Written Assignment 9

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04/29/16

1 Problem 1

Prove that if $|f(x) - f(y)| < \epsilon$ for all $x, y \in [a, b]$ then

$$\sup\{f(t) : t \in [a, b]\} - \inf\{f(t) : t \in [a, b]\} \leq \epsilon$$

Proof:

Let f be a function defined on $[a, b]$ and suppose $|f(x) - f(y)| < \epsilon$ for all $x, y \in [a, b]$. We must show that $\sup\{f(t) : t \in [a, b]\} - \inf\{f(t) : t \in [a, b]\} \leq \epsilon$. Let $\epsilon > 0$, and let $x, y \in [a, b]$. Then, $|f(x) - f(y)| < \epsilon$, which means $-\epsilon < f(x) - f(y) < \epsilon$. Then $f(y) - \epsilon < f(x) < \epsilon + f(y)$. Since $f(x) < \epsilon + f(y)$ for all $x \in [a, b]$, then $\epsilon + f(y)$ is an upper bound of f on $[a, b]$. Thus, $\sup\{f(t) : t \in [a, b]\} \leq \epsilon + f(y)$. Then, $\sup\{f(t) : t \in [a, b]\} - \epsilon \leq f(y)$ for all $f(y)$. Then, $\sup\{f(t) : t \in [a, b]\} - \epsilon$ is a lower bound of f on $[a, b]$ and $\sup\{f(t) : t \in [a, b]\} - \epsilon \leq \inf\{f(t) : t \in [a, b]\}$. Then finally, $\sup\{f(t) : t \in [a, b]\} - \inf\{f(t) : t \in [a, b]\} \leq \epsilon$, which is what we want.

2 Problem 2a

Let f be Darboux integrable on $[a, b]$, and let $c \in (a, b)$. Prove that f is Darboux integrable on $[a, c]$ and on $[c, b]$.

Proof:

Suppose f is integrable on $[a, b]$ and let $c \in (a, b)$. We will first show that f is integrable on $[a, c]$. Let $\epsilon > 0$. Then there exists a partition $P = \{x_0, ..., x_n\}$ of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Since $c \in (a, b)$, then without loss of generality, $c \in [x_{k-1}, x_k]$ for some $k \in \{1, ..., n\}$. Choose the partition $R = \{x_0, ..., x_{k-1}\} \cup \{c\}$, which is a partition of $[a, c]$. We will denote the upper sum of f on $[a, c]$ by $U'(f, R)$ and the lower sum of f on $[a, c]$ by $L'(f, R)$. Then

$$U'(f,R) - L'(f,R) = \sum_{i=1}^{k-1}(M_i - m_i)(x_i - x_{i-1}) + (\sup\{f(x) : x \in [x_{k-1}, c]\} - \inf\{f(x) : x \in [x_{i-1}, c]\})(c - x_{k-1})$$

where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$. Since $[x_{k-1}, c] \subseteq [x_{k-1}, x_k]$, then $c - x_{k-1} \leq x_k - x_{k-1}$. Also, $\sup\{f(x) : x \in [x_{k-1}, c]\} \leq \sup\{f(x) : x \in [x_{k-1}, x_k]\}$, and $\inf\{f(x) : x \in [x_{k-1}, c]\} \geq \inf\{f(x) : x \in [x_{k-1}, x_k]\}$, which means $\sup\{f(x) : x \in [x_{k-1}, c]\} - \inf\{f(x) : x \in [x_{k-1}, c]\} \leq M_k - m_k$ Thus,

$$\sum_{i=1}^{k-1}(M_i - m_i)(x_i - x_{i-1}) + (\sup\{f(x) : x \in [x_{k-1}, c]\} - \inf\{f(x) : x \in [x_{k-1}, c]\})(c - x_{k-1})$$
$$\leq \sum_{i=1}^{k-1}(M_i - m_i)(x_i - x_{i-1}) + (M_k - m_k)(x_k - x_{k-1}) = \sum_{i=1}^{k}(M_i - m_i)(x_i - x_{i-1})$$

Also since $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$, $M_i - m_i \geq 0$ for all $i \in \{1, .., n\}$. We also know that $x_i - x_{i-1} \geq 0$ for all $i \in \{1, .., n\}$. Thus, $\sum_{i=k+1}^{n}(M_i - m_i)(x_i - x_{i-1}) \geq 0$, and

$$\sum_{i=1}^{k}(M_i - m_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{k}(M_i - m_i)(x_i - x_{i-1}) + \sum_{i=k+1}^{n}(M_i - m_i)(x_i - x_{i-1})$$
$$= \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = U(f, P) - L(f, P) < \epsilon$$

which means there exists a partition R such that $U'(f, R) - L'(f, R) < \epsilon$, and f is integrable on $[a, c]$.

