Written Assignment 1
Math 290, Dr. Walnut

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1 Problem 1:

1.1 Problem:

Prove the *triangle inequality*: for all real numbers a and b, $|a| + |b| \geq |a + b|$

1.2 Proof:

Let $a, b \in \mathbb{R}$. We want to show that for all a and b, $|a + b| \leq |a| + |b|$. There will be four cases.
Case 1: Let $a, b \geq 0$. Since $a, b \geq 0$, then $|a| = a$, and $|b| = b$. Thus, $|a| + |b| = a + b$. Since $a, b \geq 0$, $a + b \geq 0$. Thus, $|a + b| = a + b$. Since $a + b = a + b$, $|a| + |b| = |a + b|$. Therefore, when $a, b \geq 0$, $|a| + |b| \geq |a + b|$.
Case 2: Let $a, b < 0$. Since $a, b \leq 0$, then $|a| = -a$ and $|b| = -b$. Thus, $|a| + |b| = -a - b$. Since $a, b < 0$, $a + b < 0$. Since $a + b < 0$, $|a + b| = -(a + b) = -a - b = |a| + |b|$. Therefore, $|a| + |b| = |a + b|$.Therefore, when $a, b < 0$, $|a| + |b| \geq |a + b|$
Case 3: Let $a < 0$, and let $b \geq 0$. Then, $|a| = -a$, and $|b| = b$. Therefore, $|a| + |b| = -a + b$.
Subcase 1: Let $|a| > |b|$. Then, $a + b < 0$. Thus, $|a + b| = -(a + b) = -a - b$. Because $b \geq 0$, $b > -b$. Therefore, $-a + b > -a - b$. Therefore, when $|a| > |b|$, $|a + b| < |a| + |b|$.
Subcase 2: Let $|a| \leq |b|$. Then, $a + b \geq 0$. Then, $|a + b| = a + b$. Since $a < 0$, then $a < -a$, and $a + b < -a + b$. Therefore, when $|a| \leq |b|$, $|a + b| < |a| + |b|$.

Therefore, when $a < 0$ and $b \geq 0$, $|a + b| \leq |a| + |b|$.

Case 4: Let $b < 0$, and let $a \geq 0$. Then, $|b| = -b$, and $|a| = b$. Therefore, $|a| + |b| = a - b$.
Subcase 1: If $|a| \geq |b|$, $a + b \geq 0$. Thus, $|a + b| = a + b$. Since $b < 0$, $b < -b$, and $a + b < a - b$. Therefore, when $|a| \geq |b|$, $|a + b| \leq |a| + |b|$.
Subcase 2: If $|b| > |a|$, $a + b < 0$, and $|a + b| = -(a + b) = -a - b$. Since $a \geq 0$, $a > -a$, and $-a - b < a - b$. Therefore, when $|b| > |a|$, $|a + b| < |a| + |b|$.

Therefore, when $b < 0$ and $a \geq 0$, $|a + b| \leq |a| + |b|$.

Since $|a + b| \leq |a| + |b|$ in all four cases, $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

2 Problem 2:

2.1 Problem:

Use the triangle inequality to prove that for all real numbers a and b, $||a| - |b|| \leq |a - b|$.

2.2 Proof:

Let $a, b \in \mathbb{R}$. We want to show that $||a| - |b|| \leq |a - b|$. We know $|a| = |(a - b) + b|$ because $b - b = 0$, and $|a + 0| = |a|$. Let $a - b = c$. $c \in \mathbb{R}$ because a and b are real numbers. Thus, $|a| = |c + b|$. Using the triangle inequality, we know $|c + b| \leq |c| + |b|$. Since $a - b = c$, $|(a - b) + b| \leq |a - b| + |b|$. By subtracting $|b|$ from both sides and simplifying $|(a - b) + b|$, we get $|a| - |b| \leq |a - b|$.

Let $|b| = |(b - a) + a|$ because $a - a = 0$, and $b - a + a = b$. Let $b - a = d$. $d \in \mathbb{R}$ because both a and b are real numbers. Thus, $|b| = |d + a|$. Using the triangle inequality, we know $|d + a| \leq |d| + |a|$. Since $b - a = d$, $|(b - a) + a| \leq |b - a| + |a|$. By subtracting $|a|$ from both sides and simplifying $|(b - a) + a|$, we get $|b| - |a| \leq |b - a|$.

We now have the two statements $|a| - |b| \leq |a - b|$ and $|b| - |a| \leq |b - a|$. Since $|a - b| = |-(a - b)| = |b - a|$, we get $|a| - |b| \leq |a - b|$ and $|b| - |a| \leq |a - b|$. Suppose $|a| - |b| = z$. By the definition of absolute value, if $z < 0$ then $|z| = -z$. Since $|a| - |b| = z$, if $|a| - |b| < 0$, $||a| - |b|| = -(|a| - |b|) = |b| - |a|$. If $z \geq 0$, then $|z| = z$. Since $|a| - |b| = z$, if $|a| - |b| \geq 0$, then $||a| - |b|| = |a| - |b|$. Therefore,

$$||a| - |b|| = \begin{cases} |a| - |b| & : |a| - |b| \geq 0 \\ |b| - |a| & : |a| - |b| < 0 \end{cases}$$

Since $|a| - |b| \leq |a - b|$, and $|b| - |a| \leq |a - b|$ for all $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

3 Problem 3:

3.1 Problem:

Let a and b be natural numbers. Prove that if $a|b$ then $a \leq b$.

