Math 478 HW 4

Lucas Bouck

10/18/16

1 Problem 2.7.12

1.1 Part a.

We want to solve $u_t = \frac{1}{4}u_{xx}$ on $(0,1)$ with the boundary conditions $u_x(t,0) = 0, u(t,1) = 0$. Expressing $u = X(x)T(t)$, we get

$$\begin{aligned} T'(t)X(x) &= \frac{1}{4}T(t)X''(x) \\ \frac{4T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = \mu \\ X''(x) - \mu X(x) &= 0 \\ T'(t) &= \frac{\mu}{4}T(t) \end{aligned}$$

Solving the Sturm-Liouville problem $X''(x) - \mu X(x) = 0$ with $X'(0) = X(1) = 0$, we get $\mu = -\frac{(2\ell-1)^2\pi^2}{4}$ and $X(x) = \alpha \cos\left(\frac{(2\ell-1)\pi}{2}x\right)$ for $\ell \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Solving the second problem $T'(t) = \frac{\mu}{4}T(t)$ now that we know μ is

$$T(t) = \beta e^{-\frac{(2\ell-1)^2\pi^2}{16}t}, \text{ for } \beta \in \mathbb{R}, \text{ and } \ell \in \mathbb{N}$$

From this, for $\ell \in \mathbb{N}$, a solution to the PDE is $u_\ell(t,x) = \gamma_\ell e^{-\frac{(2\ell-1)^2\pi^2}{16}t} \cos\left(\frac{(2\ell-1)\pi}{2}x\right)$ for some $\gamma_\ell \in \mathbb{R}$. Thus, the general solution is

$$u(t,x) = \sum_{\ell=1}^{\infty} \gamma_\ell e^{-\frac{(2\ell-1)^2\pi^2}{16}t} \cos\left(\frac{(2\ell-1)\pi}{2}x\right)$$

1.2 Part b.

Given the initial condition $u(0,x) = 7\cos(5\pi x/2)$, we want to find the solution to $u_t = \frac{1}{4}u_{xx}$. Using the general solution from part a, all we need to find are the coefficents γ_ℓ. We know

$$u(0,x) = \sum_{\ell=1}^{\infty} \gamma_\ell e^{-\frac{(2\ell-1)^2\pi^2}{16}0} \cos\left(\frac{(2\ell-1)\pi}{2}x\right) = \sum_{\ell=1}^{\infty} \gamma_\ell \cos\left(\frac{(2\ell-1)\pi}{2}x\right)$$

Therefore,

$$\begin{aligned}
\gamma_\ell &= \frac{\langle 7\cos(5\pi x/2), \cos\left(\frac{(2\ell-1)\pi}{2}x\right)\rangle}{\langle \cos\left(\frac{(2\ell-1)\pi}{2}x\right), \cos\left(\frac{(2\ell-1)\pi}{2}x\right)\rangle} \\
\gamma_3 &= \frac{\langle 7\cos(5\pi x/2), \cos\left(\frac{5\pi}{2}x\right)\rangle}{\langle \cos\left(\frac{5\pi}{2}x\right), \cos\left(\frac{5\pi}{2}x\right)\rangle} = 7 \\
\gamma_\ell &= 0, \text{ if } \ell \neq 3
\end{aligned}$$

Thus, the solution to the PDE given our initial conditions is

$$u(t,x) = 7e^{-\frac{25\pi^2}{16}t} \cos\left(\frac{5\pi}{2}x\right)$$

2 Problem 2.7.14

We want to solve the PDE $u_{xx} + u_{yy} = 0$ on $\Omega = [0,a] \times [0,a]$ with the boundary conditions $u(x,0) = u(x,a) = u(a,y) = 0$ on $x \in [0,a]$ and $y \in [0,a]$ and $u(0,y) = f(y)$ on $y \in [0,a]$. By expressing $u = X(x)Y(y)$, we get

$$\begin{aligned}
Y(y)X''(x) + Y''(y)X(x) &= 0 \\
\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} &= \mu \\
Y'' + \mu Y &= 0 \\
X'' - \mu X &= 0
\end{aligned}$$