We must now show that f is integrable on $[c, b]$. Let $\epsilon' > 0$. Then there exists a partition $P' = \{x_0, ..., x_m\}$ of $[a, b]$ such that $U(f, P') - L(f, P') < \epsilon$. Since $c \in (a, b)$, then without loss of generality, $c \in [x_{l-1}, x_l]$ for some $l \in \{1, ..., m\}$. Choose the partition $Q = \{x_l, ..., x_m\} \cup \{c\}$, which is a partition of $[c, b]$. We will denote the upper sum of f on $[c, b]$ by $U''(f, Q)$ and the lower sum of f on $[c, b]$ by $L''(f, Q)$. Then

$$U''(f, Q) - L''(f, Q) = (\sup\{f(x) : x \in [c, x_l]\} - \inf\{f(x) : x \in [c, x_l]\})(x_l - c) + \sum_{i=l}^{m}(M_i - m_i)(x_i - x_{i-1})$$

where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$. Since $[c, x_l] \subseteq [x_{l-1}, x_l]$, then $x_l - c \leq x_l - x_{l-1}$. Also, $\sup\{f(x) : x \in [c, x_l]\} \leq \sup\{f(x) : x \in [x_{l-1}, x_l]\}$, and $\inf\{f(x) : x \in [c, x_l]\} \geq \inf\{f(x) : x \in [x_{l-1}, x_l]\}$, which means $\sup\{f(x) : x \in [c, x_l]\} - \inf\{f(x) : x \in [c, x_l]\} \leq M_l - m_l$ Thus,

$$(\sup\{f(x) : x \in [c, x_l]\} - \inf\{f(x) : x \in [c, x_l]\})(x_l - c) + \sum_{i=1}^{l-1}(M_i - m_i)(x_i - x_{i-1})$$
$$\leq (M_l - m_l)(x_l - x_{l-1}) + \sum_{i=l+1}^{m}(M_i - m_i)(x_i - x_{i-1}) = \sum_{i=l}^{m}(M_i - m_i)(x_i - x_{i-1})$$

Also since $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$, $M_i - m_i \geq 0$ for all $i \in \{1, .., m\}$. We also know that $x_i - x_{i-1} \geq 0$ for all $i \in \{1, .., m\}$. Thus, $\sum_{i=1}^{l-1}(M_i - m_i)(x_i - x_{i-1}) \geq 0$, and

$$\sum_{i=l}^{m}(M_i - m_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{l-1}(M_i - m_i)(x_i - x_{i-1}) + \sum_{i=l}^{m}(M_i - m_i)(x_i - x_{i-1})$$
$$= \sum_{i=1}^{m}(M_i - m_i)(x_i - x_{i-1}) = U(f, P') - L(f, P') < \epsilon'$$

which means there exists a partition Q such that $U''(f, Q) - L''(f, Q) < \epsilon'$, and f is integrable on $[c, b]$.

3 Problem 2b

Prove that $\int_a^b f = \int_a^c f + \int_c^b f$

Proof:

Suppose f is integrable on $[a, b]$ and let $c \in (a, b)$. We must show that $\int_a^b f = \int_a^c f + \int_c^b f$. We will first show that $\inf\{U(f, P)\} \geq \inf\{U'(f, R)\} + \inf\{U''(f, Q)\}$, where $U'(f, R)$ is the upper sum of f over $[a, c]$ with some partition R and $U''(f, Q)$ is the upper sum of f over $[c, b]$ with some partition Q. Suppose not. Then there exists a partition of $[a, b]$ $P' = \{x_0, ..., x_n\}$ such that $U(f, P') < \inf\{U'(f, R)\} + \inf\{U''(f, Q)\}$. Since $c \in (a, b)$, then without loss of generality, $c \in [x_{k-1}, x_k]$ for some $k \in \{1, ..., n\}$. Then choose the partition $R' = \{x_0, ..., x_{k-1}\} \cup \{c\}$ and the partition $Q' = \{x_k, ..., x_n\} \cup \{c\}$, where R' and Q' are partitions of $[a, c]$ and $[c, b]$ respectively. Then,

