Symmetries and Optimal Multi-Dimensional Mechanism Design

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We efficiently solve the *optimal multi-dimensional mechanism design problem* for independent additive bidders with arbitrary demands when either the number of bidders is held constant or the number of items is held constant. In the first setting, we need that each bidder's values for the items are sampled from a possibly correlated, *item-symmetric* distribution, allowing different distributions for each bidder. In the second setting, we allow the values of each bidder for the items to be arbitrarily correlated, but assume that the distribution of bidder types is *bidder-symmetric*. These symmetric distributions include i.i.d. distributions, as well as many natural correlated distributions. E.g., an item-symmetric distribution can be obtained by taking an arbitrary distribution, and "forgetting" the names of items; this could arise when different members of a bidder population have various sorts of correlations among the items, but the items are "the same" with respect to a random bidder from the population.

For all $\epsilon > 0$, we obtain a computationally efficient additive ϵ-approximation, when the value distributions are bounded, or a multiplicative $(1-\epsilon)$-approximation when the value distributions are unbounded, but satisfy the Monotone Hazard Rate condition, covering a widely studied class of distributions in Economics. Our running time is polynomial in $\max\{\text{\#items,\#bidders}\}$, and *not* the size of the support of the joint distribution of all bidders' values for all items, which is typically exponential in both the number of items and the number of bidders. Our mechanisms are randomized, explicitly price bundles, and in some cases can also accommodate budget constraints.

Our results are enabled by several new tools and structural properties of Bayesian mechanisms, which we expect to find applications beyond the settings we consider here; indeed, there has already been follow-up research [Cai et al. 2012; Cai and Huang 2012] making use of our tools in both symmetric and non-symmetric settings. In particular, we provide a *symmetrization technique* that turns any truthful mechanism into one that has the same revenue and respects all symmetries in the underlying value distributions. We also prove that item-symmetric mechanisms satisfy a natural *strong-monotonicity property* which, unlike cyclic-monotonicity, can be harnessed algorithmically. Finally, we provide a technique that turns any given ϵ-BIC mechansism (i.e. one where incentive constraints are violated by ϵ) into a truly-BIC mechanism at the cost of $O(\sqrt{\epsilon})$ revenue.

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1. INTRODUCTION

How can a seller auction off a set of items to a group of interested buyers to maximize profit? This problem, dubbed the *optimal mechanism design* problem, has gained

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central importance in mathematical Economics over the past decades. The seller could certainly auction off the items sequentially, using her favorite single-item auction, such as the English auction. But this is not always the best idea, as it is easy to find examples where this approach leaves money on the table.[1] The chief challenge that the auctioneer faces is that the values of the buyers for the items, which determine how much each buyer is willing to pay for each item, is information that is private to the buyers, at least at the onset of the auction. Hence, the mechanism needs to provide the appropriate incentives for the buyers to reveal "just enough" information for the optimal revenue to be extracted.

Viewed as an optimization problem, the optimal mechanism design problem is of a rather intricate kind. First, it is a priori not clear how to evaluate the revenue of an arbitrary mechanism because it is not clear how rational bidders will play. One way to cope with this is to only consider mechanisms where rational bidders are properly incentivized to tell the designer their complete *type*, i.e. how much they would value each possible outcome of the mechanism (i.e. each bundle of items they may end up getting). Such mechanisms can be *Incentive Compatible* (IC), where each bidder's strategy is to report a type and the following *worst-case guarantee* is met: regardless of the reports of the other bidders, it is in the best interest of a bidder to truthfully report her type. Or the mechanism can be *Bayesian Incentive Compatible* (BIC), where it is assumed that the bidders' types come from a known distribution and the following *average-case guarantee* is met: in expectation over the other bidders' types, it is in the best interest of a bidder to truthfully report her type, if the other bidders report truthfully. See Sec 2 for formal definitions. We only note here that, under very weak assumptions, restricting attention to IC/BIC mechanisms in the aforementioned settings of without or with prior information over bidders' types is without loss of generality due to the revelation principle [Nisan et al. 2007].

But even once it is clear how to evaluate the revenue of a given mechanism, it is not necessarily clear what benchmark to compare it against. For example, it is not hard to see that the *social welfare*, i.e. the sum of the values of the buyers for the items they are allocated, is *not* the right benchmark to use, as in general one cannot hope to achieve revenue that is within any constant factor of the optimal social welfare: why would a buyer with a large value for an item pay an equally large price to the auctioneer to get it, if there is no competition for this item? Given the lack of a useful revenue benchmark (i.e. one that upper bounds the revenue that one may hope to achieve but is not too large to allow any reasonable approximation), the task of the mechanism designer can only be specified in generic terms as follows: come up with an IC/BIC auction whose revenue is at least as large as the revenue of any other IC/BIC auction.

Finally, even after restricting the search space to IC/BIC auctions and only comparing to the optimal revenue achievable by any IC/BIC auction, it is still easy to show that it is impossible to guarantee any finite approximation if no prior is known over the bidders' types. Instead, many solutions in the literature adopt a Bayesian viewpoint, assuming that a prior does exist and is known to both the auctioneer and the bidders, and targeting the optimal achievable *expected revenue*. Once the leap to the Bayesian setting is made the goal is typically this: *Design a BIC, possibly randomized,*

[1]A simple example is this: Suppose that an auctioneer is selling a Picasso and a Dali painting and there are two bidders of which one loves Picasso and does not care about Dali and vice versa. Running a separate English auction for each painting will result in small revenue since there is going to be no serious competition for either painting. But bundling the paintings together will induce competition and drive the auctioneer's revenue higher.

mechanism whose expected revenue is optimal among all BIC, possibly randomized, mechanisms.[2]

One of the most celebrated results in this realm is *Myerson's optimal auction* [Myerson 1981], which achieves optimal revenue via an elegant design that spans several important settings. Despite its significance, Myerson's result is limited to the case where bidders are *single-dimensional*. In simple terms, this means that each bidder can be characterized by a single number (unknown to the auctioneer), specifying the value of the bidder per item received. This is quite a strong assumption when the items are heterogeneous, so naturally, after Myerson's work, a large body of research has been devoted to the *multi-dimensional problem*, i.e. the setting where the bidders may have different values for different items/bundles of items. Even though progress has been made in certain restricted settings, it seems that we are far from an optimal mechanism, generalizing Myerson's result; see survey [Manelli and Vincent 2007] and its references for work on this problem in Economics.

Algorithmic Game Theory has also studied this problem, with an extra eye on the computational efficiency of mechanisms. Chawla et al. [2007] study the case of a single (multidimensional) unit-demand bidder with independent values for the items. They propose an elegant reduction of this problem to Myerson's single-dimensional setting, resulting in a mechanism that achieves a constant factor approximation to the optimal revenue among all BIC, possibly randomized [Chawla et al. 2010b], mechanisms. For the same problem, Cai and Daskalakis [2011] recently closed the constant approximation gap against all deterministic mechanisms by obtaining polynomial-time approximation schemes for optimal item-pricing. As for the case of correlated values, it had been known that finding the optimal pricing (i.e. deterministic mechanism) is highly inapproximable by [Briest and Krysta 2007], although no hardness results are known for randomized mechanisms. In the multi-bidder setting, Chawla et al. [2010a], Bhattacharya et al. [2010] and recently Alaei [2011] obtain constant factor approximations in the case of additive bidders or unit-demand bidders and matroidal constraints on the possible allocations.

