# Rootfinding with Chebyshev Polynomials in 2 Dimensions

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# Outline

# The Problem

## Global 2-D Rootfinding Problem

We want to find all solutions to

$$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

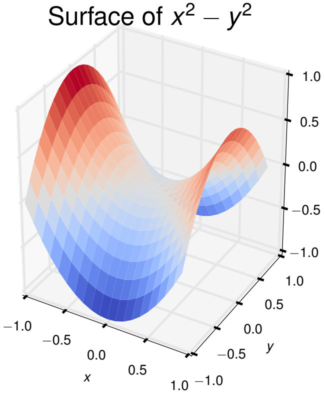
in a given bounded domain $\Omega$

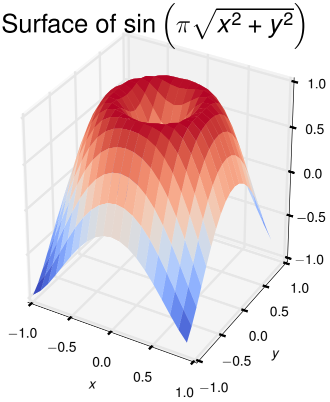
## The Big Picture: `chebtools`

- `chebtools` is a C++ library for working with Chebyshev expansions developed by Ian Bell
- Inspired by the Matlab library `chebfun`
- Solving this problem will be a significant feature for `chebtools`
- With a higher level Python interface, `chebtools` could be useful for a wide range of users

# Example

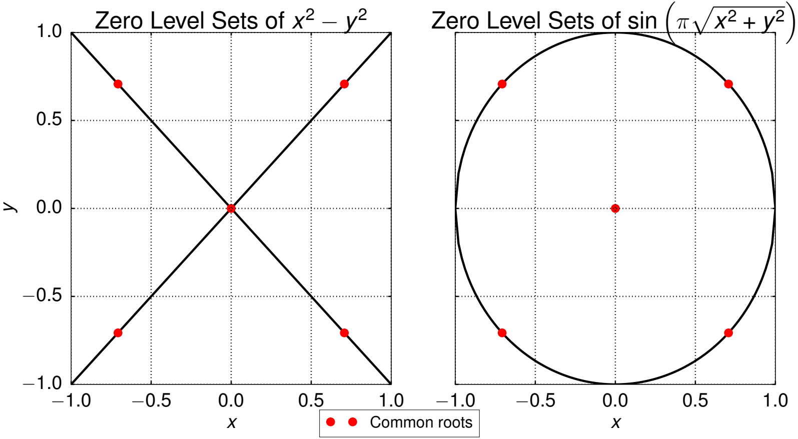
$f(x, y) = x^2 - y^2$ and $g(x, y) = \sin\left(\pi\sqrt{x^2 + y^2}\right)$ with the square domain $\Omega = [-1, 1]^2$



# Example (cont.)

Another view: We are finding where $f$ and $g$'s zero level sets intersect inside $\Omega$



# Local Methods May Not Be Good for Global rootfinding

There are actually infinitely many roots outside $\Omega$ in our example. Local methods may converge to roots outside the domain of interest.

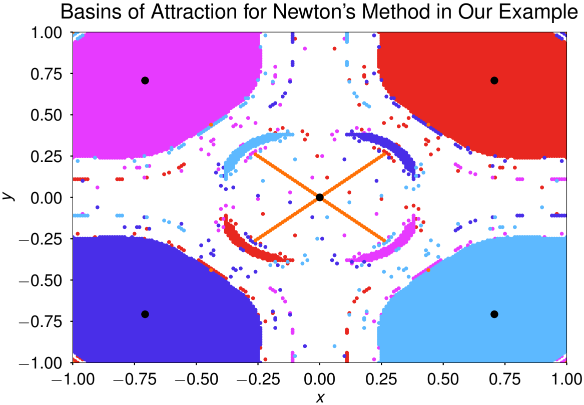



Figure: Initial guesses in white areas did not converge to roots in $\Omega$

# A Need for a Global Method

Our example illustrates a few lessons

- Many iterative and local methods like Newton's method can only guarantee convergence to a root if the initial guess is sufficiently close
- Sufficiently close depends on the problem
- This may be extremely close for some roots like $(0, 0)$ in our example

**Goal:**

To develop a truly global rootfinding method

# What are Chebyshev polynomials?

## Chebyshev polynomials

$$T_n(x) = \cos\left(n \arccos(x)\right), \; x \in [-1, 1]$$

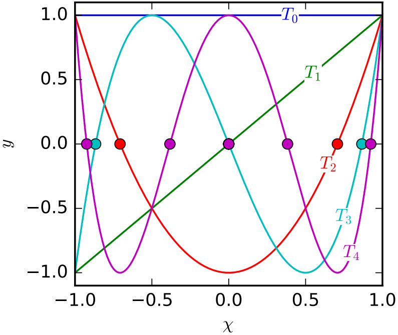
Orthogonal under a weighted inner product:

$$\int_{-1}^{1} T_n(x) T_k(x) w(x)\, dx = 0$$

when $n \neq k$
and $w(x) = \frac{1}{\sqrt{1-x^2}}$

Numerically stable for interpolating functions

Figure: First 5 Chebyshev Polynomials



# Chebyshev polynomials are numerically stable

- Interpolation on evenly spaced points is susceptible to the **Runge phenomenon**
- Chebyshev interpolation minimizes this effect.

