Written Assignment 1

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1 Problem 1

Prove that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ converges by proving the following two facts.

1.1 Part a

Prove that the sequence $x_n = \left(1+\frac{1}{n}\right)^n$ is bounded above.
Proof:
We must show that x_n is bounded from above. First consider $\left(1+\frac{1}{n}\right)^n$. By the binomial theorem, we get

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$

By doing some algebra we get,

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} &= 1 + \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k} \\
&= 1 + \sum_{k=1}^{n} \frac{n(n-1)...(n+1-k)}{k!n^k} \\
&= 1 + \sum_{k=1}^{n} \frac{1}{k!} \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)...\left(\frac{n+1-k}{n}\right)
\end{aligned}$$

If $a \in \mathbb{R}$ and $a < n$, then $\frac{a}{n} < 1$. Thus, every multiplying term in the summation is less than 1. Since $n+1 > k$ for every k in the summation, every term is positive. Since every multiplying term in the summation is positive and every multiplying term is less than 1,

$$1 + \sum_{k=1}^{n} \frac{1}{k!} \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)...\left(\frac{n+1-k}{n}\right) < 1 + \sum_{k=1}^{n} \frac{1}{k!}$$

We next need to show that $k! \geq 2^{k-1}$ for all integers $k \geq 1$. We will do this by induction on k. For the base case $k = 1$, $1! = 1$ and $2^{1-1} = 2^0 = 1$, so the result holds for $k = 1$. For the inductive step, assume $k \geq 1$ and the result holds for k. By the inductive hypothesis, $k! \geq 2^{k-1}$. Then, $(k+1)k! \geq 2^{k-1}(k+1)$. Since $k \geq 1$, $k+1 \geq 2$ and $2^{k-1}(k+1) \geq 2^{k-1}2 = 2^k$. Therefore, $(k+1)! > 2^k$. By the principle of mathematical induction, $k! \geq 2^{k-1}$ for all $k \geq 1$. Thus, $\frac{1}{k!} \leq \frac{1}{2^{k-1}}$ for all $k \geq 1$. Since $\frac{1}{k!} \leq \frac{1}{2^{k-1}}$ for every term in the sum,

$$1 + \sum_{k=1}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}}$$

We next need to show that $\sum_{k=1}^{n} \dfrac{1}{2^{k-1}} = 2 - \left(\frac{1}{2}\right)^{n-1}$ for all $n \in \mathbb{N}$. We will do this be induction on n. For the base case, let $n = 1$. Then $\sum_{k=1}^{1} \dfrac{1}{2^{k-1}} = \dfrac{1}{2^0} = 1$ and $2 - \left(\frac{1}{2}\right)^{1-1} = 2 - 1 = 1$, so the result holds for $n = 1$. For the inductive step, let $n \geq 1$ and let the result hold for n. We must show the result holds for $n+1$. By the inductive hypothesis, $\sum_{k=1}^{n} \dfrac{1}{2^{k-1}} = 2 - \left(\frac{1}{2}\right)^{n-1}$. Then, $\sum_{k=1}^{n} \dfrac{1}{2^{k-1}} + \frac{1}{2^n} = \sum_{k=1}^{n+1} \dfrac{1}{2^{k-1}} = 2 - \left(\frac{1}{2}\right)^{n-1} + \frac{1}{2^n} = 2 - \left(\frac{1}{2}\right)^{n} (1 - (1/2)^{-1}) = 2 - \left(\frac{1}{2}\right)^{n}$, which is what we want. Therefore, by mathematical induction, $\sum_{k=1}^{n} \dfrac{1}{2^{k-1}} = 2 - \left(\frac{1}{2}\right)^{n-1}$ for all $n \in \mathbb{N}$. Thus,

$$1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + 2 - \left(\frac{1}{2}\right)^{n-1}$$

. We shall now show that $1 + 2 - \left(\frac{1}{2}\right)^{n-1} < 3$. Since $\frac{1}{2} > 0$, $\left(\frac{1}{2}\right)^{n-1} > 0$ for all $n \in \mathbb{N}$. Then $-\left(\frac{1}{2}\right)^{n-1} < 0$, and $1 + 2 - \left(\frac{1}{2}\right)^{n} - 1 < 1 + 2 + 0 = 3$. Since

$$\left(1 + \frac{1}{n}\right)^n < 1 + \sum_{k=1}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + 2 - \left(\frac{1}{2}\right)^{n-1}$$

for all $n \in \mathbb{N}$, and

$$1 + 2 - \left(\frac{1}{2}\right)^{n-1} < 3$$

then

$$\left(1 + \frac{1}{n}\right)^n < 3$$

for all $n \in \mathbb{N}$. Thus, there exists a real number M such that $M \geq x_n$ for all $n \in \mathbb{N}$. Therefore, x_n is bounded from above.

