On the Polynomial Time Computation of Equilibria for Certain Exchange Economies

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Abstract

The problem of computing equilibria for exchange markets has recently started to receive a significant attention in the theoretical computer science community. It has been shown that equilibria can be computed in polynomial time in various special cases, the most important of which are when traders have utility functions that are linear, Cobb-Douglas, or a range of CES functions. These important special cases are instances when the market satisfies a property called *weak gross substitutability*. Classical results in economics, which theoretical computer scientists (including us) appear to have been hitherto unaware of, show that the price equilibria in such markets are characterized by an infinite number of linear inequalities and therefore form a convex set. In this paper, we show that under fairly general assumptions, there are polynomial-time algorithms to compute equilibria in such markets. To the best of our knowledge, these are the first poly-time algorithms for exchange markets under the general setting of weak gross substitutability. To show this result, we need to build on the proofs that characterize the equilibria as a convex set using the right assumptions and ideas.

As a consequence, we obtain alternative poly-time algorithms for computing equilibria with linear, Cobb-Douglas, a range of CES, as well as other special utility functions that satisfy weak gross substitutability, such as certain non-homogeneous CES functions. Unlike previous poly-time algorithms, our approach does not make use of the specific form of these utility functions and is in this sense more general. We expect our framework to work or be readily adaptable to handle other exchange markets, provided that the utility functions satisfy weak gross substitutability.

1 Introduction

We first describe the exchange market model and provide some basic definitions. Let us consider m economic agents which represent traders of n goods. Let $\mathbf{R}^n_+$ denote the subset of $\mathbf{R}^n$ with all nonnegative coordinates. The j-th coordinate in $\mathbf{R}^n$ will stand for good j. Each trader i has a concave utility function $u_i : \mathbf{R}^n_+ \to \mathbf{R}_+$, which represents her preferences for the different bundles of goods, and an initial endowment of goods $w_i = (w_{i1}, \ldots, w_{in}) \in \mathbf{R}^n_+$. At given prices $\pi \in \mathbf{R}^n_+$, trader i will sell her endowment, and get the bundle of goods $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbf{R}^n_+$

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which maximizes $u_i(x)$ subject to the budget constraint[1] $\pi \cdot x \leq \pi \cdot w_i$. Let $W_j = \sum_i w_{ij}$ denote the total amount of good j in the market.

An equilibrium is a vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}^n_+$ at which there is a bundle $\bar{x}_i = (\bar{x}_{i1}, \ldots, \bar{x}_{in}) \in \mathbf{R}^n_+$ of goods for each trader i such that the following two conditions hold: (i) For each good j, $\sum_i \bar{x}_{ij} \leq W_j$ and (ii) For each trader i, the vector $\bar{x}_i$ maximizes $u_i(x)$ subject to the constraints $\pi^T x \leq \pi^T w_i$ and $x \in \mathbf{R}^n_+$.

The celebrated result of Arrow and Debreu [1] states that, under quite mild assumptions, such an equilibrium exists. A special case occurs when the initial endowments are *proportional*, i.e., when $w_i = \delta_i w$, $\delta_i > 0$, so that the relative incomes of the traders are independent of the prices. This special case is equivalent to *Fisher model*, which is a market of n goods desired by m utility maximizing buyers with fixed incomes.

For any price vector π, the vector $x_i(\pi)$ that maximizes $u_i(x)$ subject to the constraints $\pi^T x \leq \pi^T w_i$ and $x \in \mathbf{R}^n_+$ is called the *demand* [2] of trader i at prices π. The *excess demand* of trader i is $z_i(\pi) = x_i(\pi) - w_i$. Then $X_k(\pi) = \sum_i x_{ik}(\pi)$ denotes the *market demand* of good k at prices π, and $Z_k(\pi) = X_k(\pi) - W_k = \sum_i z_{ik}(\pi)$ the *market excess demand* of good k at prices π. The vectors $X(\pi) = (X_1(\pi), \ldots, X_n(\pi))$ and $Z(\pi) = (Z_1(\pi), \ldots, Z_n(\pi))$ are called *market demand* (or aggregate demand) and *market excess demand*, respectively. The market is said to satisfy *positive homogeneity* if for any price vector π and any $\lambda > 0$, we have $Z(\pi) = Z(\lambda\pi)$. It is said to satisfy *Walras' Law* if for any price π, we have $\pi^T Z(\pi) = 0$. Both positive homogeneity and Walras' law are satisfied by markets in which traders have typical utility functions. We assume that these properties hold for the markets we consider; these properties will be used in many of our proofs.

When the market is defined in terms of the excess demand function, the equilibrium is defined as a vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}^n_+$ such that $Z_j(\pi) \leq 0$, for each j.

Two properties play a significant role in the theory of equilibrium and in related computational results: *gross substitutability* (GS) and the *weak axiom of revealed preferences* (WARP).

A market is said to satisfy GS (resp. weak GS) if for any two sets of prices π and π' such that $0 < \pi_j \leq \pi'_j$, for each j, and $\pi_j < \pi'_j$ for some j, we have that $\pi_k = \pi'_k$ for any good k implies $Z_k(\pi) < Z_k(\pi')$ (resp. $Z_k(\pi) \leq Z_k(\pi')$). That is, increasing the prices for some of the goods while keeping some others fixed can only cause an increase (resp. cannot cause a decrease) in demand for the goods whose price is fixed. The utility function of an individual trader is said to satisfy weak GS if for any initial endowment (the case of fixed income is included here), increasing the prices of some of the goods while keeping some others fixed cannot cause a decrease in demand for the goods whose price is fixed. It is well known and easy to see that a market satisfies weak GS if the utility function of each individual trader does.

The market excess demand is said to satisfy WARP if for any two sets of prices π and π' such that $Z(\pi) \neq Z(\pi')$ either $\pi^T Z(\pi') > 0$ or $(\pi')^T Z(\pi) > 0$. This means that if the demands at prices π and π' are different, then either $Z(\pi')$ is not within budget when the price is π or $Z(\pi)$ is not within budget when the price is π'. The excess demand function of an individual trader with any typical concave utility function satisfies WARP (precisely because of utility maximization), but the aggregate excess demand need not.

Arrow, Block, and Hurwicz [2] show the following Lemma.

[1] Given two vectors x and y, we use $x \cdot y$ or $x^T y$ to denote their inner product.

[2] In the definitions we assume that the demand is a single-valued function of the prices, which is the case with most of the commonly used utility functions. The definitions can be appropriately generalized to handle the case when the demand is a multi-valued function, which happens for instance when the utility functions are linear.

Lemma 1 *If an equilibrium price vector $\hat{\pi}$ satisfies $\hat{\pi}_j > 0$, for each good j, if the market satisfies gross substitutability, positive homogeneity, and Walras' law, then for any non-equilibrium price vector π such that $\pi_j > 0$ for each j, we have $\hat{\pi}^T Z(\pi) > 0$.*

Lemma 1 says that, under GS, WARP holds for any pair of price vectors, provided that one of them is an equilibrium price vector. The lemma generalizes to the case where there is only weak GS [3, 4] and immediately implies that the set of equilibrium prices form a convex set. It also gives, for any positive price vector π that is not an equilibrium price vector, a *separating hyperplane* [18], that is, a hyperplane that separates π from the convex set of equilibrium prices. Indeed we have $\sum_j \hat{\pi}_j Z_j(\pi) > 0$, but $\sum_j \pi_j Z_j(\pi) = 0$, by Walras' law. To compute this separating hyperplane, we need to compute the demands $Z_j(\pi)$ at the prices π.

Lemma 1 describes an important structural property of the market excess demand function under the restriction of weak GS. Without some such restriction, the excess demand function has virtually no structure. Classical results in economics [30, 31, 22, 9] essentially show that any continuous function satisfying Walras' Law can be an excess demand function of an exchange market. This is in fact true even for special markets such as those in which traders have the same non-homogeneous utility function and proportional initial endowments [21] (the Fisher setting).

A utility function $u(\cdot)$ is *homogeneous* (of degree one) if it satisfies $u(\alpha x) = \alpha u(x)$, for all $\alpha > 0$. A linear utility function has the form $u_i(x) = \sum_j a_{ij} x_{ij}$. A CES (constant elasticity of substitution) utility function has the form $u(x_i) = (\sum_j (a_{ij} x_{ij})^\rho)^{1/\rho}$, where $-\infty < \rho < 1$ but $\rho \neq 0$. ($a_{ij} \geq 0$ in both these definitions). The Cobb-Douglas utility has the form $u_i(x) = \prod_j (x_{ij})^{a_{ij}}$, where $a_{ij} \geq 0$ and $\sum_j a_{ij} = 1$. We refer to the reader to [32, 7] for a detailed discussion of these functions. All utility functions discussed in this paper satisfy $u(0) = 0$.