Solving the first Sturm-Liouville Problem $Y'' + \mu Y = 0$ with $Y(0) = Y(a) = 0$, we get $\mu = \frac{\ell^2\pi^2}{a^2}$ and $Y(y) = \alpha \sin\left(\frac{\ell\pi}{a}y\right)$ for $\ell \in \mathbb{N}, \alpha \in \mathbb{R}$. The next problem now becomes $X'' - \frac{\ell^2\pi^2}{a^2}X = 0$ with $X(a) = 0$. The solution is then $X(x) = \beta\left(e^{-\frac{\ell\pi}{a}x} - e^{\frac{\ell\pi}{a}x - 2\ell\pi}\right)$. Since $u(x,y) = X(x)Y(y)$, we get a solution

$$u_\ell(x,y) = \gamma_\ell \sin\left(\frac{\ell\pi}{a}y\right)\left(e^{-\frac{\ell\pi}{a}x} - e^{\frac{\ell\pi}{a}x - 2\ell\pi}\right)$$

The general solution is then

$$u(x,y) = \sum_{\ell=1}^{\infty} \gamma_\ell \sin\left(\frac{\ell\pi}{a}y\right)\left(e^{-\frac{\ell\pi}{a}x} - e^{\frac{\ell\pi}{a}x - 2\ell\pi}\right)$$

. We want to find a solution for $a = 1$ and $u(0, y) = 4y(1 - y)$. Then,

$$\begin{aligned}
\gamma_\ell \left(1 - e^{-2\ell\pi}\right) &= \frac{\langle 4y(1-y), \sin\left(\frac{\ell\pi}{a}y\right)\rangle}{\langle \sin\left(\frac{\ell\pi}{a}y\right), \sin\left(\frac{\ell\pi}{a}y\right)\rangle} = \frac{\int_0^1 4y(1-y)\sin\left(\frac{\ell\pi}{a}y\right)\, dy}{\int_0^1 \sin^2\left(\frac{\ell\pi}{a}y\right)\, dy} = \frac{\frac{8-8(-1)^\ell}{\ell^3\pi^3}}{1/2} \\
&= \frac{16 - 16(-1)^\ell}{\ell^3\pi^3} \\
\gamma_\ell &= \frac{16 - 16(-1)^\ell}{\left(1 - e^{-2\ell\pi}\right)\ell^3\pi^3}
\end{aligned}$$

So the final solution is

$$u(x, y) = \sum_{\ell=0}^{\infty} \frac{16 - 16(-1)^\ell}{\left(1 - e^{-2\ell\pi}\right)\ell^3\pi^3} \sin\left(\ell\pi y\right)\left(e^{-\ell\pi x} - e^{\ell\pi x - 2\ell\pi}\right)$$

3 Problem 2.7.15

We want to solve $u_{xx} + u_{yy} + u_y - u = 0$ on $(0,1)\times(0,1)$ with $u(0,y) = u(1,y) = u(x,0) = 0$ and $u(x,1) = x^2 - x^3$. By substituting $u = X(x)Y(y)$, we get

$$\begin{aligned}
X''Y + Y''X + XY' - XY &= 0 \\
\frac{X''}{X} + \frac{Y''}{Y} + \frac{Y'}{Y} - 1 &= 0 \\
\frac{X''}{X} = \frac{-Y'' - Y' + Y}{Y} &= \mu \\
X'' - \mu X &= 0 \\
Y'' + Y' + (\mu - 1)Y &= 0
\end{aligned}$$

Solving the first Sturm-Lioville problem $X'' - \mu X = 0$ with $X(0) = X(1) = 0$. We get $\mu = -\ell^2\pi^2$ and $X(x) = \alpha\sin(\ell\pi x)$ for $\ell \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. The next problem $Y'' + Y' + (\mu - 1)Y = 0$ with $Y(0) = 0$. Since we know $\mu = -\ell^2\pi^2$, the solution to this Sturm-Liouville problem will be $Y(y) = \beta\left(e^{r_1 y} - e^{r_2 y}\right)$ where $\beta \in \mathbb{R}$, $r_1 = \frac{-1+\sqrt{5+4\ell^2\pi^2}}{2}$, and $r_2 = \frac{-1-\sqrt{5+4\ell^2\pi^2}}{2}$. Thus,

$$u_\ell = \gamma_\ell \sin(\ell\pi x)\left(e^{r_1 y} - e^{r_2 y}\right)$$

and

$$u(x, y) = \sum_{\ell=1}^{\infty} \gamma_\ell \sin(\ell\pi x)\left(e^{r_1 y} - e^{r_2 y}\right).$$

