Written Assignment 3

Lucas Bouck

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1 Problem 1

Let $S \subseteq \mathbb{R}$ be a closed set, and suppose that the sequence $x_n \in S$ converges to some $x \in \mathbb{R}$. Prove that $x \in S$.

Proof: (By Contrapositive)

Let $S \subseteq \mathbb{R}$ be a closed set. Let x_n be a sequence of real numbers and suppose x_n converges to $x \in \mathbb{R}$. Suppose $x \notin S$. We want to show that there exists an $n \in \mathbb{R}$ such that $x_n \notin S$. Since $x \notin S$, $x \in S^c$. Since S is closed, S^c is open. Thus, there exists an $r > 0$ such that $(x-r, x+r) \subseteq S^c$. Choose $\epsilon = r$. Since x_n converges to x, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - x| < \epsilon$. Since $N \geq N$, $|x_N - x| < \epsilon$. Then,

$$\begin{aligned} -\epsilon < x_N - &x < \epsilon \\ -r < x_N - &x < r \\ x - r < x_N &< x + r \end{aligned}$$

This means $x_N \in (x-r, x+r) \subseteq S^c$. Therefore, $x_N \in S^c$ and $x_N \notin S$. Thus, there exists an $n \in \mathbb{R}$ such that $x_n \notin S$. By contrapositive, we have shown that if $S \subseteq \mathbb{R}$ is a closed set $x_n \in S$ converges to some $x \in \mathbb{R}$, then $x \in S$.

2 Problem 2

Prove that if $S \subseteq \mathbb{R}$ is closed and bounded, then every sequence $x_n \in S$ has a subsequence that converges to an element of S.

Proof:

Let $S \subseteq \mathbb{R}$ be a closed and bounded set. We want to show that every sequence $x_n \in S$ has a subsequence that converges to an element of S.

Let x_n be a sequence such that $x_n \in S$ for all $n \in \mathbb{N}$. We want to show that x_n has a subsequence that converges to an element of S. Since S is bounded, there exists a real number M such that for all $s \in S$, $|s| \leq M$. Since $x_n \in S$ for all n, $|x_n| \leq M$ for all n. Thus, x_n is bounded. By the Bolzano-Weierstrauss Theorem, x_n has a convergent subsequence, which will be denoted x_{n_k}. Let $\lim x_{n_k} = L$. We must now show that $L \in S$. Since x_{n_k} is a subsequence of x_n, and $x_n \in S$ for all n, $x_{n_k} \in S$ for all n_k. Then, by the result that was proven in problem 1 of this assignment, $L \in S$. Hence, if $S \subseteq \mathbb{R}$ is a closed and bounded set, then every sequence $x_n \in S$ has a subsequence that converges to an element of S.

3 Problem 3

Suppose that $S \subseteq \mathbb{R}$ is not bounded. Prove that there is a sequence $x_n \in S$ that does not have a convergent subsequence.

Proof:

Let S be a subset of $\mathbb{R}$ and suppose S is not bounded. We want to show that there exists an $x_n \in S$ that does not not have a convergent subsequence. Since S is unbounded, this means for all $M \in \mathbb{R}$, there exists a $y \in S$ such that $|y| > M$. We shall now construct our sequence x_n.

Choose x_1 to be an element of S such that $|x_1| > 1$. Choose x_{n+1} to be an element of S such that $|x_{n+1}| > x_n + 1$. We must now show that the sequence x_n has no convergent subsequence.

Suppose x_n has a convergent subsequence x_{n_k}. Let x_{n_k} converge to L. Choose $\epsilon = \frac{1}{2}$. Since x_{n_k} converges to L, there exists an M such that if $m \geq M$, $|x_{n_m} - L| < 1/2$. Consider $x_{n_{M+1}}$. Since $M + 1 \geq M$, $|x_{n_{M+1}} - L| < 1/2$. Also, $|x_{n_M} - L| < 1/2$. Adding the two inequalities together, we get

$$\begin{aligned}
|x_{n_{M+1}} - L| + |x_{n_M} - L| &< 1 \\
|x_{n_{M+1}} - L| + |L - x_{n_M}| &< 1 \\
|x_{n_{M+1}} - L + L - x_{n_M}| &\leq |x_{n_{M+1}} - L| + |L - x_{n_M}| < 1 \\
|x_{n_{M+1}} - x_{n_M}| &< 1 \\
||x_{n_{M+1}}| - |x_{n_M}|| &< 1 \\
|x_{n_{M+1}}| - |x_{n_M}| &< 1 \\
|x_{n_{M+1}}| &< |x_{n_M}| + 1
\end{aligned}$$

Since $n_{k+1} > n_k$ for all k, then $n_{M+1} > n_M$. Based on how the sequence x_n is defined, $|x_{n_{M+1}}| > |x_{n_M}| + 1$. But we have $|x_{n_{M+1}}| < |x_{n_M}| + 1$. We have a contradiction. Therefore, x_n does not have a convergent subsequence.

We have shown that if S is not bounded, then there exists a sequence $x_n \in S$ such that x_n has no convergent subsequence.

4 Problem 4

Supose that $S \subseteq \mathbb{R}$ is not closed. Prove that there is a sequence $x_n \in S$ that converges to some $x \notin S$.

Proof:

Let S be a subset of $\mathbb{R}$. Assume S is not closed. That means S^c is not open, which means there exists an $L \in S^c$ such that for all $r > 0, (L-r, L+r) \nsubseteq S^c$. We shall show that there exists a sequence $x_n \in S$ that converges to L. Since for all $r > 0, (L-r, L+r) \nsubseteq S^c$, then for all $r > 0$, there exists a $s \in (L-r, L+r)$ such that $s \notin S^c$, which means $s \in S$. We shall now construct the sequence x_n.

Choose x_1 to be an element in S such that x_1 is also an element of $(L-1, L+1)$. Choose x_{n+1} to be an element in S such that $x_{n+1} \in (L - |x_n - L|, L + |x_n - L|)$.

We shall now show that x_n converges to L. We will do this by contradiction. Suppose there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists an $n \geq N$ such that $|x_n - L| \geq \epsilon$. Since $\epsilon > 0$, there exists an element of the sequence x_n in the interval $(L-\epsilon, L+\epsilon)$. If not, then by how we defined the sequence, for all $s \in S$, $s \notin (L-\epsilon, L+\epsilon)$, which would imply that there exists an $r > 0$ such that $(L-r, L+r) \subseteq S^c$, which would contradict our original assumptions. Thus, there exists an element of the sequence x_n in the interval $(L-\epsilon, L+\epsilon)$, which will be denoted x_N. Then, there exists an $n \geq N$ such that $|x_n - L| \geq \epsilon$. This means $x_n \geq L+\epsilon$ or $x_n \leq L-\epsilon$. If $x_n \geq L+\epsilon$, since $|x_N - L| < \epsilon$, then $x_n > L + |x_N - L|$. This implies $x_n \notin (L - |x_N - L|, L + |x_N - L|)$. But the fact that $n \geq N$ implies $x_n \in (L - |x_N - L|, L + |x_N - L|)$. We have a contradiction. If $x_n \leq L - \epsilon$, since $|x_N - L| < \epsilon$, then $x_n < L - |x_N - L|$. This implies $x_n \notin (L - |x_N - L|, L + |x_N - L|)$. But the fact that $n \geq N$ implies $x_n \in (L - |x_N - L|, L + |x_N - L|)$. We have a contradiction. In both cases, we have a contradiction. Therefore, the sequence x_n converges to L.

We have shown that if S is not closed, there exists a sequence $x_n \in S$ that converges to an $L \notin S$.