3.2 Proof:

Let a and b be natural numbers. We want to show that if $a|b$ then $a \leq b$. Since $a|b$, we can write $a * z = b$ for some integer z. Since a and be are natural numbers, a and b are not 0. Since a and b are not 0, and a natural number multiplied by 0 equals 0, $z \neq 0$. Since a and b are natural numbers, a and b are positive. In order for the product of two numbers to be positive, the numbers must be either both positive or both negative. Since $a * z = b$, and a and b are positive, z must be positive. Since z must be nonzero and positive, $z \geq 1$. Since

$1 \leq z$ and a is positive, we know $1 * a \leq z * a$. Since $z * a = b$, $1 * a = a \leq b$. Therefore, if $a|b$, $a \leq b$.

3.3 Problem:

Use this result to show that for every natural number $n \geq 2$, n does not divide $n + 1$.

3.4 Proof: (by contradiction)

Assume $n \in \mathbb{R}$, and $n \geq 2$. Say $n|(n + 1)$. Then $n * z = n + 1$, where z is an integer. By subtracting n from both sides, we get $nz - n = 1$. By factoring out an n on the left side, we get $n(z - 1) = 1$. Say $z - 1 = k$. Since z and 1 are integers, then k is an integer. Then, $n * k = 1$. Since $n * k = 1$, n divides 1. Based on our previous proof in 3.2, this means $n \leq 1$. Since $n \geq 2$, then $2 \leq 1$. We have a contradiction. Therefore, for every natural number $n \geq 2$, n does not divide $n + 1$.

4 Problem 4:

4.1 Problem:

Prove that for any positive real numbers x and y, $(x + y)/2 \geq \sqrt{xy}$.

4.2 Proof:

Let $x, y \in \mathbb{R}$, and let x and y be positive. We want to show that $(x + y)/2 \geq \sqrt{xy}$. Since x and y are real numbers, $x - y$ is a real number. Because $x - y$ can only be either negative or nonnegative and the product of two numbers that carry the same sign is nonnegative, $(x - y)(x - y) \geq 0$. Then $x^2 - 2xy + y^2 \geq 0$. By adding 4xy to both sides, $x^2 - 2xy + y^2 + 4xy = x^2 + 2xy + y^2 \geq 4xy$ By factoring the left side, we get $(x + y)^2 \geq 4xy$. Then, $(x + y)^2/4 \geq xy$. Since x and y are positive, xy is positive. Then, we can take the square root of both sides and get $(x + y)/2 \geq \sqrt{xy}$. Therefore, for any positive real numbers x and y, $(x + y)/2 \geq \sqrt{xy}$.

4.3 Problem:

Prove that for any positive real numbers x and y, $(x + y)/2 = \sqrt{xy}$ if and only if $x = y$.

4.4 Proof:

($\Rightarrow$) Let x and y be positive real numbers. Assume $(x + y)/2 = \sqrt{xy}$. By squaring both sides, we get $(x + y)^2/4 = |xy|$. Because x and y are positive, xy is always positive. Then, we can simplify $|xy|$ to xy. Thus, $(x + y)^2/4 = xy$. By multiplying 4 to both sides,

$(x+y)^2 = 4xy$. By expanding the left side, we get $x^2+2xy+y^2 = 4xy$. By subtracting 4xy from both sides, we get $x^2 - 2xy + y^2 = 0$. By factoring the left side, we get $(x-y)^2 = 0$ or $(y-x)^2 = 0$. In order for $(x-y)^2 = 0$, $x - y = 0$, and $x = y$. In order for $(y-x)^2 = 0$, $y - x = 0$, and $y = x$. In either case, $x = y$. Therefore, for any positive real numbers x and y, $(x+y)/2 = \sqrt{xy}$ implies $x = y$.
($\Leftarrow$) Let x and y be positive real numbers. Assume $x = y$. Say $x = y = z$. Then $xy = z^2$, and $\sqrt{xy} = \sqrt{z^2} = |z|$. Since x and y are always positive, xy is always positive, and z is positive. Thus, $|z| = z$, and $\sqrt{xy} = z$. By multiplying 2/2, which is 1, to both sides, we get$(2/2)\sqrt{xy} = 2z/2 = (z+z)/2$. Since $z = x$, and $z = y$, $z + z = x + y$. Thus, $(z+z)/2 = (x+y)/2$. Then, $(2/2)\sqrt{xy} = (x+y)/2$. $2/2 = 1$ so, $(2/2)\sqrt{xy} = \sqrt{xy}$. Thus, $\sqrt{xy} = (x+y)/2$. Therefore, for any positive real numbers x and y, $x = y$ implies $(x+y)/2 = \sqrt{xy}$.

Since $(x+y)/2 = \sqrt{xy}$ implies $x = y$, and $x = y$ implies $(x+y)/2 = \sqrt{xy}$, $(x+y)/2 = \sqrt{xy}$ if and only if $x = y$.