1.2 Part b

Prove that x_n is increasing.
Proof:

We want to show that $x_n = \left(1 + \frac{1}{n}\right)^n$ is increasing, which means $x_n < x_{n+1}$ for all $n \in \mathbb{N}$. We must first expand $\left(1 + \frac{1}{n}\right)^n$ out using binomial theorem

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$

Then, using algebra, we get

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} &= 1 + \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k} \\
&= 1 + \sum_{k=1}^{n} \frac{n(n-1)...(n+1-k)}{k! n^k} \\
&= 1 + \sum_{k=1}^{n} \frac{1}{k!} \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) ... \left(\frac{n+1-k}{n}\right) \\
&= 1 + \sum_{k=1}^{n} \frac{1}{k!} 1 \left(1 - \frac{1}{n}\right) ... \left(1 + \frac{1-k}{n}\right)
\end{aligned}$$

Since $n < n+1$ for all n, $an < a(n+1)$ for all $a > 0$. Then, $\frac{a}{n+1} < \frac{a}{n}$, and $1 + \frac{a}{n+1} < 1 + \frac{a}{n}$. Thus, $1 - \frac{a}{n} < 1 - \frac{a}{n+1}$ for all $a > 0$. Also, since $k - 1 < n$ for each term, $\frac{k-1}{n} < 1$ and $1 - \frac{k-1}{n} > 0$. For any integer $b < k - 1$, $\frac{b}{n} < \frac{k-1}{n}$, which means $1 - \frac{b}{n} > 1 - \frac{k-1}{n} > 0$. Therefore, every $(1 + \frac{b}{n})$ in each summation term is positive. Then,

$$\frac{1}{k!} 1 \left(1 - \frac{1}{n}\right) ... \left(1 + \frac{1-k}{n}\right) < \frac{1}{k!} 1 \left(1 - \frac{1}{n+1}\right) ... \left(1 + \frac{1-k}{n+1}\right)$$

for any $k > 1$. Therefore, we get

$$\left(1 + \frac{1}{n}\right)^n = 1 + \sum_{k=1}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) ... \left(1 + \frac{1-k}{n}\right) < 1 + \sum_{k=1}^{n} \frac{1}{k!} \left(1 - \frac{1}{n+1}\right) ... \left(1 + \frac{1-k}{n+1}\right)$$

Doing some algebra again gets us,

$$
\begin{aligned}
1+\sum_{k=1}^{n}\frac{1}{k!}1\left(1-\frac{1}{n+1}\right)...\left(1+\frac{1-k}{n+1}\right) &= 1+\sum_{k=1}^{n}\frac{1}{k!}\left(\frac{n+1}{n+1}\right)\left(1-\frac{1}{n+1}\right)...\left(1+\frac{1-k}{n+1}\right)\\
&= 1+\sum_{k=1}^{n}\frac{1}{k!}\frac{(n+1)n...(n+2-k)}{(n+1)^k}\\
&= 1+\sum_{k=1}^{n}\frac{(n+1)!}{k!(n+1-k)!(n+1)^k}\\
&= \sum_{k=0}^{n}\frac{(n+1)!}{k!(n+1-k)!(n+1)^k}
\end{aligned}
$$

Then, we know

$$\sum_{k=0}^{n}\frac{(n+1)!}{k!(n+1-k)!(n+1)^k} < \frac{1}{(n+1)^{n+1}} + \sum_{k=0}^{n}\frac{(n+1)!}{k!(n+1-k)!(n+1)^k}$$

Doing some more algebra, we get

$$
\begin{aligned}
\frac{1}{(n+1)^{n+1}} + \sum_{k=0}^{n}\frac{(n+1)!}{k!(n+1-k)!(n+1)^k} &= \sum_{k=0}^{n+1}\frac{(n+1)!}{k!(n+1-k)!(n+1)^k}\\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{1}{(n+1)^k}
\end{aligned}
$$

By the binomial theorem, we get

$$\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{1}{(n+1)^k} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

Since we know

$$\left(1+\frac{1}{n}\right)^n < \sum_{k=0}^{n}\frac{(n+1)!}{k!(n+1-k)!(n+1)^k} < \sum_{k=0}^{n+1}\frac{(n+1)!}{k!(n+1-k)!(n+1)^k} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

then,

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$$

for all n. Therefore, x_n is increasing.

