Root Finding with Chebyshev Polynomials in Two Dimensions

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Summary

We want to find all isolated, common roots of $f, g : \mathbb{R}^2 \to \mathbb{R}$ in a bounded domain. We replicate the root finding algorithm found in [5], which approximates f, g with Chebyshev polynomials and finds the roots of the polynomials. We are working to extend the algorithm to non-rectangular domains.

Background, Applications, and Motivation

Background and Applications

- `ChebTools` [1] is a C++ library developed by Ian Bell
- Inspired by the Matlab library `chebfun` [3]
- Determine thermodynamic properties of a fluid [4]

Motivating Example

$f(x,y) = x^2 - y^2, g(x,y) = \sin\left(\pi\sqrt{x^2+y^2}\right)$ with the domain $[-1,1]^2$

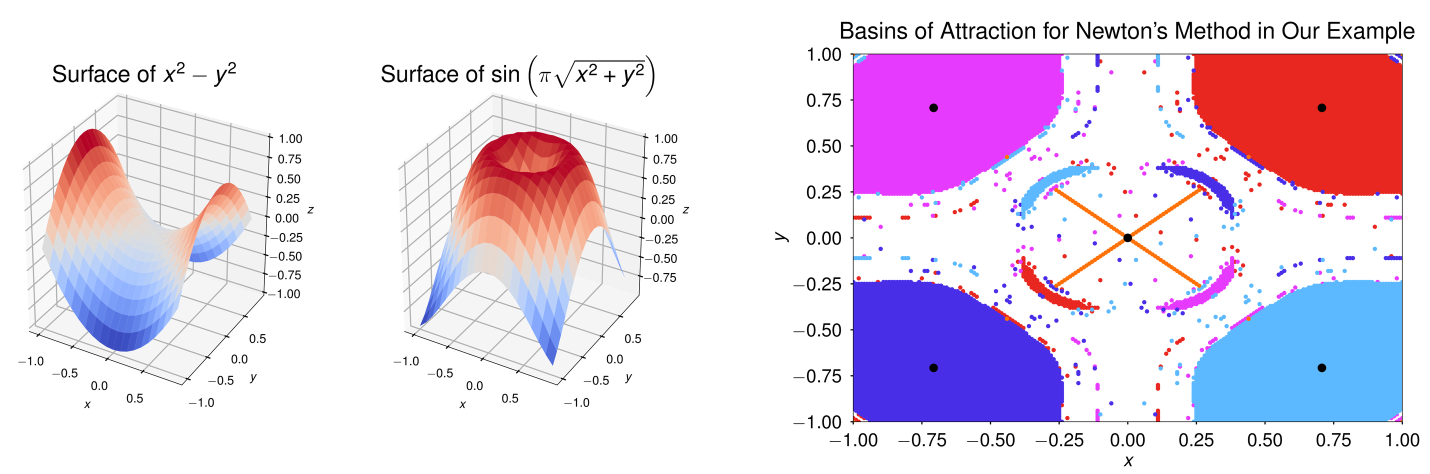


Figure: Newton's method would require multiple runs to find all roots, which would be inadequate for our application.

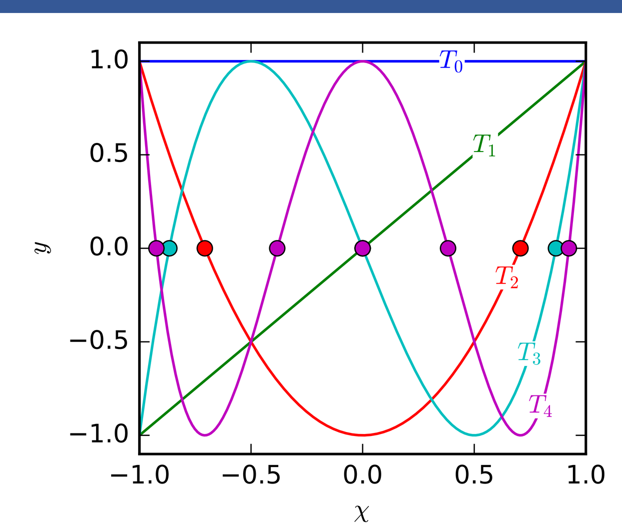
Chebyshev Polynomials

Definition

$$T_n(x) = \cos(n \arccos(x)),\ x \in [-1,1]$$

Interpolation Convergence [8, Ch. 7, 8]

- $f^{(m)}$ has bounded variation $\Longrightarrow$ polynomial convergence of order m
- f is analytic $\Longrightarrow$ geometric convergence



Summary of Algorithm from [5]

- Construct interpolations, p_f, p_g, of f, g [6, Sec. 2]
- Apply the Bézout Resultant Method for x values
- Apply 1-D root finding techniques for y values