Previous Results. Much of the previous work has focussed on poly-time algorithms for the case when traders have proportional endowments (the Fisher setting) [10, 11, 6] and on polynomial-time approximation schemes for the general setting with linear utilities [20, 17]. Generalizing an approach due to Eisenberg and Gale [14, 16] for linear utilities, Eisenberg [13] shows how to write the equilibrium conditions as the solution to a convex program, whenever the traders have proportional endowments and homogeneous utility functions. Indeed in this case, the equilibrium allocations maximize the function $\prod_i u_i(x_i)^{\delta_i}$ subject to the constraint that the market clears, where u_i is the i-th trader's utility function, and δ_i is the i-th trader's proportionality factor ($w_i = \delta_i w$).

In the mid-eighties, Eaves gave a polynomial-time algorithm for computing equilibria when traders have Cobb-Douglas utilities [12]. Recently, Jain [19] gave a polynomial-time algorithm for computing equilibria when traders have linear utility functions by formulating the problem as a convex program. Jain mentions some other utility functions for which his approach works too. It is not hard to see that his convex program describes equilibria for Cobb-Douglas and CES functions with $\rho > 0$ as well. Independently, Codenotti and Varadarajan [7], using a different convex programming formulation, gave a poly-time algorithm for computing equilibria when traders have Cobb-Douglas or CES utility functions with $\rho > 0$. In related work, Nenakov and Primak [25] (in a Russian paper) describe the equilibrium for the linear exchange model in terms of a finite set of convex inequalities and an extension to a restricted form of production as an infinite set of convex inequalities. They also observe that their approach works for Cobb-Douglas utilities.

Generalizations and extensions of Lemma 1 have been developed to markets where the demand need not be single-valued, for instance markets with linear utility functions (see, e.g., [27, 28, 26, 29]). Some of these papers also propose the ellipsoid or cutting plane methods to compute equilibria, but they give no guarantees on the quality of the output produced.

Our Results

In this paper, we address the poly-time computability of equilibria for exchange markets satisfying weak GS. Since a price equilibrium vector that is rational exists only in very special cases, we consider the question of computing an approximate equilibrium, defined below.

Definition 2

A bundle $x_i \in \mathbf{R}^n_+$ is a *μ-approximate demand*, for $\mu \geq 1$, of trader i at prices π if $u_i(x_i) \geq \frac{1}{\mu} u^*$ and $\pi^T x_i \leq \mu \pi^T w_i$, where $u^* = \max\{u_i(x) | x \in \mathbf{R}^n_+, \pi^T x \leq \pi^T w_i\}$.

Prices π and bundles x_i's are a *strong μ-approximate equilibrium* ($\mu \geq 1$) if (1) for each trader i, x_i is the demand of trader i at prices π, and (2) $\sum_i x_{ij} \leq \mu \sum_i w_{ij}$ for each good j. Prices π and bundles x_i's are a *weak μ-approximate equilibrium* ($\mu \geq 1$) if (1) for each trader i, x_i is a μ-approximate demand of trader i at prices π, and (2) $\sum_i x_{ij} \leq \mu \sum_i w_{ij}$ for each good j.

Note that our definition of weak μ-approximate equilibrium corresponds to the standard definition of approximate equilibrium used, e.g., in [20].

Under some fairly mild assumptions, we show that we can compute in polynomial time a weak $(1+\varepsilon)$-approximate equilibrium for an exchange market where the utility functions satisfy weak GS. In markets where the demand does not change too rapidly as a function of the prices, our algorithms in fact compute a strong $(1+\varepsilon)$-approximate equilibrium. Our algorithms are polynomial in the input size and $\log \frac{1}{\varepsilon}$, a running time which is usually thought of as equivalent to an exact poly-time algorithm when the desired output can be irrational. The input size is defined to be the number of traders plus the number of goods plus the maximum number of bits needed for encoding the rational numbers that describe the utility functions and initial endowments. Our algorithms are based on the ellipsoid method and the main technical contribution is the provision of an appropriate separation oracle. Indeed, for the ellipsoid algorithm to return an approximate equilibrium, the separation oracle provided by Lemma 1 is not enough. What we need is a stronger separating oracle: Given a price π that is not a $(1+\varepsilon)$-approximate equilibrium, compute a hyperplane that separates π from all points within distance $\delta > 0$ from the set of equilibria, where δ is reasonably large in terms of the input size. We construct such a separating oracle by analyzing the proof of Lemma 1 and the proof of a related characterization due to Primak [29], and building on these proofs using appropriate assumptions and extensions.

Some specific consequences of our algorithms are the following: we can compute in poly-time a weak $(1+\varepsilon)$-approximate equilibrium in exchange markets where traders have either linear, Cobb-Douglas, or CES utility functions with $\rho > 0$. We wish to emphasize that we do not need any assumptions for such specific functions. We can also allow traders to have non-homogeneous utility functions of the form $u_i(x_i) = \sum_j a_{ij}(x_{ij})^{\rho_{ij}}$, where $0 < \rho_{ij} < 1$ (we call these functions *VES*, which stands for *variable elasticity of substitution*), but we currently need to assume the traders to have a little amount of every good in their initial endowment[3]. For exchange markets

[3] We are working on getting rid of this technical restriction and it is quite likely that we will succeed. Furthermore, note that this restriction, a.k.a. *strong survival assumption*, is a standard one in equilibrium theory, and that without it an equilibrium is not guaranteed to exist. Indeed the first proof of equilibrium in [1] makes use of this assumption; in [8] this assumption is discussed, and the weaker notion of quasi-equilibrium introduced in

where traders have Cobb-Douglas or CES utility functions with $0 < \rho < 1 - c$, where $c > 0$ is any small constant, we can compute a strong $(1+\varepsilon)$-approximate equilibrium. We use the fact that the demand does not change too rapidly as a function of the prices in such markets. To our knowledge, these results are the first ones that compute a strong approximate equilibrium in poly-time (when an equilibrium that is rational need not exist). We provide these examples just to illustrate that our approach has important consequences. However, our framework makes no assumptions about the specific form of the utility functions. Any assumption, other than weak GS, is fairly mild, and our approach should generalize to markets with other reasonable utility functions satisfying weak GS.

The rest of the paper is organized as follows. Section 2 presents useful properties of approximate demands and approximate equilibria. It also states two assumptions about exchange markets that we use in some of our results, along with brief justifications for these assumptions. Section 3 contains the main results of the paper: four separation proofs that can be turned into polynomial time algorithms, two for strong and two for weak $(1+\varepsilon)$-approximate equilibria. Section 4 provides details to illustrate how one of our separation results can be turned a polynomial time algorithm for computing a strong $(1+\varepsilon)$-approximation price.

2 Preliminaries

In this section we prove some simple but important properties of approximate demands and approximate equilibria. We also state and justify some assumptions that the algorithms of the subsequent sections make.

An important property of approximate demands, their resilience to price change, is captured by the following lemma.

Lemma 3 *Let π and π' be two sets of prices in $\mathbf{R}_+^n$ such that $|\pi_j - \pi'_j| \le \varepsilon \cdot \min\{\pi_j, \pi'_j\}$ for each j, where $\varepsilon > 0$. Let x_i be a $(1+\delta)$-approximate demand for trader i at prices π. Then x_i is a $(1+\varepsilon)^2(1+\delta)$-approximate demand for trader i at prices π'.*

Proof :

The constraint $|\pi_j - \pi'_j| \le \varepsilon \cdot \min\{\pi_j, \pi'_j\}$ implies that

$$(1-\varepsilon)\cdot \pi_j \le \pi'_j \le (1+\varepsilon)\cdot \pi_j \tag{1}$$

$$(1-\varepsilon)\cdot \pi'_j \le \pi_j \le (1+\varepsilon)\cdot \pi'_j \tag{2}$$

We will first show that x_i approximately satisfies the budget constraint, to within a factor of $(1+\varepsilon)^2(1+\delta)$, at prices π'. Using (1) and the fact that $\pi^T x_i \le (1+\delta)\pi^T w_i$, which follows from the fact that x_i is a $(1+\delta)$-approximate demand for trader i at prices p, we get

$$(\pi')^T x_i \;=\; \sum_j \pi'_j x_{ij} \le (1+\varepsilon)\sum_j \pi_j x_{ij} \;\le\; (1+\varepsilon)(1+\delta)\pi^T w_i. \tag{3}$$

Using (2) we get

$$\pi \cdot w_i \;=\; \sum_j \pi_j w_{ij} \;\le\; (1+\varepsilon)\sum_j \pi'_j w_{ij} \;=\; (1+\varepsilon)(\pi')^T w_i. \tag{4}$$

order to get rid of it.

Substituting (4) in (3), we get

$$(\pi')^T x_i \leq (1+\varepsilon)^2(1+\delta)(\pi')^T w_i.$$

We will now show that x_i approximately maximizes trader i's utility function. Let $U = u_i(x^*)$, $y^* = \text{argmax}\{u_i(x) | x \in \mathbf{R}_+^n, (\pi')^T x \leq (\pi')^T w_i\}$, and $U' = u_i(y^*)$.