$$U'(f, R') + U''(f, Q') = \sum_{i=1}^{k-1} M_i(x_i - x_{i-1}) + \sup\{f(x) : x \in [x_{k-1}, c]\}(c - x_{k-1})$$
$$+ \sup\{f(x) : x \in [c, x_k]\}(x_k - c) + \sum_{i=k+1}^{n} M_i(x_i - x_{i-1})$$

where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$. Since $\sup\{f(x) : x \in [x_{k-1}, c]\} \leq \sup\{f(x) : x \in [x_{k-1}, x_k]\} = M_k$ and $\sup\{f(x) : x \in [c, x_k]\} \leq \sup\{f(x) : x \in [x_{k-1}, x_k]\} = M_k$,

$$\sup\{f(x) : x \in [x_{k-1}, c]\}(c - x_{k-1}) + \sup\{f(x) : x \in [c, x_k]\}(x_k - c) \leq M_k(c - x_{k-1}) + M_k(x_k - c)$$

Since $M_k(c - x_{k-1}) + M_k(x_k - c) = M_k(x_k - x_{k-1})$, we have

$$\sum_{i=1}^{k-1} M_i(x_i - x_{i-1}) + \sup\{f(x) : x \in [x_{k-1}, c]\}(c - x_{k-1})$$
$$+ \sup\{f(x) : x \in [c, x_k]\}(x_k - c) + \sum_{i=k+1}^{n} M_i(x_i - x_{i-1})$$
$$\leq \sum_{i=1}^{k-1} M_i(x_i - x_{i-1}) + M_k(x_k - x_{k-1}) + \sum_{i=k+1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

Since $\sum_{i=1}^{n} M_i(x_i - x_{i-1}) = U(f, P)$, we have $U'(f, R') + U''(f, Q') \leq U(f, P)$, which contradicts our assumption that $\inf\{U(f, P)\} < \inf\{U'(f, R)\} + \inf\{U''(f, Q)\}$, so

$$\inf\{U(f, P)\} \geq \inf\{U'(f, R)\} + \inf\{U''(f, Q)\}$$

We must now show that $\sup\{L(f, P)\} \leq \sup\{L'(f, R)\} + \sup\{L''(f, Q)\}$, where $L'(f, R)$ is the lower sum of f over $[a, c]$ with some partition R and $L''(f, Q)$ is the upper sum of f over $[c, b]$ with some partition Q. Suppose not. Then there exists a partition of $[a, b]$ $P'' = \{x_0, ..., x_z\}$ such that $L(f, P') > \sup\{L'(f, R)\} + \sup\{L''(f, Q)\}$. Since $c \in (a, b)$, then without loss of generality, $c \in [x_{l-1}, x_l]$ for some $l \in \{1, ..., z\}$. Then choose the partition $R'' = \{x_0, ..., x_{l-1}\} \cup \{c\}$ and the partition $Q'' = \{x_l, ..., x_z\} \cup \{c\}$, where R'' and Q'' are partitions of $[a, c]$ and $[c, b]$ respectively. Then,

$$L'(f, R'') + L''(f, Q'') = \sum_{i=1}^{l-1} m_i(x_i - x_{i-1}) + \inf\{f(x) : x \in [x_{l-1}, c]\}(c - x_{l-1})$$
$$+ \inf\{f(x) : x \in [c, x_l]\}(x_l - c) + \sum_{i=l+1}^{z} m_i(x_i - x_{i-1})$$

where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$. Since $\inf\{f(x) : x \in [x_{l-1}, c]\} \geq \inf\{f(x) : x \in [x_{l-1}, x_l]\} = m_l$ and $\inf\{f(x) : x \in [c, x_l]\} \geq m_l$,

$$\inf\{f(x) : x \in [x_{l-1}, c]\}(c - x_{l-1}) + \inf\{f(x) : x \in [c, x_l]\}(x_l - c) \geq m_l(c - x_{l-1}) + m_l(x_l - c)$$

