Written Assignment 8

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1 Problem 1

Prove that f_n converges to f uniformly on a set D if and only if $\lim_{n\to\infty} ||f_n - f||_{\text{sup}} = 0$.

Proof:
($\Rightarrow$)

Let f_n be a sequence of functions on a set D and suppose f_n converges uniformly to f. Let $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that for all $x \in D$ and $n \geq N$, $|f_n(x) - f(x)| < \frac{\epsilon}{2}$. Choose such N. Since $|f_n(x) - f(x)| < \frac{\epsilon}{2}$, for all $x \in D$, $\frac{\epsilon}{2}$ is an upper bound on $|f_n(x) - f(x)|$. Since $||f_n - f||_{\text{sup}} = \sup\{|f_n(x) - f(x)| : x \in D\}$ and $\frac{\epsilon}{2}$ is an upper bound, then $||f_n - f||_{\text{sup}} \leq \frac{\epsilon}{2}$. Then if $n \geq N$, $||f_n - f||_{\text{sup}} \leq \frac{\epsilon}{2} < \epsilon$. Thus, there exists a $N \in \mathbb{N}$ such that if $n \geq N$, then $||f_n - f||_{\text{sup}} < \epsilon$. Therefore $\lim_{n\to\infty} ||f_n - f||_{\text{sup}} = 0$.

($\Leftarrow$)

Let f_n be a sequence of functions on a set D and suppose $\lim_{n\to\infty} ||f_n - f||_{\text{sup}} = 0$. Let $\epsilon > 0$. Then there exists a $N \in \mathbb{N}$ such that if $n \geq N$, then $||f_n - f||_{\text{sup}} < \epsilon$. Choose such N. Let $x \in D$. If $n \geq N$, then $|f_n(x) - f(x)| \leq ||f_n - f||_{\text{sup}} < \epsilon$, which means $|f_n(x) - f(x)| < \epsilon$. Therefore for all $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for all $x \in D$ and $n \geq N$, $|f_n(x) - f(x)| < \epsilon$, which means f_n converges uniformly to f on D.

2 Problem 2a

Consider $f_n(x) = x^n$ defined on (0,1). Prove that $f_n \to 0$ pointwise on (0,1) but not uniformly on (0,1).

Proof of pointwise convergence.
Proof:

Let $x \in (0,1)$. From a result proven in class (see notes from 1/27/16), since $x \in (0,1)$, then $\lim_{n\to\infty} x^n = 0$. Thus for every $x \in (0,1)$, $f_n(x) \to 0$. Therefore, $f_n \to 0$ pointwise on (0,1).

Proof that f_n doesn't converge uniformly to 0 on (0,1).
Proof:
Let $\epsilon = \frac{1}{3}$. Let $N \in \mathbb{N}$. We must show that there exists a $n \geq N$ and a $x \in (0,1)$ such that $|x^n - 0| \geq \epsilon$. Choose $n = N$, which means $n \geq N$. Choose $x = \frac{n}{n+1} \in (0,1)$. Based on 1a from Written Assignment 1, I proved that $\left(1 + \frac{1}{m}\right)^m < 3$ for all natural m. Since $n \in \mathbb{N}$, $\left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n < 3$. Then, $\left(\frac{n}{n+1}\right)^n > \frac{1}{3}$. Thus, $\left(\frac{n}{n+1}\right)^n = |x^n - 0| > \frac{1}{3}$. Therefore f_n doesn't converge uniformly to 0 on (0,1).

3 Problem 2b

Prove that for any $0 < b < 1$, $f_n \to 0$ uniformly on $(0, b)$.

Proof:
Let $0 < b < 1$. Let $\epsilon > 0$. From a result proven in class (see notes from 1/27/16), since $b \in (0,1)$, $\lim_{n\to\infty} b^n = 0$. Then there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $b^n = |b^n - 0| < \epsilon$. Choose such N. Let $n \geq N$ and let $x \in (0, b)$. Then $x^n < b^n < b^N < \epsilon$. Thus, $|x^n - 0| < \epsilon$, and $f_n \to 0$ uniformly on $(0, b)$.

4 Problem 3

Prove that if f_n converges to f uniformly on a set D then f_n converges to f pointwise on D.

Proof:
Let $x \in D$ and let $\epsilon > 0$. Since f_n converges to f uniformly on D. There exists a $N \in \mathbb{N}$ such that for all $y \in D$ and $n \geq N$, $|f_n(y) - f(y)| < \epsilon$. Choose such N. Then because $x \in D$, if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$. This means $f_n(x) \to f(x)$ and f_n converges pointwise to f on D.

5 Problem 4

Prove that if f_n uniformly converges to f on D then f_n is uniformly Cauchy on D.

Proof:
Let f_n be a sequence of functions defined on D and suppose f_n uniformly converges to f. Let $\epsilon > 0$. Then there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $||f_n - f||_{\text{sup}} < \frac{\epsilon}{2}$. Choose such N. Let $n, m \geq N$. Then $||f_m - f||_{\text{sup}} + ||f_n - f||_{\text{sup}} = ||f_m - f||_{\text{sup}} + ||f - f_n||_{\text{sup}} < \epsilon$. By the triangle inequality, $||f_m - f_n||_{\text{sup}} < \epsilon$, which means f_n is uniformly Cauchy on D.