Math 478 HW 5

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We'll need all the following Taylor expansions for the following problems, so let's get it out of the way.

$$\begin{aligned}
\alpha u(x+2h) &= \alpha u(x) + 2\alpha h u'(x) + \frac{\alpha 4h^2}{2}u''(x) + \frac{\alpha 8h^3}{6}u'''(x) + \frac{\alpha 16h^4}{4!}u^{(4)}(x) + \frac{\alpha 32h^5}{5!}u^{(5)}(x) + ... \\
\beta u(x+h) &= \beta u(x) + \beta h u'(x) + \frac{\beta h^2}{2}u''(x) + \frac{\beta h^3}{6}u'''(x) + \frac{\beta h^4}{4!}u^{(4)}(x) + \frac{\beta h^5}{5!}u^{(5)}(x) + ... \\
\gamma u(x) &= \gamma u(x) \\
\delta u(x-h) &= \delta u(x) - \delta h u(x) + \delta\frac{h^2}{2}u''(x) - \delta\frac{h^3}{6}u'''(x) + \delta\frac{h^4}{4!}u^{(4)}(x) - \delta\frac{h^5}{5!}u^{(5)}(x) + ... \\
\epsilon u(x-2h) &= \epsilon u(x) - \epsilon 2h u(x) + \epsilon\frac{4h^2}{2}u''(x) - \epsilon\frac{8h^3}{6}u'''(x) + \epsilon\frac{16h^4}{4!}u^{(4)}(x) - \epsilon\frac{32h^5}{5!}u^{(5)}(x) + ...
\end{aligned}$$

$$\begin{aligned}
\alpha u(x+2h) + \beta u(x+h) + \gamma u(x) + \delta u(x-h) + \epsilon u(x-2h) &= (\alpha+\beta+\gamma+\delta+\epsilon)u(x) \\
&+ (2\alpha+\beta-\delta-2\epsilon)hu'(x) + (4\alpha+\beta+\delta+4\epsilon)\frac{h^2}{2}u''(x) \\
&+ (8\alpha+\beta-\delta-8\epsilon)\frac{h^3}{6}u'''(x) + (16\alpha+\beta+\delta+16\epsilon)\frac{h^4}{4!}u^{(4)}(x) \\
&+ (32\alpha+\beta-\delta-32\epsilon)\frac{h^5}{5!}u^{(5)}(x) + (64\alpha+\beta+\delta+64\epsilon)\frac{h^6}{6!}u^{(6)}(x) + ...
\end{aligned}$$

1 Problem 3.7.1

So if $\alpha = -1, \beta = 8, \gamma = 0, \delta = -8, \epsilon = 1$, then

$$\begin{aligned}
\frac{-u(x+2h) + 8u(x+h) - 8u(x-h) + u(x-2h)}{12h} &= \frac{0}{12h}u(x) + u'(x) + 0h^2u''(x) \\
&+ 0h^3u'''(x) + 0h^4u^{(4)}(x) - \frac{4h^4}{5!}u^{(5)}(x) + ... \\
&= u'(x) + \mathcal{O}(h^4)
\end{aligned}$$

so $\frac{-u(x+2h)+8u(x+h)-8u(x-h)+u(x-2h)}{12h}$ is a 4th order approximation for $u'(x)$.

2 Problem 3.7.2

2.1 Part a

Suppose $\delta = \epsilon = 0$. Then,

$$\frac{\alpha u(x+2h) + \beta u(x+h) + \gamma u(x)}{(2\alpha+\beta)h} = \frac{\alpha+\beta+\gamma}{(2\alpha+\beta)h}u(x) + u'(x) + \frac{(4\alpha+\beta)h}{2(2\alpha+\beta)}u''(x)$$
$$+ \frac{(8\alpha+\beta)h^2}{6(2\alpha+\beta)}u'''(x) + \frac{(16\alpha+\beta)h^3}{4!(2\alpha+\beta)}u^{(4)}(x) + \frac{(32\alpha+\beta)h^4}{5!(2\alpha+\beta)}u^{(5)}(x) + ...$$

We want to get the highest order we can, so we should choose the coefficients such that $(2\alpha+\beta) \neq 0$ and as many coefficients as possible are zero. Then consider the system

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$$

The solution is then

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$$

and our approximation is

$$\frac{u(x+2h) - 4u(x+h) + 3u(x)}{-2h} = u'(x) + \frac{h^2}{-3}u'''(x) + \mathcal{O}(h^3) = u'(x) + \mathcal{O}(h^2)$$

Suppose we want to do better and get $16\alpha + \beta = 0$. Then our system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 1 & 0 \\ 16 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The only solution is the zero vector, which would make $2\alpha + \beta = 0$, which we don't want. Thus, we cannot get a third order approximation. Thus

$$\frac{u(x+2h) - 4u(x+h) + 3u(x)}{-2h} = u'(x) + \frac{h^2}{-3}u'''(x) + \mathcal{O}(h^3) = u'(x) + \mathcal{O}(h^2)$$

Is our best approximation, which is order 2.