While our algorithmic understanding of the optimal mechanism design problem is solid, at least as far as constant factor approximations go, there has been virtually no result in designing computationally efficient revenue-optimal mechanisms for multi-dimensional settings, besides the single-bidder result of [Cai and Daskalakis 2011]. In particular, one can argue that the previous approaches [Alaei 2011; Bhattacharya et al. 2010; Chawla et al. 2007, 2010a] are inherently limited to constant factor approximations, as ultimately the revenue of these mechanisms is compared against the optimal revenue in a related single-dimensional setting [Chawla et al. 2007, 2010a], or a convex programming relaxation of the problem [Alaei 2011; Bhattacharya et al. 2010]. Our focus in this work is to fill this important gap in the algorithmic mechanism design literature, i.e. to *obtain computationally efficient near-optimal multi-dimensional mechanisms*, coming ϵ-close to the optimal revenue in polynomial time, for any desired accuracy $\epsilon > 0$. We obtain a Polynomial-Time Approximation Scheme (PTAS) for the following two important cases of the general problem.

The BIC k-items problem. Given as input an arbitrary (possibly correlated) distribution $\mathcal{F}$ over valuation vectors for k items, a demand bound C, and an integer m, the number of bidders, output a BIC mechanism M whose expected revenue is optimal relative to any other, possibly randomized, BIC mechanism, when played by m addi-

[2]In view of the results of [Briest et al. 2010; Chawla et al. 2010b], to achieve optimal, or even near-optimal, revenue in correlated settings, or even i.i.d. multi-item settings, we are forced to explore randomized mechanisms.

tive bidders with demand C whose valuation vectors are sampled independently from $\mathcal{F}$.

The BIC k-bidders problem. Given as input an integer n representing the number of items, k item-symmetric[3] distributions $\mathcal{F}_1, \ldots, \mathcal{F}_k$, and demand bounds $C_1, \ldots, C_k$ (one for each bidder), output a BIC mechanism M whose expected revenue is optimal relative to any other, possibly randomized, BIC mechanism, when played by k additive bidders with demands $C_1, \ldots, C_k$ respectively whose valuation vectors for the n items are sampled independently from $\mathcal{F}_1, \ldots, \mathcal{F}_k$.

In other words, the problems we study are where either the number of bidders is large, but they come from the same population, i.e. each bidder's value vector is sampled from the same, arbitrary, possibly correlated distribution, or the number of items is large, but each bidder's value distribution is item-symmetric (possibly different for each bidder). While these do not capture the problem of Bayesian mechanism design in its complete generality, they certainly represent important special cases of the general problem and indeed the first interesting cases for which computationally efficient near-optimal mechanisms have been obtained. Before stating our main result, it is worth noting that:

• When the number of bidders is large, it does not make sense to expect that the auctioneer has a separate prior distribution for the values of each individual bidder for the items. So our assumption in the k-items problem that the bidders are drawn from the same population of bidders is a realistic one, and—in our opinion—the practically interesting case of the general problem. Indeed, there are hardly any practical examples of auctions using bidder-specific information (think, e.g., eBay, Sotheby's etc.) A reasonable extension of our model would be to assume that bidders come from a constant number of different sub-populations of bidders, and that the auctioneer has a prior for each sub-population. Our results extend to this setting.

• When the number of items is large, it is still hard to imagine that the auctioneer has a distribution for each individual item. In the k-bidders problem, we assume that each bidder's value distribution is item-symmetric. This certainly contains the case where each bidder has i.i.d. values for the items, but there are realistic applications where values are correlated, but still item-symmetric. Consider the following scenario. Suppose that the auctioneer has the same number of Yankees, Red Sox, and White Sox baseball caps to sell. Moreover, suppose that k bidders are sampled from a large population of bidders with the following characteristics: Each bidder in the population is a fan of one of the three teams and has non-zero value for exactly one of the three kinds of caps, but it is unknown to the auctioneer which kind that is and what the value of the bidder for that kind is. (Hence, the values of a random bidder for the caps are certainly non-i.i.d., as if the bidder likes a Red Sox cap then she will equally like another Red Sox cap, but will have zero value for a Yankees cap.) Suppose also that we are willing to make the assumption that all teams have approximately the same number of fans in the population and those fans have statistically the same passion for their team. Then a random bidder's values for the items is drawn from an item-symmetric distribution, or close to one, so we can handle such distributions. In this case too, our techniques still apply if we deviate from the item-symmetric model to models where there is a constant number of types of objects, e.g. caps and jerseys, and symmetries do not permute types, but permute objects within the same type.

[3]A distribution over $\mathbb{R}^n$ is symmetric if, for all $\vec{v} \in \mathbb{R}^n$, the probability it assigns to $\vec{v}$ is equal to the probability it assigns to any permutation of $\vec{v}$.

THEOREM 1.1. *(Additive approximation) For all k, if $\mathcal{F}$ samples values from $[0,1]^k$ there exists a PTAS with additive error ϵ for the BIC k-items problem. For all k, if $\mathcal{F}_i$ samples vectors from $[0,1]^n$, there exists a PTAS with additive error $\epsilon \cdot \max\{C_i\}$ for the BIC k-bidders problem.*

Remark 1.2. Some qualifications on Theorem 1.1 are due.

- The mechanism output by our PTAS is truly BIC, not ϵ-BIC, and there are no extra assumptions necessary to achieve this.
- We make no assumptions about the size of the support of $\mathcal{F}_i$ or $\mathcal{F}$, as the runtime of our algorithms *does not* depend on the size of the support. This is an important distinction between our work and the literature where it is folklore knowledge that if one is willing to pay computational time polynomial in the size of the support of the value distribution, then the optimal mechanism can be easily computed via an LP (see, e.g., [Bhattacharya et al. 2010; Briest et al. 2010; Dobzinski et al. 2011]). However, exponential size supports are easy to observe. Take, e.g., our k-bidders problem and assume that every bidder's value for each of the items is i.i.d. uniform in $\{\$5, \$10\}$. The naïve LP based approach would result in time polynomial in 2^n, while our solution needs time polynomial in n.
- If we are willing to replace BIC by ϵ-BIC (or ϵ-IC) in Thm 1.1 and compare our revenue to the best revenue achievable by any BIC (or respectively IC) mechanism, then we can also accommodate budget constraints for our bidders. The only step of our algorithm that does not respect budgets or does not apply to IC mechanisms is the ϵ-BIC to BIC reduction (Thm 3.3). For space considerations, we restrict our attention to BIC throughout the main body of the paper and prove the related claims for IC in Appendix K of the full version [Daskalakis and Weinberg 2011].
- If the value distributions are discrete and every marginal has constant-size support, then our algorithms achieve *exactly optimal revenue* in polynomial time, even though the support of such a distribution may well be exponential. For instance, in the example given in the second bullet our algorithm obtains exactly optimal revenue in time polynomial in n. In these cases, we can find optimal truly BIC or truly IC mechanisms that also accommodate budget constraints.
- The mechanisms produced by our techniques satisfy the demand bounds of each bidder in a strong sense (and not in expectation). Moreover, the user of our theorem is free to choose whether they want to satisfy *ex-interim individual rationality*, that the expected value of a bidder for the received bundle of items is larger than the expected price she pays, or *ex-post individual rationality*, where this constraint is true with probability 1 (and not just in expectation). We focus the main presentation on producing mechanisms that are ex-interim IR. In Appendix D of the full version we explain the required modification for producing ex-post IR mechanisms *without any loss in revenue.*
- The assumption that $\mathcal{F}_i, \mathcal{F}$ sample from $[0,1]^n$ as opposed to some other bounded set is w.l.o.g. and previous work has made the same assumption on the input distributions [Hartline and Lucier 2010; Hartline et al. 2011].
- The point of the additive error of $\epsilon \cdot \max\{C_i\}$ is not to set ϵ so small that it cancels out the factor of $\max\{C_i\}$, but rather to accept the factor of $\max\{C_i\}$ as lost revenue. Notice that, if each v_{ij} has expectation that is bounded away from 0, the maximum attainable revenue scales with $\max\{C_i\}$. In this case, an additive error of $\epsilon \cdot \max\{C_i\}$ corresponds to a $(1 - O(\epsilon))$-factor approximation to the optimal revenue. As another example, when each marginal satisfies the Monotone Hazard Rate condition, an additive error of $\epsilon \cdot \max\{C_i\}$ as obtained by Theorem 1.1 can be readily used to obtain

a multiplicative $(1 - O(\epsilon))$-factor approximation. (For details see the proof of Corollary 1.3 below in Appendix J of the full version.)