Bézout Resultant Method

With p_f, p_g of degree $(m_f, n_f), (m_g, n_g)$, we can construct a square **Bézout Matrix Polynomial** of degree $M \le m_f + m_g$ and size $n = \max(n_f, n_g)$:

$$B(x) = \sum_{i=0}^{M} B_i T_i(x),$$

- $\det(B(x_0)) = 0 \iff p_f(x_0,\cdot)$ and $p_g(x_0,\cdot)$ have a common root
- Solving $\det(B(x_0)) = 0$ involves linearizing $B(x)$

Challenges with the Algorithm

Poor Conditioning

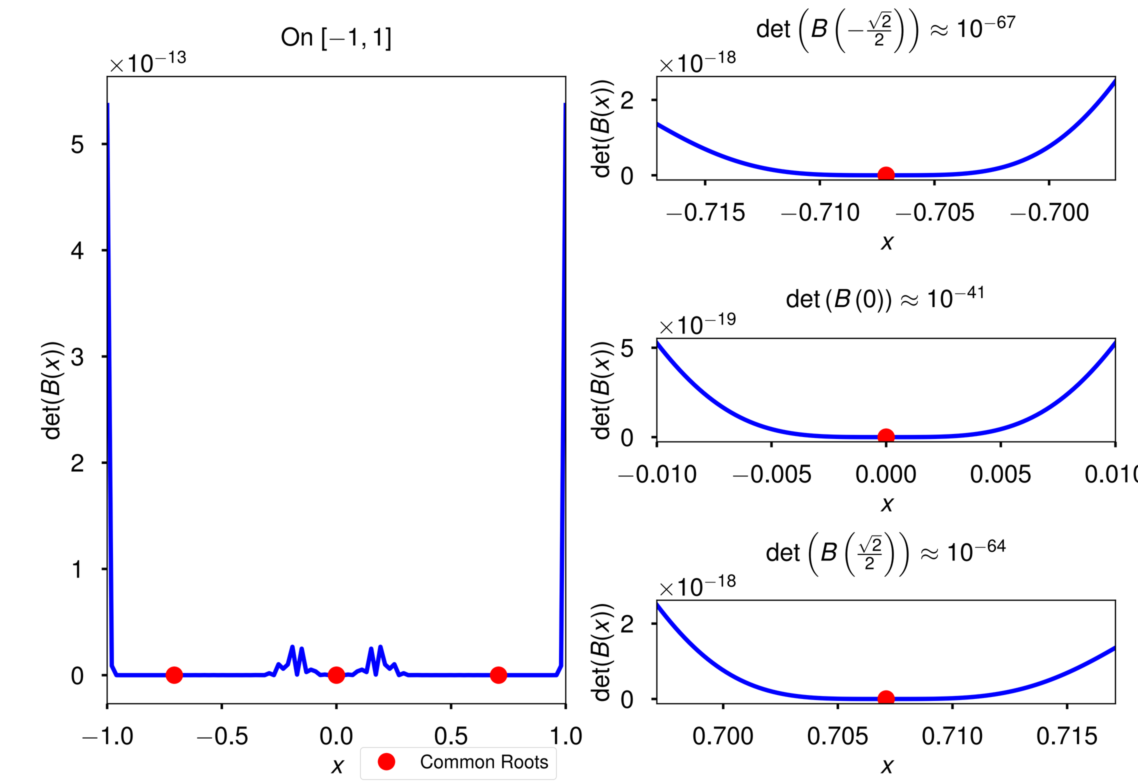


Figure: $\det(B(x))$ from our example. The Bézout technique can square the condition number. [5, Sec. 5]

Poor Scaling

- Have to solve a generalized eigenvalue problem [5, Sec. 3]
- Computational complexity of $\mathcal{O}(M^3 n^3)$ [5, Sec. 4]

Our Implementation

Refinement

- `chebfun2` refines roots by repeating the Bézout method in a smaller region surrounding the root [5, Sec. 7]
- We refine the roots with Newton's method

1-D Root Finding

- Our 1-D algorithm is a mix of a bisection and secant method
- Faster than companion matrix methods [2]

Domain Subdivision

- `chebfun2` divides the domain to reduce the matrix polynomial problem size [5, Sec. 4]
- We have yet to implement a domain subdivision method

Current Results

	ChebTools		chebfun	
Functions	Max $\|\cdot\|_2$ Error	Time (s)	Max $\|\cdot\|_2$ Error	Time (s)
$F_1(x,y), F_2(x,y)$	6.97×10^{-32}	0.004268	7.7×10^{-16}	0.522
$G_1(x,y), G_2(x,y)$	6.21×10^{-16}	214.059	2.92×10^{-10}	0.294
$H_1(x,y), H_2(x,y)$	3.24×10^{-16}	195.903	2.77×10^{-11}	0.296

$$F_1(x,y) = T_3(x) - 13T_1(x) \qquad F_2(x,y) = T_3(y) - 13T_1(y)$$
$$G_1(x,y) = \cos(\pi x)(y-2) \qquad G_2(x,y) = (y - .9)(x-2)$$
$$H_1(x,y) = \cos\left(\pi x - \frac{\pi}{10}\right)(y-2) \qquad H_2(x,y) = (y-.1)(y-.9)(x-2)$$

