Written Assignment 5
Math 290, Dr. Walnut

Lucas Bouck

11/2/15

1 Problem 1:

Let a and b be natural numbers. Use the WOP to prove that $gcd(a, b)$ is the smallest natural number d with the property that for some integers x and y, $ax + by = d$.
Proof:

Let a and b be natural numbers. Define the set S to contain the natural number n such that there exist integers x and y such that $ax + by = n$. Since this set only contains natural numbers, all the elements of the set S are in the natural numbers. Thus, $S \subseteq \mathbb{N}$. For all natural numbers a and b, $a*1+b*1 = a+b$. Since 1 and 1 are integers, $(a+b) \in S$. Thus, S is not empty. Since $S \subseteq \mathbb{N}$ and $S \neq \emptyset$, by the well ordering principle, S has a least element d. Thus, by the way it is defined, there does exist a smallest natural number d with the property that for some integers x and y, $ax + by = d$.

We want to show that d is a common divisor of a and b. We will show this by contrapositive. There will be two cases. For the first case, let $d \not| a$. Thus, by the division algorithm, $a = dq+r$, where $0 < r < d$. Then, $r = a-dq = a-(ax+by)q = a-aqx-bqy = a(1-qx)-b(qy)$. Since $q, x, y \in \mathbb{Z}$, then $(1-qx)$ and qy are integers. Since $r > 0$ and there exist integers $(1-qx)$ and qy such that $r = a(1-qx)-b(qy)$, $r \in S$. Since $r \in S$ and $r < d$, d is not the least element of S. For case 2, let $d \not| b$. Thus, by the divisor algorithm, $b = dk+n$, where $0 < n < d$. Then, $n = b - dk = b - (ax + by)k = b - akx - bky = b(1 - ky) - a(kx)$. Since $k, x, y \in \mathbb{Z}$, then $(1 - ky)$ and kx are integers. Since $n > 0$ and there exist integers $(1 - ky)$ and kx such that $n = b(1 - ky) - a(kx)$, $n \in S$. Since $n \in S$ and $n < d$, d is not the least element of S. In both cases, if d is not a common divisor of a and b, then d is not the least element of S. Thus, by contrapositive, since d is the least element of S, d is a common divisor of a and b.

We know that d as defined exists and is a common divisor of a and b. We now want to show that d is greater than or equal to all other common divisors of a and b. Let z be a common divisor of a and b. Then, there exist integers m and p such that $zm = a$ and $zp = b$. Since $ax + by = d$, then $zmx + zpy = d$. Then, $z(mx + py) = d$. Let $mx + py = l$.

Since $m, x, p, y \in \mathbb{Z}$, then $l \in \mathbb{Z}$. Hence, there exists an integer l such that $zl = d$. Then, $z|d$. Since $z|d$, $z \leq d$. Therefore, any common divisor of a and b is less than or equal to d.

Since we have proven the the smallest natural number d with the property that for some integers x and y, $ax + by = d$ does exist, is a common divisor of a and b, and is greater than or equal to all other common divisors of a and b, then d as it is defined is the greatest common divisor of a and b.

2 Problem 2:

Use induction to prove that for all natural numbers $n \geq 2$, $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$.
Proof:

Let $n = 2$.Since $2 > \sqrt{2}$, then $\sqrt{2} + \sqrt{2} = 2\sqrt{2} < 2 + \sqrt{2}$. Then, $\sqrt{2} < \frac{2+\sqrt{2}}{2} = \frac{\sqrt{2}+1}{\sqrt{2}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$. Thus, $\sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$. The result holds for $n = 2$.

Let $n \in \mathbb{N}$. Assume $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$ is true. Since $n \in \mathbb{N}$, $n + 1 > n$. Since n is a natural number, $\sqrt{n+1} > \sqrt{n}$. Then, $\sqrt{n+1}\sqrt{n} > n$. Then, $1 + \sqrt{n+1}\sqrt{n} > n + 1$. Then, $\frac{1+\sqrt{n+1}\sqrt{n}}{\sqrt{n+1}} > \frac{n+1}{\sqrt{n+1}}$. Then, $\frac{1}{\sqrt{n+1}} + \sqrt{n} > \sqrt{n+1}$. By the inductive hypothesis, $\frac{1}{\sqrt{n+1}} + \sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}$. Since $\frac{1}{\sqrt{n+1}} + \sqrt{n} > \sqrt{n+1}$, then $\sqrt{n+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}$. Thus, the result holds for $n + 1$.

3 Problem 3:

Use induction to prove that for all $x > 0$ and natural numbers $n \geq 2$, $(1 + x)^n > 1 + nx$.
Proof:

Let $n = 2$. Since $x > 0$, $x^2 > 0$. Thus, $x^2 + 2x + 1 > 2x + 1$. Thus, $(1 + x)^2 > 1 + 2x$. Therefore, the result holds for $n = 2$.

Let $n \in \mathbb{N}$. Assume $(1 + x)^n > 1 + nx$ is true. Since $x > 0$ and $n > 0$, $nx^2 > 0$. Thus, $nx^2 + (n + 1)x + 1 > (n + 1)x + 1$. By factoring the left side, we get $(1 + nx)(1 + x) > (n + 1)x + 1$. By the induction hypothesis, which is $(1 + x)^n > 1 + nx$, $(1 + x)^n(1 + x) > (1 + nx)(1 + x) > (n + 1)x + 1$. Thus, $(1 + x)^{n+1} > 1 + (n + 1)x$. Therefore, the result holds for $n + 1$.

4 Problem 4

Let a and b be natural numbers. Use induction to prove that if $gcd(a, b) = 1$ then for all natural numbers n, $gcd(a, b^n) = 1$. (Hint: You already proved the base case, $n = 2$, in a previous written assignment, so you do not have to do it again here.)
Proof:

We already did the base case in a previous assignment, so we will just do the inductive step. Let $n \in \mathbb{N}$. Let $gcd(a, b) = 1$ then for all natural numbers n, $gcd(a, b^n) = 1$ be true. We want to show that if $gcd(a, b) = 1$, then $gcd(a, b^{n+1}) = 1$. Let $gcd(a, b) = 1$. By the inductive hypothesis, $gcd(a, b^n) = 1$. Then, there exist integers x and y such that $ax + b^n y = 1$. Since $gcd(a, b) = 1$, then there exist integers m and l such that $am + bl = 1$. Then, $bl = 1 - am$. By multiplying bl to both sides of $ax + b^n y = 1$, we get $axbl + bb^n ly = axbl + b^{n+1}(ly) = bl = 1 - am$. Then, $axbl + am + b^{n+1}(ly) = a(xbl + m) + b^{n+1}(ly) = 1$. Let $(xbl + m) = c$ and let $ly = d$. Since $x, b, l, m, y \in \mathbb{Z}$, then $c, d \in \mathbb{Z}$. Then, there exist integers c and d such that $ac + b^{n+1}d = 1$. By a theorem proven earlier in the course, $gcd(a, b^{n+1}) = 1$. Therefore, the result holds for $n + 1$.