmetric point-mass in sum constraint. Denote $p =$ VAL(D).

Note that each D_i is supported on $[0, p]$, and has expected revenue 1 (because SREV$(D) = n$ and D is symmetric). So,

$$\text{VAL}(D_i) = \int_0^p Pr[v_i > x]dx \leq \int_0^1 dx + \int_1^p (1/x)dx = 1 + \ln p.$$

We now observe that we have two estimates of VAL(D). First, we know that VAL$(D) = p$. And second, we know that VAL$(D) = \sum_i \text{VAL}(D_i) \leq n + n \ln p$. Putting these together, we get that p must satisfy:

$$p \leq n + n \ln p.$$

For all $n \geq 2$, this implies that $p \leq 5n \ln n$ (as all $p > 5n \ln n$ violate the above inequality). □

F.2   Proof of Proposition 10

PROOF OF PROPOSITION 10.   Partition n into $\ln n$ sets $S_1, \ldots, S_{\ln n}$, where $|S_k| = n/(\ln n)$ for each k. Construct D as follows: for each set S_k independently, with probability $1 - n^{-2k}$ the value of every item is 0, and with probability n^{-2k} the value of each item is n^{2k} times an independent draw from $\mathrm{ER}_{n^{1/8}}$. It is clear that $\mathrm{SREV}(D) = n$, since the optimal expected revenue of each item is 1. To prove the claim, we show that $\mathrm{BREV}(D) \in O(n)$ and that $\mathrm{PREV}(D) \in \Omega(n \ln n)$.

To bound $\mathrm{BREV}(D)$ from above, consider the revenue generated by a grand-bundle price of p. If $p \leq n$, then this revenue is at most n, so assume $p > n$. Choose $\ell \geq 0$ such that $p \in (n^{2\ell}, n^{2\ell+2}]$. First, observe that clearly the total value for $S_1 \sqcup \ldots \sqcup S_{\ell-1}$ is at most $n^{2\ell}$, as there are only n items, and the value for each item is no more than $n^{2\ell-15/8}$. Next, observe that with probability at least $1 - 2n^{-2\ell-2}$, no S_k for $k > \ell$ contributes any value. This is because by the union bound, the probability that any $S_k, k > \ell$, contributes non-zero value is at most $\sum_{k>\ell} n^{-2\ell} \leq 2n^{-2\ell-2}$. Together, this means that for any price $p \in (n^{2\ell}, n^{2\ell+2}]$, the probability of sale is completely determined, up to $\pm 2n^{-2\ell-2}$, by the probability that the value for S_ℓ exceeds $p - n^{2\ell}$. So, we now determine the probability that the value for S_ℓ exceeds $p - n^{2\ell}$.

Here, we make use of Lemma 17 to claim that the buyer's value for S_ℓ concentrates around $n^{2\ell+1}$ whenever it is non-zero. To see this, observe that the buyer's value, when non-zero, is $n^{2\ell}$ times sum of $n/\ln n$ i.i.d. draws from $\mathrm{ER}_{n^{1/8}}$. By Lemma 17, the probability that the sum (without the $n^{2\ell}$ multiplier) exceeds $3n$ is therefore $e^{-\Omega(\sqrt{n})}$, and the probability that the value for the bundle exceeds $3n^{2\ell+1}$ is also at most $e^{-\Omega(\sqrt{n})}$. Therefore, the probability that the bundle sells at any price $p > 3n^{2\ell+1} + n^{2\ell}$ is at most $e^{-\Omega(\sqrt{n})} + 2n^{-2\ell-2}$, and therefore any such $p \leq n^{2\ell+2}$ generates expected revenue $O(1)$. Moreover, for any $p \in (2n^{2\ell}, 3n^{2\ell+1} + n^{2\ell})$, the probability of sale is at most $2n^{-2\ell-2} + n^{-2\ell}$, as the buyer must have non-zero value for *some* bundle $S_k, k \geq \ell$. So, such a price would generate revenue $O(n)$. As this argument holds for any ℓ, we have shown that $\mathrm{BREV}(D) \in O(n)$.

Now, we show that $\mathrm{PREV}(D) \in \Omega(n \ln n)$. Consider the mechanism that partitions the items into S_k, and sets price $n^{2k+1} \cdot \ln n/2$ on S_k. By Lemma 17, conditioned on having non-zero value for bundle S_k, the buyer's value exceeds $n^{2k} \cdot (n/\ln n) \cdot \ln(n)/2$ with probability $1 - e^{-\Omega(\sqrt{n})}$. Thus the total revenue of this mechanism, and hence $\mathrm{PREV}(D)$, is at least $\ln(n) \cdot (n^{-2k}) \cdot (n^{2k+1} \cdot \ln(n)/2) = \Omega(n \ln n)$.   □

G   COMPUTATIONAL CONSIDERATIONS

Our main result, Theorem 2, shows that $6 \cdot \max\{\mathrm{SREV}(D), \mathrm{BREV}(D)\} \geq \mathrm{REV}(D)$ for a single buyer. This suggests a simple mechanism that obtains a constant approximation to the optimal revenue: estimate $\mathrm{SREV}(D)$ and $\mathrm{BREV}(D)$, then run whichever of the two mechanisms obtains higher revenue estimate. In this section, we argue that a slight modification of this approach can be implemented in polynomial time, given appropriate access to the distribution D.