Set $z = \frac{y^*}{(1+\varepsilon)^2}$. By definition of y^*, we have

$$(\pi')^T y^* \leq (\pi')^T w_i$$

Using (1) and (2), we transform this into

$$\frac{1}{(1+\varepsilon)} \pi^T y^* \leq (1+\varepsilon)\pi^T w_i.$$

This implies that $\pi^T z \leq \pi^T w_i$. Since $z \in \mathbf{R}_+^n$ and $\pi^T z \leq \pi^T w_i$, $U \geq u_i(z)$.

Therefore,

$$\begin{aligned}
U &\geq u_i(z) \\
&= u_i\left(\frac{y^*}{(1+\varepsilon)^2}\right) = u_i\left(\frac{y^*}{(1+\varepsilon)^2} + (1 - \frac{1}{(1+\varepsilon)^2}) \cdot 0\right) \\
&\geq \frac{1}{(1+\varepsilon)^2} u_i(y^*) + \left(1 - \frac{1}{(1+\varepsilon)^2}\right) u_i(0) \qquad \text{by concavity of } u_i. \\
&= \frac{1}{(1+\varepsilon)^2} u_i(y^*) \qquad \text{since } u_i(0) = 0. \\
&= \frac{1}{(1+\varepsilon)^2} U'.
\end{aligned}$$

Since $u_i(x_i) \geq \frac{1}{(1+\delta)} U$, it follows that

$$u_i(x_i) \geq \frac{1}{(1+\delta)(1+\varepsilon)^2} U'.$$

This completes the proof.

□

The following lemma shows how to translate a weak approximate equilibrium into another one which exactly satisfies all budget constraints and in which all markets clear exactly.

Lemma 4 *Let prices π and bundles x_i's be a weak μ-approximate equilibrium. We can compute, in polynomial time, a weak μ^2-approximate equilibrium consisting of prices π and bundles y_i's such that (i) $\pi^T y_i = \pi^T w_i$ for each trader i and (ii) $\sum_i y_{ij} = \sum_i w_{ij}$ for each good j.*

Proof :

Set $\hat{y}_{ij} = \frac{1}{\mu} x_{ij}$ for each i and j. Then $\pi^T \hat{y}_i = \frac{1}{\mu}\pi^T x_i \leq \pi^T w_i$. The latter inequality follows from the fact that x_i is a μ-approximate demand at prices π. Also, $\sum_i \hat{y}_{ij} = \frac{1}{\mu}\sum_i x_{ij} \leq \sum_i w_{ij}$ for each j. By concavity of the u_i's it follows that

$$
\begin{aligned}
u_i(\hat{y}_i) &\geq \frac{1}{\mu} u_i(x_i) + \Big(1 - \frac{1}{\mu}\Big) u_i(0) \\
&\geq \frac{1}{\mu} u_i(x_i) \\
&\geq \frac{1}{\mu^2} \max\{u_i(x) \mid x \in \mathbf{R}^n_+, \pi^T x \leq \pi^T w_i\}.
\end{aligned}
$$

Let $Y_j = \sum_i \hat{y}_{ij}$. Now we redistribute the "surplus" $W_j - Y_j$, for each j, so that each trader uses up her money and the market clears exactly. To see that this can be done, assume that $W_j - Y_j > 0$ for some good j. We know that $W_k - Y_k \geq 0$ for all goods k. The surplus money available to a trader i is $\pi^T w_i - \pi^T \hat{y}_i$. This is non-negative for every trader. Summing this over all traders we get

$$
\sum_i (\pi^T w_i - \pi^T \hat{y}_i) = \pi^T (W_1 - Y_1, W_2 - Y_2, \ldots, W_n - Y_n) > 0.
$$

The last inequality follows from the fact that $W_j - Y_j > 0$ for some j and $W_k - Y_k \geq 0$ for all k. Thus some trader has surplus money.

A similar argument suffices to show that if $W_j - Y_j = 0$ for every good j then every trader satisfies her budget constraint exactly. So a possible redistribution scheme is the following: pick an arbitrary good j with surplus, that is, $W_j - Y_j > 0$ and an arbitrary trader i with excess money. Increase $\hat{y}_{ij}$ until either the trader has no excess money or there is no surplus of good j. Repeat this until all surplus is exhausted.

Let y_i denote the new allocations. By construction, we have $\pi^T y_i = \pi^T w_i$ for each i, and $\sum_i y_{ij} = \sum_i w_{ij}$ for each j. Since increasing the amount of goods in her bundle does not decrease a trader's utility, we have $u(y_i) \geq u_i(\hat{y}_i) \geq \frac{1}{\mu^2} \max\{u_i(x) \mid x \in \mathbf{R}^n_+, \pi^T x \leq \pi^T w_i\}$. □

The following lemma gives a useful transformation from a market with arbitrary initial endowments to the case where the vector of initial endowments has all positive components.

Lemma 5 *Let M be a market where the traders have utility functions u_i that are homogeneous of degree 1 and initial allocations $w_i \in \mathbf{R}^n_+$. Let $\hat{M}$ be a market where the traders' initial allocations are given by $\hat{w}_{ij} = w_{ij} + \frac{\varepsilon}{m} \sum_i w_{ij}$, where $\varepsilon > 0$. Let π be a weak (resp. strong) $(1+\delta)$-approximate equilibrium, for some $\delta > 0$, for $\hat{M}$, with each price being positive, and let $\hat{x}_i$ be the corresponding allocations. Then π is a weak (resp. strong) $(1+\delta)(1+\varepsilon)$-approximate equilibrium for M with corresponding allocations $x_i = \frac{\pi^T w_i}{\pi^T \hat{w}_i} \hat{x}_i$.*

Proof :

We present the proof for the weak approximate equilibrium, the proof for the other case is very similar. Now

$$
\pi^T x_i = \frac{\pi^T w_i}{\pi^T \hat{w}_i} \pi^T \hat{x}_i \leq (1+\delta) \pi^T w_i,
$$

since $\hat{x}_i$ is a $(1+\delta)$-approximate demand for market $\hat{M}$ at prices π.

Let α_i be the optimal utility of trader i at prices π and budget $\pi^T w_i$, and let $\hat{\alpha}_i$ be her optimal utility with budget $\pi^T \hat{w}_i$. Since the utility function is homogeneous of degree 1, we have $\hat{\alpha}_i = \frac{\pi^T w_i}{\pi^T \hat{w}_i}\alpha_i$. Also,

$$u_i(x_i) = \frac{\pi^T w_i}{\pi^T \hat{w}_i} u_i(\hat{x}_i) \geq \frac{\pi^T w_i}{\pi^T \hat{w}_i}\frac{\hat{\alpha}_i}{1+\delta} = \frac{\alpha_i}{1+\delta},$$

where the inequality follows because $\hat{x}_i$ is a $(1+\delta)$-approximate demand for $\hat{M}$. Finally,

$$\sum_i x_{ij} \leq \sum_i \hat{x}_{ij} \leq (1+\delta)\sum_i \hat{w}_{ij} \leq (1+\delta)(1+\varepsilon)\sum_i w_{ij},$$

completing the proof.

□

We now state some assumptions about the market M whose price equilibrium we wish to compute. These are used in the separation results and algorithms of the subsequent sections. The first assumption says that there is a price equilibrium where the price ratios are bounded. This is a fairly mild assumption, and some such assumption is evidently necessary for the general approach developed in the subsequent sections. Moreover, we argue that for interesting specific markets this assumption entails no loss of generality. The second assumption, which is only needed by the algorithms that output a strong approximate equilibrium, says that the demand does not change too rapidly as a function of price. It is also clear that such an assumption is unavoidable in a general approach to efficiently compute a strong approximate equilibrium.

Assumption 1: The market M has a price equilibrium π^* in the region $\Delta = \{\pi \in \mathbf{R}^n_+ | 2^{-L} \leq \pi_j \leq 1 \text{ for each } j\}$, where L is bounded by a polynomial in the input size. For markets with utility functions that are either CES with $\rho > 0$, Cobb-Douglas, or VES, this assumption is satisfied [24] whenever the market has an equilibrium and every good is desired by some trader[4]. Goods that are not desired by any trader are just discarded from the problem. Since we handle linear utility functions by approximating them with CES functions with ρ close to 1 (see Section 4), such utility functions are also taken care of. This assumption is a mild one that holds quite generally.

We define $\Delta^+ = \{\pi \in \mathbf{R}^n_+ | 2^{-L} - \frac{2^{-L}}{2} \leq \pi_j \leq 1 + \frac{2^{-L}}{2} \text{for each } j\}$, and $\Delta^{++} = \{\pi \in \mathbf{R}^n_+ | 2^{-L} - \frac{2^{-L}}{2} - \frac{2^{-L}}{4} \leq \pi_j \leq 1 + \frac{2^{-L}}{2} + \frac{2^{-L}}{4} \text{for each } j\}$.

Assumption 2: For any price $\pi \in \Delta^{++}$, we have $|\frac{\partial Z^k}{\partial \pi_j}| \leq 2^D$ for each j and k, where D is bounded by a polynomial in the input size.

This assumption is satisfied by many natural markets, for instance markets with CES utilities with $0 < \rho < 1-c$, where $c > 0$ is any small constant. The statement of the assumption requires that the demands be differentiable, but this requirement can be relaxed.