One might prefer to assume that the value distributions are not upper bounded, but satisfy some tail condition, such as the Monotone Hazard Rate (MHR) condition. (The class of MHR distributions includes such familiar distributions as the Normal, Exponential and Uniform distributions, and is widely used in Mechanism Design. See Appendix J of the full version for a precise definition.) Using techniques from [Cai and Daskalakis 2011], we can extend our results to MHR distributions. All the relevant remarks still apply.

COROLLARY 1.3. *(Multiplicative approximation for MHR distributions) For all k, if the k marginals of $\mathcal{F}$ all satisfy the MHR condition, there exists a PTAS obtaining at least a $(1-\epsilon)$-fraction of the optimal revenue for the BIC k-items problem (whose runtime does not depend on $\mathcal{F}$ or C). Likewise, for all k, if every marginal of $\mathcal{F}_i$ is MHR for all i, there exists a PTAS obtaining at least a $(1-\epsilon)$-fraction of the optimal revenue for the BIC k-bidders problem (whose runtime does not depend on $\mathcal{F}_i$ or C_i).*

1.1. Tools and Techniques

Our main result (Theorem 1.1) is enabled by establishing several new tools and structural results for multi-dimensional mechanism design, which are of applicability far beyond the symmetric auction-settings we focus on. (See, e.g., next section.)

Our first contribution is a *symmetrization technique*, stated informally below and formally as Theorem 3.1. This technique enables one to save on the description complexity of the optimal mechanism by exploiting any kind of symmetry in the underlying joint distribution $\mathcal{D}$ from which the bidders' values for the items are sampled (no matter how pervasive that symmetry may be in $\mathcal{D}$). In the highly symmetric settings of Theorem 1.1, our symmetrization technique bears witness to a succinct (i.e. poly-size) description of the revenue-optimal mechanism.

INFORMAL THEOREM 1. *For all distributions $\mathcal{D}$ and all mechanisms M, M can be symmetrized into a new mechanism M', such that M' obtains the same expected revenue as M, retains the truthfulness of M (such as IC, BIC, ϵ-IC, or ϵ-BIC.), and satisfies all symmetries that $\mathcal{D}$ has.*

In addition, we prove a structural property of item-symmetric mechanisms which we call *strong-monotonicity*.[4] This structure enables one to efficiently find an optimal mechanism, as well as efficiently implement the ϵ-BIC to BIC reduction (described below) in item-symmetric settings. The theorem is stated informally below, and formally as Theorem 3.2.

INFORMAL THEOREM 2. *Every BIC, item-symmetric mechanism for an item-symmetric distribution $\mathcal{D}$ is strongly-monotone. That is, each bidder receives a higher valued item more probably than a lower valued item, where the probability is with respect to the randomness in the other bidders' values and the mechanism.*

Finally, we provide an *ϵ-BIC to BIC reduction* that turns an ϵ-BIC mechanism into a BIC mechanism, while approximately preserving the revenue of the mechanism. We note here that the rounding procedure attributed to Nisan in [Chawla et al. 2007] doesn't suffice in multi-bidder scenarios. Simply put, incentivizing a single bidder to purchase a more expensive option affects other bidders as well, and this procedure may completely destroy any truthfulness the original mechanism had. Therefore, a

[4]The term strong monotonicity has also been used in other contexts. There is no connection between our use and other usages.

new technique that considers the interaction between the bidders is necessary. We prove the following theorem, stated informally below and formally as Theorem 3.3.

INFORMAL THEOREM 3. *Any ϵ-BIC mechanism can be converted to a BIC mechanism at a cost of $O(\sqrt{\epsilon})$ revenue. Furthermore, the conversion runs in polynomial time for inputs to the k-bidders or k-items problem.*

1.2. Related Work

As discussed earlier, several polynomial-time constant-factor approximation algorithms have been recently discovered for multi-dimensional settings [Bhattacharya et al. 2010; Chawla et al. 2010a; Alaei 2011]. Our work differs from these in that we obtain nearly-optimal revenue, i.e. a PTAS.

Other recent results have also obtained a PTAS for settings related to ours [Cai and Daskalakis 2011; Cai and Huang 2012; Cai et al. 2012]. However, the settings covered here are more general in that we allow arbitrary demand bounds and possibly budget constraints. Instead, the results in [Cai and Daskalakis 2011] apply to a single, unit-demand bidder with no budget constraints, while those of [Cai and Huang 2012] and [Cai et al. 2012] apply to additive bidders with no demand or budget constraints.

In terms of techniques, these papers provide tools that interface nicely with the tools developed here. Specifically, Corollary 1.3 in our paper makes use of an extreme value theorem from [Cai and Daskalakis 2011]. In the other direction, our ϵ-BIC to BIC reduction (Theorem 3.3) is used in both [Cai and Huang 2012] and [Cai et al. 2012]. In addition, [Cai et al. 2012] makes use of our structural theorems (Theorems 3.1 and Theorem 3.2) as well. In other words, the tools provided in this paper are orthogonal to the tools developed in [Cai and Daskalakis 2011; Cai and Huang 2012; Cai et al. 2012], and our results here are a necessary prerequisite for comparable theorems in [Cai and Huang 2012; Cai et al. 2012].

1.3. Structure

The rest of the paper is organized as follows: Sec 2 provides a few standard definitions from Mechanism Design. Sec 3 gives an overview of our proof of Thm 1.1, explaining the different components that get into our proof and guiding through the rest of the paper. The rest of the main body and the appendix of the full version [Daskalakis and Weinberg 2011] provide all technical details. Appendix J of the full version provides the proof of Corollary 1.3.

2. PRELIMINARIES AND NOTATION

We assume that the seller has a single copy of n (heterogeneous) items that she wishes to auction to m bidders. Each bidder i has some non-negative value for item j which we denote v_{ij}. We can think of bidder i's *type* as an n-dimensional vector $\vec{v}_i$, and denote the entire profile of bidders as $\vec{v}$, or sometimes $(\vec{v}_i\ ;\ \vec{v}_{-i})$ if we want to emphasize its decomposition to the type $\vec{v}_i$ of bidder i and the joint profile $\vec{v}_{-i}$ of all other bidders. We denote by $\mathcal{D}$ the distribution from which $\vec{v}$ is sampled. We also denote by $\mathcal{D}_i$ the distribution of types for bidder i, and by $\mathcal{D}_{-i}$ the distribution of types for every bidder except i. The *value of an additive bidder* with demand C for any subset of items is the sum of her values for her favorite C items in the subset; that is, we assume bidders are *additive up to their demand*.

Since we are shooting for BIC/IC mechanisms, we will only consider (direct revelation) mechanisms where each bidder's strategy is to report a type. When the reported bidder types are $\vec{v}$, we denote the (possibly randomized) *outcome of mechanism M* as $M(\vec{v})$. The outcome can be summarized in: the *expected price* charged to each bidder (denoted $p_i(\vec{v})$), and a collection of *marginal probabilities* $\vec{\phi}(\vec{v}) = (\phi_{ij}(\vec{v}))_{ij}$, where $\phi_{ij}(\vec{v})$ denotes the marginal probability that bidder i receives item j.

