Math 478 HW 1

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1 Problem 1.5.1

Use the gradient method to find the outward normal unit vector to a point on the boundary of the following regions.

1.1 Part a

$\Omega = \{(x, y) \in \mathbb{R}^2 : x^2/25 + y^2/36 < 1\}$

Answer: Our $G(x, y) = x^2/25 + y^2/36$, so $\nabla G(x, y) = (2x/25, y/18)$. To see if it is inward or outward, I'll just use a test point $\nabla G(0, 6) = (0, 1/3)$, so the gradient is pointing out. By dividing by $||\nabla G||$, the vector is now a unit vector and

$$n(x, y) = \frac{1}{\sqrt{\left(\frac{2x}{25}\right)^2 + \left(\frac{y}{18}\right)^2}} \left(\frac{2x}{25}, \frac{y}{18}\right)$$

1.2 Part b

$\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2/4 + y^2/9 + z^2/81 < 1\}$

Answer: Our $G(x, y, z) = x^2/4 + y^2/9 + z^2/81$, so $\nabla G(x, y, z) = \left(\frac{x}{2}, \frac{2y}{9}, \frac{2z}{81}\right)$. To see if it is inward or outward, I'll just use a test point $\nabla G(2, 0, 0) = (1, 0, 0)$, which is away from the domain, so the gradient is pointing out. By dividing by $||\nabla G||$, the vector is now a unit vector and

$$n(x, y, z) = \frac{1}{\sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{2y}{9}\right)^2 + \left(\frac{2z}{81}\right)^2}} \left(\frac{x}{2}, \frac{2y}{9}, \frac{2z}{81}\right)$$

1.3 Part c

$\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 16\}$

Answer: Our $G(x, y, z) = x^2 + y^2$, so $\nabla G(x, y, z) = (2x, 2y, 0)$. To see if it is inward or outward, I'll just use a test point $\nabla G(4, 0, 0) = (8, 0, 0)$, which is away from the domain, so the gradient is pointing out. By dividing by $||\nabla G||$, the vector is now a unit vector and

$$n(x, y, z) = \frac{1}{\sqrt{4x^2 + 4y^2}}(2x, 2y, 0)$$

1.4 Part d

$\Omega = \{(x, y) \in \mathbb{R}^2 : y - 3x^2 > 0\}$

Answer: Our $G(x, y, z) = y - 3x^2$, so $\nabla G(x, y) = (-6x, 1)$. To see if it is inward or outward, I'll just use a test point $\nabla G(0, 0) = (0, 1)$, which is pointing into the domain, so we need to multiply the gradient by -1 to get an outward vector. Now $-\nabla G(x, y) = (6x, -1)$. By dividing by $||\nabla G||$, the vector is now a unit vector and

$$n(x, y) = \frac{1}{\sqrt{36x^2 + 1}}(6x, -1)$$

2 Problem 1.5.2

2.1 Part a

Find $\frac{\partial F}{\partial n}$ for $F(x, y) = x^2 + y^2$ and $\Omega = \{(x, y) : x^2/4 + y^2/9 < 1\}$ at (2,0).

Answer: $\nabla F(x, y) = (2x, 2y)$ and $n(x, y) = \frac{1}{\sqrt{(x/2)^2 + (2y/9)^2}}(x/2, 2y/9)$. Then, $\nabla F(2, 0) = (4, 0)$ and $n(2, 0) = (1, 0)$. Then,

$$\frac{\partial F}{\partial n}(2, 0) = \nabla F(2, 0) \cdot n(2, 0) = (4, 0) \cdot (1, 0) = 4$$

2.2 Part b

Find $\frac{\partial F}{\partial n}$ for $F(x, y) = \ln x + y^2$ and $\Omega = \{(x, y) : 3y - x^2 < 17\}$ at (2,7).

Answer: $\nabla F(x, y) = (1/x, 2y)$ and $n(x, y) = \frac{1}{\sqrt{4x^2+9}}(-2x, 3)$. Then, $\nabla F(2, 7) = (1/2, 14)$ and $n(2, 7) = (-4/5, 3/5)$. Then,

$$\frac{\partial F}{\partial n}(2, 7) = \nabla F(2, 7) \cdot n(2, 7) = (1/2, 14) \cdot (-4/5, 3/5) = -4/10 + 3 * 14/5 = -2/5 + 42/5 = 8$$

2.3 Part c

Find $\frac{\partial F}{\partial n}$ for $F(x, y) = 5x^2 + y^2 + xyz$ and $\Omega = \{(x, y, z) : x^2 + y^2 < 16\}$ at $(2, 2\sqrt{3}, 3)$.