We will assume that we are given a sample access to the distributions $\{D_i\}_i$. When each D_i is regular,[25] one sample in fact suffices to find a price p that guarantees revenue $\mathrm{REV}(D_i)/2$ [34], and $\mathrm{POLY}(1/\varepsilon)$ suffice to find a price that guarantees $(1 - \varepsilon) \cdot \mathrm{REV}(D_i)$ [28]. Unfortunately, sample access alone can be insufficient even when there is just $n = 1$ item for arbitrary distributions (consider for instance a distribution that is x with probability $1/2^{2^n}$ and 0 otherwise, for some unknown x—there is no hope of learning a good price from few samples). In such cases, we thus make the additional weak assumption that for each D_i, we also have access to a price p_i guaranteeing revenue at least $\alpha \cdot \mathrm{REV}(D_i)$, and the probability q_i that $v_i \geq p_i$. Calculating $\mathrm{SREV}(D)$ (up to a factor of

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[25]A one-dimensional distribution is regular if $x - \frac{1-F(x)}{f(x)}$ is monotone non-decreasing.

α) and implementing a mechanism that sells items separately (and guarantees a α-approximation to $\textsc{SRev}(D)$) is then trivial: simply set the price p_i for item i.

Given that we can compute and implement $\textsc{SRev}(D)$, what we would like is to also estimate $\textsc{BRev}(D)$ and compute an approximately optimal price for the grand bundle. One approach would be to take samples from the distribution D, then optimize revenue for the observed empirical distribution. However, this strategy again suffers from the sample-complexity issues described above: the number of samples required to estimate $\textsc{BRev}(D)$ might, in principle, be quite large (again, certain items may have exponentially large value with exponentially small probability).

Instead, recall that the *only* reason for $\textsc{SRev}(D)$ to not itself yield a 6-approximation to $\textsc{Rev}(D)$ is if $\textsc{Val}(D_{\emptyset}^C)$ concentrates around its expectation, in which case there exists a price $p^* = 2\textsc{Val}(D_{\emptyset}^C)/5$ for the grand bundle that sells with probability at least 47/72. Importantly, we conclude that if $\textsc{SRev}(D)$ is not a 6-approximation, then some price p^* for the grand bundle sells with probability at least 47/72 and guarantees a 6-approximation.

So, from here, we do not actually need to evaluate $\textsc{BRev}(D)$, we just need to estimate the revenue guaranteed by any bundle price that sells with probability at least 47/72. To test this, we simply take $\ln(1/\delta)/\varepsilon^2$ samples from each distribution, and let $G(p)$ denote the fraction of samples that exceed price p. The Dvoretsky-Kiefer-Wolfowitz inequality then immediately guarantees that except with probability $2/\delta^2$, $G(p)$ is within $\pm\varepsilon$ of the probability that the buyer's value for the grand bundle exceeds p. In particular, for any price p that (really) sells with probability $q \geq 47/72$, we will have $G(p) \geq q - \varepsilon \geq q(1 - 2\varepsilon)$. Therefore, if we simply enumerate over all prices p with $G(p) \geq 47/72 - \varepsilon$, then we will find (with probability at least $1 - 1/\delta^2$) the best price for the grand bundle among all prices that sell with probability at least 47/72. Moreover, we know (up to a factor of $(1 - 2\varepsilon)$, with probability at least $1 - 1/\delta^2$) the revenue generated by this price, so we can simply compare it to $\sum_i p_i q_i$ and pick whichever is better.

To summarize, we have shown that the following algorithm, with probability at least $1 - \delta$, obtains a 6α-approximation in time $\textsc{poly}(\ln(1/\delta), 1/\varepsilon, n)$, for any $\alpha \geq (1 + 2\varepsilon)$.

(1) Given as input p_i, q_i such that item i sells with probability q_i at price p_i and that $p_i q_i \geq \alpha\textsc{Rev}(D_i)$.
(2) Take $\ln(1/\delta)/\varepsilon^2$ samples from each D_i, and index them by $v_{i\ell}$. For all ℓ, observe that $\sum_i v_{i\ell}$ form indepndent samples from the buyer's value for the grand bundle (so, now we have $\ln(1/\delta)/\varepsilon^2$ samples from the buyer's value for the grand bundle).
(3) Let $p^* = \arg\max_{p, G(p) \geq 47/72 - \varepsilon}\{p \cdot G(p)\}$. Note that p^* requires enumerating over only the $\ln(1/\delta)/\varepsilon^2$ samples from Step 2.
(4) If $p^* \cdot G(p^*) \geq \sum_i p_i q_i$, then bundle the items together at price p^*. Otherwise, sell the items separately at prices $\vec{p}$.

H LIST OF CONCENTRATION INEQUALITIES

This section contains a list of concentration inequalities used throughout the article.

THEOREM 11 (CHEBYSHEV'S INEQUALITY). *For any random variable X, $\Pr[|X - \mathbb{E}[X]| \geq k] \leq \frac{var(X)}{k^2}$.*

THEOREM 12 (MULTIPLICATIVE CHERNOFF BOUND). *If $X_1, \ldots, X_n$ are independent random variables supported on $[0, 1]$, and $X := \sum_i X_i$, then for any $\varepsilon \in (0, 1)$, $\Pr[|X - \mathbb{E}[X]| \geq \varepsilon \cdot \mathbb{E}[X]] \leq e^{-\varepsilon^2 \mathbb{E}[X]/3}$.*

THEOREM 13 (MULTIPLICATIVE CHERNOFF BOUND FOR LARGE DEVIATIONS). *If $X_1, \ldots, X_n$ are independent random variables supported on $[0, 1]$, and $X := \sum_i X_i$, then for any $\varepsilon \geq 1$, $\Pr[X - \mathbb{E}[X]| \geq \varepsilon \cdot \mathbb{E}[X]] \leq e^{-\varepsilon \mathbb{E}[X]/3}$.*

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