A collection of marginal probabilities $\vec{\phi}(\vec{v}) := (\phi_{ij}(\vec{v}))_{ij}$ is *feasible* iff there exists a consistent with them joint distribution over allocations of items to bidders so that in addition, with probability 1, no item is allocated to more than one bidder, and no bidder receives more items than her demand. A straightforward application of the Birkhoff-von Neumann theorem [Johnson et al. 1960] reveals that a sufficient condition for the above to hold is that *in expectation* no item is given more than once, and all bidders receive an *expected number of items* less than or equal to their demand. Note that this sufficient condition is expressible in terms of the ϕ_{ij}'s only. Moreover, under the same conditions, we can efficiently sample a joint distribution with the desired ϕ_{ij}'s. (See Appendix C of the full version [Daskalakis and Weinberg 2011] for details.)

The outcome of mechanism M restricted to bidder i on input $\vec{v}$ is denoted $M_i(\vec{v}) = (\vec{\phi}_i(\vec{v}), p_i(\vec{v}))$. Assuming that bidder i is additive (up to her demand) and risk-neutral and that the mechanism is feasible (so in particular it does not violate the bidder's demand constraint) the *value* of bidder i for outcome $M_i(\vec{w})$ is just (her expected value) $\vec{v}_i \cdot \vec{\phi}_i(\vec{w})$, while the bidder's *utility* for the same outcome is $U(\vec{v}_i, M_i(\vec{w})) := \vec{v}_i \cdot \vec{\phi}_i(\vec{w}) - p_i(\vec{w})$. Such bidders subtracting price from expected value are called *quasi-linear.* Moreover, for a given value vector $\vec{v}_i$ for bidder i, we write: $\pi_{ij}(\vec{v}_i) = \mathbb{E}_{\vec{v}_{-i} \sim \mathcal{D}_{-i}}[\phi_{ij}(\vec{v}_i \ ; \ \vec{v}_{-i})]$.

We proceed to formally define incentive compatibility of mechanisms in our notation:

Definition 2.1. (BIC/ϵ-BIC/IC/ϵ-IC Mechanism) A mechanism M is called ϵ-BIC iff the following inequality holds for all $i, \vec{v}_i, \vec{w}_i$:

$$\mathbb{E}_{\vec{v}_{-i} \sim \mathcal{D}_{-i}} \left[U(\vec{v}_i, M_i(\vec{v}))\right] \geq \mathbb{E}_{\vec{v}_{-i} \sim \mathcal{D}_{-i}} \left[U(\vec{v}_i, M_i(\vec{w}_i \ ; \ \vec{v}_{-i}))\right] - \epsilon v_{\max} \cdot \sum_j \pi_{ij}(\vec{w}_i),$$

where $v_{\max}$ is the maximum possible value of any bidder for any item in the support of the value distribution. In other words, M is ϵ-BIC iff when a bidder lies by reporting $\vec{w}_i$ instead of $\vec{v}_i$, they do not expect to gain more than $\epsilon v_{\max}$ times the expected number of items that $\vec{w}_i$ receives. Similarly, M is called ϵ-IC iff for all i, $\vec{v}_i, \vec{w}_i, \vec{v}_{-i}$: $U(\vec{v}_i, M_i(\vec{v})) \geq U(\vec{v}_i, M_i(\vec{w}_i \ ; \ \vec{v}_{-i})) - \epsilon v_{\max} \cdot \sum_j \phi_{ij}(\vec{w}_i \ ; \ \vec{v}_{-i})$. A mechanism is called BIC iff it is 0-BIC and IC iff it is 0-IC.[5]

In our proof of Thm 1.1 throughout this paper we assume that $v_{\max} = 1$. If $v_{\max} < 1$, we can scale the value distribution so that this condition is satisfied. We also define individual rationality of BIC/ϵ-BIC mechanisms:

Definition 2.2. A BIC/ϵ-BIC mechanism M is called *ex-interim individually rational (ex-interim IR)* iff for all i, $\vec{v}_i$:

$$\mathbb{E}_{\vec{v}_{-i} \sim \mathcal{D}_{-i}} \left[U(\vec{v}_i, M_i(\vec{v}))\right] \geq 0.$$

It is called *ex-post individually rational (ex-post IR)* iff for all i, $\vec{v}_i$ and $\vec{v}_{-i}$, $U(\vec{v}_i, M_i(\vec{v})) \geq 0$ with probability 1 (over the randomness in the mechanism).

While we focus the main presentation on obtaining ex-interim IR mechanisms, in Appendix D of the full version [Daskalakis and Weinberg 2011] we describe how without any loss in revenue we can turn these mechanisms into ex-post IR.

[5]Any feasible mechanism that we call ϵ-BIC, respectively ϵ-IC, by our definition is certainly an $\epsilon \cdot \max\{C_i\}$-BIC, respectively $\epsilon \cdot \max\{C_i\}$-IC, mechanism by the more standard definition, which omits the factors $\sum_j \pi_{ij}(\vec{w}_i)$, respectively $\sum_j \phi_{ij}(\vec{w}_i \ ; \ \vec{v}_{-i})$, from the incentive error. We only include these factors here for convenience.

For a mechanism M, we denote by $R^M(\mathcal{D})$ the expected revenue of the mechanism when bidders sampled from $\mathcal{D}$ play M truthfully. We also let $R^{OPT}(\mathcal{D})$ (resp. $R_\epsilon^{OPT}(\mathcal{D})$) denote the maximum possible expected revenue attainable by any BIC (resp. ϵ-BIC) mechanism when bidders are sampled from $\mathcal{D}$ and play truthfully. For all cases we consider, these terms are well-defined.

We state and prove our results assuming that we can exactly sample from all input distributions efficiently and exactly evaluate their cumulative distribution functions. Our results still hold *even if we only have oracle access to sample from the input distributions*, as this is sufficient for us to approximately evaluate the cumulative functions to within the right accuracy in polynomial time (by making use of our symmetry and discretization tools, described in the next section). The approximation error on evaluating the cumulative functions is absorbed into loss in revenue. See discussion in Appendix A of the full version.

Finally, we denote by S_m, S_n the symmetric groups over the sets $[m] := \{1, \dots, m\}$ and $[n]$ respectively. Moreover, for $\sigma = (\sigma_1, \sigma_2) \in S_m \times S_n$, we assume that σ maps element $(i, j) \in [m] \times [n]$ to $\sigma(i, j) := (\sigma_1(i), \sigma_2(j))$. We extend this definition to map a value vector $\vec{v} = (v_{ij})_{i \in [m], j \in [n]}$ to the vector $\vec{w}$ such that $\vec{w}_{\sigma(i,j)} = \vec{v}_{ij}$, for all i, j. Likewise, if $\mathcal{D}$ is a value distribution, $\sigma(\mathcal{D})$ is the distribution that first samples $\vec{v}$ from $\mathcal{D}$ and then outputs $\sigma(\vec{v})$.