Assumption 2 may not be satisfied in some markets, for instance when some traders have CES functions with large ρ. This assumption is needed only by the algorithms that compute a strong $(1+\varepsilon)$-approximate equilibrium.

[4]There is a related notion of equilibrium called quasi-equilibrium, introduced by [8], that may exist even when an equilibrium as defined by us may not [1, 15, 23]. In a quasi-equilibrium, traders with zero income are not required to optimize their utility. Our definition of equilibrium is the same as the competitive equilibrium of Arrow and Debreu [1]. In markets with the specific utility functions we mention, the equilibrium price of any good that is desired by some trader cannot be zero, whereas, in a quasi-equilibrium, the price can be zero.

3 Computationally Tractable Proofs of Separation

Consider an exchange market satisfying Assumption 1 and a vector $\pi \in \mathbf{R}_+^n$ that is not a weak $(1+\varepsilon)$-approximate equilibrium. It is not hard to see, using Lemma 3, that the Euclidean distance from π to any equilibrium in Δ is at least some $\delta > 0$, where δ is bounded below as a function of the input size and ε. Using the fact that the set of price equilibria form a convex set, we can see that there is a hyperplane that separates π from all points that are within distance δ of any equilibrium in Δ. This argument only establishes the existence of such a separating hyperplane but says nothing about how it can be efficiently computed. In this section, we present two proofs of existence that can be converted into efficient algorithms. The first one is Lemma 7, which is based on the proof of Lemma 1. Lemma 7 says that the separating hyperplane normal to the direction $Z(\pi)$ is itself "the right one". Thus its computational version requires nothing more than the ability to compute the demand at π, but the lemma itself places a restriction on the initial endowments. The second one, Lemma 9, is based in [29] and places no such restrictions; however its computational version is more demanding.

Similarly, for a vector $\pi \in \mathbf{R}_+^n$ that is not a strong $(1+\varepsilon)$-approximate equilibrium, the existence of a hyperplane that separates π from all points within δ of equilibria in Δ follows easily if we have Assumption 2. The computationally tractable proofs of the existence of such a separating hyperplane are Lemma 6 and Lemma 8. Lemma 6, based on the proof of Lemma 1, also says that the separating hyperplane normal to the direction $Z(\pi)$ is itself the right one. Thus its computational version again requires only the computation of the demand at π. Lemma 8 is based on [29]. Its computational version, which we describe at length in the next section, is more intricate but also only requires the ability to compute the demand.

The proofs of Lemmas 6 and 8 involve a careful analysis of the original proofs to verify that the proofs do result in sufficient separation using some additional assumptions. The proofs of Lemma 7 and 9 are more delicate and involve new ideas.

Lemma 6 *Let M be an exchange market satisfying weak gross substitutability and Assumptions 1 and 2. Let $\pi \in \Delta^+$ be a price vector that is not a strong $(1+\varepsilon)$-approximate equilibrium, for some $\varepsilon > 0$. Then for any equilibrium $\hat{\pi} \in \Delta$, we have $\hat{\pi}^T Z(\pi) \geq \delta$, where $\delta \geq 1/2^{E_1}$, and E_1 is bounded by a polynomial in the input size and $\log \frac{1}{\varepsilon}$. Moreover $||Z(\pi)||_2 \leq 2^{E_2}$, where E_2 is bounded by a polynomial in the input size. (Note that $\pi^T Z(\pi) = 0$ by Walras' Law.)*

Proof :

The bound on $||Z(\pi)||_2$ follows easily from the fact that $\pi \in \Delta^+$. For the other inequality, the first step of the proof, following [2], is to apply a change of units so that we may assume that the equilibrium price $\hat{\pi}$ is $\mathbf{1} = (1, \ldots, 1) \in \mathbf{R}^n$ while $2^{-L_1} \leq \pi_j \leq 2^{L_1}$ for some L_1 that is polynomial in the input size. The change of units, which is not explicitly spelt out in [2], is as follows. Let w_i^* and u_i^* denote the endowment and utility function of the i'th trader in the original market, and let $\hat{\pi}^* \in \Delta$ denote an equilibrium in the original market, and let $\pi^* \in \Delta^+$ be the price vector given in the lemma. Let $w_i = \hat{\pi}_k^* w_i^*$ be the new endowments and let the new utility function of the i'th trader be

$$u_i(x_{i1}, \ldots, x_{in}) = u_i^*(\frac{x_{i1}}{\hat{\pi}_1^*}, \ldots, \frac{x_{in}}{\hat{\pi}_n^*}).$$

For a price vector $\sigma^* \in \mathbf{R}^n_+$, let

$$\sigma = (\frac{\sigma^*_1}{\hat{\pi}^*_1}, \dots, \frac{\sigma^*_n}{\hat{\pi}^*_n})$$

be the corresponding price vector in the new market. It is easy to check that $(x_{i1}, \dots, x_{in})$ satisfies the budget constraint in the new market at price σ if and only if $(\frac{x_{i1}}{\hat{\pi}^*_1}, \dots, \frac{x_{in}}{\hat{\pi}^*_n})$ satisfies the budget constraint in the old market at price σ^*. Moreover $(y_{i1}, \dots, y_{in})$ is the demand in the new market of the i'th trader at price σ if and only if $(\frac{y_{i1}}{\hat{\pi}^*_1}, \dots, \frac{y_{in}}{\hat{\pi}^*_n})$ is her demand in the old market at price σ^*. It follows that σ is an equilibrium (resp. weak $(1+\varepsilon)$-approximate equilibrium, strong $(1+\varepsilon)$-approximate equilibrium) in the new market if and only if σ^* is an equilibrium (resp. weak $(1+\varepsilon)$-approximate equilibrium, strong $(1+\varepsilon)$-approximate equilibrium) in the old market.

Moreover, $Z_k(\sigma) = \hat{\pi}^*_k Z^*_k(\sigma^*)$. Since $\hat{\pi} = (1, \dots, 1)$, we have $\hat{\pi}_k Z_k(\sigma) = \hat{\pi}^*_k Z^*_k(\sigma^*)$, and so $\hat{\pi}^T Z(\sigma) = (\hat{\pi}^*)^T Z^*(\sigma^*)$. To prove the lemma, it therefore suffices to show that

$$\hat{\pi}^T Z(\pi) = \mathbf{1}^T Z(\pi) \geq \delta.$$

Note that if $\sigma^* \in \Delta^+$, then $2^{-L_1} \leq \sigma_j \leq 2^{L_1}$ where L_1 is bounded by a polynomial in the input size. Moreover $\frac{\partial Z_k(\sigma)}{\partial \sigma_j} \leq 2^{D_1}$ for any σ such that $\sigma^* \in \Delta^+$, where D_1 is bounded by a polynomial in the input size.

Assume without loss of generality that the goods are ordered so that $\pi_1 \leq \pi_2 \cdots \leq \pi_n$. We define a sequence of prices as follows: for $1 \leq s \leq n$, let $\pi^s = (\pi_1, \pi_2, \dots, \pi_{s-1}, \pi_s, \dots, \pi_s)$. That is, $\pi^s_j = \pi_j$ for the first $s-1$ goods, and $\pi^s_j = \pi_s$ for the remaining goods. Note that $\pi^1 = (\pi_1, \dots, \pi_1)$ and $\pi^n = (\pi_1, \dots, \pi_n) = \pi$. The sequence of prices is a gradual transformation from the equilibrium price π^1 to the given price π.

Arrow, Block, and Hurwicz show that $\mathbf{1}^T(Z(\pi^{s+1}) - Z(\pi^s)) \geq 0$ for $1 \leq s \leq n-1$ and that this inequality is strict for at least one s if π is not an equilibrium. We will show below that $\mathbf{1}^T(Z(\pi^{s+1}) - Z(\pi^s)) \geq \delta$ for some s if π is not a strong $(1+\varepsilon)$-approximate equilibrium. The Lemma then follows because

$$\begin{aligned}
\mathbf{1}^T Z(\pi) &= \mathbf{1}^T Z(\pi^n) \\
&= \sum_{1 \leq s \leq n-1} \mathbf{1}^T (Z(\pi^{s+1}) - Z(\pi^s)) + \mathbf{1}^T Z(\pi^1) \\
&\geq \delta + \mathbf{1}^T Z(\pi^1) \\
&= \delta
\end{aligned}$$

The last equality follows because π^1 is an equilibrium.

We refer to the change in price from π^s to π^{s+1} as the s'th step. Let $G^h_s = \{1, \dots, s\}$ and $G^t_s = \{s+1, \dots, n\}$. G^h_s is the subset of goods whose prices remain fixed during the s'th step, and G^t_s is the complement step. Using weak GS, Arrow et al. [2] argue that $Z_j(\pi^{s+1}) \geq Z_j(\pi^s)$ for $j \in G^h_s$, and $Z_j(\pi^{s+1}) \leq Z_j(\pi^s) \leq 0$ for $j \in G^h_t$.