2.2 Part b

If $\alpha = \epsilon = -1/12, \beta = \delta = 4/3$ and $\gamma = -5/2$, then

$$
\begin{aligned}
\frac{-u(x+2h)}{12} + \frac{4u(x+h)}{3} + \frac{-5u(x)}{2} + \frac{4u(x-h)}{3} + \frac{-5u(x-2h)}{2} &= h^2u''(x) - \frac{8h^6}{6!}u^{(6)}(x) + ... \\
\frac{\frac{-u(x+2h)}{12} + \frac{4u(x+h)}{3} + \frac{-5u(x)}{2} + \frac{4u(x-h)}{3} + \frac{-5u(x-2h)}{2}}{h^2} &= u''(x) - \frac{8h^4}{6!}u^{(6)}(x) + ... \\
\frac{\frac{-u(x+2h)}{12} + \frac{4u(x+h)}{3} + \frac{-5u(x)}{2} + \frac{4u(x-h)}{3} + \frac{-5u(x-2h)}{2}}{h^2} &= u''(x) + \mathcal{O}(h^4)
\end{aligned}
$$

The order of this approximation is 4.

2.3 Part c

From the previous part, we have a fourth order approximation. Let's see if we can get a higher order approximation. To get a higher order approximation, we would need $64\alpha + \beta + \delta + 64\epsilon = 0$. The resulting linear system would be

$$
\begin{pmatrix} 2 & 1 & -1 & -2 \\ 8 & 1 & -1 & -8 \\ 16 & 1 & 1 & 16 \\ 32 & 1 & -1 & -32 \\ 64 & 1 & 1 & 64 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \delta \\ \epsilon \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

The only solution to this would be the zero vector, which would cause $4\alpha + \beta + \delta + 4\epsilon = 0$, which we don't want. This means that we cannot get a higher order approximation, and order 4 is the best approximation we can get.

2.4 Part d

We can get an approximation for $u'''(x)$. Let $\alpha = 1, \beta = -2, \gamma = 0, \delta = 2, \epsilon = -1$. Then,

$$
\begin{aligned}
\frac{u(x+2h) - 2u(x+h) + 2u(x-h) - u(x-2h)}{2h^3} &= u'''(x) + \frac{30h^2}{5!}u^{(5)}(x) + ... \\
&= u'''(x) + \mathcal{O}(h^2)
\end{aligned}
$$

creates a second order approximation for $u'''(x)$

3 Problem 3.7.3

3.1 Part a

If $\alpha = -1/2, \beta = 2, \gamma = -3/2, \delta = \epsilon = 0$, then

$$\begin{aligned}
\frac{-u(x+2h)/2 + 2u(x+h) - 3u(x)/2}{h} &= (-\frac{1}{2} + 2 - \frac{3}{2} + 0 + 0)\frac{u(x)}{h} + u'(x) \\
&+ (-\frac{4}{2} + 2)\frac{hu''(x)}{2} + \frac{(-4+2)h^2}{6}u'''(x) + ... \\
&= u'(x) - \frac{h^2}{3}u'''(x) + ... \\
&= u'(x) + \mathcal{O}(h^2)
\end{aligned}$$

Thus, we have a second order approximation for $u'(x)$

3.2 Part b

If $\alpha = \beta = 0$, choose $\gamma = -3, \delta = 4, \epsilon = -1$. Then,

$$\begin{aligned}
\frac{-3u(x) + 4u(x-h) - u(x-2h)}{-2hh} &= (-3 + 4 - 1)\frac{u(x)}{-2h} + u'(x) \\
&+ (4-4)\frac{hu''(x)}{-4} + \frac{(-4+8)h^2}{-12}u'''(x) + ... \\
&= u'(x) - \frac{h^2}{3}u'''(x) + ... \\
&= u'(x) + \mathcal{O}(h^2)
\end{aligned}$$

so we have a second order approximation for $u'(x)$