Since $m_l(c - x_{l-1}) + m_l(x_l - c) = m_l(x_l - x_{l-1})$, we have

$$\sum_{i=1}^{l-1} m_i(x_i - x_{i-1}) + \inf\{f(x) : x \in [x_{l-1}, c]\}(c - x_{l-1})$$
$$+ \inf\{f(x) : x \in [c, x_l]\}(x_l - c) + \sum_{i=l+1}^{z} m_i(x_i - x_{i-1})$$
$$\geq \sum_{i=1}^{l-1} m_i(x_i - x_{i-1}) + m_l(x_l - x_{l-1}) + \sum_{i=l+1}^{z} m_i(x_i - x_{i-1}) = \sum_{i=1}^{z} m_i(x_i - x_{i-1})$$

Since $\sum_{i=1}^{z} m_i(x_i - x_{i-1}) = L(f,P')$, we have $L'(f,R'') + L''(f,Q'') \geq L(f,P')$, which contradicts our assumption that $\sup\{L(f,P)\} > \sup\{L'(f,R)\} + \sup\{L''(f,Q)\}$, so

$$\sup\{L(f,P)\} \leq \sup\{L'(f,R)\} + \sup\{L''(f,Q)\}$$

Now that we have, $\inf\{U(f,P)\} \geq \inf\{U'(f,R)\} + \inf\{U''(f,Q)\}$ and $\sup\{L(f,P)\} \leq \sup\{L'(f,R)\} + \sup\{L''(f,Q)\}$, we get the following by using the fact that f is integrable on $[a,b], [a,c]$, and $[c,b]$.

$$\sup\{L'(f,R)\}+\sup\{L''(f,Q)\} = \inf\{U'(f,R)\}+\inf\{U''(f,Q)\} \leq \inf\{U(f,P)\} = \sup\{L(f,P)\}$$

So we now have $\sup\{L'(f,R)\} + \sup\{L''(f,Q)\} \leq \sup\{L(f,P)\}$ and $\sup\{L(f,P)\} \leq \sup\{L'(f,R)\} + \sup\{L''(f,Q)\}$, which means

$$\sup\{L(f,P)\} = \sup\{L'(f,R)\} + \sup\{L''(f,Q)\}$$

and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

which is what we want.

4 Problem 3

Suppose that f and g are both Darboux integrable on $[a,b]$ and that $f(x) \leq g(x)$ for all $x \in [a,b]$. Prove that $\int_a^b f \leq \int_a^b g$.

Proof:

Suppose that f and g are both Darboux integrable functions on $[a,b]$ and that $f(x) \leq g(x)$ for all $x \in [a,b]$. We want to show that $\int_a^b f \leq \int_a^b g$. We will first show that for any closed interval $[c,d] \subseteq [a,b]$, $\sup\{f(x) : x \in [c,d]\} \leq \sup\{g(x) : x \in [c,d]\}$. Suppose not. then $\sup\{g(x) : x \in [c,d]\}$ is not an upper bound of f on $[c,d]$, which means there exists a $x' \in [c,d]$ such that $f(x') > \sup\{g(x) : x \in [c,d]\}$. Then $f(x') > g(x)$ for all $x \in [c,d]$, which contradicts our assumption that $f(x) \leq g(x)$ for all $x \in [a,b]$. Thus, for any closed interval $[c,d] \subseteq [a,b]$, $\sup\{f(x) : x \in [c,d]\} \leq \sup\{g(x) : x \in [c,d]\}$.

Let $P = \{x_0, ..., x_n\}$ be a partition of $[a,b]$. Then for all $i \in \{1, ..., n\}$, $\sup\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \leq \sup\{g(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1})$. Then,

$$U(f,P) = \sum_{i=1}^{n} \sup\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \leq \sum_{i=1}^{n} \sup\{g(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) = U(g,P)$$

Thus for all partitions P of $[a,b]$, $U(f,P) \leq U(g,P)$. Since $\inf\{U(f,P)\} \leq U(f,P)$ for all partitions P, $\inf\{U(f,P)\}$ is also a lower bound of $\{U(g,P)\}$, so $\inf\{U(f,P)\} \leq$

$\inf\{U(g, P)\}$. Since both f and g are both Darboux integrable on $[a, b]$,

$$\int_a^b f \leq \int_a^b g$$

which is what we want.