Approximating Nash Social Welfare by Matching and Local Search*

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ABSTRACT

For any $\varepsilon > 0$, we give a simple, deterministic $(4+\varepsilon)$-approximation algorithm for the Nash social welfare (NSW) problem under submodular valuations. The previous best approximation factor was 380 via a randomized algorithm. We also consider the asymmetric variant of the problem, where the objective is to maximize the weighted geometric mean of agents' valuations, and give an $(\omega + 2 + \varepsilon)$e-approximation if the ratio between the largest weight and the average weight is at most ω.

We also show that the $\frac{1}{2}$-EFX envy-freeness property can be attained simultaneously with a constant-factor approximation. More precisely, we can find an allocation in polynomial time which is both $\frac{1}{2}$-EFX and a $(8+\varepsilon)$-approximation to the symmetric NSW problem under submodular valuations. The previous best approximation factor under $\frac{1}{2}$-EFX was linear in the number of agents.

CCS CONCEPTS

• **Theory of computation → Approximation algorithms analysis**; • **Mathematics of computing → Combinatorial optimization**.

KEYWORDS

Nash Social Welfare, Envy-freeness, Fairness, Local Search, Approximation Algorithms, Combinatorial Optimization

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1 INTRODUCTION

We consider the problem of allocating a set G of m indivisible items among a set A of n agents, where each agent $i \in A$ has a valuation function $v_i : 2^G \to \mathbb{R}_{\geq 0}$ and weight (entitlement) $w_i > 0$ such that $\sum_{i \in A} w_i = 1$. The Nash social welfare (NSW) problem asks for an allocation $\mathcal{S} = (S_i)_{i \in A}$ that maximizes the weighted geometric mean of the agents' valuations,

$$\mathrm{NSW}(\mathcal{S}) = \prod_{i \in A} (v_i(S_i))^{w_i}.$$

We refer to the special case when all agents have equal weight (i.e., $w_i = 1/n$) as the *symmetric* NSW problem, and call the general case the *asymmetric* NSW problem. Throughout, we let $w_{\max} := \max_{i \in A} w_i$. For $\alpha > 1$, an *α-approximate solution* to the NSW problem is an allocation $\mathcal{S}$ with $\mathrm{NSW}(\mathcal{S}) \geq \mathrm{OPT}/\alpha$, where OPT denotes the optimum value of the NSW-maximization problem.

Allocating resources among agents in a fair and efficient manner is a fundamental problem in computer science, economics, and social choice theory; we refer the reader to the monographs [5, 10, 11, 41, 45, 46, 48] on the background. A common measure of efficiency is *utilitarian social welfare*, i.e., the sum of the utilities $\sum_{i \in A} v_i(S_i)$ for an allocation $(S_i)_{i \in A}$. In contrast, fairness is often measured by *max-min fairness*, i.e., $\min_{i \in A} v_i(S_i)$; maximizing this objective is also known as the *Santa Claus problem* [4].

Symmetric NSW provides a balanced tradeoff between the often conflicting requirements of fairness and efficiency. It has been introduced independently in a variety of contexts. It is a discrete analogue of the Nash bargaining game [33, 42]; it corresponds to the notion of competitive equilibrium with equal incomes in economics [47]; and arises as a proportional fairness notion in networking [34]. The more general asymmetric objective has also been well-studied since the seventies [31, 32]. It has found many applications in different areas, such as bargaining theory [15, 35], water resource allocation [22, 30], and climate agreements [49].

A distinctive feature of the NSW problem is invariance under scaling of the valuation functions v_i by independent factors λ_i, i.e., each agent can express their preference in a "different currency"

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without changing the optimization problem (see [41] for additional characteristics).

$\frac{1}{2}$-EFX Allocations. Envy-freeness up to any item (EFX) is considered the most compelling fairness notion in the discrete setting with equal entitlements [14], where an allocation $\mathcal{S} = (S_i)_{i\in A}$ is said to be EFX if

$$v_i(S_i) \geq v_i(S_k - j), \quad \forall i, k \in A, \forall j \in S_k .$$

That is, no agent envies another agent's bundle after the removal of any single item from the envied agent's bundle. It is not known whether EFX allocations always exists or not, and it is regarded as the "fair division's biggest open question" [44]. This motivated the study of its relaxation α-EFX for an $\alpha \in (0, 1)$, where an allocation $\mathcal{S}$ is said to be α-EFX if

$$v_i(S_i) \geq \alpha \cdot v_i(S_k - j), \quad \forall i, k \in A, \forall j \in S_k .$$

The best-known α, for which the existence is known, is $\frac{1}{2}$ for submodular valuations, albeit with the efficiency guarantee of $O(n)$-approximation to the symmetric NSW problem [18, 43].

For NSW, without loss of generality we can assume that the allocations $\mathcal{S} = (S_i)_{i\in A}$ partition the set of items, i.e., $\cup_{i\in A} S_i = G$. We call such an allocation a *complete allocation*; an allocation $\mathcal{S}$ with $\cup_{i\in A} S_i \subsetneq G$ will be called a *partial allocation.*

In the context of envy-free allocations, it might be beneficial not to allocate some items: the allocation with $S_i = \emptyset$ for each agent is in fact envy-free. The two challenges are to find a complete allocation that satisfies certain envy-freeness property, and to guarantee efficiency, such as high NSW value at the same time.

Submodular and Subadditive Valuation Functions. A set function $v : 2^G \to \mathbb{R}$ is *monotone* if $v(S) \leq v(T)$ whenever $S \subseteq T$. A monotone set function with $v(\emptyset) = 0$ is also called a *valuation function* or simply *valuation.* The function $v : 2^G \to \mathbb{R}$ is *submodular* if

$$v(S) + v(T) \geq v(S \cap T) + v(S \cup T) \quad \forall S, T \subseteq G ,$$

and *subadditive* if

$$v(S) + v(T) \geq v(S \cup T) \quad \forall S, T \subseteq G .$$

We assume the valuation functions are given by value oracles that return $v(S)$ for any $S \subseteq G$ in $O(1)$ time.

Our Contributions. Our main theorem on NSW is the following.

Theorem 1.1. *For any $\varepsilon > 0$, there is a deterministic polynomial-time $(nw_{\max} + 2 + \varepsilon)$e-approximation algorithm for the asymmetric Nash social welfare problem with submodular valuations. For symmetric instances, the algorithm returns a $(4 + \varepsilon)$-approximation. The number of arithmetic operations and value oracle calls is polynomial in n, m, and $1/\varepsilon$.*

Algorithm 1 in Section 2.1 presents the algorithm asserted in the theorem. Note that $nw_{\max}$ is the ratio between the maximum weight $w_{\max}$ and the average weight $(1/n)$. In the symmetric case, when all weights are $w_i = 1/n$, this bound gives $(3 + \varepsilon)\mathrm{e} < 8.2$. In this case, we can improve the analysis to obtain a $(4 + \varepsilon)$-approximation algorithm.

As our second main result, we show that a $\frac{1}{2}$-EFX allocation with high NSW value exists and can also be efficiently found. We give a general reduction for subadditive valuations. In the context of $\frac{1}{2}$-EFX allocations, $\mathrm{NSW}(\mathcal{S})$ will always refer to the NSW value of allocation $\mathcal{S}$ in the symmetric case ($w_i = 1/n$ for all $i \in A$).

Theorem 1.2. *There is a deterministic strongly polynomial-time algorithm that given a symmetric NSW instance with subadditive valuations and given a (complete or partial) allocation $\mathcal{S}$ of the items, it returns a complete allocation $\mathcal{T}$ that is $\frac{1}{2}$-EFX and $\mathrm{NSW}(\mathcal{T}) \geq \mathrm{NSW}(\mathcal{S})/2$.*

The above algorithm is strongly polynomial in the value oracle model: number of basic arithmetic operations and oracle calls is polynomially bounded in n and m. Together with Theorem 1.1, we obtain the following corollary.

Corollary 1.3. *For any $\varepsilon > 0$, there is a deterministic polynomial algorithm that returns a $\frac{1}{2}$-EFX complete allocation that is $(8 + \varepsilon)$-approximation to the symmetric NSW problem under submodular valuations. The number of arithmetic operations and value oracle calls is polynomial in n, m, and $1/\varepsilon$.*

1.1 Related Work

Prior Work on Approximating NSW. Let us first consider *additive valuations*, i.e., when $v_i(S) = \sum_{j\in S} v_{ij}$ for nonnegative values v_{ij}. Maximizing symmetric NSW is NP-hard already in the case of two agents with identical additive valuations, by a reduction from the Subset-Sum problem. It is NP-hard to approximate within a factor better than 1.069 for additive valuations [26], and better than 1.5819 for submodular valuations [29].

On the positive side, a number of remarkably different constant-factor approximations are known for additive valuations. The first such algorithm with the factor of $2 \cdot e^{1/e} \approx 2.889$ was given by Cole and Gkatzelis [21] using a continuous relaxation based on a particular market equilibrium concept. Later, [20] improved the analysis of this algorithm to achieve the factor of 2. Anari, Oveis Gharan, Saberi, and Singh [2] used a convex relaxation that relies on properties of real stable polynomials. The current best factor is $\mathrm{e}^{1/\mathrm{e}} + \varepsilon \simeq 1.45$ by Barman, Krishnamurthy, and Vaish [8]; the algorithm uses a different market equilibrium based approach.

For the general class of subadditive valuations [6, 18, 29], $O(n)$-approximations are known. This is the best one can hope for in the value oracle model [6], for the same reasons that this is impossible for the utilitarian social welfare problem [23]. Sublinear approximation $O(n^{53/54})$ is possible for XOS valuations if we are given access to both demand and XOS oracles [7]. Recall that all submodular valuations are XOS, and all XOS valuations are subadditive.

Constant-factor approximations were also obtained beyond additive valuation functions: capped-additive [27], separable piecewise-linear concave (SPLC) [3], and their common generalization, capped-SPLC [16] valuations; the approximation factor for capped-SPLC valuations matches the $\mathrm{e}^{1/\mathrm{e}} + \varepsilon$ factor for additive valuations. All these valuations are special classes of submodular. Subsequently, Li and Vondrák [37] designed an algorithm that estimates the optimal value within a factor of $\frac{\mathrm{e}^3}{(\mathrm{e}-1)^2} \simeq 6.8$ for a broad class of submodular valuations, such as coverage and summations of matroid rank functions, by extending the techniques of [2] using real stable polynomials. However, this algorithm only estimates the optimum value but does not find a corresponding allocation in polynomial time.

In [28], Garg, Husić, and Végh developed a constant-factor approximation for a broader subclass of submodular valuations called *Rado-valuations*. These include weighted matroid rank functions and many others that can be obtained using operations such as induction by network and contractions. An important example outside this class is the coverage valuation. They attained an approximation ratio 772 for the symmetric case and $772(w_{\max}/w_{\min})^3$ for the asymmetric case. Most recently, Li and Vondrák [38] obtained a randomized 380-approximation for symmetric NSW under submodular valuations by extending the the approach of [28].

We significantly improve and simplify the approach used in [28] and [38]; we give a comparison to these works in Section 2.2.