3. OVERVIEW OF OUR APPROACH

A Naïve LP Formulation. Let $\mathcal{D}$ be the distribution of all bidders' values for all items (supported on a subset of $\mathbb{R}^{m \times n}$, where m is the number of bidders and n is the number of items). For a mechanism design problem with unit-demand bidders whose values are distributed according to $\mathcal{D}$, it is folklore knowledge how to write a linear programming formulation of size polynomial in $|\text{supp}(\mathcal{D})|$ optimizing revenue. The relaxation keeps track of the (marginal) probability $\phi_{ij}(\vec{v}) \in [0, 1]$ that item j is given to bidder i if the bidders' values are $\vec{v}$, and enforces feasibility constraints (no item is given more than once in expectation, no bidder gets more than one item in expectation), incentive compatibility constraints (in expectation over the other bidders' values for the items, no bidder has incentive to misreport her values for the items, if the other bidders don't), while optimizing the expected revenue of the mechanism. Notice that all constraints and the objective function can be written in terms of the marginals ϕ_{ij}. Moreover, using the Birkhoff-von Neumann decomposition theorem, it is possible to convert the solution of this LP to a mechanism that has the same revenue and satisfies the feasibility constraints strongly (i.e. not just in expectation, but also with probability 1). We give the details of the linear program in Appendix B of the full version [Daskalakis and Weinberg 2011], and also describe how to generalize this LP to incorporate publicly known demand and budget constraints.

Despite its general applicability, the naïve LP formulation has a major drawback in that $|\text{supp}(\mathcal{D})|$ could in general be infinite, and when it is finite it is usually exponential in both m and n. For the settings we consider, this is always the case. For example, in the very simple setting where $\mathcal{D}$ samples each value i.i.d. uniformly from $\{\$5, \$10\}$, the support of the distribution becomes $2^{m \times n}$. Such support size is obviously prohibitive if we plan to employ the naïve LP formulation to optimize revenue.

A Comparison to Myerson's Setting. *What enables succinct and computationally efficient mechanisms in the single-item setting of Myerson?* Indeed, the curse of dimensionality discussed above arises even when there is a single item to sell; e.g., if every bidder's distribution has support 2 and the bidders are independent, then the number of different bidder profiles is already 2^m. What drives Myerson's result is the realization that there is structure in a BIC mechanism coming in the form of *monotonicity*:

for all i, for all $v_{i1} \geq v'_{i1}$: $\mathbb{E}_{\vec{v}_{-i}}(\phi_{i1}(v_{i1}\ ;\ \vec{v}_{-i})) \geq \mathbb{E}_{\vec{v}_{-i}}(\phi_{i1}(v'_{i1}\ ;\ \vec{v}_{-i}))$, i.e. the expected probability that bidder i gets the single item for sale in the auction increases with the value of bidder i, where the expectation is taken over the other bidders' values. Unfortunately, such a crisp monotonicity property of BIC mechanisms fails to hold if there are multiple items, and even if it were present it would still not be sufficient in itself to reduce the naïve LP to a manageable size.

So what next? We argued earlier that the symmetric distributions considered in the BIC k-items and the BIC k-bidders problems are very natural cases of the general optimal mechanism design problem. We argue next that they are natural for another reason: they enable enough structure for (i) the optimal mechanism to have small description complexity, instead of being an unusable, exponentially long list of what the mechanism ought to do for every input value vector $\vec{v}$; and (ii) the succinct solution to be efficiently computable, bypassing the exponentially large naïve LP. Our structural results are discussed in the following paragraphs. The first is enabled by exploiting randomization to transfer symmetries from the value distribution to the optimal mechanism. The second is enabled by proving a *strong-monotonicity property* of all BIC mechanisms. Our notion of monotonicity is more powerful than the notion of *cyclic-monotonicity*, which holds more generally but can't be exploited algorithmically. Together our structural results bring to light how the item- and bidder-symmetric settings are mathematically more elegant than general settings with no apparent structure.

Structural Result 1: *The Interplay Between Symmetries and Randomization.* Since the inception of Game Theory scientists were interested in the implications of symmetries in the structure of equilibria [Gale et al. 1950; Brown and Neumann 1950; Nash 1951]. In his seminal paper [Nash 1951], Nash showed a rather interesting structural result, informally reading as follows: "If a game has any symmetry, there exists a Nash equilibrium satisfying that symmetry." Indeed, something even more powerful is true: "There always exists a Nash equilibrium that simultaneously satisfies all symmetries that the game may have."

Inspired by Nash's symmetry result, albeit in our different setting, we show a similar structural property of **randomized** mechanisms.[6] Our structural result is rather general, applying to settings beyond those addressed in Thm 1.1, and even beyond MHR or regular distributions. The following theorem holds for *any* (arbitrarily correlated) joint distribution $\mathcal{D}$.

THEOREM 3.1. *Let $\mathcal{D}$ be the distribution of bidders' values for the items (supported on a subset of $\mathbb{R}^{m\times n}$). Let also $\mathcal{S} \subseteq S_m \times S_n$ be an arbitrary set such that $\mathcal{D} \equiv \sigma(\mathcal{D})$, for all $\sigma \in \mathcal{S}$; that is, assume that $\mathcal{D}$ is invariant under all permutations in $\mathcal{S}$. Then any BIC mechanism M can be symmetrized into a mechanism M' that respects all symmetries in $\mathcal{S}$ without any loss in revenue. I.E. for all bid vectors $\overrightarrow{v}$ the behavior of M' under $\overrightarrow{v}$ and $\sigma(\overrightarrow{v})$ is identical (up to permutation by σ) for all $\sigma \in \mathcal{S}$. The same result holds if we replace BIC with ϵ-BIC, IC, or ϵ-IC.*

While we postpone further discussion of this theorem and what it means for M to behave "identically" to Sec 4, we give a quick example to illustrate the symmetries that randomization enables in the optimal mechanism. Consider a single unit-demand bidder and two items. Her value for each item is drawn i.i.d. from the uniform distribution on $\{4, 5\}$. It is easy to see that the only optimal deterministic mechanism assigns price 4 to one item and 5 to the other. However, there is an optimal randomized

[6]We emphasize 'randomized', since none of the symmetries we describe holds for deterministic optimal mechanisms.

mechanism that offers each item at price $4\frac{1}{2}$, and the uniform lottery ($1/2$ chance of getting item 1, $1/2$ chance of getting item 2) at price 4. While item 1 and item 2 need to be priced differently in the deterministic mechanism to achieve optimal revenue, they can be treated identically in the optimal randomized mechanism. Thm 3.1 applies in an extremely general setting: distributions can be continuous with arbitrary support and correlation, bidders can have budgets and demands, we could be maximizing social welfare instead of revenue, etc.

Structural Result 2: *Strong-Monotonicity.* Even though the naïve LP formulation is not computationally efficient, Thm 3.1 certifies the existence of a compact solution for the cases we consider. This solution lies in the subspace of $\mathbb{R}^{m\times n}$ spanned by the symmetries induced by $\mathcal{D}$. Still Thm 3.1 does not inform us how to locate such a symmetric optimal solution. Indeed, the symmetry of the optimal solution is not a priori capable in itself to decrease the size of our naïve LP to a manageable one. For this purpose we establish a strong monotonicity property of item-symmetric BIC mechanisms (an item-symmetric mechanism is one that respects every item symmetry; see Sec 4 for a definition).

THEOREM 3.2. *If $\mathcal{D}$ is item-symmetric, every item-symmetric BIC mechanism is* strongly monotone*:*

$$\text{for all bidders } i\text{, and items } j, j'\text{: } v_{ij} \geq v_{ij'} \implies \mathbb{E}_{\vec{v}_{-i}}(\phi_{ij}(\vec{v})) \geq \mathbb{E}_{\vec{v}_{-i}}(\phi_{ij'}(\vec{v})).$$

i.e., if i likes item j more than item j', her expected probability (over the other bidders' values) of getting item j is higher. We give an analogous monotonicity property of IC mechanisms in Appendix K of the full version [Daskalakis and Weinberg 2011].