We then have

$$
\begin{aligned}
0 &= \pi^{s+1} \cdot Z(\pi^{s+1}) - \pi^s \cdot Z(\pi^s) \\
&= \sum_{j \in G_s^h} (\pi_j^{s+1} Z_j(\pi^{s+1}) - \pi_j^s Z_j(\pi^s)) + \sum_{j \in G_s^t} (\pi_j^{s+1} Z_j(\pi^{s+1}) - \pi_j^s Z_j(\pi^s)) \\
&= \sum_{j \in G_s^h} (\pi_j^{s} Z_j(\pi^{s+1}) - \pi_j^s Z_j(\pi^s)) + \sum_{j \in G_s^t} (\pi_j^{s+1} Z_j(\pi^{s+1}) - \pi_j^s Z_j(\pi^s)) \\
&= \sum_{j \in G_s^h} \pi_j^s (Z_j(\pi^{s+1}) - Z_j(\pi^s)) + \sum_{j \in G_s^t} \pi_j^{s+1} (Z_j(\pi^{s+1}) - Z_j(\pi^s)) + \sum_{j \in G_s^t} (\pi_j^{s+1} - \pi_j^s) Z_j(\pi^s) \\
&= -\sum_{j \in G_s^h} (\pi_{s+1} - \pi_j^s)(Z_j(\pi^{s+1}) - Z_j(\pi^s)) + \sum_{j \in G_s^h} \pi_{s+1} (Z_j(\pi^{s+1}) - Z_j(\pi^s)) \\
&\quad + \sum_{j \in G_s^t} \pi_{s+1} (Z_j(\pi^{s+1}) - Z_j(\pi^s)) + \sum_{j \in G_s^t} (\pi_j^{s+1} - \pi_j^s) Z_j(\pi^s)
\end{aligned}
$$

Rearranging terms, we get

$$
\pi_{s+1} \sum_j (Z_j(\pi^{s+1}) - Z_j(\pi^s)) = \sum_{j \in G_s^h} (\pi_{s+1} - \pi_j^s)(Z_j(\pi^{s+1}) - Z_j(\pi^s)) - \sum_{j \in G_s^t} (\pi_j^{s+1} - \pi_j^s) Z_j(\pi^s)
$$

Since $Z_j(\pi^s) \leq 0$ for $j \in G_s^t$, and $\pi_{s+1} - \pi_j^s \geq \pi_{s+1} - \pi_s$ for each $j \in G_s^h$, we have

$$
\pi_{s+1} \sum_j (Z_j(\pi^{s+1}) - Z_j(\pi^s)) \geq \sum_{j \in G_s^h} (\pi_{s+1} - \pi_s)(Z_j(\pi^{s+1}) - Z_j(\pi^s)),
$$

which gives

$$
\mathbf{1}^T (Z(\pi^{s+1}) - Z(\pi^s)) \geq \frac{(\pi_{s+1} - \pi_s)}{\pi_{s+1}} \sum_{j \in G_s^h} (Z_j(\pi^{s+1}) - Z_j(\pi^s)) \tag{5}
$$

Since π^n is not a strong $(1+\varepsilon)$-approximate equilibrium, we must have $Z_\ell(\pi^s) \geq \varepsilon W_\ell$ for some good ℓ. On the other hand, $Z_\ell(\pi^1) = 0$ at equilibrium π^1. So there exists a k such that $1 \leq k \leq n-1$ such that $Z_\ell(\pi^{k+1}) - Z_\ell(\pi^k) \geq \frac{\varepsilon}{n} W_\ell$. Since in the k'th step the excess demand increases only for goods in G_k^h, it must be that $\ell \in G_k^h$. By the assumption on the boundedness of the derivative, such a change in demand can happen only if prices change significantly in the k'th step, that is

$$
\pi_{k+1} - \pi_k \geq \frac{\varepsilon W_\ell}{n^2 2^{D_1}}.
$$

Using Equation 5, and the fact that $Z_j(\pi^{k+1}) \geq Z_j(\pi^k)$ for all $j \in G_k^h$, we get

$$
\begin{aligned}
\mathbf{1}^T (Z(\pi^{k+1}) - Z(\pi^k)) &\geq \frac{(\pi_{k+1} - \pi_k)}{\pi_{k+1}} \sum_{j \in G_k^h} (Z_j(\pi^{k+1}) - Z_j(\pi^k)) \\
&\geq \frac{\varepsilon W_\ell}{n^2 2^{D_1} 2^{L_1}} (Z_\ell(\pi^{k+1}) - Z_\ell(\pi^k)) \\
&\geq \frac{\varepsilon W_\ell}{n^2 2^{D_1} 2^{L_1}} \cdot \frac{\varepsilon}{n} W_\ell = \delta,
\end{aligned}
$$

completing the proof.

□

Lemma 7 *Let M be an exchange market satisfying Assumption 1, and suppose that the utility function of each trader is homogeneous, satisfies weak GS and that $w_{ij} > 0$ for each trader i and good j. Let $\pi \in \Delta^+$ be a price vector that is not a weak $(1+\varepsilon)$-approximate equilibrium, for some $\varepsilon > 0$. Then for any equilibrium $\hat{\pi} \in \Delta$, we have $\hat{\pi}^T Z(\pi) \geq \delta$, where $\delta \geq 1/2^{E_1}$, and E_1 is bounded by a polynomial in the input size and $\log\frac{1}{\varepsilon}$. Moreover $||Z(\pi)||_2 \leq 2^{E_2}$, where E_2 is bounded by a polynomial in the input size. (Note that $\pi^T Z(\pi) = 0$.)*

Proof :

The proof is similar to the previous lemma up to the derivation of Equation 5. The rest of the argument is as follows. At equilibrium price π^1, some trader, say i, demand at least W_1/m of good 1. That is, $x_{i1} \geq W_1/m$.

Since the price of good 1 is never increased at each of the $n-1$ steps, and the prices of the other goods never decrease at each of these steps, we have from weak GS of i's utility function that $x_{in} \geq \cdots \geq x_{i1}$.

We say that the k'th step is a *big* step if $\pi_{k+1} - \pi_k \geq \frac{\varepsilon\pi_1}{3n}$. We first claim that there must be a big step. For otherwise, $\pi_n - \pi_1 \leq \varepsilon\pi_1/3$, which implies that $\pi_j^n - \pi_j^1 \leq \varepsilon\pi_1^j/3 = \varepsilon \min\{\pi_j^1, \pi_j^n\}/3$. Using Lemma 3, we can conclude that the demand at π^1 is a $(1+\varepsilon)$-approximate demand at π^n. Since the market clears with the demand at equilibrium π^1, this implies that π^n is a weak $(1+\varepsilon)$-approximate equilibrium.

Suppose that the s'th step is a big step. Let β be the fraction of her income that i spends on goods in G_s^h at prices π^s. Since good 1 is in G_s^h, we have

$$\beta \geq \frac{\pi_1^s x_{is}}{\sum_j \pi_j^s w_{ij}} \geq \frac{\pi_1 x_{i1}}{\sum_j \pi_j W_j} \geq \frac{\pi_1 W_1}{m \sum_j \pi_j W_j}$$

Since the prices for goods in G_s^h remain the same in the s'th step whereas the prices for goods in G_s^t increase, we conclude from weak GS of trader i's homogeneous utility function[5] that she spends at least a fraction β of her income on goods in G_s^h at price π^{s+1}. Therefore the increase in the money she spend on goods in G_s^h at price π^{s+1} as compared to π^s is at least

$$\beta(\pi^{s+1} - \pi^s)^T w_i \geq \beta(\pi_j^{s+1} - \pi_j^s) w_{ij} \geq \beta \frac{\varepsilon\pi_1}{3n} w_{ij}$$

for some $j \in G_s^t$.

Weak GS also implies that $x_{ij}(\pi^{s+1}) \geq x_{ij}(\pi^s)$ for each $j \in G_s^h$. Since the prices remain the same for goods in G_s^h in the s'th step, we can conclude that

$$\sum_{j \in G_s^h} (x_{ij}(\pi^{s+1}) - x_{ij}(\pi^s)) \geq \frac{\beta\varepsilon\pi_1 w_{ij}}{3n \max_{j \in G_s^h} \pi_j^s} \geq \frac{\beta\varepsilon\pi_1 w_{ij}}{3n\pi_n}.$$

This implies that

$$\sum_{j \in G_s^h} (Z_j(\pi^{s+1}) - Z_j(\pi^s)) \geq \frac{\beta\varepsilon\pi_1 w_{ij}}{3n\pi_n}.$$

[5]This is the only place in the proof where homegeneity is used

Plugging this inequality into Equation 5, we have

$$\begin{aligned}\mathbf{1}^T(Z(\pi^{s+1}) - Z(\pi^s)) &\geq \frac{(\pi_{s+1} - \pi_s)}{\pi_{s+1}} \sum_{j \in G_s^h} (Z_j(\pi^{s+1}) - Z_j(\pi^s)) \\ &\geq \frac{\varepsilon \pi_1}{3n\pi_{s+1}} \cdot \frac{\beta \varepsilon \pi_1 w_{ij}}{3n\pi_n} \geq \frac{\varepsilon \pi_1}{3n\pi_n} \cdot \frac{\beta \varepsilon \pi_1 w_{ij}}{3n\pi_n} = \delta,\end{aligned}$$

completing the proof. □

Remark : The lemma can be shown to be true with minor changes to the proof even if traders have non-homogeneous VES functions. What is needed is not homogeneity but a much weaker property that, informally stated, says that if we increase the income of a trader while keeping the prices fixed, the demand increases significantly, in relation to the original demand, for every good. This property is true for VES and several other non-homogeneous functions that satisfy weak GS. We postpone to the final version a formal statement of this property.