Prior Work on EFX and Related Notions. The existence of EFX allocations has not been settled despite significant efforts [14, 17, 43, 44]. This problem is open for more than two agents with general monotone valuations (including submodular), and for more than three agents with additive valuations. This necessitated the study of its relaxations α-EFX for $\alpha \in (0, 1)$ and partial EFX allocations. For the notion of α-EFX, the best-known α is 0.618 for additive [1] and 0.5 for general monotone valuations (including submodular) [43].

For the notion of partial EFX allocations, the existence is known for general monotone valuations if we do not allocate at most $n - 2$ items [9, 19, 40], albeit without any efficiency guarantees. For additive valuations, although $n - 2$ is still the best bound known, there exist partial EFX allocations with 2-approximation to the NSW problem [13].

A well-studied weaker notion is envy-freeness up to one item (EF1), where no agent envies another agent after the removal of *some* item from the envied agent's bundle. EF1 allocations are known to exist for general monotone valuations and can also be computed in polynomial-time [39]. However, an EF1 allocation alone is not desirable because it might be highly inefficient in terms of any welfare objective. For additive valuations, the allocations maximizing NSW are EF1 [14]. Although the NSW problem is APX-hard [36], there exists a pseudopolynomial time algorithm to find an allocation that is EF1 and 1.45-approximation to the NSW problem under additive valuations [8]. For capped-SPLC valuations, [16] shows the existence of an allocation that is $\frac{1}{2}$-EF1 and 1.45-approximation to the NSW problem. The existence of an EF1 allocation with high NSW is open for submodular valuations.

Subsequent to our work, [24] improves Theorem 1.2 to show the existence of an allocation $\mathcal{T}$ that is $\frac{1}{2}$-EFX and $\mathrm{NSW}(\mathcal{T}) \geq \frac{2}{3}\,\mathrm{NSW}(\mathcal{S})$ for a given allocation $\mathcal{S}$.

1.2 Notation

We will also use monotone set functions with $v(\emptyset) > 0$; we refer to these as *endowed valuation functions*. We use $\log(x)$ for the natural logarithm throughout. For set $S \subseteq G$ and $j \in G$, we use $S + j$ to denote $S \cup \{j\}$ and $S - j$ for $S \setminus \{j\}$ and we write $v(j)$ for $v(\{j\})$. For a vector $p \in \mathbb{R}^G$ and $S \subseteq G$, we denote $p(S) = \sum_{i \in S} p_i$.

By a *matching* from A to G we mean a mapping $\tau : A \to G \cup \{\perp\}$ where $\tau(i) \neq \tau(j)$ if $\tau(i) \neq \perp$; $\perp$ is a special symbol representing unmatched agents.

2 OVERVIEW OF THE ALGORITHMS

2.1 Approximation Algorithm for Nash Social Welfare

Algorithm 1 is our new proposed algorithm for the Nash social welfare problem. We start with an overview of the algorithm. The analysis is given in Section 3.

Algorithm 1: Approximating the submodular NSW problem

```
Input: Valuations $(v_i)_{i \in A}$ over $G$, weights $w \in \mathbb{R}^A_{>0}$ such that
        $\sum_{i \in A} w_i = 1$, and $\varepsilon > 0$.
Output: Allocation $\mathcal{S} = (S_i)_{i \in A}$.
Find a matching $\tau : A \to G$ maximizing $\prod_{i \in A} v_i(\tau(i))^{w_i}$ and
  set $H := \tau([n]), J := G \setminus H$
$\mathcal{R} = (R_i)_{i \in A}$ :=LocalSearch($J, (v_i)_{i \in A}$)
Find a matching $\sigma : A \to H$ maximizing $\prod_{i=1}^{n} v_i(R_i + \sigma(i))^{w_i}$
return $\mathcal{S} = (R_i + \sigma(i))_{i \in A}$
```

Phase 1: Initial Matching. We find an optimal assignment of one item to each agent, i.e., a matching $\tau : A \to G$ maximizing $\prod_{i \in A} v_i(\tau(i))^{w_i}$. This can be done using a max-weight matching algorithm with weights $w_i \log v_i(j)$ in the bipartite graph between A and G with edge set $\{(i, j) : v_i(j) > 0\}$. If no matching of size n exists, then we can conclude that there is no allocation with positive NSW value, and return an arbitrary allocation. For the rest of the paper, we assume there is a matching covering A, and let $H := \tau([n])$ be the set of matched items.

Phase 2: Local Search. In the second phase, we let $J := G \setminus H$ denote the set of items not assigned in the first phase. We let $\bar{A} := \{i \in A : v_i(J) > 0\}$ denote the set of agents that have a positive value on the items in J. For every $i \in \bar{A}$, we select

$$\ell(i) \in \operatorname*{argmax}_{j \in J} v_i(j)$$

as a *favorite* item of agent i in J. By submodularity, $v_i(\ell(i)) > 0$. For each $i \in \bar{A}$, we define the endowed valuation function $\bar{v}_i : 2^J \to \mathbb{R}_{>0}$ as

$$\bar{v}_i(S) := v_i(\ell(i)) + v_i(S) \quad \forall S \subseteq J\,.$$

Thus, $\bar{v}_i(\emptyset) = v_i(\ell(i))$, and $\bar{v}_i(j) \leq 2\bar{v}_i(\emptyset)$ for any $j \in J$. Further, we set the accuracy parameter

$$\bar{\varepsilon} := -1 + \sqrt[m]{1 + \varepsilon}\,.$$

(Instead of this exact value, we can set a lower value within a constant factor range.)

Our local search starts with allocating all items to a single agent in $\bar{A}$. As long as moving one item to a different agent increases the potential function

$$\prod_{i \in \bar{A}} (\bar{v}_i(R_i))^{w_i}$$

by at least a factor $(1 + \bar{\varepsilon})$, we perform such an exchange. Phase 2 terminates when no more such exchanges are possible, and returns the current allocation. For all agents $i \in A \setminus \bar{A}$, we let $R_i = \emptyset$.

Algorithm 2: $\texttt{LocalSearch}(J, (v_i)_{i \in A})$

```
$\bar{A} \leftarrow \{i \in A : v_i(J) > 0\}$
$\ell(i) \leftarrow \mathrm{argmax}\{v_i(\ell) : \ell \in J\}$ for $i \in \bar{A}$
Define $\bar{v}_i(S) := v_i(\ell(i)) + v_i(S)$
$R_k \leftarrow J$ for some $k \in \bar{A}$ and $R_i \leftarrow \emptyset$ for $i \in A - k$
while $\exists i, k \in \bar{A}$ and $j \in R_i$ such that $\left(\frac{\bar{v}_i(R_i - j)}{\bar{v}_i(R_i)}\right)^{w_i} \cdot \left(\frac{\bar{v}_k(R_k + j)}{\bar{v}_k(R_k)}\right)^{w_k} > 1 + \bar{\varepsilon}$ do
    $R_i \leftarrow R_i - j$ and $R_k \leftarrow R_k + j$
return $\mathcal{R} := (R_i)_{i \in A}$
```

Phase 3: Rematching. In the final phase, we match the items in H to the agents optimally, considering allocation $\mathcal{R} = (R_i)_{i \in A}$ of J. This can be done by again solving a maximum-weight matching problem, now with weights $w_{ij} = w_i \log v_i(R_i + j)$.

2.2 Our Techniques and Comparison with Previous Approaches

We now compare our algorithm to those in [28] and in [38]. At a high level, all three algorithms proceed in three phases, with Phases 1 and 3 being the same as outlined above. However, they largely differ in how the allocation $\mathcal{R}$ of $J = G \setminus H$ is obtained in Phase 2.

Garg, Husić, and Végh [28] use a rational convex relaxation, based on the concave extension of Rado valuations. After solving the relaxation exactly, they use combinatorial arguments to sparsify the support of the solution and construct an integral allocation.

Li and Vondrák [38] allow arbitrary submodular valuations. For submodular functions, the concave extension is NP-hard to evaluate. Instead, they work with the multilinear extension. This can be evaluated with random sampling, but it is not convex. To solve the relaxation (approximately), they use an iterated continuous greedy algorithm. The allocation $\mathcal{R}$ is obtained by independent randomized rounding of this fractional solution. Whereas the algorithm is simple, the analysis is somewhat involved. The main tool to analyze the rounding is the Efron–Stein concentration inequality; but this only works well if every item in the support of the fractional solution has bounded value. This is not true in general, and the argument instead analyzes a two-stage randomized rounding that gives a lower bound on the performance of the actual algorithm. First, a set of 'large' fractional items is preserved, and a careful combinatorial argument is needed to complete the allocation.

Our approach for the second part is radically different and much simpler. We do not use any continuous relaxation, but $\mathcal{R}$ is obtained by a simple local search with respect to the modified valuation functions. Because of using these modified valuations, we can first guarantee a high NSW value of the infeasible allocation $(R_i + \ell(i))_{i \in A}$ of J in the analysis. Our analysis of the local search is inspired by the *conditional equilibrium* notion introduced by Fu, Kleinberg, and Lavi [25]. They show that any conditional equilibrium 2-approximates the utilitarian social welfare and give an auction algorithm for finding such an equilibrium under submodular valuations.

We note that local search applied directly to the NSW problem cannot yield a constant factor approximation algorithm even if we allow changing an arbitrary fixed number k of items. This can be seen already when $m = n$, i.e., every allocation with positive NSW value is a matching. Also, some other natural variants of local search do not work, or the analysis is not clear; for example, our analysis does not seem to work for local search applied to the (seemingly more natural) choice of $\bar{v}_i(S) = v_i(S + \tau(i))$. The idea of defining $\ell(i)$ and using the modified valuation functions is inspired by rounding of the fractional solution from previous approaches; the role of the $\ell(i)$ items is similar to the large items in [38], but we obtain much better guarantees using a more direct deterministic approach.

The last part of the analysis concerns the rematching in Phase 3. Here, we convert the infeasible allocation $(R_i + \ell(i))_{i \in A}$ to a feasible allocation by an alternating path argument, combining the initial matching τ and an (unknown) optimal matching g. While the rematching phase was already present (and essentially identical) in [28] and [38], it is implemented and analyzed differently here. We show the existence of a matching ρ that together with $\mathcal{R}$ gives good approximation of the optimum. The papers [28] and [38] find such ρ by first showing that there is matching π that has high NSW together with $\mathcal{R}$ and the items $\ell(i)$. Then, they show in a convoluted way that we can remove the items $\ell(i)$ and find a matching ρ (as a combination of π and the initial matching τ) while only losing only a constant in objective when compared to the solution consisting of $\pi, \mathcal{R}$ and the $\ell(i)$'s.

We prove the existence of a good matching ρ by carefully analyzing the alternating cycles in the union of the optimal allocation of H and the initial matching τ of Phase 1. Our proof is much simpler than the previous analysis of [28] and [38], and facilitates the improved approximation factor. (The exact numbers are difficult to compare as the loss depends on the properties of solutions obtained in Phase 2, and since in the current paper the analysis of Phase 2 and Phase 3 is done in a more synchronous way.) We note that the particular matching ρ mentioned here is not needed; the algorithm finds the most profitable matching with respect to the $\mathcal{R}$. This provides a solution at least as good as the one in the analysis.

2.3 1/2-EFX Guarantee

The algorithm asserted in Theorem 1.2 is Algorithm 3 in Section 4.1. Our first key tool is a subroutine that finds a partial allocation that is $\frac{1}{2}$-EFX and preserves a large fraction of the NSW value.