From ϵ- to truly-BIC. Exploiting the aforementioned structural theorems we are able to efficiently compute *exactly optimal mechanisms* for value distributions $\mathcal{D}$ whose marginals on every item have constant-size support. ($\mathcal{D}$ itself can easily have exponentially-large support if, e.g., the items are independent.) To adapt our solution to continuous distributions or distributions whose marginals have non-constant support, we attempt the obvious rounding idea that changes $\mathcal{D}$ by rounding all values sampled from $\mathcal{D}$ down to the nearest multiple of some accuracy ϵ, and solves the problem on the resulting distribution $\mathcal{D}_\epsilon$. While we can argue that the optimal BIC mechanism for $\mathcal{D}_\epsilon$ is also approximately optimal for $\mathcal{D}$, we need to also give up on the incentive compatibility constraints, resulting in an approximately-BIC mechanism where bidders may have an incentive to misreport their values, but the incentive to misreport is always smaller than some function of ϵ. A natural approach to eliminate those incentives to misreport is to appropriately discount the prices for items or bundles of items charged by the mechanism computed for $\mathcal{D}_\epsilon$, generalizing the single-bidder rounding idea attributed to Nisan in [Chawla et al. 2007]. Unfortunately, this approach fails to work in the multi-bidder settings, destroying both the revenue and truthfulness. Simply put, even though the discounts encourage bidders to choose more expensive options, these choices affect not only the price they pay us, but the prices paid by other bidders as well as the incentives of other bidders. Once we start rounding the prices, we could completely destroy any truthfulness the original mechanism had, leaving us with no guarantees on revenue.

Our approach is entirely different, comprising a non-trivial extension of the main technique of [Hartline et al. 2011]. We run simultaneous VCG auctions, one per bidder, where each bidder competes with make-believe replicas of himself, whose values are drawn from the same value distribution where his own values are drawn from. The goods for sale in these per-bidder VCG auctions are replicas of the bidder drawn from the modified distribution $\mathcal{D}_\epsilon$. These replicas are called surrogates. The intention is that the surrogates bought by the bidders in the per-bidder VCG auction will compete with each other in the optimal mechanism M designed for the modified distribution $\mathcal{D}_\epsilon$. Accordingly, the value of a bidder for a surrogate is the expected value of the bidder for the items that the surrogate is expected to win in M *minus* the price the surrogate is expected to pay. This is exactly our approach, except we modify mechanism M to discount all these prices by a factor of $1 - O(\epsilon)$. This is necessary to argue that bidders choose to purchase a surrogate with high probability, as otherwise we cannot hope to make good revenue. There are several technical ideas coming into the design and analysis of our two-phase auction (surrogate sale, surrogate competition). We describe these ideas in detail in Sec 6.2, emphasizing several important complications departing from the setting of [Hartline et al. 2011]. Importantly, the approach of [Hartline et al. 2011] is brute force in $|\text{supp}(\mathcal{D}_i)|$. While this is okay for k-items, this takes exponential time for k-bidders. In addition to showing the following theorem, we show how to make use of Thm 3.2 to get the reduction to run in polynomial time in both settings.

THEOREM 3.3. *Consider a generic setting with n items and m bidders who are additive up to some capacity. Let $\mathcal{D} := \times_i \mathcal{D}_i$ and $\mathcal{D}' := \times_i \mathcal{D}'_i$ be product distributions, sampling every bidder independently from $[0,1]^n$. Suppose that, for all i, $\mathcal{D}_i$ and $\mathcal{D}'_i$ can be coupled so that, with probability 1, a value vector $\vec{v}_i$ sampled from $\mathcal{D}_i$ and a value vector $\vec{v}'_i$ sampled from $\mathcal{D}'_i$ satisfy that $v_{ij} \geq v'_{ij} \geq v_{ij} - \delta, \forall j$. Then, for all $\eta, \epsilon > 0$, any ϵ-BIC mechanism M_1 for $\mathcal{D}'$ can be transformed into a BIC mechanism M_2 for $\mathcal{D}$ such that $R^{M_2}(\mathcal{D}) \geq (1-\eta) \cdot R^{M_1}(\mathcal{D}') - \frac{\epsilon + 4\delta}{\eta} T$, where T is the maximum number of items that can be awarded by a feasible mechanism. Furthermore, if $\mathcal{D}$ and $\mathcal{D}'$ are both valid inputs to the BIC k-bidders or k-items problem, the transformation runs in time polynomial in n and m. Moreover, for the BIC k-items problem, $T = k$ and, for the BIC k-bidders problem, $T \leq k \max_i C_i$, where C_i is the demand of bidder i.*

Figure 1 shows how the various components discussed above interact with each other to prove Theorem 1.1. The modifications required to establish Corollary 1.3 are provided in Appendix J of the full version [Daskalakis and Weinberg 2011].

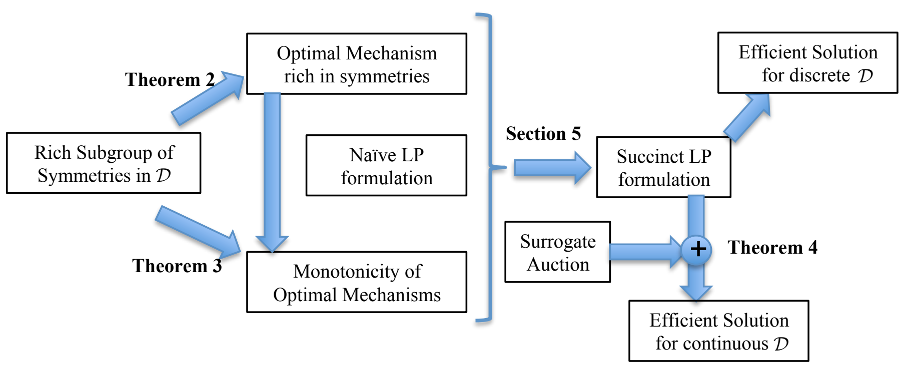

Fig. 1. Our Proof Structure

4. SYMMETRY THEOREM

We provide the necessary definitions to understand exactly what our symmetry result is claiming.

Definition 4.1. (Symmetry in a Distribution) We say that a distribution $\mathcal{D}$ has symmetry $\sigma \in S_m \times S_n$ if, for all $\vec{v} \in \mathbb{R}^{m\times n}$, $Pr_{\mathcal{D}}[\vec{v}] = Pr_{\mathcal{D}}[\sigma(\vec{v})]$. We also write $\mathcal{D} \equiv \sigma(\mathcal{D})$.

Definition 4.2. (Symmetry in a Mechanism) We say that a mechanism respects symmetry $\sigma \in S_m \times S_n$ if, for all $\vec{v} \in \mathbb{R}^{m\times n}$, $M(\sigma(\vec{v})) = \sigma(M(\vec{v}))$.

Definition 4.3. (Permutation of a Mechanism) For any $\sigma \in S_m \times S_n$, and any mechanism M, define the mechanism $\sigma(M)$ as $[\sigma(M)](\vec{v}) = \sigma(M(\sigma^{-1}(\vec{v})))$.

We proceed to state our symmetry theorem; its proof can be found in Appendix E of the full version [Daskalakis and Weinberg 2011].

Theorem 3.1. (Restated from Sec 3) *For all $\mathcal{D}$, any BIC (respectively IC, ϵ-IC, ϵ-BIC) mechanism M can be symmetrized into a BIC (respectively IC, ϵ-IC, ϵ-BIC) mechanism M' such that, for all $\sigma \in S_m \times S_n$, if $\mathcal{D}$ has symmetry σ, M' respects σ, and $R^M(\mathcal{D}) = R^{M'}(\mathcal{D})$.*

We note that Thm 3.1 is an extremely general theorem. $\mathcal{D}$ can have arbitrary correlation between bidders or items, and can be continuous. One might wonder why we had to restrict our theorem to symmetries in $S_m \times S_n$ and not arbitary permutations of the set $[m] \times [n]$. In fact, after reading through our proof, one can see that the same inequalities that make symmetries in $S_m \times S_n$ work also hold for symmetries in $S_{[m]\times[n]}$. However, the mechanism resulting from our proof is not a feasible one, since our transformation can violate feasibility constraints for symmetries $\sigma \notin S_m \times S_n$.

