In-Class Assignment 1

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1 Problem 1a

Let s_n and t_n be convergent sequences such that $\lim_n s_n = L$ and $\lim_n t_n = M$. Assume that $s_n < t_n$ for all n. Prove that $L \leq M$.
Proof: (By contrapositive)

Let s_n and t_n be convergent sequences such that $\lim_n s_n = L$ and $\lim_n t_n = M$. Assume $M < L$. We want to show that there exists a natural number n such that $t_n \leq s_n$. Fix $\epsilon = \frac{L-M}{2}$. Since $\lim_n s_n = L$ and $\lim_n t_n = M$, there exists $N_1, N_2 \in \mathbb{N}$ such that $|s_l - L| < \epsilon$ for all $l \geq N_1$ and $|t_m - M| < \epsilon$ for all $m \geq N_2$. Let $n = \max\{N_1, N_2\}$. Because $n \geq N_1$ and $n \geq N_2$, $|s_n - L| < \epsilon$ and $|t_n - M| < \epsilon$. Since both $|s_n - L| < \epsilon$ and $|t_n - M| < \epsilon$, we get

$$|s_n - L| + |t_n - M| < \epsilon + \epsilon = 2\epsilon = 2\frac{L-M}{2} = L - M$$

We know that $|s_n - L| = |L - s_n|$, so $|L - s_n| + |t_n - M| < L - M$. By the triangle inequality, we know $|(L - s_n) + (t_n - M)| \leq |s_n - L| + |t_n - M|$. Then, $|(L - s_n) + (t_n - M)| \leq |L - s_n| + |t_n - M| < L - M$, which means $|(L - s_n) + (t_n - M)| < L - M$. We also know that $(L - s_n) + (t_n - M) \leq |(L - s_n) + (t_n - M)|$. Then,

$$\begin{aligned}
(L - s_n) + (t_n - M) &< L - M \\
L - s_n + t_n - M &< L - M \\
L - s_n + t_n &< L \\
-s_n + t_n &< 0 \\
t_n &< s_n
\end{aligned}$$

Since $t_n < s_n$, $t_n \leq s_n$. Since we have shown that there exists an n such that $t_n \leq s_n$. By contrapositive, if $\lim_n s_n = L$, $\lim_n t_n = M$, and $s_n < t_n$, then $L \leq M$.

2 Problem 1b

Prove that under the above hypothesis it is not necessarily true that $L < M$.
Proof: (By example)

In order to prove that it is not necessarily true that $L < M$, an example will be provided to show that $L = M$ still satisfies the above hypothesis. Let $t_n = \frac{1}{n}$ and $s_n = \frac{-1}{n}$, and let $L = \lim_n s_n$ and $M = \lim_n t_n$. We want to show that $t_n > s_n$ for all natural n and that $L = M = 0$.

We first want to prove that $s_n < t_n$ for all natural n. Let n be a natural number. We know $-1 < 1$. By multiplying $\frac{1}{n}$ to both sides, we get $\frac{-1}{n} < \frac{1}{n}$. Since n is a natural number, the inequality is preserved, and $s_n < t_n$ for all natural n.

We must now show that $\lim_n s_n = 0$. Let $\epsilon < 0$. We want to show that there exists a natural N such that $\left|\frac{-1}{n} - 0\right| < \epsilon$ for all $n \geq N$. Choose N so that $N\epsilon > 1$. Since $\epsilon, 1 \in \mathbb{R}$ and $\epsilon > 0$, by the Archimedean principle, there exists a natural number N such that $1 < N\epsilon$. Let $n \in \mathbb{N}$ such that $n \geq N$. Then, $1 < N\epsilon \leq n\epsilon$. Then, $1 < n\epsilon$. Since n is natural, inequality is preserved from multiplication and division, and $\frac{1}{n} < \epsilon$. Since n is natural, $\left|\frac{1}{n}\right| = \frac{1}{n}$. Also, we know that $\left|\frac{1}{n}\right| = \left|\frac{-1}{n}\right|$. Then, $\left|\frac{-1}{n}\right| < \epsilon$ for all $n \geq N$. Then, $\left|\frac{-1}{n} - 0\right| < \epsilon$ for all $n \geq N$, which means $\lim_n s_n = 0$.

Next, we want to show that $\lim_n t_n = 0$. Let $\epsilon < 0$. We want to show that there exists a natural N such that $\left|\frac{1}{n} - 0\right| < \epsilon$ for all $n \geq N$. Choose N so that $N\epsilon > 1$. Since $\epsilon, 1 \in \mathbb{R}$ and $\epsilon > 0$, by the Archimedean principle, there exists a natural number N such that $1 < N\epsilon$. Let $n \in \mathbb{N}$ such that $n \geq N$. Then, $1 < N\epsilon \leq n\epsilon$. Then, $1 < n\epsilon$. Since n is natural, inequality is preserved from multiplication and division, and $\frac{1}{n} < \epsilon$. Since n is natural, $\left|\frac{1}{n}\right| = \frac{1}{n}$. Then, $\left|\frac{1}{n}\right| < \epsilon$. Then, $\left|\frac{1}{n} - 0\right| < \epsilon$. Therefore, $\lim_n t_n = 0$.

Since $\lim_n t_n = 0$ and $\lim_n s_n = 0$, $L = M$. Because $L = M$ and $s_n < t_n$ for all natural n, we have provided an example that shows that it is not necessarily true that $L < M$ under the above hypothesis.