Lemma 2.1. *There exists a deterministic strongly polynomial algorithm* $\texttt{MakeFairOrEfficient}(\mathcal{T})$*, that, for any partial allocation* $\mathcal{T}$*, returns another partial allocation* $\mathcal{R}$ *that satisfies one of the following properties*

(i) $\mathrm{NSW}(\mathcal{R}) \geq \mathrm{NSW}(\mathcal{T})$ *and* $\cup_{i \in A} R_i \subsetneq \cup_{i \in A} T_i$*, or*
(ii) $\mathrm{NSW}(\mathcal{R}) \geq \frac{1}{2}\,\mathrm{NSW}(\mathcal{T})$ *and* $\mathcal{R}$ *is* $\frac{1}{2}$*-EFX.*

This is shown by modifying the approach of Caragiannis, Gravin, and Huang [13]. For additive valuations, their algorithm takes an input allocation $\mathcal{T}$ and returns a partial allocation $\mathcal{R}$ that is EFX and $\mathrm{NSW}(\mathcal{R}) \geq \frac{1}{2}\,\mathrm{NSW}(\mathcal{T})$. We simplify and extend this approach from additive to subadditive valuations, but prove only the weaker $\frac{1}{2}$-EFX property.

The key subroutine for them provides a similar alternative as in Lemma 2.1. In outcome (ii), they have the stronger EFX guarantee, while in outcome (i), they show that the NSW value increases by a certain factor. In outcome (i), it is not clear how an increase in the NSW value could be shown for subadditive valuations. However, arguing about the support decrease leads to a simpler argument.

In [13], only a partial EFX allocation is found. Theorem 1.2 shows the existence of a complete allocation, albeit with the weaker $\frac{1}{2}$-EFX property. To derive Theorem 1.2, we start by repeatedly calling `MakeFairOrEfficient` until outcome (ii) is reached. Note that the outcome (i) can only happen at most m times because the number of items in $\mathcal{R}$ reduces by at least one after each call.

The allocation at this point may be partial. We show that the remaining items can be allocated using the classical *envy-free cycle procedure* by Lipton, Markakis, Mossel, and Saberi [39]. Even though this procedure is known for the weaker EF1 property [12], we show that—after a suitable preprocessing step—it can produce an $\frac{1}{2}$-EFX allocation while not decreasing the NSW value of the allocation.

3 ANALYSIS OF THE NSW ALGORITHM

In this section, we prove Theorem 1.1. In Section 3.1, we formulate simple properties of approximate local optimal solution found in Phase 2. This is followed by a technical bound comparing the approximate local optimal solution to the optimal solution. In this step, we present two different analyses: in Section 3.2 for the asymmetric case, and in Section 3.3 for the symmetric case. Section 3.4 gives a lower bound on the weight of the final matching found in Phase 3 of the algorithm; this argument is the same for the asymmetric and symmetric cases. This completes the proof of Theorem 1.1.

3.1 Local Optima

Throughout this section, we work with the item set J, set of agents $\bar{A}$, favourite items $\ell(i)$, endowed valuations $\bar{v}_i(S) = v_i(\ell(i)) + v_i(S)$, and $\bar{\varepsilon} = -1 + \sqrt[m]{1+\varepsilon}$.

Definition 3.1 ($\bar{\varepsilon}$-local optimum)**.** A complete allocation $\mathcal{R} = (R_i)_{i\in A}$ is an *$\bar{\varepsilon}$-local optimum with respect to valuations $\bar{v}_i$*, if for all pairs of different agents $i, k \in \bar{A}$ and all $j \in R_i$ it holds

$$\left(\frac{\bar{v}_i(R_i - j)}{\bar{v}_i(R_i)}\right)^{w_i} \cdot \left(\frac{\bar{v}_k(R_k + j)}{\bar{v}_k(R_k)}\right)^{w_k} \le (1 + \bar{\varepsilon}) .$$

A 0-local optimum will be simply called *local optimum.*

Lemma 3.2. *Consider an NSW instance with submodular valuations, and let $\varepsilon > 0$. Then,* `LocalSearch`$(J, v_1, \dots, v_n)$ *returns an $\bar{\varepsilon}$-local maximum with respect to the endowed valuations $\bar{v}_i$ in $O\left(\frac{m}{\varepsilon}\log m\right)$ exchange steps.*

Proof. It is immediate that the algorithm terminates with an $\bar{\varepsilon}$-local maximum. Recalling that $\bar{v}_i(j) \le 2\bar{v}_i(\emptyset)$ for any $j \in J$, submodularity implies $v_i(J) < (|J| + 1)\bar{v}_i(\emptyset) \le m\bar{v}_i(\emptyset)$ for every $i \in \bar{A}$. Hence,

$$\prod_{i\in\bar{A}} \bar{v}_i(J)^{w_i} \le m \prod_{i\in\bar{A}} \bar{v}_i(\emptyset)^{w_i} ,$$

and therefore the product $\prod_{i\in\bar{A}} \bar{v}_i(R_i)^{w_i}$ may grow by at most a factor m throughout all exchange steps. Every swap increases this product by at least a factor $(1 + \bar{\varepsilon})$. Thus, the total number of swaps is bounded by $\log_{(1+\bar{\varepsilon})} m = m \log_{1+\varepsilon} m = O\left(\frac{m}{\varepsilon}\log m\right)$. □

We need the following two properties of submodular valuations.

Proposition 3.3. *Let $\bar{v} : 2^J \to \mathbb{R}_{>0}$ be a submodular endowed valuation. Let $S \subseteq T \subseteq J$ and $j \in J$. Then,*

$$\frac{\bar{v}(T + j)}{\bar{v}(T)} \le \frac{\bar{v}(S + j)}{\bar{v}(S)} .$$

Proof. By the monotonicity, and submodularity of v we have

$$\begin{aligned}\frac{\bar{v}(T + j)}{\bar{v}(T)} = \frac{\bar{v}(T) + \bar{v}(T + j) - \bar{v}(T)}{\bar{v}(T)} &\le \frac{\bar{v}(S) + \bar{v}(T + j) - \bar{v}(T)}{\bar{v}(S)} \\ &\le \frac{\bar{v}(S) + \bar{v}(S + j) - \bar{v}(S)}{\bar{v}(S)} \\ &= \frac{\bar{v}(S + j)}{\bar{v}(S)} .\end{aligned}$$

□

Proposition 3.4. *Let $\bar{v} : 2^J \to \mathbb{R}_{>0}$ be a submodular endowed valuation. For any $j \in R$,*

$$\bar{v}(R - j) \ge \sum_{k\in R} (\bar{v}(R) - \bar{v}(R - k)).$$

Proof. Let us denote $R - j := \{r_1, \dots, r_s\}$. By submodularity, we have

$$\begin{aligned}\bar{v}(R - j) &= \bar{v}(\emptyset) + \sum_{k=1}^{s} (\bar{v}(\{r_1, \dots, r_k\}) - \bar{v}(\{r_1, \dots, r_{k-1}\})) \\ &\ge \bar{v}(\emptyset) + \sum_{k=1}^{s} (\bar{v}(R) - \bar{v}(R - r_k)) \ge \sum_{k\in R} (\bar{v}(R) - \bar{v}(R - r_k))\end{aligned}$$

where in the last step, we used the fact that $\bar{v}(\emptyset) = v(\ell(i)) \ge v(j) \ge \bar{v}(R) - \bar{v}(R - j)$. □

We analyze our local search in slightly different ways in the symmetric case (where $w_1 = \ldots = w_n = 1/n$) and the general asymmetric case. We consider the asymmetric case first.

3.2 Local Equilibrium Analysis for Asymmetric NSW

Let $\bar{\varepsilon} \ge 0$, and let $\mathcal{R} = (R_i)_{i\in A}$ be an $\bar{\varepsilon}$-local optimum with respect to the endowed valuations $\bar{v}_i$. Let $j \in J$ and let $i \in \bar{A}$ be the agent such that $j \in R_i$. We define the price of j as

$$p_j := w_i \log \frac{\bar{v}_i(R_i)}{\bar{v}_i(R_i - j)} .$$

Lemma 3.5. *For an $\bar{\varepsilon}$-local optimum $\mathcal{R} = (R_i)_{i\in\bar{A}}$ and prices p_j defined as above, for every item $j \in R_i$ and every agent $k \in \bar{A}$, we have*

$$\frac{\bar{v}_k(R_k + j)}{\bar{v}_k(R_k)} \le (1 + \bar{\varepsilon})^{1/w_k} \mathrm{e}^{p_j/w_k} .$$

Moreover, if the valuation $\bar{v}_k$ is submodular, then for all $T \subseteq J$, we have

$$\frac{\bar{v}_k(R_k \cup T)}{\bar{v}_k(R_k)} \le (1 + \bar{\varepsilon})^{|T|/w_k} \cdot \mathrm{e}^{\sum_{j\in T} p_j/w_k} .$$

Proof. By definition, $\mathrm{e}^{p_j/w_i} = \frac{\bar{v}_i(R_i)}{\bar{v}_i(R_i - j)}$. If $k = i$ the first statement is trivial. Otherwise, for $k \ne i$, the first statement follows from the $\bar{\varepsilon}$-optimality of $\mathcal{R}$; if false, we would swap item j to agent k.

For the second statement, assume w.l.o.g. $T = \{t_1, t_2, \dots, t_{|T|}\} \subseteq J$. Since $\bar{v}_k$ is submodular, by Proposition 3.3 we have

$$\begin{aligned}\frac{\bar{v}_k(R_k \cup T)}{\bar{v}_k(R_k)} &= \prod_{a=1}^{|T|} \frac{\bar{v}_k(R_k \cup \{t_1,\ldots,t_a\})}{\bar{v}_k(R_k \cup \{t_1,\ldots,t_{a-1}\})} \\ &\le \prod_{a=1}^{|T|} \frac{\bar{v}_k(R_k + t_a)}{\bar{v}_k(R_k)} \\ &\le (1+\bar{\varepsilon})^{|T|/w_k}\, \mathrm{e}^{\sum_{j\in T} p_j/w_k}\,. \qquad \square\end{aligned}$$

The following lemma shows that the spending of agent i, $p(R_i)$, is at most their weight w_i.

Lemma 3.6 (Bounded spending). *For an $\bar{\varepsilon}$-local optimum $\mathcal{R} = (R_i)_{i\in\bar{A}}$ and prices p_j defined as above, $p(R_i) \le w_i$ for every agent $i \in \bar{A}$, and consequently, $p(J) \le 1$.*

Proof. From the definition of p_j, we have

$$p(R_i) = w_i \sum_{j\in R_i} \log \frac{\bar{v}_i(R_i)}{\bar{v}_i(R_i - j)} \le w_i \sum_{j\in R_i} \frac{\bar{v}_i(R_i) - \bar{v}_i(R_i - j)}{\bar{v}_i(R_i - j)} \le w_i$$

due to the elementary inequality $\log x \le x - 1$, and by Proposition 3.4 we know that $\sum_{j\in R_i}(\bar{v}_i(R_i) - \bar{v}_i(R_i - j)) \le \bar{v}_i(R_i - j')$ for $j' \in \operatorname{argmin}_{j\in R_i} \bar{v}_i(R_i - j)$.