We also emphasize a subtle property of our symmetrizing transformation: the transformation takes as input a set of symmetries satisfied by $\mathcal{D}$ and a mechanism, and symmetrizes the mechanism so that it satisfies all symmetries in the given set of symmetries. Our transformation *does not work if the given set of symmetries is not a subgroup.* Luckily the maximal subset of symmetries in $S_m \times S_n$ satisfied by a value distribution is *always a subgroup,* and this enables our result.

5. OPTIMAL SYMMETRIC MECHANISMS FOR DISCRETE DISTRIBUTIONS

In this section, we solve the following problem: “Given a distribution $\mathcal{D}$ with constant support per dimension and a subgroup of symmetries $S \subseteq S_m \times S_n$ satisfied by $\mathcal{D}$, find a BIC mechanism M that respects all symmetries in S and maximizes $R^M(\mathcal{D})$.” By Thm 3.1, such M will in fact be optimal with respect to all mechanisms. Intuitively, optimizing over symmetric mechanisms should require less work than over general mechanisms, since we should be able to exploit the symmetry constraints in our optimization. Indeed, suppose that every bidder can report c different values for each item, where c is some absolute constant. Then the naïve LP of Section 3/Appendix B of the full version [Daskalakis and Weinberg 2011] has size polynomial in c^{mn}, where m, n are the number of bidders and items respectively. In Sec 5.1 we give a simple observation that reduces the number of variables and constraints of this LP for any given S. This observation in itself is sufficient to provide an efficient solution to the BIC k-items problem (in our constant-support-per-dimension setting), but falls short from solving the BIC k-bidders problem. For the latter, we need another structural consequence of symmetry, which comes in the form of a *strong-monotonicity* property satisfied by all symmetric BIC mechanisms. Strong-monotonicity and symmetry together enable us to obtain an efficient solution to the BIC k-bidders problem in Sec 5.2 (still for our constant-support-per-dimension setting). We explicitly write the LPs that find the optimal BIC mechanism. Simply tacking on a $-\epsilon \cdot \sum_j \pi_{ij}(\vec{w}_i)$ to the correct side of the BIC constraints yields an LP to find the optimal ϵ-BIC mechanism for any ϵ. Efficiently solving non-constant/infinite supports per dimension is postponed to Sec 6.

5.1. Reducing the LP size for any Subgroup of Symmetries, and Solving the discrete BIC k-items Problem

We provide an LP formulation that works for any S. Our LP is the same as the naïve LP of Figure 2 (Appendix B of the full version), except we drop some constraints of that LP and modify its objective function as follows. Since our mechanism needs to respect every symmetry in S, it must satisfy

$$\phi_{ij}(\vec{v}) = \phi_{\sigma(i,j)}(\sigma(\vec{v})), \forall i, j, \vec{v}, \sigma \in S \text{ and } p_i(\vec{v}) = p_{\sigma(i)}(\sigma(\vec{v})), \forall i, \vec{v}, \sigma \in S.$$

Therefore, if we define an equivalence relation by saying that $\vec{v} \sim_S \sigma(\vec{v})$, for all $\sigma \in S$, we only need to keep variables $\phi_{ij}(\vec{v}), p_i(\vec{v})$ for a single representative from each equivalence class. We can then use the above equalities to substitute for all non-representative $\vec{v}$'s into the naïve LP. This will cause some constraints to become duplicates. If we let E denote the set of representatives, then we are left with the LP of Figure 3 in Appendix G of the full version, after removing duplicates. In parentheses at the end of each type of variable/constraint is the number of distinct variables/constraints of that type.

LEMMA 5.1. *The LP of Fig. 3 in Appendix G of the full version has polynomial size for the BIC k-items problem, if the support of every marginal of the value distribution is an absolute constant.*

5.2. Strong-Monotonicity, and Solving the discrete BIC k-bidders Problem

Unfortunately, the reduction of the previous section is *not* strong enough to make the LP polynomial in the number of items n, even if S contains all item permutations and there is a constant number of bidders. This is because a bidder can deviate to an exponential number c^n of types, and our LP needs to maintain an exponential number of BIC constraints. To remedy this, we prove that every item-symmetric BIC mechanism for bidders sampled from an item-symmetric distribution satisfies a natural monotonicity property:

Definition 5.2. (Strong-Monotonicity of a BIC mechanism) A BIC or ϵ-BIC mechanism is said to be *strongly monotone* if for all i, j, j', $v_{ij} \geq v_{ij'} \Rightarrow \pi_{ij}(\vec{v_i}) \geq \pi_{ij'}(\vec{v_i})$. That is, bidders expect to receive their favorite items more often.

Theorem 3.2. (Restated from Sec 3) *If M is BIC and $\mathcal{D}$ and M are both item-symmetric, then M is strongly monotone. If M is ϵ-BIC and $\mathcal{D}$ and M are both item-symmetric, there exists a ϵ-BIC mechanism of the same expected revenue that is strongly monotone.*

The proof of Thm 3.2 can be found in Appendix F of the full version and is a direct consequence of 2-cycle monotonicity and item symmetry. We note again that our notion of strong-monotonicity is different than the notion of cyclic-monotonicity that holds more generally, but is not sufficient for obtaining efficient algorithms. Instead strong-monotonicity suffices due to the following:

OBSERVATION 1. *When playing an item-symmetric, strongly monotone BIC mechanism, bidder $\vec{v_i}$ has no incentive to report any $\vec{w_i}$ with $w_{ij} > w_{ij'}$ unless $v_{ij} \geq v_{ij'}$.*

LEMMA 5.3. *There exists a polynomial-size LP for the BIC k-bidders problem, if the support of every marginal of the value distribution is an absolute constant. The LP is shown in Fig. 4 of Appendix G of the full version.*

We note that Theorem 3.2 is also true for IC and ϵ-IC mechanisms with the appropriate definition of strong-monotonicity. The definition and proof are given in Appendix K of the full version.

6. EFFICIENT MECHANISMS FOR GENERAL DISTRIBUTIONS

We use the results of Sec 5 to prove Thm 1.1. First, it is not hard to see that discretizing the value distribution to multiples of δ, for sufficiently small $\delta = \delta(\epsilon)$, and applying Lemmas 5.1 and 5.3 yields an algorithm for computing an ϵ-BIC ϵ-optimal mechanism for the k-items and k-bidders problems. The resulting technical difficulty is turning these mechanisms into being 0-BIC. To do this, we employ a non-trivial modification of the construction in [Hartline et al. 2011] to improve the truthfulness of the mechanism at the cost of a small amount of revenue. We present our construction and its challenges in Sec 6.2.

6.1. A Warmup: ϵ-Truthful Near-Optimal Mechanisms

Discretization:. Let $\mathcal{D}$ be a valid input to the BIC k-items or the BIC k-bidders problem. For each i, create a new distribution $\mathcal{D}'_i$ that first samples a bidder from $\mathcal{D}_i$, and rounds every value down to the nearest multiple of δ. Let $\mathcal{D}'$ be the product distribution of all $\mathcal{D}'_i$'s. Let also T denote the maximum number of items that can be awarded by a feasible mechanism. We show the following lemma whose proof can be found in Appendix H of the full version.