Lemma 8 *Let M be an exchange market satisfying Assumptions 1 and 2. Let $\pi \in \Delta^+$ be a price vector that is not a strong $(1+\varepsilon)$-approximate equilibrium, for some $\varepsilon > 0$. Then there exists a $q \in \Delta^{++}$ such that $\pi^T \frac{Z(q)}{\|Z(q)\|_\infty} < -\delta$, where $\delta \geq 1/2^E$, and E is bounded by a polynomial in the input size and $\log \frac{1}{\varepsilon}$.*

Proof :

The fact that $\pi \in \Delta^+$ is a price vector that is not a strong $(1+\varepsilon)$-approximate equilibrium implies, by definition, that for some good j, $\sum_i x_{ij} > (1+\varepsilon)\sum_i w_{ij}$. Therefore, the excess demand of good j at prices π, $Z_j(\pi)$ satisfies $Z_j(\pi) > \varepsilon W_j$. Recall that $W_j = \sum_i w_{ij}$ is the total endowment of good j. Therefore, the excess demand vector $Z(\pi) = (Z_1(\pi), Z_2(\pi), \ldots, Z_n(\pi))$ satisfies $(Z(\pi))^T Z(\pi) > \varepsilon^2 (W_j)^2$.

Now let $q = \pi + tZ(\pi)$ for some $t > 0$ to be determined later. Our choice of t will guarantee that

(A) $q \in \Delta^{++}$ and

(B) $(Z(\pi))^T Z(q) > \frac{\varepsilon^2 (W_j)^2}{4}$.

We now make two claims about what value of t will guarantee (A) and (B). The proofs of these claims are somewhat technical and appear in the Appendix.

Claim 1: If $0 < t < \frac{1}{2^{L+2} W n 2^D}$ then $q \in \Delta^{++}$. Here $W = \max_k \{W_k\}$.

Claim 2: If $0 < t < \frac{\varepsilon}{n^3 2^{2D+3}}$ then $(Z(\pi))^T Z(q) > \frac{\varepsilon^2 (W_j)^2}{4}$.

Setting

$$t = \frac{\varepsilon}{n^3 W 2^{L+2D+3}}$$

satisfies the bounds in Claim 1 and 2 and therefore guarantees conditions (A) and (B).

The inequality in (B) has the following consequences. By Walras' Law, $q^T Z(q) = 0$. This implies

$$(\pi + tZ(\pi))^T Z(q) = \pi^T Z(q) + t(Z(\pi))^T Z(q) = 0.$$

Therefore,

$$\pi^T Z(q) = -t(Z(\pi))^T Z(q) < \frac{-t\varepsilon^2 (W_j)^2}{4} = -\frac{\varepsilon^3 (W_j)^2}{Wn^3 2^{L+2D+5}}. \tag{6}$$

For the final step we need bounds on $Z_k(q)$ for each good k. A lower bound is obtained by noting that $Z_k(q) = X_k(q) - W_k \geq -W_k$. To obtain an upper bound we first note that by Assumption 1, the market has an equilibrium price vector $\pi^* \in \Delta$. Since $Z_k(\pi^*) = 0$, $|\pi^*_\ell - q_\ell| \leq 1$ for all ℓ, and $|\frac{\partial Z_k(r)}{\partial r_\ell}| \leq 2^D$ for all goods k and ℓ and any $r \in \Delta^{++}$ (by Assumption 2), we can use the mean value theorem to get $|Z_k(q) - Z_k(\pi^*)| \leq n2^D$. Therefore, $Z_k(q) \leq n2^D$.

Using these bounds we conclude that $\|Z(q)\|_\infty \leq Wn2^D$ and therefore

$$\pi^T \frac{Z(q)}{\|Z(q)\|_\infty} < -\frac{\varepsilon^3 (w_j)^2}{W^2 n^4 2^{3D+L+5}} \equiv -\delta.$$

Note that $\|Z(q)\|_\infty \neq 0$ since $\pi^T Z(q) < -\delta < 0$. Because Assumptions 1 and 2 guarantee that L and D are bounded above by polynomials in the input size, it follows that $\delta \geq 1/2^E$ where E is bounded above by a polynomial in the input size and $\log(1/\varepsilon)$.

□

Lemma 9 *Let M be an exchange market and let $\pi \in \Delta^+$ be a price vector that is not a weak $(1+\varepsilon)$-approximate equilibrium, for some $\varepsilon > 0$. Then there exists a $q \in \Delta^{++}$ such that $\pi^T Z(q) \leq -\delta$, where $\delta \geq 1/2^{E_1}$, and E_1 is bounded by a polynomial in the input size and $\log \frac{1}{\varepsilon}$. Moreover $||Z(q)||_2 \leq 2^{E_2}$, where E_2 is bounded by a polynomial in the input size.*

Proof :

Let $\ell \in \mathbf{R}^n_+$ be the vector that minimizes the L_2 norm over the set S of all $(1+\varepsilon)$-approximate aggregate excess demands at price π. That is,

$$S = \{\sum_i (y_i - w_i) \mid y_i \text{ is a } (1+\varepsilon)\text{-approx demand for trader } i \text{ at } \pi\ \}$$

Since π is not a weak $(1+\varepsilon)$-approximate equilibrium, we have $\sum_j |l_j|^2 \geq \varepsilon^2 \min_j W_j^2$. Since $\pi \in \Delta^+$, it is not hard to see that $|\ell_j| \leq 2^E$, where E is bounded by a polynomial in the input size.

We note that the set S forms a convex set. Thus for any vector ℓ' in this set, we have using the definition of ℓ that $\ell^T \ell' \geq \ell^T \ell \geq \varepsilon^2 \min_j W_j^2$.

Let $\mu = \frac{\varepsilon 2^{-L}}{4 2^E}$, and let $q = p + \mu\ell$. Since $|\pi_j - q_j| \leq \frac{\varepsilon 2^{-L}}{4}$, we have that $q \in \Delta^{++}$ and $|\pi_j - q_j| \leq \varepsilon \min\{\pi_j, q_j\}$. From Lemma 3, it follows that the demand at q is a $(1+\varepsilon)$-approximate demand at prices π. Thus $Z(q) \in S$ and $Z(q)^T \ell \geq \varepsilon^2 \min_j W_j^2$.

By Walras' law, we have $0 = q^T Z(q) = (\pi + \mu\ell)^T Z(q)$, which implies that $\pi^T Z(q) = -\mu \ell^T Z(q) \leq -\mu\varepsilon^2 \min_j W_j^2 = -\delta$. The bound on δ follows from a simple calculation. Also since $q \in \Delta^{++}$, the bound on $||Z(q)||_2$ follows immediately.

□

Note that if M satisfies weak GS and $\hat{\pi}$ is any equilibrium, then Lemma 1 implies that $\hat{\pi}^T Z(q) \geq 0$ for q in Lemmas 8 and 9.

4 Algorithmic Implications of the Separation Lemmas

In this section we show how to turn the existence of separation results from the previous section into polynomial time algorithms using the ellipsoid method. As an illustration, we do this for one of the four results (Lemma 8). Specifically, we define a convex set K that contains all strong $(1+\varepsilon)$-approximate equilibrium price vectors and construct a polynomial time separation oracle that either asserts that the given price vector π is a strong $(1+\varepsilon)$-approximate equilibrium or produces a hyperplane that separates π from K. Plugging this separation oracle into the ellipsoid method gives a polynomial time algorithm for computing a strong $(1+\varepsilon)$-approximate equilibrium price vector. The same basic approach applies to Lemmas 6 and 7. The computational requirements for Lemma 9 are somewhat different and so we postpone a discussion of their computational implications to the full version.

In this section, we use $\mathbf{Q}$ to denote the set of rationals, $\mathbf{Q}^n$ to denote the set of all n-tuples of rationals, and $\mathbf{Q}^n_+$ to denote the set of all n-tuples of non-negative rationals.

Let $F = \{x \in \mathbf{R}^n_+ \mid x^T Z(\pi) \geq 0 \text{ for all } \pi \in \mathbf{R}^n_+\} \cap \Delta$. Let $\gamma = \delta/\sqrt{n}$, where δ is from Lemma 8. Using notation from [18], for any $A \subseteq \mathbf{R}^n$ and non-negative real a, let $S(A, a)$ denote the set $\{x \in \mathbf{R}^n \mid \|x - y\|_2 \leq a \text{ for some } y \in A\}$. If A is the singleton $\{c\}$ then we use $S(c, a)$ to denote $S(\{c\}, a)$. Let $K = S(F, \gamma)$. Note that K is a convex set. Also note that $\delta < 2^{-L}/2$ and therefore $\gamma < 2^{-L}/2$. Since $F \subseteq \Delta$, it follows that $S(F, \gamma) \in \Delta^+$. Let C_ε denote the set of all strong $(1+\varepsilon)$-approximate equilibrium price vectors in Δ^+.