Adding up the prices over all the sets R_i, whose union is J, we obtain $p(J) = \sum_{i\in\bar{A}} p(R_i) \le \sum_{i\in\bar{A}} w_i \le 1$. □

We recall the First Welfare Theorem: any Walrasian equilibrium allocation maximizes the utilitarian social welfare. For conditional equilibrium, [25, Proposition 1] give an approximate version of the first welfare theorem: the utilitarian social welfare in any conditional equilibrium is at least half of the maximal welfare. Analogously, if we interpret local optimum as equilibrium, then the following proposition states that such an equilibrium gives an e-approximation of the optimal *Nash* social welfare with respect to the *endowed valuations*. Recall that, by definition of $\bar{A}$, $\bar{v}_i(S) = 0$ for any $i \in A \setminus \bar{A}$ and any $S \subseteq J$.

Proposition 3.7. *Let $\mathcal{R} = (R_i)_{i\in A}$ be a local optimum and $\mathcal{S} = (S_i)_{i\in A}$ be an optimal NSW allocation with respect to the endowed submodular valuations $\bar{v}_i$. Then*

$$\prod_{i\in\bar{A}} \bar{v}_i(R_i)^{w_i} \ge \frac{1}{\mathrm{e}} \cdot \prod_{i\in\bar{A}} \bar{v}_i(S_i)^{w_i}\,.$$

Proof. By Lemma 3.6, $\sum_{i\in\bar{A}} p(S_i) \le p(J) \le 1$. Then, by Lemma 3.5,

$$\begin{aligned}\prod_{i\in\bar{A}} \bar{v}_i(S_i)^{w_i} &\le \prod_{i\in\bar{A}} \bar{v}_i(R_i \cup S_i)^{w_i} \le \prod_{i\in\bar{A}} \bar{v}_i(R_i)^{w_i} \cdot \mathrm{e}^{p(S_i)} \\ &= \mathrm{e}^{\sum_{i\in\bar{A}} p(S_i)} \cdot \prod_{i\in\bar{A}} \bar{v}_i(R_i)^{w_i} \le \mathrm{e} \cdot \prod_{i\in\bar{A}} \bar{v}_i(R_i)^{w_i}\,. \quad \square\end{aligned}$$

Proposition 3.7 is included solely for the intuition. We cannot really use it as such, because it doesn't deal with the allocation of items in H. For this, we need the final rematching phase (Section 3.4). We will need a bound in the following form. The parameters h_i will represent the number of items that agent i takes from the set H in the optimum solution.

Lemma 3.8. *Let $\bar{\varepsilon} \ge 0$, and let $\mathcal{R} = (R_i)_{i\in\bar{A}}$ be an $\bar{\varepsilon}$-local optimum with respect to the endowed valuations $\bar{v}_i$ that are submodular. Let $(S_1, S_2, \ldots, S_n)$ denote any partition of the set J, and let $h_i \ge 0$ such that $\sum_{i\in A} h_i \le n$. Then,*

$$\prod_{i\in A\setminus\bar{A}} h_i^{w_i} \prod_{i\in\bar{A}} \left(\frac{v_i(S_i)}{\max\{v_i(\ell(i)), v_i(R_i)\}} + h_i\right)^{w_i} \le (1+\varepsilon)(2+nw_{\max})\mathrm{e}.$$

Proof. By Lemma 3.5, for each $i \in \bar{A}$ we can bound

$$\begin{aligned}\frac{v_i(S_i)}{\max\{v_i(\ell(i)), v_i(R_i)\}} &\le \frac{v_i(R_i \cup S_i)}{\frac{1}{2}[v_i(\ell(i)) + v_i(R_i)]} \\ &\le \frac{2\bar{v}_i(R_i \cup S_i)}{\bar{v}_i(R_i)} \\ &\le 2(1+\bar{\varepsilon})^{|S_i|/w_i}\mathrm{e}^{p(S_i)/w_i}\,.\end{aligned}$$

Thus,

$$\begin{aligned}&\prod_{i\in A\setminus\bar{A}} h_i^{w_i} \prod_{i\in\bar{A}} \left(\frac{v_i(S_i)}{\max\{v_i(\ell(i)), v_i(R_i)\}} + h_i\right)^{w_i} \\ &\le \prod_{i\in A\setminus\bar{A}} h_i^{w_i} \prod_{i\in\bar{A}} \left(2(1+\bar{\varepsilon})^{|S_i|/w_i}\mathrm{e}^{p(S_i)/w_i} + h_i\right)^{w_i} \\ &\le \prod_{i\in A\setminus\bar{A}} h_i^{w_i} \prod_{i\in\bar{A}} \left((2+h_i)(1+\bar{\varepsilon})^{|S_i|/w_i} \cdot \mathrm{e}^{p(S_i)/w_i}\right)^{w_i} \\ &\le (1+\bar{\varepsilon})^m \mathrm{e}^{p(J)} \prod_{i\in A} (2+h_i)^{w_i}\,.\end{aligned}$$

By the choice of $\bar{\varepsilon}$, $(1+\bar{\varepsilon})^m = 1+\varepsilon$. From Lemma 3.6, we get $p(J) \le 1$. The proof of the lemma is complete by showing that the last product is at most $(2 + nw_{\max})$. This follows by the AM-GM inequality:

$$\prod_{i\in A}(2+h_i)^{w_i} \le \sum_{i\in A} w_i(2+h_i) \le 2 + w_{\max}\sum_{i\in A} h_i \le 2 + nw_{\max}\,.$$

□

3.3 Local Equilibrium Analysis for Symmetric NSW

Let $\bar{\varepsilon} \ge 0$, and let $\mathcal{R} = (R_i)_{i\in A}$ be an $\bar{\varepsilon}$-local optimum with respect to the endowed valuations $\bar{v}_i$, in the symmetric case. Define $\hat{\varepsilon} := (1+\bar{\varepsilon})^n - 1$; we have $1 + \hat{\varepsilon} = (1+\bar{\varepsilon})^n \le (1+\bar{\varepsilon})^m = 1 + \epsilon$ since $n \le m$. In particular, $0 \le \bar{\varepsilon} \le \hat{\varepsilon} \le \varepsilon \le 1$.

Let $j \in J$ and let $i \in \bar{A}$ be the agent such that $j \in R_i$. We define the *price* of j as

$$p_j := \frac{\bar{v}_i(R_i)}{\bar{v}_i(R_i - j)} - 1 = \frac{\bar{v}_i(R_i) - \bar{v}_i(R_i - j)}{\bar{v}_i(R_i - j)}\,.$$

The following lemma gives the basic properties of these prices that we will need in the following.

Lemma 3.9. *Given an $\bar{\varepsilon}$-local optimum $\mathcal{R} = (R_i)_{i\in A}$, and the prices p_j defined as above, we have*

- *For every item $j \in J$, $p_j \le 1$.*
- *For every item $j \in J \setminus R_i$,*

$$\frac{\bar{v}_k(R_k + j)}{\bar{v}_k(R_k)} \le (1+\hat{\varepsilon})(1+p_j).$$

- *For every $T \subseteq J$,*

$$\frac{\bar{v}_i(R_k \cup T)}{\bar{v}_k(R_k)} \le 1 + \sum_{j\in T}(2\hat{\varepsilon} + p_j)\,.$$

Proof. By construction of $\bar{v}_i$, $\bar{v}_i(R_i) - \bar{v}_i(R_i - j) \le \bar{v}_i(\emptyset) \le \bar{v}_i(R_i - j)$. Hence, $p_j = \frac{\bar{v}_i(R_i) - \bar{v}_i(R_i - j)}{\bar{v}_i(R_i - j)} \le 1$.

From the $\bar{\varepsilon}$-optimality of $\mathcal{R}$, we get $\frac{\bar{v}_k(R_k + j)}{\bar{v}_k(R_k)} \le (1+\bar{\varepsilon})^n \frac{\bar{v}_i(R_i)}{\bar{v}_i(R_i - j)} = (1+\hat{\varepsilon})(1+p_j)$, because otherwise we could swap item j to agent k.

For the third statement, by submodularity, we have

$$\begin{aligned}\frac{\bar{v}_k(R_k \cup T)}{\bar{v}_k(R_k)} &\le \frac{\bar{v}_k(R_k) + \sum_{j\in T}(\bar{v}_k(R_k + j) - \bar{v}_k(R_k))}{\bar{v}_k(R_k)} \\ &\le 1 + \sum_{j\in T}((1+\hat{\varepsilon})(1+p_j) - 1) \le 1 + \sum_{j\in T}(2\hat{\varepsilon} + p_j)\end{aligned}$$

using the first and second statement. □

The following lemma shows that the *spending* of each agent i, $p(R_i) = \sum_{j\in R_i} p_j$, is at most 1.

Lemma 3.10 (Bounded spending). *Let $\mathcal{R} = (R_i)_{i\in\bar{A}}$ be an $\bar{\varepsilon}$-local optimum with respect to the endowed valuations $\bar{v}_i$. Then, $p(R_i) \le 1$ for every agent $i \in \bar{A}$, and consequently, $p(J) \le |\bar{A}|$.*

Proof. From the definition of the prices p_j, and by Proposition 3.4, we have

$$\begin{aligned}p(R_i) &= \sum_{j\in R_i} \frac{\bar{v}_i(R_i) - \bar{v}_i(R_i - j)}{\bar{v}_i(R_i - j)} \\ &\le \frac{\sum_{j\in R_i}(\bar{v}_i(R_i) - \bar{v}_i(R_i - j))}{\min_{k\in R_i} \bar{v}_i(R_i - k)} \le 1\,.\end{aligned}$$

Since $(R_1, \dots, R_n)$ is a partition of J (every item is allocated throughout our local search), we have

$$p(J) = \sum_{j\in J} p_j = \sum_{i\in\bar{A}} \sum_{j\in R_i} p_j \le |\bar{A}|\,. \qquad \square$$

The next lemma bounds the value of any set relative to our local optimum in terms of prices.

Proposition 3.11. *Let $\mathcal{R} = (R_i)_{i\in A}$ be an $\bar{\varepsilon}$-local optimum and $S \subseteq J$ any set of items. Then,*

$$\frac{v_i(S)}{\max\{v_i(R_i), v_i(\ell(i))\}} \le 1 + 2\sum_{j\in S}(2\hat{\varepsilon} + p_j).$$

Proof. By Lemma 3.9,

$$\frac{v_i(\ell(i)) + v_i(S)}{v_i(\ell(i)) + v_i(R_i)} = \frac{\bar{v}_i(S)}{\bar{v}_i(R_i)} \le \frac{\bar{v}_i(R_i \cup S)}{\bar{v}_i(R_i)} \le 1 + \sum_{j\in S}(2\hat{\varepsilon} + p_j).$$

Let $\lambda = \frac{v_i(R_i)}{v_i(\ell(i))}$. We can rewrite the inequality above as follows:

$$\frac{1 + \frac{v_i(S)}{v_i(\ell_i)}}{1+\lambda} \le 1 + \sum_{j\in S}(2\hat{\varepsilon} + p_j).$$

From here,

$$\frac{v_i(S)}{v_i(\ell_i)} \le (1+\lambda)(1 + \sum_{j\in S}(2\hat{\varepsilon} + p_j)) - 1 = \lambda + (1+\lambda)\sum_{j\in S}(2\hat{\varepsilon} + p_j).$$

We use this inequality if $0 \le \lambda \le 1$. If $\lambda > 1$, we divide by λ to obtain:

$$\frac{v_i(S)}{v_i(R_i)} \le 1 + (1/\lambda + 1)\sum_{j\in S}(2\hat{\varepsilon} + p_j).$$

Either way, the worst case is $\lambda = 1$, which gives

$$\frac{v_i(S)}{\max\{v_i(\ell(i)), v_i(R_i)\}} \le 1 + 2\sum_{j\in S}(2\hat{\varepsilon} + p_j)\,. \qquad \square$$

Again, the bounds in this section do not deal with the allocation of the items in H. This will be handled by the final rematching phase (Section 3.4), where we will need a bound in the following form.