LEMMA 6.1. *For all δ, let M' be the optimal δ-BIC mechanism for $\mathcal{D}'$. Then $R^{M'}(\mathcal{D}') \geq R^{OPT}(\mathcal{D}) - \delta T$. Moreover, let M be the mechanism that on input $\vec{v}$ rounds every v_{ij} down to the nearest multiple v'_{ij} of δ and implements the outcome $M'(\vec{v}')$. Then M is 2δ-BIC for bidders sampled from $\mathcal{D}$, and has revenue at least $R^{OPT}(\mathcal{D}) - \delta T$.*

Now notice that our algorithms of Sec 5 allow us to find an optimal δ-BIC mechanism M' for $\mathcal{D}'$. So an application of Lemma 6.1 allows us to obtain a 2δ-BIC mechanism for $\mathcal{D}$ whose revenue is at least $R^{OPT}(\mathcal{D}) - \delta T$.

6.2. Truthful Near-Optimal Mechanisms: Proof of Theorems 3.3 and 1.1

We start this section with describing our ϵ-BIC to BIC transformation result (Thm 3.3) arguing that it can be implemented efficiently in the BIC k-items and k-bidders settings.[7] Combining our transformation with the results of the previous sections we obtain Thm 1.1 in the end of this section. Our transformation is inspired by [Hartline et al. 2011], but has several important differences. We explicitly describe our transformation, point out the key differences between our setting and that considered in [Hartline et al. 2011], and outline the proof of correctness, postponing the complete proof of Theorem 3.3 to Appendix I of the full version.

Algorithm Phase 1: Surrogate Sale

(1) Recall from the statement of Theorem 3.3 that $\mathcal{D}$ and $\mathcal{D}'$ can be coupled so that, whenever we have $\vec{v}$ sampled from $\mathcal{D}$ and $\vec{v}'$ sampled from $\mathcal{D}'$, we have $v_{ij} \geq v'_{ij} \geq v_{ij} - \delta$, for all i, j. Moreover, M_1 is an ϵ-BIC mechanism for $\mathcal{D}'$, for some ϵ.
(2) Modify M_1 to give each bidder a rebate of δ whenever they receive an item. Further modify M_1 to multiply all prices it charges by a factor of $(1 - \eta)$. Call M the mechanism resulting from these modifications. Interpret the η-fraction of the prices given back as additonal rebates.

[7] We will explicitly describe the transformation for the BIC k-items and k-bidders settings. For an arbitrary setting where m bidders sample their valuation vectors for n items independently (but not necessarily identically) from $[0, 1]^n$ (allowing correlation among items), simply employ the BIC k-items transformation, replacing k with n.

(3) For each bidder i, create $r-1$ *replicas* sampled i.i.d. from $\mathcal{D}_i$ and r *surrogates* sampled i.i.d. from $\mathcal{D}'_i$. Use $r = (\frac{\eta}{\delta})^2 \cdot m^2 \cdot \hat{\beta}$, where $\hat{\beta} = (\frac{1}{\delta}+1)^k$, for the k-items transformation, and $\hat{\beta} = (n+1)^{1/\delta+1}$, for the k-bidders transformation.
(4) Ask each bidder to report $\vec{v_i}$. For k-bidders only: Fix a permutation σ such that $v_{i\sigma(j)} \geq v_{i\sigma(j+1)}, \forall j$. For each surrogate and replica $\vec{w_i}$, permute $\vec{w_i}$ into $\vec{w_i}'$ satisfying $w'_{i\sigma(j)} \geq w'_{i\sigma(j+1)}, \forall j$.
(5) Create a weighted bipartite graph with replicas (and bidder i) on the left and surrogates on the right. The weight of an edge between replica (or bidder i) with type $\vec{r_i}$ and surrogate of type $\vec{s_i}$ is $\vec{r_i}$'s utility for the expected outcome of $\vec{s_i}$ when playing M (where the expectation is taken over the randomness of M and of the other bidders assuming they are sampled from $\mathcal{D}'_{-i}$).
(6) Compute the VCG matching and prices. If a replica (or bidder i) is unmatched in the VCG matching, add an edge to a random unmatched surrogate. The surrogate selected for bidder i is whoever she is matched to.

Algorithm Phase 2: Surrogate Competition

(1) Let $\vec{s_i}$ denote the surrogate chosen to represent bidder i in phase one, and let $\vec{s}$ denote the entire surrogate profile. Have the surrogates $\vec{s}$ play M.
(2) If bidder i was matched to their surrogate through VCG, charge them the VCG price and award them $M_i(\vec{s})$. (Recall that this has both an allocation and a price component; the price is added onto the VCG price.) If bidder i was matched to a random surrogate after VCG, award them nothing and charge them nothing.

There are several differences between our transformation and that of [Hartline et al. 2011]. First, observe that, because $\mathcal{D}'$ and M_1 are explicitly given as input to our transformation (via an exact sampling oracle from $\mathcal{D}'_i$ and explicitly specifying the outcome awarded to every type $\vec{v}_i$ sampled from $\mathcal{D}'_i$, for all i), we do not have to worry about approximation issues in calculating the edge weights of our VCG auctions in Phase 1. Second, in [Hartline et al. 2011], the surrogates are taking part in an algorithm rather than playing a mechanism, and every replica has non-negative value for the outcome of an algorithm because there are no prices charged. Here, however, replicas may have negative value for the outcome of a mechanism because there are prices charged. Therefore, some edges may have negative weights, and the VCG matching may not be perfect. We have modified M to give rebates (phase 1, step 2) so that the VCG matching cannot be far from perfect, and show that we do not lose too much revenue from unmatched bidders. Finally, in the k-bidders problem, the vanilla approach that does not permute sampled replicas and surrogates (like we do in phase 1, step 4 of our reduction) would require exponentially many replicas and surrogates to preserve revenue. To maintain the computational efficiency of our reduction, we resort to sampling only polynomially many replicas/surrogates and permuting them according to the permutation induced by the bidder's reported values. This may seem like it is giving a bidder control over the distribution of replicas and surrogates sampled for her. We show, exploiting the monotonicity results of Sec 5, that our construction is still BIC despite our permuting the replicas and surrogates. We overview the main steps of the proof of Thm 3.3 and give its complete proof in Appendix I of the full version. We conclude this section with the proof of Thm 1.1.

Proof of Theorem 1.1: Choose $\mathcal{D}'$ to be the distribution that samples from $\mathcal{D}$ and rounds every v_{ij} down to the nearest multiple of δ. Let then M_1 be the optimal δ-BIC mechanism for $\mathcal{D}'$ as computed by the algorithms of Section 5. By Lemma 6.1, $R^{M_1}(\mathcal{D}') \geq R^{OPT}(\mathcal{D}) - \delta T$. Applying Thm 3.3 we obtain a BIC mechanism M_2 such that

$$R^{M_2}(\mathcal{D}) \geq (1-\eta)\cdot R^{M_1}(\mathcal{D}') - \frac{5\delta}{\eta}T \tag{1}$$

$$\geq R^{OPT}(\mathcal{D}) - \eta \cdot R^{OPT}(\mathcal{D}) - (1-\eta)\delta T - \frac{5\delta}{\eta}T. \tag{2}$$

Notice that $R^{OPT}(\mathcal{D}) \leq T$. Hence, choosing $\eta = \epsilon$ and $\delta = \epsilon^2$, (2) gives

$$R^{M_2}(\mathcal{D}) \geq R^{OPT}(\mathcal{D}) - O(\epsilon \cdot k) \quad \text{(for } k\text{-items); and} \tag{3}$$

$$R^{M_2}(\mathcal{D}) \geq R^{OPT}(\mathcal{D}) - O(\epsilon \cdot \sum_i C_i) \quad \text{(for } k\text{-bidders).} \tag{4}$$

The proof of Theorem 1.1 is concluded by noticing that $\sum_i C_i \leq k \max_i C_i$ and k is an absolute constant. □

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