Now we state (without proof) three simple properties of K: (i) For all $x \in K$, for all $\pi \in \mathbf{R}^n_+$, $x^T \frac{Z(\pi)}{\|Z(\pi)\|_\infty} \geq -\delta$, (ii) $K \subseteq C_\varepsilon$, and (iii) $\mathrm{Vol}(K) \geq \gamma^n$.

We are now almost ready to prove the algorithmic version of Lemma 8. We first need to introduce the notion of a demand oracle.

Definition 10

An exchange market M is said to be equipped with a *demand oracle* if there is an algorithm that takes as input a price vector $\pi \in \mathbf{Q}^n_+$ and a positive rational σ, and returns a vector $Y = (Y_1, Y_2, \ldots, Y_n)$ such that $|Y_j - Z_j(\pi)| \leq \sigma$ for all j. The algorithm is required to run in polynomial time in the input size and in $\log(1/\sigma)$.

In the algorithm below, we will use $\mathtt{DEM}_M(\pi, \sigma)$ to denote a call to the demand oracle for the market M with inputs π and σ. We will use the notation $a :\approx_p b$ to mean that what is assigned to a is the binary expansion of the right hand side, cut off p bits after the binary point. Assuming the existence of a demand oracle for a market M, we can construct a polynomial time separation oracle for the market, that separates any price vector π that is not a strong $(1+\varepsilon)$-approximate equilibrium from K, which is the set of interest to us.

Separating Hyperplane Construction Algorithm

Input: (i) Description of the market M satisfying Assumptions 1, 2, weak GS, and equipped with a demand oracle, (ii) a price vector $\pi \in \mathbf{Q}^n_+$, and (iii) a positive rational σ.

Output: A vector $Y \in \mathbf{R}^n$ such that either $Y = 0$ and $\pi \in C_\varepsilon$ or the following three properties hold: (i) $\|Y\|_\infty = 1$, (ii) $x^T Y \geq -\delta - \sigma/2$ for all $x \in K$, and (iii) if $\pi \notin \Delta^+$ or if $\pi \in \Delta^+$, but π is not a strong $(1+\varepsilon)$-approximate equilibrium price vector then $\pi^T Y < -\delta + \sigma/2$. Here $\delta = \frac{\varepsilon^3 (W_j)^2}{W^2 n^4 2^{3D+L+5}}$ (as in the proof of Lemma 8). Recall that L is from Assumption 1, D from Assumption 2, and $W = \max_k\{W_k\}$.

1. $p := L + 4 + \lceil \log_2 \left(\frac{n^2 2^D + W}{\sigma} \right) \rceil$;

2. $Z := \mathtt{DEM}_M(\pi, 1/2^p)$;
3. $t := \frac{\varepsilon}{Wn^3 2^{2D+L+3}}$;
4. $q :\approx_p \pi + t * Z$;
5. $Y := \mathtt{DEM}_M(q, 1/2^p)$;
5. **if** $(Y \neq 0)$ **then**
6. $\quad\quad Y := Y/\|Y\|_\infty$;
7. **return** Y;

Let us denote a call to this algorithm by $\mathtt{HYP}(M, \pi, \sigma)$. The next lemma, which is the algorithmic version of Lemma 8, proves the correctness of the above algorithm. The proof of this lemma is similar to the proof of Lemma 8, except that now we need to show that the claimed separation holds despite using limited precision to compute the quantities $Z(\pi)$, q, and $Z(q)$.

Lemma 11 *Let M be an exchange market satisfying Assumptions 1, 2, and weak GS. Further suppose that M is equipped with a demand oracle. Let $\pi \in \mathbf{Q}^n_+$ be a price vector. Let $\sigma, \varepsilon > 0$ be rationals. Then $\mathtt{HYP}(M, \pi, \sigma)$ returns a vector $Y \in \mathbf{R}^n_+$ such that either $Y = 0$ or $\|Y\|_\infty = 1$ and $x^T Y \geq -\delta - \sigma/2$ for all $x \in K$. Furthermore, if $\pi \notin \Delta^+$ or $\pi \in \Delta^+$, but π is not a strong $(1+\varepsilon)$-approximate equilibrium then $\pi^T Y < -\delta + \sigma/2$. $\mathtt{HYP}(M, \pi, \sigma)$ terminates in time that is polynomial in the input size, $\log(1/\varepsilon)$, and $\log(1/\sigma)$.*

¿From this separating hyperplane construction algorithm, we can construct a separation oracle for K, which we will call $\mathtt{SEP}_K$, following the notation in [18].

Separation Oracle for K
Input: (i) A price vector $\pi \in \mathbf{Q}^n_+$ and (ii) a rational $\sigma > 0$.
Output: Either (i) an assertion that $\pi \in C_\varepsilon$ or (ii) a vector $Y \in \mathbf{Q}^n_+$, $\|Y\|_\infty = 1$, such that $\pi^T Y < x^T Y + \sigma$ for all $x \in K$.

1. $Y := \mathtt{HYP}(M, \pi, \sigma)$;
2. **if** $(\pi^T Y < -\delta + \sigma/2)$ **then return** Y;
3. **else return** the assertion that $\pi \in C_\varepsilon$;

Correctness of $\mathtt{SEP}_K$ is easy to establish. If the condition in (2.) is true and $\pi^T Y < -\delta + \sigma/2$ then since $x^T Y \geq -\delta - \sigma/2$ for all $x \in K$, we have $\pi^T Y < x^T Y + \sigma$ for all $x \in K$. Otherwise, by correctness of $\mathtt{HYP}$, $\pi \in \Delta^+$ and π is a strong $(1+\varepsilon)$-approximate equilibrium. Using $\mathtt{SEP}_K$, the lower bound of γ^n on $\mathrm{Vol}(K)$, and Theorem 3.2.1 on Page 87 in [18] on the *central-cut ellipsoid method*, we obtain the following theorem.

Theorem 12 *Let M be an exchange market satisfying Assumptions 1, 2, and weak GS. Further suppose that M is equipped with a demand oracle. There exists an algorithm that takes as input a description of M and any rational $\varepsilon > 0$ and returns a strong $(1+\varepsilon)$-approximate equilibrium price vector in time that is polynomial in the input size and in $\log(1/\varepsilon)$.*

By similarly constructing a separation oracle based on Lemma 7 (and the remark following it), we obtain the following theorem.

Theorem 13 *Let M be an exchange market satisfying Assumption 1, and suppose that each utility function satisfies weak GS and is either homogeneous or a VES function. Further suppose that $w_{ij} > 0$ for each trader i and good j, that M is equipped with a demand oracle, and also* *that there is a poly-time algorithm that, given any $\pi \in \mathbf{R}^n_+$ and $\mu > 0$, either asserts that π is a weak $(1+\mu)$-approximate equilibrium or that π is not a weak $(1+\mu/2)$-approximate equilibrium. There exists an algorithm that takes as input a description of M and any rational $\varepsilon > 0$ and returns a weak $(1+\varepsilon)$-approximate equilibrium price vector in time that is polynomial in the input size and in* $\log(1/\varepsilon)$.

Note that we can handle other utility functions that are non-homoneneous provided they satisfy the weak requirement that is described in the remark following Lemma 7.

Consequences for specific utility functions

In this section, we describe the consequences of the general algorithm described above to markets with linear, CES ($\rho > 0$), Cobb-Douglas, and VES utility functions. The fact that these utility functions satisfy weak GS is readily seen from Proposition 2 of [27]. It is also not difficult to verify the existence of a demand oracle in each of these cases.

Since the demand of a trader with a linear utility function $u_i(x_i) = \sum_j a_{ij}x_{ij}$ is not single valued, we pick a rational number $\rho > 1 - \varepsilon/c\log n$, for a sufficiently large constant c that is independent of the a_{ij}'s, and replace the linear utility function by the CES utility function $v_i(x_i) = (\sum_j (a_{ij}x_{ij})^\rho)^{1/\rho}$. Since for any $x \in \mathbf{R}^n_+$, we have $u_i(x) \leq v_i(x) \leq (1+\varepsilon)u_i(x)$, this transformation is good enough for the computation of a weak approximate equilibrium. We will henceforth assume that the linear utility function is covered by the class of CES utility functions with $\rho > 0$.

For markets with Cobb-Douglas or CES utility functions with $\rho > 0$, we compute a weak $(1+\varepsilon)$-approximate equilibrium by applying Lemma 5 and then using the algorithm of Theorem 13 (Assumption 1 holds). Since after applying the transformation of Lemma 5 the market always has an equilibrium while the original market may not, we compute a weak $(1+\varepsilon)$-approximate equilibrium for the input market even when it does not have an equilibrium.

