Written Assignment 2

Lucas Bouck

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1 Problem 1

Let x_n be a sequence of real numbers and denote the *nth tail* of x_n by $T_n = \{x_j : j \geq n\}$.

1.1 Problem 1a

Suppose that for each $n \in \mathbb{N}, T_n$ has a least element, that is, that for each n there exists an $x \in T_n$ such that $x \leq y$ for all $y \in T_n$. Prove that x_n has a increasing subsequence, that is, there exists a subsequence x_{n_k} such that for all $k, x_{n_k} \leq x_{n_{k+1}}$.
Proof:

Let x_n be a sequence of real numbers such that every tail of x_n have a least element. We want to show that there exists an increasing subsequence x_{n_k} of x_n. Let x_{n_k} be a subsequence of x_n and define the sequence of natural numbers n_k to be $n_k = \{n : x_n \leq y$ for all $y \in T_n\}$. We now have a subsequence x_{n_k} that contains all the least elements that are also the first element of the tail that they are in.

We shall now show that $x_{n_k} \leq x_{n_{k+1}}$ for all k. Let $k \in \mathbb{N}$. Since x_{n_k} is a subsequence, n_k is increasing and $n_{k+1} > n_k$. Thus, $x_{n_{k+1}} \in T_{n_k}$. Since $x_{n_k} \leq y$ for all $y \in T_{n_k}$, then $x_{n_{k+1}} \geq x_{n_k}$ for all k. Therefore, we have shown that there exists an increasing subsequence of x_n. We are done.

1.2 Problem 1b

Suppose that there exists an $N \in \mathbb{N}$ such that T_N does not have a least element. Prove that x_n has a decreasing subsequence.
Proof:

Let x_n be a sequence of real numbers, and assume that there exists an $N \in \mathbb{N}$ such that T_N does not have a least element. This means there exists an $N \in \mathbb{N}$ such that for all $x \in T_N$ there exists a $y \in T_N$ such that $y < x$. We must show that there exists a subsequence x_{n_k} such that x_{n_k} is decreasing.

Let x_{n_k} be a subsequence of x_n and define the first term of the sequence of natural numbers n_k to be $n_1 = N$, where T_N is a tail of x_n that has no least element. For $k > 1$, choose n_k such that x_{n_k} is the first term in T_N such that $x_{n_k} < x_{n_{k-1}}$ and $n_k > n_{k-1}$. Suppose there doesn't exist a x_{n_k} such that $x_{n_k} < x_{n_{k-1}}$. That means $x_{n_{k-1}} \leq x_j$ for all $j > n_{k-1}$. Based on how the sequence is defined, $x_{n_{k-1}} \leq x_{n_i}$ for all $N \leq n_i < n_{k-1}$. For a natural number $m < n_{k-1}$ that is not in the sequence n_k, $x_m \geq x_{n_{k-1}}$ because if $x_m > x_{n_{k-1}}$, n_{k-1} would not be in the sequence as the way we defined it. This makes $x_{n_{k-1}}$ the least element of T_N, but T_N has no least element, which creates a contradiction. Therefore, there does exist a x_{n_k} such that $x_{n_k} < x_{n_{k-1}}$, and our subsequence can be defined as specified above.

We must now show that x_{n_k} is decreasing. Since the sequence is defined such that $x_{n_k} < x_{n_{k-1}}$ for all $k > 1$, $x_{n_k+1} < x_{n_k}$ for all $k \geq 1$. Therefore, x_{n_k} is decreasing, and there exists a subsequence, x_{n_k}, of x_n such that x_{n_k} is decreasing.

1.3 Problem 1c

Prove that if x_n is a bounded sequence of real numbers, then x_n has a convergent subsequence.

Proof:

Let x_n be a bounded sequence of real numbers. We want to show that x_n has a convergent subsequence. Considering the nth tails of x_n, every tail of x_n has a least element or there exists an $N \in \mathbb{N}$ such that T_N does not have a least element. We will do the proof by cases.