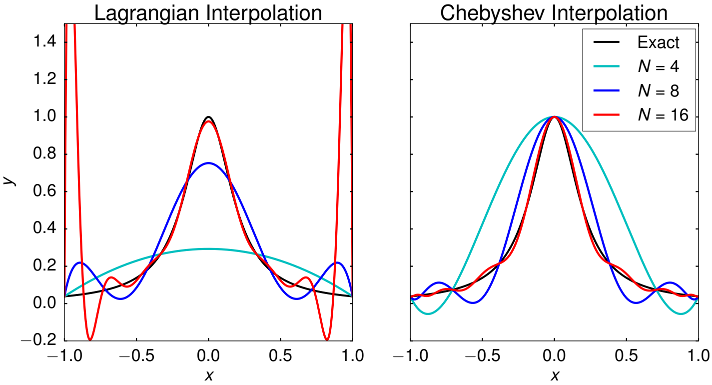



Figure: Interpolating $\frac{1}{1+25x^2}$

1. The Problem

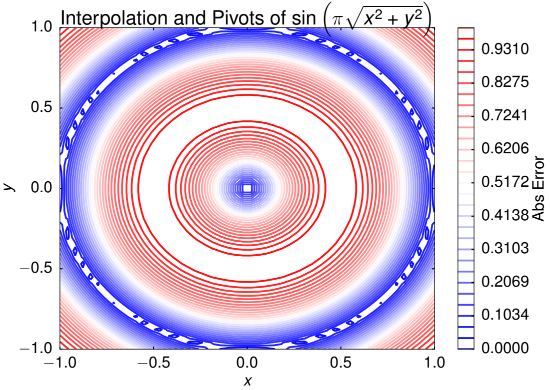
2. What are Chebyshev Polynomials and why use them?

3. The Algorithm

# 2-D Interpolation

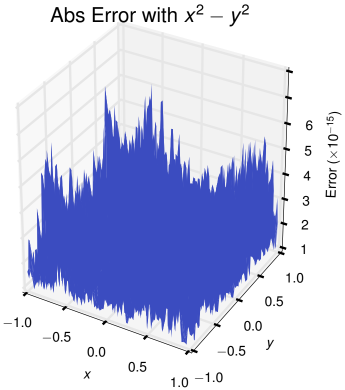
## Idea:

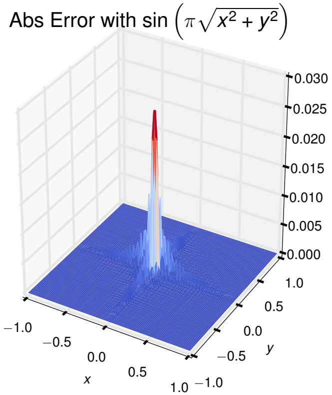
- Pick point of maximum error
- Do 1-D interpolations in $x$ and $y$ directions



# Our Algorithm Can Provide Accurate Approximations

- Almost machine precision accuracy with $x^2 - y^2$
- Struggles with $\sin\left(\pi\sqrt{x^2+y^2}\right)$ due to the lack of differentiability at $(0,0)$

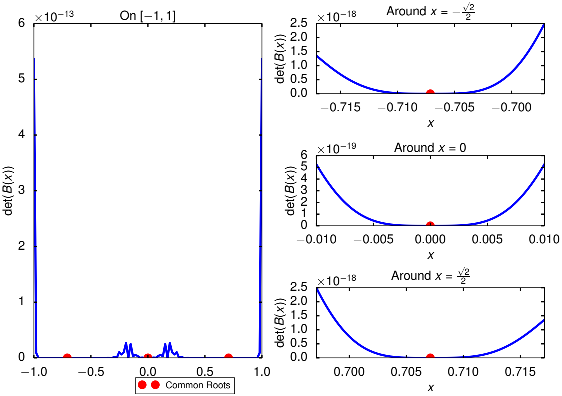


# Bézout Matrix Polynomials

## We create a matrix polynomial $B(x)$ from our Chebyshevs

- $B(x)$ is a matrix with polynomial entries
- $\det(B(x)) = 0 \iff$ common roots along the specific $x$ value
- Solving $\det(B(x)) = 0$ is a **matrix polynomial eigenvalue problem**



# 1-D Rootfinding and Local Refinement

- We now have the possible $x$ values for where there are common roots
- Employ a well known 1-D rootfinding method again using Chebyshev Polynomials
- **Problem:** Poor conditioning of matrix polynomial problem $\implies$ need for local refinement of roots
- Refine the roots with Newton's method

# Results:

# Conclusions and Future Work

Key Contributions:

- Made significant progress in replicating the 2-D rootfinding algorithm in `chebfun` with some modifications in C++11, which will be fully open source
- Developed ideas and have a strategy for extending the algorithm to non-rectangular domains

Future Work:

- Further develop ideas on non-rectangular domains and subdivision strategies
- Introduce GPU/parallel computing to the rootfinding process

Other Contributions while at NIST:

- Idea for using Gram-Schmidt process to create orthogonal terms for developing equations of state

# Acknowledgements


- Dr. Ian Bell for the mentorship and guidance
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# References

References here