For markets that include the VES functions, we cannot apply Lemma 5. Under the assumption that each trader's initial endowment includes something of every good, we compute a weak $(1+\varepsilon)$-equilibrium using the algorithm of Theorem 13 (Assumption 1 holds). We are currently investigating if the computational version of Lemma 9 is available for the utility functions discussed here. This is essentially a question of whether a certain explicit convex program is solvable in poly-time up to a specified precision. If the answer is in the affirmative, as we believe it is, the assumption on the initial endowments can be dispensed with.

For markets with Cobb-Douglas or CES utilities with ρ bounded away from 1 by some positive constant, the demand does not change too rapidly as a function of the prices. Thus Assumption 2 is valid in these situations and we can use the algorithm of Theorem 12 to get a *strong* $(1+\varepsilon)$-approximate equilibrium whenever the input market has an equilibrium (Assumption 1 holds). If the input market does not have an equilibrium, we can detect this in poly-time [24].

5 Ongoing Work

Interestingly enough, the approach described in this paper also works when the utility functions are homogeneous of degree one (they need not satisfy weak GS) but the endowments are required to be proportional (the Fisher setting). This is because of the phenomenon of *aggregation*, i.e., the market excess demand behaves as though it were the excess demand of an individual trader with an appropriate concave utility function [13, 5]. This implies that the market excess demand function satisfies WARP ([32], pp. 399-400), which is easily seen to imply the inequality of Lemma 1. We then obtain an approach for the computation of equilibria which is an alternative to the one implied by Eisenberg's construction [13] (see also [6]). In order to make specific claims on the kind of homogeneous utility functions that can be accommodated, we have to check that Assumption 1 holds and a demand oracle is available.

Lemmas 6 and 7 open the door to the analysis of an algorithm based on *tâtonnement* [2]: if the current price π is not close to equilibrium, increase the prices of goods for which the excess demand is positive and decrease the prices of goods for which the excess demand is negative. Indeed, suppose that $\pi \in \Delta^+$ and that π is not an approximate equilibrium. Then let $\pi' = \pi + \mu Z(\pi)$, where $\mu > 0$. Lemmas 6 and 7 imply that π' is closer than π to every equilibrium in Δ if μ is chosen small enough. On the other hand, the Lemmas also imply that μ can be chosen large enough that the distance to every equilibrium in Δ falls significantly. Now if π is not in Δ^+ then we simply set π' to be the point in Δ closest to π; the distance falls significantly in this case too.

This approach gives us an algorithm that is exponential as stated but in many interesting situations runs in pseudo-polynomial time. A more refined analysis may even give a poly-time approximation scheme, and we postpone a discussion of this to the full version.

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Appendix

Claim 1 from the proof of Lemma 8

If $0 < t < \frac{1}{2^{L+2}Wn2^D}$ then $q \in \Delta^{++}$. Here $W = \max_k\{W_k\}$.

Proof : From the definition of the set Δ^{++} and the fact that $q = \pi + tZ(\pi)$, we see that we need to ensure

$$2^{-L} - \frac{2^{-L}}{2} - \frac{2^{-L}}{4} \le \pi_k + tZ_k(\pi) \le 1 + \frac{2^{-L}}{2} + \frac{2^{-L}}{4} \text{ for all goods } k. \tag{7}$$

Since $\pi \in \Delta^+$, we have the following bounds on π_k: $2^{-L} - \frac{2^{-L}}{2} \le \pi_k \le 1 + \frac{2^{-L}}{2}$. As shown in the proof of Lemma 8, $-W_k \le Z_k(\pi) \le n2^D$. Using the bounds on π_k and the bounds on $Z_k(\pi)$ we note that to ensure condition (7), it suffices to ensure

$$2^{-L} - \frac{2^{-L}}{2} - \frac{2^{-L}}{4} \le 2^{-L} - \frac{2^{-L}}{2} - tW_k \text{ for all } k$$

and

$$1 + \frac{2^{-L}}{2} + t(n2^D) \le 1 + \frac{2^{-L}}{2} + \frac{2^{-L}}{4}.$$

Since $W = \max_k\{W_k\}$, we see that any value of t,

$$t \le \frac{1}{2^{L+2}Wn2^D}. \tag{8}$$

satisfies all of the above inequalities and will ensure that $q \in \Delta^{++}$.

□

Claim 2 from the proof of Lemma 8

If $0 < t < \frac{\varepsilon}{n^3 2^{2D+3}}$ then $(Z(\pi))^T Z(q) > \frac{\varepsilon^2(W_j)^2}{4}$.

Proof : We will now show how to pick a value of t that will guarantee condition (B), that is, $(Z(\pi))^T Z(q) > \frac{\varepsilon^2(W_j)^2}{4}$. Let $\gamma = \frac{\varepsilon W_j}{\sqrt{6n}}$. We will pick a value of t that will guarantee for all k

(i) If $|Z_k(\pi)| \le \gamma$ then $|Z_k(\pi)Z_k(q)| < \frac{3}{2}\gamma^2$.

(ii) If $|Z_k(\pi)| > \gamma$ then $Z_k(\pi)Z_k(q) > \frac{(Z_k(\pi))^2}{2}$.

If these two conditions hold, then

$$(Z(\pi))^T Z(q) = \sum_k Z_k(\pi)Z_k(q) > \frac{\varepsilon^2(W_j)^2}{2} - \frac{3}{2}n\gamma^2. \tag{9}$$

The latter inequality holds because for good j, $Z_j(\pi) > \varepsilon W_j > \frac{\varepsilon W_j}{\sqrt{6n}}$. Therefore, by condition (ii), $Z_j(\pi)Z_j(q) > \frac{\varepsilon^2(W_j)^2}{2}$. By conditions (i) and (ii), the remaining terms of the summation $\sum_k Z_k(\pi)Z_k(q)$ are each at least $-\frac{3}{2}\gamma^2$. This establishes the inequality in (9). Now substituting $\gamma = \frac{\varepsilon W_j}{\sqrt{6n}}$ in the right hand side of (9), we get that $(Z(\pi))^T Z(q) > \frac{\varepsilon^2(W_j)^2}{4}$.

To guarantee condition (i), it suffices to ensure that $|Z_k(q) - Z_k(\pi)| \le \gamma/2$ for all k for which $|Z_k(\pi)| \le \gamma$. This will ensure that $|Z_k(q)| \le 3\gamma/2$ and therefore $|Z_k(\pi)Z_k(q)| \le 3\gamma^2/2$. To ensure $|Z^k(q) - Z^k(\pi)| \le \gamma/2$, it suffices to ensure that $|q_k - \pi_k| \le \frac{1}{n2^D}\frac{\gamma}{2}$. This follows by using the bound $|\frac{\partial Z_k(r)}{\partial r^\ell}| \le 2^D$ for all goods k and ℓ and all $r \in \Delta^{++}$ and by using the mean value theorem. To ensure that $|q_k - \pi_k| \le \frac{1}{n2^D}\frac{\gamma}{2}$, we need to ensure that $|tZ_k(\pi)| \le \frac{1}{n2^D}\frac{\gamma}{2}$ for all k because $q_k = \pi_k + tZ_k(\pi)$. Since condition (i) has $|Z_k(\pi)| \le \gamma$ as its hypothesis, picking a value of t,

$$t \le \frac{1}{n2^{D+1}} \tag{10}$$

suffices.

Let $K = \{k \mid |Z^k(\pi)| > \gamma\}$. To guarantee condition (ii), it suffices to ensure that $|Z_k(\pi) - Z_k(q)| \le \frac{|Z_k(\pi)|}{2}$ for all $k \in K$. By using the bound $|\frac{\partial Z_k(r)}{\partial r_\ell}| \le 2^D$ for all goods k and ℓ and all $r \in \Delta^{++}$ and the mean value theorem, we see that this can be ensured by guaranteeing that

$$|\pi_\ell - q_\ell| \le \frac{|Z_k(\pi)|}{n2^{D+1}} \text{ for all goods } \ell \text{ and for all } k \in K.$$

Since $|Z_k(\pi)| > \gamma$ for all $k \in K$, the above condition is ensured if

$$|\pi_\ell - q_\ell| \le \frac{\gamma}{n2^{D+1}}. \tag{11}$$

In ensuring that condition (i) holds, we have already ensured the above condition for all $\ell \notin K$. Now let $\ell \in K$. To ensure (11), it suffices to ensure that $|tZ_\ell(\pi)| \le \frac{\gamma}{n2^{D+1}}$ because $q_\ell = \pi_\ell + tZ_\ell(\pi)$. Using the upper bound $|Z_\ell(\pi)| \le n2^D$ calculated earlier, we note that $|tZ_\ell(\pi)| \le \frac{\gamma}{n2^{D+1}}$ is ensured by choosing $t \le \frac{\gamma}{n^2 2^{2D+1}}$. Since $\gamma = \frac{\varepsilon W_j}{\sqrt{6n}}$, we see that picking a value of t, $t \le \frac{\varepsilon}{n^3 2^{2D+3}}$ will ensure that $(Z(\pi))^T Z(q) > \frac{\varepsilon^2 (W_j)^2}{4}$. □