Lemma 3.12. *Let $\bar{\varepsilon} \ge 0$, and let $\mathcal{R} = (R_i)_{i\in A}$ be an $\bar{\varepsilon}$-local optimum with respect to the endowed valuations $\bar{v}_i$. Let $(S_1, S_2, \dots, S_n)$ denote any allocation of the set J, and let $h_i \ge 0$ be such that $\sum_{i\in A} h_i \le n$. Then,*

$$\prod_{i\in A\setminus\bar{A}} h_i \prod_{i\in\bar{A}} \left(\frac{v_i(S_i)}{\max\{v_i(\ell(i)), v_i(R_i)\}} + h_i\right) \le (1+\varepsilon)^n\, 4^n\,.$$

Proof. By Proposition 3.11,

$$\prod_{i\in\bar{A}} \left(\frac{v_i(S_i)}{\max\{v_i(\ell(i)), v_i(R_i)\}} + h_i\right) \le \prod_{i\in\bar{A}} \left(1 + 2\sum_{j\in S_i}(2\hat{\varepsilon} + p_j) + h_i\right).$$

So by the AM-GM inequality we have

$$\begin{aligned}&\prod_{i\in A\setminus\bar{A}} h_i \prod_{i\in\bar{A}} \left(\frac{v_i(S_i)}{\max\{v_i(\ell(i)), v_i(R_i)\}} + h_i\right) \\ &\le \prod_{i\in A\setminus\bar{A}} h_i \prod_{i\in\bar{A}} \left(1 + 2\sum_{j\in S_i}(2\hat{\varepsilon} + p_j) + h_i\right) \\ &\le \frac{1}{n^n}\left(\sum_{i\in A\setminus\bar{A}} h_i + \sum_{i\in\bar{A}}\left(1 + 2\sum_{j\in S_i}(2\hat{\varepsilon} + p_j) + h_i\right)\right)^n \\ &= \left(\frac{\sum_{i\in A} h_i}{n} + \frac{\sum_{i\in\bar{A}} 1}{n} + \frac{\sum_{j\in J} 4\hat{\varepsilon}}{n} + \frac{\sum_{j\in J} 2p_j}{n}\right)^n\,.\end{aligned}$$

We upper-bound each of these two summands. First, using the fact that $\sum_{i\in A} h_i \le n$. Second, using $|\bar{A}| \le n$. Third, using $|J| \le m$. Fourth, using

$$\sum_{i\in\bar{A}} p(S_i) \le \sum_{j\in J} p_j \le |\bar{A}| \le n$$

from Lemma 3.10. We obtain,

$$\begin{aligned}&\left(\frac{\sum_{i\in A} h_i}{n} + \frac{\sum_{i\in\bar{A}} 1}{n} + \frac{\sum_{j\in J} 4\hat{\varepsilon}}{n} + \frac{\sum_{j\in J} 2p_j}{n}\right)^n \\ &\le \left(1 + 1 + \frac{4m\hat{\varepsilon}}{n} + 2\right)^n = 4^n\left(1 + \frac{m\hat{\varepsilon}}{n}\right)^n\,.\end{aligned}$$

Since $m \ge |H| = n$, by Bernoulli's inequality $4^n\left(1 + \frac{m\hat{\varepsilon}}{n}\right)^n \le 4^n\,(1+\hat{\varepsilon})^{\frac{mn}{n}} = (1+\varepsilon)^n 4^n$. □

3.4 Rematching

Throughout, let OPT denote the optimum NSW value of the instance. For sets $\mathcal{R} = (R_i)_{i\in A}$, and a matching $\pi : A \to H \cup \{\perp\}$, we let

$$\mathrm{NSW}(\mathcal{R}, \pi) \coloneqq \prod_{i\in A} v_i(R_i + \pi(i))^{w_i} .$$

In Phase 3, we select a matching $\rho : A \to H$ that maximizes $\mathrm{NSW}(\mathcal{R}, \rho)$, where $\mathcal{R} = (R_i)_{i\in A}$ denotes the $\bar{\varepsilon}$-local optimum with respect to the endowed valuations $\bar{v}_i$ from Phase 2. The following lemma completes the proof of Theorem 1.1.[1]

Lemma 3.13. *Let $\bar{\varepsilon} \geq 0$, and let $\mathcal{R} = (R_i)_{i\in A}$ be an $\bar{\varepsilon}$-local optimum with respect to the endowed valuations $\bar{v}_i$ that are submodular. Then, there exists a matching $\rho : A \to H$ such that, for the symmetric problem, it holds*

$$\mathrm{NSW}(\mathcal{R}, \rho) \geq \frac{\mathrm{OPT}}{4(1+\varepsilon)} ,$$

and, for the asymmetric problem, it holds

$$\mathrm{NSW}(\mathcal{R}, \rho) \geq \frac{\mathrm{OPT}}{(2 + nw_{\max})\mathrm{e}(1+\varepsilon)} .$$

Proof. Consider an optimal solution $(S_1 \cup H_1, \ldots, S_n \cup H_n)$ to the NSW problem where S_i is the set of items allocated to i from $J = G \setminus H$, and H_i is the set of items allocated to i from H. For $i \in A \setminus \bar{A}$, we must have $H_i \neq \emptyset$, and we can assume $S_i = \emptyset$. Let $h_i \coloneqq |H_i|$. We define a matching $g : A \to H \cup \{\perp\}$ as follows. If $h_i > 0$, let $g(i) \in \mathrm{argmax}_{j\in H_i} v_i(S_i + j)$ be one of the items in H_i providing the largest marginal gain to agent i. Otherwise, let $g(i) \coloneqq \perp$. Submodularity implies

$$v_i(S_i \cup H_i) \leq v_i(S_i) + h_i v_i(g(i)) \quad \forall i \in A . \tag{1}$$

Let us partition the set of agents A as

$$A_\pi \coloneqq \{i \in A : v_i(g(i)) \geq \max\{v_i(R_i), v_i(\ell(i))\}\} ,$$
$$A_R \coloneqq \{i \in A \setminus A_\pi : v_i(R_i) \geq \max\{v_i(g(i)), v_i(\ell(i))\}\} ,$$
$$A_\ell \coloneqq \{i \in A \setminus (A_\pi \cup A_R) : v_i(\ell(i)) \geq \max\{v_i(R_i), v_i(g(i))\}\} .$$

As an intermediate step in the construction of the claimed matching ρ, we first define an allocation $\mathcal{T} = (T_i)_{i\in A}$ and matching $\pi : A \to H \cup \{\perp\}$ as follows.

- For $i \in A_\pi$, let $T_i \coloneqq \emptyset$ and $\pi(i) \coloneqq g(i)$.
- For $i \in A_R$, let $T_i \coloneqq R_i$ and $\pi(i) \coloneqq \perp$.
- For $i \in A_\ell$, let $T_i \coloneqq \{\ell(i)\}$ and $\pi(i) \coloneqq \perp$.

Note that $A \setminus \bar{A} \subseteq A_\pi$. Note that this allocation is not feasible: $\ell(i) = \ell(i')$ is possible for different agents, and the same item may even be contained in R_i for some $i \in A_R$. We complete the proof in two steps. First, we lower bound $\mathrm{NSW}(\mathcal{T}, \pi)/\mathrm{OPT}$. Then, we show that π and the initial matching τ from Phase 1 can be recombined into a matching ρ such that $\mathrm{NSW}(\mathcal{R}, \rho) \geq \mathrm{NSW}(\mathcal{T}, \pi)$.

Claim. *For the symmetric problem*

$$\mathrm{NSW}(\mathcal{T}, \pi) \geq \frac{\mathrm{OPT}}{4(1+\varepsilon)} .$$

For the asymmetric problem

$$\mathrm{NSW}(\mathcal{T}, \pi) \geq \frac{\mathrm{OPT}}{(2 + nw_{\max})\,\mathrm{e}(1+\varepsilon)} .$$

Proof. Our goal is to upper bound $\frac{\mathrm{OPT}}{\mathrm{NSW}(\mathcal{T},\pi)}$, which is

$$\prod_{i\in A_\pi}\left(\frac{v_i(S_i \cup H_i)}{v_i(\pi(i))}\right)^{w_i} \prod_{i\in A_R}\left(\frac{v_i(S_i \cup H_i)}{v_i(R_i)}\right)^{w_i} \prod_{i\in A_\ell}\left(\frac{v_i(S_i \cup H_i)}{v_i(\ell(i))}\right)^{w_i} .$$

In order to do so, we first upper bound the loss of each agent depending in which set they belong. If $i \in A \setminus \bar{A}$ then $i \in A_\pi$, by (1) and submodularity, we have

$$\frac{v_i(S_i \cup H_i)}{v_i(\pi(i))} \leq \frac{h_i v_i(g(i))}{v_i(\pi(i))} = h_i .$$

If $i \in A_\pi \cap \bar{A}$, by (1), as well as using the definition of A_π and submodularity, we can bound

$$\begin{aligned}\frac{v_i(S_i \cup H_i)}{v_i(\pi(i))} &\leq \frac{v_i(S_i) + h_i v_i(g(i))}{v_i(\pi(i))}\\ &= \frac{v_i(S_i)}{v_i(\pi(i))} + h_i\\ &\leq \frac{v_i(S_i)}{\max\{v_i(\ell(i)), v_i(R_i)\}} + h_i .\end{aligned}$$

Similarly, if $i \in A_R$, we get

$$\begin{aligned}\frac{v_i(S_i \cup H_i)}{v_i(R_i)} &\leq \frac{v_i(S_i) + h_i v_i(g(i))}{v_i(R_i)}\\ &\leq \frac{v_i(S_i)}{v_i(R_i)} + h_i\\ &= \frac{v_i(S_i)}{\max\{v_i(\ell(i)), v_i(R_i)\}} + h_i .\end{aligned}$$

Finally, if $i \in A_\ell$, the bound is

$$\begin{aligned}\frac{v_i(S_i \cup H_i)}{v_i(\ell(i))} &\leq \frac{v_i(S_i) + h_i v_i(g(i))}{v_i(\ell(i))}\\ &\leq \frac{v_i(S_i)}{v_i(\ell(i))} + h_i\\ &= \frac{v_i(S_i)}{\max\{v_i(\ell(i)), v_i(R_i)\}} + h_i .\end{aligned}$$

Consequently,

$$\frac{\mathrm{OPT}}{\mathrm{NSW}(\mathcal{T},\pi)} \leq \prod_{i\in A\setminus\bar{A}} h_i^{w_i} \prod_{i\in A}\left(\frac{v_i(S_i)}{\max\{v_i(\ell(i)), v_i(R_i)\}} + h_i\right)^{w_i} .$$

The proof of the claim is complete by Lemmas 3.12 and 3.8. □

It remains to construct a matching $\rho : A \to H \cup \{\perp\}$ such that $\mathrm{NSW}(\mathcal{R}, \rho) \geq \mathrm{NSW}(\mathcal{T}, \pi)$. First, note that if $A_\ell = \emptyset$, then $\rho = \pi$ is a suitable choice. In case $A_\ell \neq \emptyset$, we construct alternating paths from the initial matching τ from Phase I and ρ to eliminate the $\ell(i)$ items from $\mathcal{T}$. A critical property for the argument is as follows.

