Group Written Assignment 4

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1 Problem 1

Use induction to prove that for all real numbers $x > 0$ and natural numbers n,

$$(1+x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2$$

.

Proof:

Let x be a real number greater than 0 and let n be a natural number. We want to show that $(1+x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2$ for all natural numbers n and real numbers x that are greater than 0. We must first show that the result holds for the base case $n = 1$. Let $n = 1$. Then, $(1+x)^1 = 1 + x$, and $1 + (1)x + \frac{(1)(1-1)}{2}x^2 = 1 + x$. Since $1 + x = 1 + x$, the result holds for $n = 1$.

Let n be a natural number. Assume that the result holds for n. We want to show that the result holds for $n+1$, which means we want to show $(1+x)^{n+1} \geq 1 + (n+1)x + \frac{(n+1)(n)}{2}x^2$. By the induction hypothesis, $(1+x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2$. Since $x > 0$, we can multiply $(1+x)$ to both sides and maintain the inequality. We then get, $(1+x)^{n+1} \geq \left(1 + nx + \frac{n(n-1)}{2}x^2\right)(1+x)$. Then,

$$\begin{aligned}
\left(1 + nx + \frac{n(n-1)}{2}x^2\right)(1+x) &= (1+x) + (1+x)nx + (1+x)\frac{n(n-1)}{2}x^2 \\
&= 1 + x + nx + nx^2 + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)}{2}x^3 \\
&= 1 + (n+1)x + \frac{2n + n^2 - n}{2}x^2 + \frac{n(n-1)}{2}x^3 \\
&= 1 + (n+1)x + \frac{(n+1)n}{2}x^2 + \frac{n(n-1)}{2}x^3
\end{aligned}$$

Since $x > 0$ and n is a natural number, which means $n \geq 1$, $\frac{n(n-1)}{2}x^3 \geq 0$. Since $\frac{n(n-1)}{2}x^3 \geq 0$, then $1 + (n+1)x + \frac{(n+1)n}{2}x^2 + \frac{n(n-1)}{2}x^3 \geq 1 + (n+1)x + \frac{(n+1)n}{2}x^2$. Since

$(1+x)^{n+1} \geq 1+(n+1)x+\frac{(n+1)n}{2}x^2+\frac{n(n-1)}{2}x^3$, then $(1+x)^{n+1} \geq 1+(n+1)x+\frac{(n+1)n}{2}x^2$, which is what we want. Therefore, by the principle of mathematical induction, $(1+x)^n \geq 1+nx+\frac{n(n-1)}{2}x^2$ for all natural n and all real $x > 0$.

2 Problem 2

Prove for all real numbers x, y and integer $n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

.

Proof:

For this proof, we will first prove that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ for all integers n and k with $0 \leq k \leq n$. Let n and k be integers with $0 \leq k \leq n$. Using the formula for a combination, $\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{k!(n-(k-1))!} + \frac{n!}{k!(n-k)!} = \frac{n!k}{k!(n+1-k)!} + \frac{(n+1-k)n!}{k!(n+1-k)!} = \frac{n!(k+n+1-k)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!}$. Using the combination formula again, $\frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$. Therefore, for all integers n and k with $0 \leq k \leq n$, $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Let's now move to the actual proof of the result above. We must first show that the result holds for integer $n = 0$. Let $n = 0$ and let x and y be real numbers. Then, $(x+y)^0 = 1$, and $\sum_{k=0}^{0} \binom{0}{0} x^k y^{0-k} = \binom{0}{0} x^0 y^0 = 1$. Therefore, the result holds for $n = 0$.

Let n be an integer with $n \geq 0$. Assume the result holds for n. We want to show that the result holds for $n+1$, which means we want to show $(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$. By the induction hypothesis, $(x+y)^{n+1} = (x+y)(x+y)^n = (x+y)\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. Then,

$$\begin{aligned}
(x+y)\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} &= x\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} + y\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\
= &\binom{n}{0} x^1 y^n + \binom{n}{1} x^2 y^{n-1} + ... + \binom{n}{n-1} x^n y^1 + \binom{n}{n} x^{n+1} \\
&+ \binom{n}{0} y^{n+1} + \binom{n}{1} x^1 y^n + ... + \binom{n}{n-1} x^{n-1} y^2 + \binom{n}{n} x^n y^1
\end{aligned}$$

Matching like terms together, we get

$$\left[\binom{n}{0} + \binom{n}{1}\right] x^1 y^n + \left[\binom{n}{1} + \binom{n}{2}\right] x^2 y^{n-1} + ... + \left[\binom{n}{n-1} + \binom{n}{n}\right] x^n y^1 + \binom{n}{n} x^{n+1} + \binom{n}{0} y^{n+1}$$

. Using the fact that for all integers n and k with $0 \leq k \leq n$, $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, we get

$$\binom{n+1}{1} x^1 y^n + \binom{n+1}{2} x^2 y^{n-1} + ... + \binom{n+1}{n} x^n y^1 + x^{n+1} + y^{n+1}$$

This then becomes,

$$\sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k} + \binom{n+1}{n+1} x^{n+1} y^0 + \binom{n+1}{0} x^0 y^{n+1}$$

Adding the extra terms into as the last and first terms of the sum respectively, we get

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$$

This is what we wanted. Therefore, by the principle of mathematical induction, for all real numbers x, y and integer $n \geq 0$, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.