Discussion

- `chebfun` is much faster than `ChebTools`
- This speed difference can be explained by the lack of a subdivision strategy in our implementation
- A new subdivision strategy is part of current and future work

Current Work

Our Application Motivates Current Work

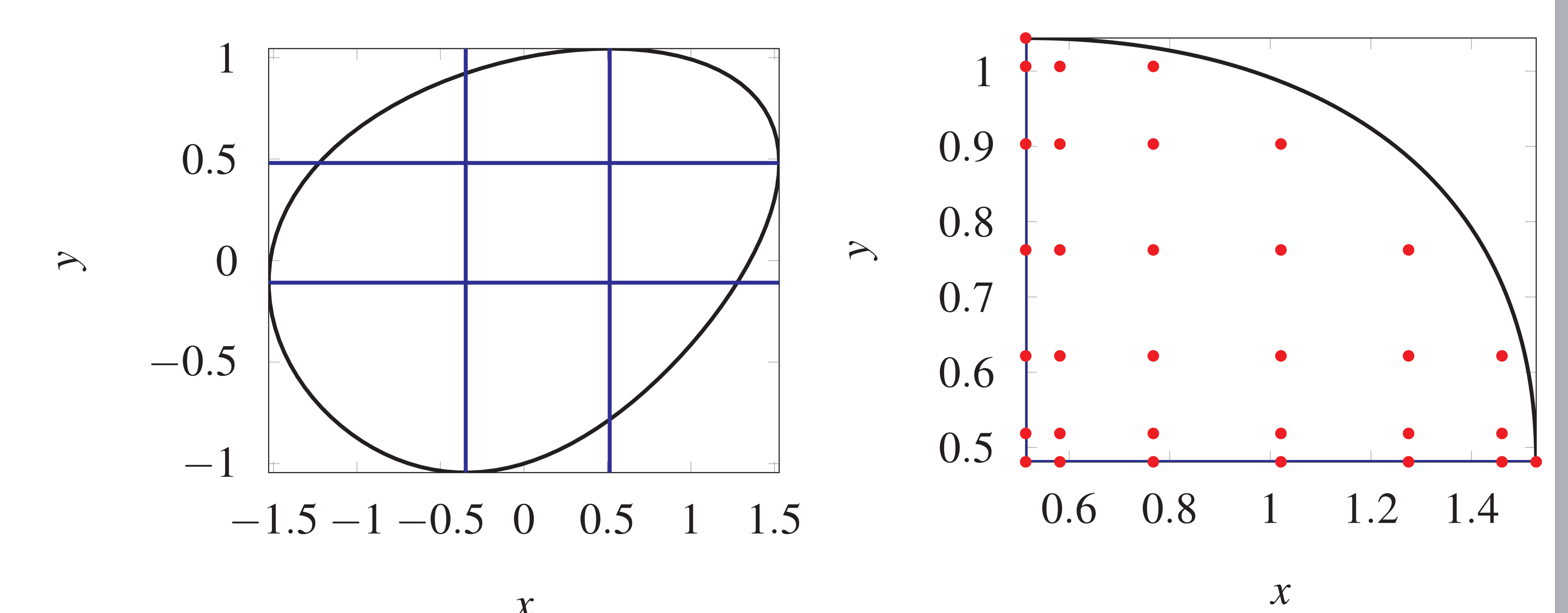
- Empirical equations of state in thermodynamics have ranges of validity which are not rectangular domains
- Have to use coordinate transformations [7] or to work on non-rectangular domains in `chebfun2`

Our Subdivision Idea

- Subdivide where domain boundary is horizontal, vertical, or has a derivative discontinuity
- Interior subdomains can proceed with rectangular subdivision like in the `chebfun` algorithm
- Approximate on exterior subdomains using interior Chebyshev points

Example Domain

Interior of the closed curve $(1.5\cos(t) + .15\sin(2t), \sin(t) + .3\cos(t))$ with $t \in [0, 2\pi]$



(a) Boundary in black and initial subdivisions in blue.

(b) Interior 7x7 Chebyshev nodes in red.

Figure: Example domain on the left and the upper right subdomain on the right

Future Work

- Parallelize the `ChebTools` library
- Adaptive capabilities for approximations

Add Features To Construct a Chebyshev expansion from:

- linear least squares
- linear boundary value problems

References

[1] I.H. Bell, L. Bouck, and B.K. Alpert. Chebtools: A c++11 based library for working with chebyshev expansions. *Journal of Open Source Software*, 2018. To be submitted.

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[7] Alex Townsend. Chebfun2 objects on non-rectangular domains. http://www.chebfun.org/examples/approx2/Other2DDomains.html, 2013.

[8] L.N. Trefethen. *Approximation Theory and Approximation Practice (Other Titles in Applied Mathematics)*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2012.

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