Mechanism design: Multi-parameter environments

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- **Our goals:**
  - Incentivize the agents **to truthfully report their values**
  - Choose an outcome that **maximizes the social welfare**

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- In general, the agents might have different values for the possible winners of the item

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- Each agent i has a private value $v_i(S)$ for every possible bundle $S \subseteq M$ of items
  - Each agent i has 2^m parameters

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- **Allocation rule:** Maximize the social welfare according to the input

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- **Payment rule:** For a set of functions $h_1, \dots, h_n$ such that h_i is independent of the bid of agent i,

$$p_i(\boldsymbol{b}) = h_i(\boldsymbol{b}_{-i}) - \sum_{j \neq i} b_j\big(\boldsymbol{x}(\boldsymbol{b})\big)$$

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&= \underbrace{v_i\big(\boldsymbol{x}(\boldsymbol{b})\big) + \sum_{j \neq i} b_j\big(\boldsymbol{x}(\boldsymbol{b})\big)}_{\text{The social welfare according to the true value of agent } i \text{ and the bids of the other agents}} - h_i(\boldsymbol{b}_{-i})
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- Agent i cares about the welfare of all agents (based on the reported valuations) and aims to maximize the quantity

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- The mechanism is designed so that the incentives of the agents are aligned with the goal of maximizing the social welfare $\square$

Clarke payments

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- There are a lot of different VCG mechanisms, depending on how we choose the h-functions
- We would like to have reasonable payment rules, that satisfy a couple of properties:
  - **Individual rationality:** Every agent has non-negative utility, and therefore incentive to participate
  - **No positive transfers:** The mechanism does not pay the agents, the agents pay the mechanism

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- The payment of agent i is the difference between the maximum social welfare of the other agents when she does not participate, and the social welfare when she participates
- Agent i pays the loss in welfare due to her participation

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$\square$

Drawbacks of VCG mechanisms

- **Preference elicitation:** VCG mechanisms demand from each agent to communicate her values for every possible outcome
  - Not practical in many situations: communicating 2^m parameters in the case of combinatorial auctions is impossible, even for small m

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- **Social welfare maximization might be a hard problem**
- Knapsack auctions:
  - each agent i demands w_i items and has a private value v_i
  - the seller has a total amount of W items
  - Even though every agent has only one private parameter, maximizing the social welfare is equivalent to the Knapsack problem, which is NP-hard

Single-Minded Bidders

- Each bidder has a (private) set $T_i \in M$ and there is a private parameter $v_i \in \mathbb{R}^+$ s.t.

$$v_i(S) = \begin{cases} v_i & \text{if } S \supseteq T_i \\ 0 & \text{otherwise} \end{cases}$$

- A bid in this context is a pair (b_i, S_i).

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- Next we will talk about the greedy mechanism ..

The Greedy Mechanism

- Sort the bidders s.t.

$$\frac{b_1}{\sqrt{|S_1|}} \geq \frac{b_2}{\sqrt{|S_2|}} \geq \cdots \frac{b_n}{\sqrt{|S_n|}}.$$

- $W = \emptyset$ (W is the set of agents which can get their S_i's.)
- For i from 1 to n do: if $S_i \cap (\cup_{j \in W} S_j) = \emptyset$, add i to W.
- Return the allocation which gives S_i to player i iff $i \in W$.

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$$p_i = \frac{b_{\alpha(i)}}{\sqrt{|S_{\alpha(i)}|/|S_i|}} = b_{\alpha(i)} \sqrt{\frac{|S_i|}{|S_{\alpha(i)}|}}$$

The Analysis

- The mechanism is polynomial time and outputs a valid allocation.
- Incentive Compatibility: monotonicity and critical payment
- Social Welfare Approximately.

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  When $\alpha(i)$ doesn't exists. Then there is no other bidder who fails to win due to bidder i. So she is charged 0, the critical payment.
  Suppose $\alpha(i)$ does exist. Then i will still win as long as it appears before $\alpha(i)$ in the ordering. Thus the critical payment is x such that $\frac{x}{\sqrt{|S_i|}} = \frac{b_{\alpha(i)}}{\sqrt{|S_{\alpha(i)}|}}$, so we have the critical payment is

$$b_{\alpha(i)} \sqrt{\frac{|S_i|}{|S_{\alpha(i)}|}} = p_i.$$

Both Properties are Enough

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Any mechanism where losers pay 0 *which has both the monotonicity and critical payment properties is incentive compatible.*

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Proof.

Fix player $i \in [n]$ and all bids other than i's. Let $u(b, S)$ be the utility that player i would get by bidding (b, S), so $u(b, S) = v_i(S) - p_i(b, S)$. By the critical payment, we know $p_i(b, S) = \inf\{x : i \text{ wins with bid } (x, S)\}$.

- non-negative utility: 0 or $v_i - p_i(b, T_i) \geq 0$.
- dominant strategy: $u(v_i, T_i) \geq u(b, T_i) \geq u(b, S)$ for all (b, S) with $T_i \subseteq S$ and (b, S) is a winning bid. (Why?)

$\square$

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- So for the first part, we just need to show $p_i(b, T_i) \leq p_i(b, S)$. Holds by the monotonicity property and $T_i \subseteq S$.
- $u(v_i, T_i) = u(b, T_i)$ when (v_i, T_i) is a winning bid, since (b, T_i) is a winning bid; $p_i(b, T_i) \geq v_i$ if (v_i, T_i) is not a winning bid.

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$$\sum_{i \in \mathrm{OPT}} v_i \leq \sum_{i \in W} \sum_{j \in \mathrm{OPT}_i} v_j \leq \sum_{i \in W} v_i \sqrt{m} = \sqrt{m} \sum_{i \in W} v_i.$$