Claim 3.14. *For every $i \in \bar{A}$, $v_i(\tau(i)) \geq v_i(\ell(i))$.*

Proof. Consider the matching $\overline{\tau} : A \to G$ defined as $\overline{\tau}(i) \coloneqq \ell(i)$, and $\overline{\tau}(h) \coloneqq \tau(h)$ for $h \neq i$. $\overline{\tau}$ is a matching since $\ell(i) \notin H$. By the choice of τ, $\prod_{h\in A} v_h(\overline{\tau}(h))^{w_h} \leq \prod_{h\in A} v_h(\tau(h))^{w_h}$, implying the claim. □

In order to construct the matching ρ, we define an auxiliary directed graph $D = (A_\ell \cup A_\pi \cup H, E)$, where the arc set is defined as

$$E = \{(\tau(i), i) : i \in A_\ell \cup A_\pi\} \cup \{(i, \pi(i)) : i \in A_\pi\} .$$

[1] One needs to select a smaller parameter ε to obtain the bounds in Theorem 1.1.

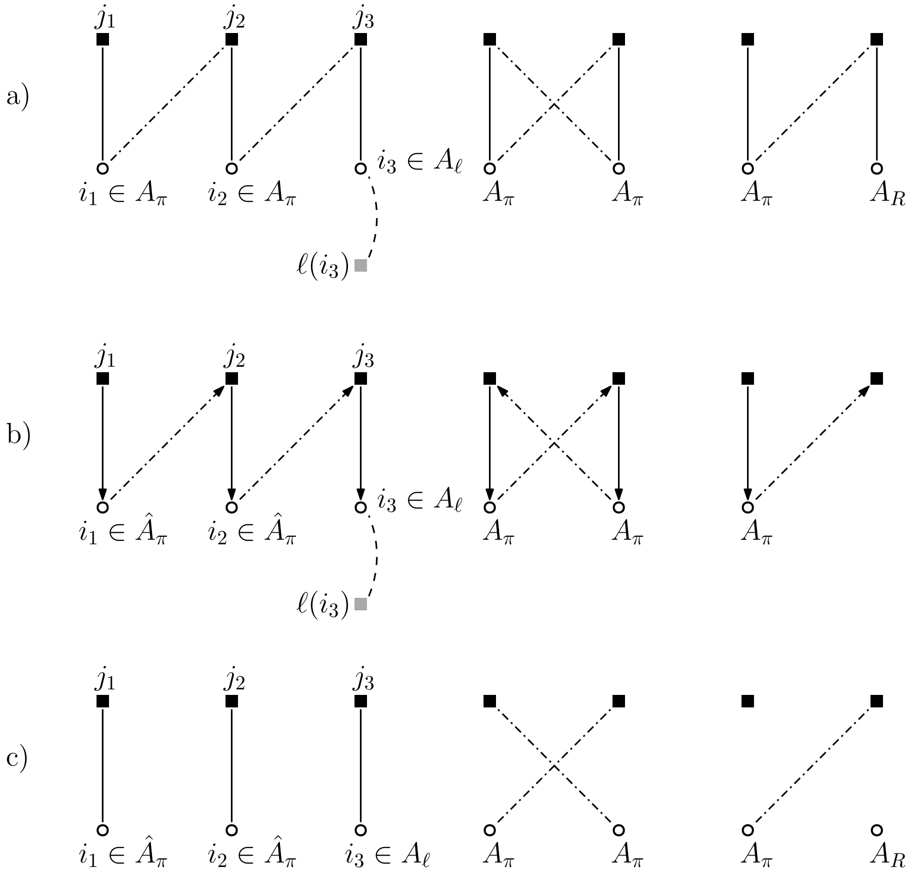

Figure 1: White circles represent the agents, black squares the item set H, and grey squares the favorite items. Solid lines represent matching τ, while dashed-dotted lines represent a subset of matching π. Figure a) shows matching τ, matching π for the agents in A_π, and the $\ell(i)$ items for the agents in A_ℓ. Figure b) shows graph D (and the $\ell(i)$'s). Figure c) shows matching ρ.

See Figure 1 for an example. Note that $\pi(i) \neq \perp$ if $i \in A_\pi$. Thus, each node in A_π has exactly one outgoing and exactly one incoming arc, each node in A_ℓ has exactly one incoming arc and no outgoing arcs, and each item node in H has at most one incoming and at most one outgoing arc.

Let $\hat{A}_\pi \subseteq A_\pi$ be the set of nodes that can reach A_ℓ in the digraph D. By construction, each $i \in A_\pi$ is either contained in a cycle inside $A_\pi \cup H$, or on a directed path ending in $A_\ell \cup H$; these paths start in H and may terminate in either H or A_ℓ. We choose $\hat{A}_\pi$ as the set of nodes where the path terminates in A_ℓ.

We define the matching $\rho : A \to H \cup \{\perp\}$ as

$$\rho(i) := \begin{cases} \perp, & \text{if } i \in A_R\,, \\ \tau(i)\,, & \text{if } i \in A_\ell \cup \hat{A}_\pi\,, \\ \pi(i)\,, & \text{if } i \in A_\pi \setminus \hat{A}_\pi\,. \end{cases}$$

Claim. *ρ is a matching.*

Proof. For a contradiction, assume $j = \pi(i') = \tau(i)$ for $i' \in A_\pi \setminus \hat{A}_\pi$ and $i \in A_\ell \cup \hat{A}_\pi$. Then, $(i', j), (j, i)$ forms 2-hop directed path from i' to i in D. Since $i \in \hat{A}_\pi$, there is a directed path P from i to a node in A_ℓ. Concatenating these two paths gives a directed path from i' to a node in A_ℓ. Thus, $i' \in \hat{A}_\pi$, a contradiction. □

It remains to show

$$\prod_{i \in A_\ell \cup \hat{A}_\pi} v_i(\tau(i))^{w_i} \geq \prod_{i \in A_\ell} v_i(\ell(i))^{w_i} \prod_{i \in \overline{A}_\pi} v_i(\pi(i))^{w_i}\,. \quad (2)$$

The set of nodes in $A_\ell \cup \hat{A}_\pi$ are covered by maximal directed paths in D terminating in A_ℓ. First, consider a length one path $P = (j, i)$ that comprises an item node $j \in H$ and an agent node $i \in A_\pi$ such that $j = \tau(i)$, and j has no incoming arcs in D. Then, $v_i(\tau(i)) \geq v_i(\ell(i))$ by Claim 3.14.

Consider now a longer path $P = (j_1, i_1, j_2, i_2, \ldots, j_k, i_k)$ for $k > 1$, where $j_t \in H$ are item nodes, $i_t \in \hat{A}_\pi$ for $t < k$ and $i_k \in A_\ell$. Thus, $\tau(i_t) = j_t$ for $t \in [k]$ and $\pi(i_t) = j_{t+1}$ for $t \in [k-1]$. We claim that

$$\prod_{t=1}^{k} v_{i_t}(\tau(i_t))^{w_{i_t}} \geq v_{i_k}(\ell(i_k))^{w_{i_k}} \prod_{t=1}^{k-1} v_{i_t}(\pi(i_t))^{w_{i_t}}\,.$$

The proof follows the same lines as the proof of Claim 3.14. Indeed, if this equality does not hold, then there would exist a better matching $\overline{\tau} : A \to G$ defined as $\overline{\tau}(i_k) := \ell(i_k)$, and $\overline{\tau}(i_t) := j_{t+1} = \pi(i_t)$ for $t = 1, 2, \ldots, k-1$, and $\overline{\tau}(h) := \tau(h)$ for $h \neq i$.

Algorithm 3: Guaranteeing $\frac{1}{2}$-EFX for the symmetric NSW problem

```
Input: Allocation $\mathcal{S}$ that is $\alpha$-approximation to the NSW
       problem $(A, G, (v_i)_{i \in A})$.
Output: Allocation $\mathcal{T}$ that is $\frac{1}{2}$-EFX and $2\alpha$-approximation to
        the symmetric NSW problem.
$\mathcal{T} \leftarrow \mathcal{S}$
repeat
    $\mathcal{T} \leftarrow$ MakeFairOrEfficient($\mathcal{T}$)        // Algorithm 4
until $\mathcal{T}$ is not $\frac{1}{2}$-EFX
$U \leftarrow G \setminus \cup_i T_i$                // set of unallocated items
repeat
    Let $j \in U$ be such that $v_i(T_i) < v_i(j)$ for some agent $i$
    $T_i \leftarrow \{j\}$
    $U \leftarrow (U \cup T_i) - j$
until $v_i(T_i) \geq v_i(j), \forall i \in A, \forall j \in U$
$\mathcal{T} \leftarrow$ EnvyFreeCycle($\mathcal{T}, U$)
return $\mathcal{T}$
```

The inequality (2) follows by multiplying these inequalities over all maximal directed paths in D that terminate in A_ℓ. This completes the proof. □

4 FINDING FAIR AND EFFICIENT ALLOCATIONS

4.1 Completing the Partial Allocation

In this section, we derive Theorem 1.2 from Lemma 2.1. The proof of Lemma 2.1, describing the subroutine MakeFairOrEfficient is given in Section 4.2. The algorithm described in Theorem 1.2 is Algorithm 3. It uses two subroutines: MakeFairOrEfficient, and the envy-free cycle procedure EnvyFreeCycle from [39], described below.

The input of Algorithm 3 is an allocation $\mathcal{S}$ that is α-approximation to the symmetric NSW problem. It then repeatedly calls MakeFairOrEfficient($\mathcal{T}$) (Algorithm 4) until the final allocation is $\frac{1}{2}$-EFX and 2α-approximation to the symmetric NSW problem. Recall that the output of this subroutine is either a partial allocation $\mathcal{T}'$ that satisfies either $\text{NSW}(\mathcal{T}') \geq \text{NSW}(\mathcal{T})$ and $\cup_i T'_i \subsetneq \cup_i T_i$, or $\text{NSW}(\mathcal{T}') \geq \frac{1}{2}\,\text{NSW}(\mathcal{T})$ and $\mathcal{T}'$ is $\frac{1}{2}$-EFX. Since $\cup_i T'_i \subsetneq \cup_i T_i$ in each call in the first case, the number of calls to Algorithm 4 is at most m.

At this point, we have an $\frac{1}{2}$-EFX partial allocation $\mathcal{T}$ with $\text{NSW}(\mathcal{T}) \geq \frac{1}{2}\,\text{NSW}(\mathcal{S})$. The rest of Algorithm 3 allocates the remaining items $U = G \setminus \cup_{i \in A} T_i$ so that $\text{NSW}(\mathcal{T})$ does not decrease, and the $\frac{1}{2}$-EFX property is maintained.