Answer: $\nabla F(x, y, z) = (10x+yz, 2y+xz, xy)$ and $n(x, y, z) = \frac{1}{\sqrt{4x^2+4y^2}}(2x, 2y, 0)$. Then, $\nabla F(2, 2\sqrt{3}, 0) = (20 + 6\sqrt{3}, 4\sqrt{3} + 6, 4\sqrt{3})$ and $n(2, 2\sqrt{3}, 0) = \frac{1}{\sqrt{4(2)^2+(2\sqrt{3})^2}}(4, 4\sqrt{3}, 0) = (1/2, \sqrt{3}/2, 0)$. Then,

$$\frac{\partial F}{\partial n}(2, 2\sqrt{3}, 0) = \nabla F((2, 2\sqrt{3}, 0) \cdot n(2, 2\sqrt{3}, 0) = (20 + 6\sqrt{3}, 4\sqrt{3} + 6, 4\sqrt{3}) \cdot (1/2, \sqrt{3}/2, 0)$$
$$= (10 + 3\sqrt{3}) + (6 + 3\sqrt{3}) = 16 + 6\sqrt{3}$$

3 Problem 1.5.6

Assume that u satifsies Laplace's equation on a domain $\Omega \subset \mathbb{R}^3$ and that v is a continuously differentiable function such that $v(x) =$ for all $x \in \partial\Omega$. Use the divergence theorem to show that

$$\int_\Omega \nabla u(x) \cdot \nabla v(x) dx = 0$$

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Answer: Let u satisfy Laplace's equation, $\Delta u = 0$, on $\Omega \subset \mathbb{R}^3$ and let v be a continuously differentiable function such that $v(x) = 0$ for all $x \in \partial\Omega$. Then, by the integration by parts formula, which comes from divergence theorem,

$$\int_\Omega \nabla u(x) \cdot \nabla v(x) dx = \int_{\partial\Omega} v(x)(\nabla u(x) \cdot n(s))ds + \int_\Omega v(x)\mathrm{div}(\nabla u(x))dx$$

The function $v(x)$ is 0 on $\partial\Omega$, so

$$\int_{\partial\Omega} v(x)(\nabla u(x) \cdot n(s))ds = 0.$$

We know that $\mathrm{div}(\nabla u(x)) = \Delta u$, and we assumed $u(x)$ satisfied $\Delta u = 0$ in Ω. Therefore,

$$\int_\Omega v(x)\mathrm{div}(\nabla u(x))dx = \int_\Omega v(x)\Delta u(x)dx = 0.$$

Thus,

$$\int_\Omega \nabla u(x) \cdot \nabla v(x) dx = \int_{\partial\Omega} v(x)(\nabla u(x) \cdot n(s))ds + \int_\Omega v(x)\mathrm{div}(\nabla u(x))dx = 0 + 0 = 0,$$

which is what we want.

4 Problem 1.5.9

Let u denote a solution to the heat equation $u_t = ku_{xx}$ subject to homogeneous Dirichlet boundary conditions on $\Omega = (0, L)$ and define

$$E(t) = \int_0^L u(t,x)^2 dx \text{ for } t \geq 0.$$

Show that $E(t)$ is a decreasing function of time by showing that the derivative of this function satisfies $dE(t)/dt \leq 0$ for all $t > 0$.

Answer: Let $t > 0$. Then,

$$\begin{aligned}
\frac{dE(t)}{dt} &= \frac{d}{dt}\int_0^L u(t,x)^2 dx \\
&= \int_0^L 2u(t,x)u_t(t,x)dx \\
&= 2k\int_0^L u(t,x)u_{xx}(t,x)dx \\
&= 2ku_x(t,x)u(t,x)\Big|_0^L - 2k\int_0^L u_x(t,x)^2 dx \\
&= 2k[u_x(t,L)u(t,L) - u_x(t,0)u(t,0)] - 2k\int_0^L u_x(t,x)^2 dx
\end{aligned}$$

The solution u satisfies homogeneous Dirichlet boundary conditions, so $u(t,0) = u(t,L) = 0$ for all $t \geq 0$. This means $2k[u_x(t,L)u(t,L) - u_x(t,0)u(t,0)] = 0$ and

$$\frac{dE(t)}{dt} = -2k\int_0^L u_x(t,x)^2 dx$$

The function $u_x(t,x)^2 \geq 0$, for all x, t, so $\int_0^L u_x(t,x)^2 dx \geq 0$ and

$$\frac{dE(t)}{dt} = -2k\int_0^L u_x(t,x)^2 dx \leq 0,$$

which is what we want. Therefore, $E(t)$ is a decreasing function of time.