Since we proven that x_n is bounded above and increasing, x_n is convergent.

2 Problem 2

Prove the Squeeze Theorem: Suppose that s_n and t_n are convergent sequences with $\lim s_n = \lim t_n = L$. If x_n satisfies $t_n \leq x_n \leq s_n$ for all n, then $\lim x_n = L$.
Proof:

Let s_n and t_n be convergent sequences with $\lim s_n = \lim t_n = L$. Let x_n be a sequence such that $t_n \leq x_n \leq s_n$ for all n. We want to show that $\lim x_n = L$.

Let $\epsilon > 0$. We want to show that there exists an $M \in \mathbb{N}$ such that if $m \geq M$, then $|x_m - L| < \epsilon$. Since $\lim s_n = \lim t_n = L$, there exists $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$ and $k \geq N_2$, $|s_k - L| < \epsilon$ and $|t_n - L| < \epsilon$. Choose $M = \max\{N_1, N_2\}$. Then if $m \geq M$, $|s_m - L| < \epsilon$ and $|t_m - L| < \epsilon$. Because $|s_m - L| < \epsilon$ and $|t_m - L| < \epsilon$, we know $-\epsilon < s_m - L < \epsilon$ and $-\epsilon < t_m - L < \epsilon$. Because $t_n \leq x_n \leq s_n$ for all n, $t_m \leq x_m \leq s_m$ and $t_m - L \leq x_m - L \leq s_m - L$. Since $\epsilon > s_m - L$ and $-\epsilon < t_m - L$, then $-\epsilon < t_m - L \leq x_m - L \leq s_m - L < \epsilon$. Thus, $-\epsilon < x_m - L < \epsilon$ and $|x_m - L| < \epsilon$ for all $m \geq M$. Therefore, there exists an $M \in \mathbb{N}$ such that if $m \geq M$, then $|x_m - L| < \epsilon$, and $\lim x_n = L$.

3 Problem 3

Let A be a nonempty bounded subset of $\mathbb{R}$. Prove that $\inf(A) = -\sup(-A)$ where $-A = \{-x : x \in A\}$.
Proof:

Let A be a nonempty subset of $\mathbb{R}$. Define $-A = \{-x : x \in A\}$. We want to show that $\inf(A) = -\sup(-A)$.

Let $\inf(A) = d$. Then for all $x \in A$, $d \leq x$. By multiplying both sides by -1, we get $d \geq -x$ for all $x \in A$, which means that $-d$ is an upper bound of $-A$. We must now show that if c is an upper bound of $-A$, then $c \geq -d$.

Let c be an upper bound of $-A$. That means that for all $x \in A$, $c \geq -x$. Then for all $x \in A$, $-c \leq x$, which makes c a lower bound of A. Since c is a lower bound of A, $c \leq d = \inf(A)$. Then $c \geq -d$. Thus, $\sup(-A) = -d$, and $\inf(A) = d = -\sup(-A)$. Therefore, for A, a nonempty subset of $\mathbb{R}$, $\inf(A) = -\sup(-A)$.

4 Problem 4

Let $x \geq 0$ be such that for all $\epsilon > 0, 0 \leq x < \epsilon$. Prove that $x = 0$.
Proof: (By Contradiction)

Let $x \geq 0$ be such that for all $\epsilon > 0, 0 \leq x < \epsilon$. Suppose that $x \neq 0$. Since $x \geq 0$ and $x \neq 0$, then $x > 0$. Let $\epsilon = \frac{x}{2}$. Because $x > 0$, $\frac{x}{2} > 0$ and $\frac{x}{2} < x$. Thus, $\epsilon < x$, which contradicts our assumption that $x < \epsilon$. Therefore, by contradiction, if for all $\epsilon > 0, 0 \leq x < \epsilon$, then $x = 0$.