First, we modify the allocation in the second repeat loop to ensure that each agent's value for their bundle is at least their value for each remaining item in U. This is done by swapping an agent's bundle T_i with a singleton item $j \in U$ whenever i values j more than the entire bundle T_i.

Finally, we run the envy-cycle procedure EnvyFreeCycle($\mathcal{T}, U$) from [39] to allocate the remaining items in U, starting with the allocation $\mathcal{T}$. The envy-cycle procedure maintains the directed (envy) graph $D = (A, E)$, where $(i, j) \in E$ if i envies j's bundle, i.e., $v_i(Y_i) < v_i(Y_j)$. If there is a cycle in G, then we can circulate bundles along the cycle to improve each agent's utility. Otherwise, there must be a source agent in G, whom no agent envies. We then assign an arbitrary item from U to a source agent. We update the envy graph, and iterate until U is fully assigned.

We now verify the correctness and efficiency of this algorithm.

Lemma 4.1. *The second repeat loop of Algorithm 3 is repeated at most nm times. It maintains the $\frac{1}{2}$-EFX and* $\text{NSW}(\mathcal{T})$ *is non-decreasing.*

Proof. The bound on the number of swaps follows since every agent $i \in A$ may swap their bundle at most m times. After the first swap, they maintain a singleton bundle, and they can swap their bundle for the same item j only once, since their valuation $v_i(T_i)$ strictly increases in each swap.

It is immediate that $\text{NSW}(\mathcal{T})$ is non-decreasing. It is left to show that the $\frac{1}{2}$-EFX property is maintained. Let $i \in A$ be the agent who swapped their bundle T_i for $T'_i = \{j\}$ in the current iteration. Then, the value of i's own bundle increased while the allocation of everyone else remained the same. Hence, agent i cannot violate the $\frac{1}{2}$-EFX property. For the other agents $k \neq i$, $v_k(T_k) \geq \frac{1}{2} \cdot v_k(T'_i - g)$ for all $g \in T'_i$ trivially holds, since T'_i is a singleton. □

The property (3) below is satisfied after the second repeat loop. Hence, the next lemma completes the analysis of Algorithm 3.

Lemma 4.2. *The subroutine* EnvyFreeCycle($\mathcal{T}, U$) *terminates in* $O(n^3 m)$ *time, and* $\text{NSW}(\mathcal{T})$ *is non-decreasing. Assume that* $\mathcal{T} = (T_i)_{i \in A}$ *is* $\frac{1}{2}$*-EFX, and*

$$v_i(T_i) \geq v_i(j) \quad \forall i \in A, \forall j \in U\,. \tag{3}$$

Then, EnvyFreeCycle($\mathcal{T}, U$) *also maintains the* $\frac{1}{2}$*-EFX property.*

Proof. The running time analysis is the same as in [39]. Finding and removing a cycle in the envy-graph can be done in $O(n^2)$ time. Further, whenever swapping around a cycle, at least one edge is removed from the envy graph. New edges can only be added when we allocate new items from U, with at most n edges every time. Since $|U| \leq m$, the total number of new edges added throughout is nm. This yields the overall $O(n^3 m)$ bound.

Again, it is immediate that $\text{NSW}(\mathcal{T})$ is non-decreasing in every step. We need to show that the $\frac{1}{2}$-EFX property is maintained both when swapping around cycles and when adding new items from U. When swapping around a cycle, this follows since the set of bundles remains the same, and no agent's value decreases.

Consider the case when a source agent say i, gets a new item j: their new bundle becomes $T'_i = T_i + j$. Note that i is the only agent whose value increases; all other bundles remain the same. We need to show that for any $k \neq i$,

$$v_k(T_k) \geq \tfrac{1}{2} v_k(T'_i - g) \quad \forall g \in T'_i\,.$$

We show that

$$v_k(T'_i - g) = v_k(T_i + j - g) \leq v_k(T_i) + v_k(j) \leq 2 v_k(T_k)\,.$$

Here, the first inequality follows by subadditivity and monotonicity. The second inequality uses (3), and that $v_k(T_k) \geq v_k(T_i)$, since i was a source node in the envy graph. □

4.2 Finding a Fair or an Efficient Allocation

In this Section, we prove Lemma 2.1. The subroutine `MakeFairOrEfficient(`$\mathcal{T}$`)` is shown in Algorithm 4, and generalizes an algorithm by Caragiannis, Gravin, and Huang [13] from additive to subadditive valuations. We begin with defining the notions of $\frac{1}{2}$*-EFX feasible bundles and graph.*

Definition 4.3 ($\frac{1}{2}$-EFX feasible bundles and graph)**.** Given a partial allocation $\mathcal{T} = (T_i)_{i\in A}$, we say that T_k is a $\frac{1}{2}$*-EFX feasible bundle for agent* i, if $v_i(T_k) \geq \frac{1}{2}\max_{\ell\in A, j\in T_\ell} v_i(T_\ell - j)$. The $\frac{1}{2}$*-EFX feasibility graph* of $\mathcal{T}$ is a bipartite graph $\mathcal{K} = (A \cup \mathcal{T}, E)$ where the edge set E is defined as:

$$E = \left\{(i,T_i) \mid T_i \text{ is } \frac{1}{2}\text{-EFX feasible for } i\right\} \cup \left\{(i,T_k) \mid v_i(T_k) > 2v_i(T_i) \text{ and } v_i(T_k) \geq \max_{\ell\in A, j\in T_\ell} v_i(T_\ell - j)\right\}. \tag{4}$$

The following claim can be easily verified using the definition.

Claim 4.4. *The degree of every node $i \in A$ is at least 1 in the graph $\mathcal{K} = (A \cup \mathcal{T}, E)$.*

In this section, a *matching* will refer to a matching between agents and bundles (and not between agents and items as in previous sections). Thus, a matching is a mapping $\rho : A \to \mathcal{T} \cup \{\perp\}$ such that $\rho(i) = \rho(k)$ implies $\rho(i) = \rho(k) = \perp$. A *perfect matching* has $\rho(i) \neq \perp$ for every $i \in A$. Matchings may use pairs (i, T_k) that are not in E; we say that ρ is a matching in the bipartite graph $\mathcal{K} = (A \cup \mathcal{T}, E)$ if $(i, \rho(i)) \in E$ whenever $\rho(i) \neq \{\perp\}$. For two matchings ρ and τ, an *alternating path between ρ and τ* is a path $P = (i_1, S_{i_1}, i_2, \ldots, S_{i_{k-1}}, i_\ell, S_{i_\ell})$ such that $\rho(i_t) = S_{i_t}$, $t = 1, \ldots, \ell$, $\tau(i_{t+1}) = S_{i_t}$, $t = 1, \ldots, \ell - 1$. The following lemma is immediate from the definition of the $\frac{1}{2}$-EFX feasibility graph.

Lemma 4.5. *If the $\frac{1}{2}$-EFX feasibility graph $\mathcal{K} = (A \cup \mathcal{T}, E)$ of an allocation $\mathcal{T}$ contains a perfect matching ρ, then $(i, \rho(i))_{i\in A}$ is a $\frac{1}{2}$-EFX allocation.*

We now give an overview of Algorithm 4. For an input partial allocation $\mathcal{T} = (T_i)_{i\in A}$, it returns a partial allocation $\mathcal{R}$ that satisfies one of the alternatives in Lemma 2.1: either (i) $\mathrm{NSW}(\mathcal{R}) \geq \mathrm{NSW}(\mathcal{T})$ and $\cup_i R_i \subsetneq \cup_i T_i$, or (ii) $\mathrm{NSW}(\mathcal{R}) \geq \frac{1}{2}\,\mathrm{NSW}(\mathcal{T})$ and $\mathcal{R}$ is $\frac{1}{2}$-EFX.

The algorithm gradually *'trims down'* the bundles $\mathcal{T}$. That is, we maintain a partial allocation $\mathcal{S} = (S_i)_{i\in A}$ with $S_i \subseteq T_i$ throughout. Every main loop of the algorithm either terminates by constructing an allocation $\mathcal{R}$ satisfying (ii), or removes an item from one of the S_h sets. The other possible termination option is when the $\frac{1}{2}$-EFX feasibility graph of $\mathcal{S}$ contains a perfect matching ρ. In this case, we return $\mathcal{R} = (\rho(i))_{i\in A}$. This is a $\frac{1}{2}$-EFX allocation by Lemma 4.5; Lemma 4.8 shows it also satisfies $\mathrm{NSW}(\mathcal{R}) \geq \frac{1}{2}\,\mathrm{NSW}(\mathcal{T})$ and is thus a suitable output of type (ii).

At the beginning of each main loop, we define two matchings. The first is the perfect matching τ that simply defines $\tau(i) = S_i$ for all $i \in A$. The second is a matching ρ in $\mathcal{K}$. This is required to satisfy three properties: First, it matches all trimmed down bundles, i.e., all bundles S_i with $S_i \subsetneq T_i$. Second, $|\{i : \rho(i) = S_i\}|$ is maximized subject to the first requirement. Third, subject to these requirements, ρ is chosen as a maximal matching. (The existence of such a matching is guaranteed by Lemma 4.6 below).

Algorithm 4: `MakeFairOrEfficient(`$\mathcal{T}$`)`

```
Input: Partial allocation $\mathcal{T}$.
Output: Partial allocation $\mathcal{R}$ such that either
        $\mathrm{NSW}(\mathcal{R}) \geq \mathrm{NSW}(\mathcal{T})$ and $\cup_i R_i \subsetneq \cup_i T_i$, or
        $\mathrm{NSW}(\mathcal{R}) \geq \frac{1}{2}\mathrm{NSW}(\mathcal{T})$ and $\mathcal{R}$ is $\frac{1}{2}$-EFX.
1  $\mathcal{S} \leftarrow \mathcal{T}$
2  repeat
3      $\mathcal{K} = (A \cup \mathcal{S}, E) \leftarrow \frac{1}{2}$-EFX feasibility graph of $\mathcal{S}$
        // Definition 4.3
4      $\mathcal{L} \leftarrow \{S_i, i \in A \mid S_i \subsetneq T_i\}$      // set of trimmed down
        bundles
5      Define matching $\tau$ with $\tau(i) = S_i$ for all $i \in A$
        // candidate matching
6      $\rho \leftarrow$ matching in $\mathcal{K}$ where                // Lemma 4.6
          $(a)$ all bundles in $\mathcal{L}$ are matched,
          $(b)$ $|\{i : \rho(i) = S_i\}|$ is maximized subject to $(a)$, and
          $(c)$ $\rho$ is maximum subject to $(a)$ and $(b)$
7      if $\exists i_1 \in A$ not matched in $\rho$ then
8          $(S_h, g_h) \leftarrow \arg\max_{k\in A, g\in S_k} v_{i_1}(S_k - g)$
9          if $v_h(S_h - g_h) \geq \frac{1}{2} \cdot v_h(T_h)$ then
10             $S_h \leftarrow S_h - g_h$
11         else
12             $P = (i_1, S_{i_1}, i_2, \ldots, S_{i_{\ell-1}}, i_\ell, S_{i_\ell}) \leftarrow$ alternating
                path between $\tau$ and $\rho$ starting at $i_1$ and ending at
                either $S_{i_\ell} = S_h$ or an unmatched $S_{i_\ell} \neq S_h$
                // Lemma 4.7
13             Construct $\mathcal{R}$:
14               $R_{i_1} \leftarrow S_h - g_h$
15               for $f \leftarrow 2$ to $\ell$ do $R_{i_f} \leftarrow S_{i_{f-1}}$
16               for $i \in A \setminus (\{i_1, \ldots, i_\ell\} \cup \{h\})$ do $R_i \leftarrow T_i$
17             if $P$ ends at an unmatched bundle $S_{i_\ell} \neq S_h$ then
18                 $R_h \leftarrow T_h \setminus (S_h - g_h)$
19             return $\mathcal{R}$
20 until $\rho$ is a perfect matching in $\mathcal{K}$
21 return $\mathcal{R} = (\rho(i))_{i\in A}$
```

