# Written Assignment 5

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## 1 Problem 1a

Let $f$ and $g$ be functions with a common domain $D$. Show that if $f$ and $g$ are both continuous at a point $a \in D$ then the function $fg$ is continuous at $a$.

**Proof:**

Let $f$ and $g$ be functions with a common domain $D$. Assume that $f$ and $g$ are both continuous at a point $a \in D$. We must show that $fg$ is continuous at $a$. There will be two cases.

*Case 1:* Suppose $a$ is an isolated point. Since $fg$ has the domain $D$, and $a \in D$, then $fg$ is continuous at $a$.

*Case 2:* Suppose $a$ is a cluster point. Since $f$ and $g$ are continuous at $a$, $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. Let $x_n \in D \setminus \{a\}$ be a sequence with $x_n \to a$. By the sequential characterization of limits, $f(x_n) \to f(a)$ and $g(x_n) \to g(a)$. Because $f(x_n) \to f(a)$ and $g(x_n) \to g(a)$, $fg(x_n) \to f(a)g(a)$. This means $\lim_{x\to a} fg(x) = fg(a)$ and $fg$ is continuous at $a$.

## 2 Problem 1b

Let $f$ and $g$ be functions with a common domain $D$. Show that if $f$ and $g$ are both continuous at a point $a \in D$ then the function $\alpha f + \beta g$ is continuous at $a$.

**Proof:**

Let $f$ and $g$ be functions with a common domain $D$. Assume that $f$ and $g$ are both continuous at a point $a \in D$. We must show that $\alpha f + \beta g$ is continuous at $a$. There will be two cases.

*Case 1:* Suppose $a$ is an isolated point. Since $\alpha f + \beta g$ has the domain $D$, and $a \in D$, then $\alpha f + \beta g$ is continuous at $a$.

*Case 2:* Suppose $a$ is a cluster point. Since $f$ and $g$ are continuous at $a$, $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. Let $x_n \in D \setminus \{a\}$ be a sequence with $x_n \to a$. By the sequential characterization of limits, $f(x_n) \to f(a)$ and $g(x_n) \to g(a)$. Because $f(x_n) \to f(a)$ and $g(x_n) \to g(a)$, $\alpha f(x_n) \to \alpha f(a)$, and $\beta g(x_n) \to \beta g(a)$. Then $\alpha f + \beta g \to \alpha f(a) + \beta g(a)$. This means $\lim_{x\to a}(\alpha f + \beta g)(x) = \alpha f(a) + \beta g(a)$ and $\alpha f + \beta g$ is continuous at $a$.

# 3 Problem 2

Let $f$ and $g$ be continuous functions on $\mathbb{R}$. Prove that the composite function $f \circ g$ is also continuous on $\mathbb{R}$.

**Proof:**

Let $f$ and $g$ be continuous functions on $\mathbb{R}$. Let $a \in \mathbb{R}$. We want to show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ if $|x - a| < \delta$ then $|f(g(x)) - a| < \epsilon$. Since both $f$ is continuous at $a$, there exist $\delta_1 > 0$ such that for all $y \in \mathbb{R}$ if $|y - a| < \delta$ then $|f(y) - f(a)| < \epsilon$. We know that $g$ is continuous at $a$ and $\delta_1 > 0$, so there exists a $\delta_2 > 0$ such that for all $x \in \mathbb{R}$ if $|x - a| < \delta_2$ then $|g(x) - g(a)| < \delta_1$. This means if $|x - a| < \delta_2$, then $|g(x) - g(a)| < \delta_1$, which implies that $|f(g(x)) - f(g(a))| < \epsilon$. Therefore, there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ if $|x - a| < \delta$ then $|f(g(x)) - a| < \epsilon$, and $f \circ g$ is continuous at $a$.