Case 1: Suppose every tail of x_n has a least element. Then by the theorem in 1a, x_n has an increasing subsequence x_{n_k}. Since x_n is bounded, x_n is bounded from above. Since x_{n_k} is a subsequence of x_n, x_{n_k} is bounded from above. Because x_{n_k} is bounded from above and increasing, x_{n_k} is convergent.

Case 2: Suppose there exists an $N \in \mathbb{N}$ such that T_N does not have a least element. By theorem 1b, x_n has a decreasing subsequence x_{n_j}. Since x_n is bounded, x_n is bounded from below. Since x_{n_j} is a subsequence of x_n, x_{n_j} is bounded from below. Because x_{n_j} is bounded from below and decreasing, x_{n_j} is convergent.

In both cases, x_n has a convergent subsequence. Therefore, if x_n is a bounded sequence of real numbers, then x_n has a convergent subsequence.

2 Problem 2

Prove that a sequence x_n converges to $L \in \mathbb{R}$ if and only if every subsequence of x_n has in turn a subsequence that converges to L.

Proof:

$(\Rightarrow)$

Let x_n be a sequence of real numbers that converges to $L \in \mathbb{R}$. We must show that every subsequence of x_n has a subsequence that converges to L. Since x_n converges to L, we know that every subsequence of x_n converges to L.

Let x_{n_k} be a subsequence of x_n. We know x_{n_k} converges to L. Since x_{n_k} is a subsequence of itself, there exists a subsequence of x_{n_k} such that the subsequence converges. Therefore, if x_n converges to $L \in \mathbb{R}$, then every subsequence of x_n has in turn a subsequence that converges to L.

($\Leftarrow$) (By contrapositive)

Let x_n be a sequence of real numbers that doesn't converge to $L \in \mathbb{R}$. This means that there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists a $n \geq N$ such that $|x_n - L| \geq \epsilon$. We will refer to this ϵ later in the proof. We must now show that there exists a subsequence, x_{n_k} of x_n such that all subsequences of x_{n_k} do not converge to L.

Define n_k as $n_k = \{n : |x_n - L| \geq \epsilon\}$. We shall now show that as defined, x_{n_k} exists. Let N be a natural number and suppose that there does not exist a x_{n_i} such that $n_i \geq N$ and $|x_{n_i} - L| \geq \epsilon$. Then, for all $m \geq n_i$, $|x_m - L| < \epsilon$. But our first assumption was that for this particular ϵ, for all $N \in \mathbb{N}$ there exists an $n \geq N$ such that $|x_n - L| \geq \epsilon$, which contradicts the hypothesis that there does not exist a x_{n_i} such that $|x_{n_i} - L| \geq \epsilon$. Therefore, there exists an x_{n_i} such that $|x_{n_i} - L| \geq \epsilon$ for any natural N, and our subsequence exists as defined.

We must now show that no subsequences of x_{n_k} converge to L. Let $x_{n_{k_j}}$ be a subsequence of x_{n_k}. Because x_{n_k} satisfies $|x_{n_k} - L| \geq \epsilon$ and every member of $x_{n_{k_j}}$ is also a member of x_{n_k}, then $|x_{n_{k_j}} - L| \geq \epsilon$ for all n_{k_j}. Let $M \in \mathbb{N}$. Since $|x_{n_{k_j}} - L| \geq \epsilon$ for all n_{k_j}, there exists a $n_{k_j} \geq M$ such that $|x_{n_{k_j}} - L| \geq \epsilon$. Thus, for all natural M there exists a $n_{k_j} \geq M$ such that $|x_{n_{k_j}} - L| \geq \epsilon$. Therefore, every subsequence of x_{n_k} does not converge to L. Moreover, there exists a subsequence of x_n such that all of its subsequences do not converge to L.

We have shown that if x_n does not converge to L, there exists a subsequence, x_{n_k} of x_n such that all subsequences of x_{n_k} do not converge to L. By contrapositive, this means we have proven that if every subsequence of x_n has a subsequence that converges to L, then x_n converges to L.