If ρ is not perfect, then we consider an unmatched agent i_1, and find the bundle that maximizes i_1's utility after removal of one item. Let $(S_h, g_h) \in \arg\max_{k\in A, g\in S_k} v_{i_1}(S_k - g)$. If agent h's value of $S_h - g_h$ is at least $\frac{1}{2}$ times their value for the original bundle T_h, then we remove g_h from S_h and the main loop finishes. Otherwise, we construct an alternating path between ρ and τ, denoted as $P = (i_1, S_{i_1}, i_2, S_{i_2}, \ldots, S_{i_{\ell-1}}, i_\ell, S_{i_\ell})$, starting with i_1 and ending with either $S_{i_\ell} = S_h$ or an unmatched bundle $S_{i_\ell} \neq S_h$. Lemma 4.7 shows that such a P exists. Using P, we construct an allocation $\mathcal{R}$ in line 13. Lemma 4.8 shows that this is a suitable output of type (i).

Analysis. The number of iterations of the repeat loop is at most m, because the algorithm remove one item from some bundle in each iteration, in which it does not terminate. Since we can find the maximum matching in line 6 and alternating path in line 12 in strongly polynomial-time, Algorithm 4 runs in strongly polynomial-time.

The next lemma guarantees that the matching ρ is well-defined. The proof follows similarly as in [13].

Lemma 4.6. *In each iteration of the repeat loop in Algorithm 4, a matching exists in $\mathcal{K}$ where all bundles of $\mathcal{L}$ are matched.*

Proof. Let $E^{(t)}$ denote the edge set of the $\frac{1}{2}$-EFX feasibility graph and $\mathcal{L}^{(t)}$ the set of trimmed down bundles, and $\rho^{(t)}$ the maximum matching in the t-th iteration.

We show by induction that there exists a matching $\rho^{(t)}$ such that all bundles in $\mathcal{L}^{(t)}$ are matched. At the beginning of the first iteration, $\mathcal{L}^{(1)}$ is empty, so the claim is clearly true. Suppose the claim is true until the beginning of $(t+1)$-st iteration. Let $\mathcal{S}$ denote the trimmed down bundles in the t-th iteration, and let i_1 be the unmatched agent, and (S_h, g_h) the bundle and item selected in line 8.

By the requirement that $|\{i : \rho(i) = S_i\}|$ is maximized subject to all trimmed down bundles being matched, we have $(i_1, S_{i_1}) \notin E^{(t)}$. By the choice of h, we have $(i, S_h) \in E^{(t)}$ in the t-th iteration.

Note that $\mathcal{L}^{(t+1)} = \mathcal{L}^{(t)} \cup \{h\}$. Consider the $\frac{1}{2}$-EFX feasibility graph in the $(t+1)$-st iteration. Since all bundles different from $S'_h := S_h - g_h$ remained unchanged, for every edge $(i, S_k) \in E^{(t)}$ with $k \neq h$ it follows that $(i, S_k) \in E^{(t+1)}$. According to Definition 4.3, $(i_1, S'_h) \in E^{(t+1)}$. Let us define ρ' as

$$\rho'(i) := \begin{cases} S'_h & \text{if } i = i_1\,, \\ \rho^{(t)}(i) & \text{if } i \neq i_1 \text{ and } \rho^{(t)}(i) \neq S_h\,, \\ \perp & \text{otherwise}\,. \end{cases}$$

By the above, this gives a matching in $E^{(t+1)}$, and it matches all bundles in $\mathcal{L}^{(t+1)} = \mathcal{L}^{(t)} \cup \{h\}$. □

Lemma 4.7. *The alternating path P, as described in line 12 of Algorithm 4, exists.*

Proof. Since i_1 is an unmatched agent and the requirement that $|\{i : \rho(i) = S_i\}|$ is maximized subject to all trimmed down bundles being matched in the maximum matching ρ in line 6, we must have $(i_1, S_{i_1}) \notin E$. If $\rho(i_2) = S_{i_1}$ for an agent $i_2 \in A$, then we continue with S_{i_2}, otherwise we stop. Continuing this way, we eventually reach either $S_{i_\ell} = S_h$ or an unmatched bundle $S_{i_\ell} \neq S_h$. □

Lemma 4.8. *If Algorithm 4 returns an allocation $\mathcal{R}$ in line 21, then* $\mathrm{NSW}(\mathcal{R}) \geq \frac{1}{2}\,\mathrm{NSW}(\mathcal{T})$ *and $\mathcal{R}$ is $\frac{1}{2}$-EFX. If it returns $\mathcal{R}$ in line 19, then* $\mathrm{NSW}(\mathcal{R}) \geq \mathrm{NSW}(\mathcal{T})$ *and* $\cup_i R_i \subsetneq \cup_i T_i$.

Proof. Let us start with the case when a perfect matching $\mathcal{R} = (\rho(i))_{i \in A}$ is returned in line 21. The $\frac{1}{2}$-EFX property follows by Lemma 4.5. Let us show $\mathrm{NSW}(\mathcal{R}) \geq \frac{1}{2}\,\mathrm{NSW}(\mathcal{T})$.

Throughout the algorithm, $v_i(S_i) \geq \frac{1}{2} v_i(T_i)$ is maintained according to the condition on bundle trimming. By Claim 4.4, either $R_i = S_i$, or $v_i(R_i) > 2v_i(S_i)$. Therefore, we have

$$\forall i : v_i(R_i) \geq v_i(S_i) \geq \frac{1}{2} v_i(T_i)\,. \tag{5}$$

Consequently,

$$\mathrm{NSW}(\mathcal{R}) = \prod_{i \in A} (v_i(R_i))^{1/n} \geq \frac{1}{2} \cdot \prod_{i \in A} (v_i(T_i))^{1/n} \geq \tfrac{1}{2}\,\mathrm{NSW}(\mathcal{T})\,.$$

Consider now the case when the algorithm terminated with $\mathcal{R}$ in line 19. We need to show $\mathrm{NSW}(\mathcal{R}) \geq \mathrm{NSW}(\mathcal{T})$ and $\cup_i R_i \subsetneq \cup_i T_i$. Two cases here depend on whether $S_{i_\ell} = S_h$ or $S_{i_\ell} (\neq S_h)$ is an unmatched bundle. For the first case, we have

$$\begin{aligned} &v_{i_f}(R_{i_f}) = v_{i_f}(S_{i_{f-1}}) > 2v_{i_f}(S_{i_f}) \geq v_{i_f}(T_{i_f}),\ \forall f \in \{2, \ldots, \ell\}\,, \\ &v_{i_1}(R_{i_1}) = v_{i_1}(S_h - g_h) > 2v_{i_1}(S_{i_1}) \geq v_{i_1}(T_{i_1}) \end{aligned}$$

This implies

$$\mathrm{NSW}(\mathcal{R}) = \prod_{i \in A} \big(v_i(R_i)\big)^{1/n} > \prod_{i \in A} \big(v_i(T_i)\big)^{1/n} = \mathrm{NSW}(\mathcal{T})\,.$$

Since we do not assign g_h to any agent in $\mathcal{R}$, we must have $\cup_i R_i \subsetneq \cup_i T_i$.

For the second case, since S_{i_ℓ} is an unmatched bundle in ρ by the choice of the path P, we have $S_{i_\ell} \notin \mathcal{L}$ by the requirements on ρ. That is, $S_{i_\ell} = T_{i_\ell}$. By Claim 4.4, we have

$$\begin{aligned} &v_{i_\ell}(R_{i_\ell}) = v_{i_\ell}(S_{i_{\ell-1}}) > 2v_{i_\ell}(S_{i_\ell}) = 2v_{i_\ell}(T_{i_\ell})\,, \\ &v_{i_f}(R_{i_f}) = v_{i_f}(S_{i_{f-1}}) > 2v_{i_f}(S_{i_f}) \geq v_{i_f}(T_{i_f}),\ \forall f \in \{2, \ldots, \ell - 1\}\,, \\ &v_{i_1}(R_{i_1}) = v_{i_1}(S_h - g_h) > 2v_{i_1}(S_{i_1}) \geq v_{i_1}(T_{i_1}) \\ &v_h(R_h) = v_h(X_h \setminus (S_h - g_h)) > \tfrac{1}{2} v_h(T_h)\,. \end{aligned} \tag{6}$$

The last inequality follows from subadditivity using $v_h(T_h) \leq v_h(S_h - g_h) + v_h(T_h \setminus (S_h - g_h))$. Using (6), we get

$$\mathrm{NSW}(\mathcal{R}) = \prod_{i \in A} \big(v_i(R_i)\big)^{1/n} > \prod_{i \in A} \big(v_i(T_i)\big)^{1/n} = \mathrm{NSW}(\mathcal{T})\,.$$

Finally, since we do not assign items in T_{i_l} to any agent in $\mathcal{R}$, we must have $\cup_i R_i \subsetneq \cup_i T_i$. Note that if $T_{i_l} = \emptyset$, then $\mathrm{NSW}(\mathcal{T}) = 0$ and $R_i = \emptyset, \forall i$ is a suitable output of type (ii). □

5 CONCLUSION

We have shown a $(4+\varepsilon)$-approximation algorithm for the symmetric NSW problem with submodular valuations, which is the largest natural class of valuations that allows a constant-factor approximation (using value queries) even for utilitarian social welfare. Moreover, our algorithm gives an $\mathrm{e}(2 + nw_{\max} + \varepsilon)$-approximation algorithm for the asymmetric NSW problem under submodular valuations. However, there are still several directions and open problems to explore. An obvious one is to improve the approximation ratio for the symmetric case. The current hardness of approximation stands at $\frac{\mathrm{e}}{\mathrm{e}-1} \simeq 1.58$ for submodular valuations, which is the same as the optimal factor for maximizing utilitarian social welfare. It would be interesting to prove a separation between the two optimization objectives for submodular valuations.

Another open problem is the asymmetric NSW problem. The goal is to get a constant-factor approximation independent of the weights w_i. For the asymmetric problem, getting a universal constant factor is open even in the basic case of additive valuations. The simplest case not covered by our algorithm is when one agent has weight $1/2$ and all other agents have weight $1/2n$.

There are several open questions on the existence of EFX and its relaxations for submodular valuations. We mention two: First, does there exist a (complete) α-EFX allocations for $\alpha > 1/2$? Here, we do not make any efficiency requirements. Second, does there exist an EF1 allocation with high NSW value? Note that [14] shows that for additive valuations, the optimal NSW allocation is EF1.

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