Given $u(x,1) = x^2 - x^3$ we can use this to calculate γ_ℓ. First, $u(x,1) = \sum_{\ell=1}^{\infty} \gamma_\ell \sin(\ell\pi x)\,(e^{r_1} - e^{r_2})$, so

$$\begin{aligned}
\gamma_\ell &= \frac{1}{(e^{r_1} - e^{r_2})} \frac{\langle \sin(\ell\pi x), x^2 - x^3 \rangle}{\langle \sin(\ell\pi x), \sin(\ell\pi x) \rangle} = \frac{1}{(e^{r_1} - e^{r_2})} \frac{\int_0^1 \sin(\ell\pi x)(x^2 - x^3)\, dx}{\int_0^1 \sin^2(\ell\pi x)\, dx} \\
&= \frac{1}{(e^{r_1} - e^{r_2})} \frac{-4\ell\pi - 8\ell\pi(-1)^\ell}{\ell^4\pi^4} \\
&= \frac{1}{(e^{r_1} - e^{r_2})} \frac{-4 - 8(-1)^\ell}{\ell^3\pi^3}
\end{aligned}$$

Therefore, the solution is

$$u(x,y) = \sum_{\ell=1}^{\infty} \frac{-4 - 8(-1)^\ell}{\ell^3\pi^3\,(e^{r_1} - e^{r_2})} \sin(\ell\pi x)\,(e^{r_1 y} - e^{r_2 y})$$

where $r_1 = \frac{-1+\sqrt{5+4\ell^2\pi^2}}{2}$ and $r_2 = \frac{-1-\sqrt{5+4\ell^2\pi^2}}{2}$.

4 Problem 2.7.18

We want to solve $u_{tt} + u_t = u_{xx}$ for $t \geq 0$ and $x \in (0,1)$ with homogenous Dirichlet boundary conditions and $u(0,x) = \sin(2\pi x)$ and $u_t(0,x) = \sin(\pi x) + 5\sin(3\pi x)$. Using $u = T(t)X(x)$, we get

$$\begin{aligned}
T''(t)X(x) + T'(t)X(x) &= T(t)X''(x) \\
\frac{T''(t) + T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = \mu \\
X'' - \mu X &= 0 \\
T'' + T' - \mu T &= 0
\end{aligned}$$

With homogenous Dirichlet boundary conditions, the solution to the first Sturm-Liouville problem, $X'' - \mu X = 0$, is $\mu = -\ell^2\pi^2$ and $X = \alpha\sin(\ell\pi x)$ for $\ell \in \mathbb{N}$ and real number α. The next problem becomes $T'' + T' + \ell^2\pi^2 T = 0$, and the solution to this becomes

$$T(t) = e^{-t/2}\left(c_1 \cos\left(\frac{\sqrt{4\ell^2\pi^2 - 1}}{2} t\right) + c_2 \sin\left(\frac{\sqrt{4\ell^2\pi^2 - 1}}{2} t\right)\right)$$

and the general solution becomes

$$u(t,x) = \sum_{\ell=1}^{\infty} \sin(\ell\pi x)\, e^{-t/2}\left(\alpha_\ell \cos\left(\frac{\sqrt{4\ell^2\pi^2 - 1}}{2} t\right) + \beta_\ell \sin\left(\frac{\sqrt{4\ell^2\pi^2 - 1}}{2} t\right)\right)$$

with

$$u_t(t,x) = \sum_{\ell=1}^{\infty} \frac{-\alpha_\ell}{2} \sin\left(\ell\pi x\right) e^{-t/2} \left(\cos\left(\frac{\sqrt{4\ell^2\pi^2-1}}{2} t \right) + \sqrt{4\ell^2\pi^2-1} \sin\left(\frac{\sqrt{4\ell^2\pi^2-1}}{2} t \right) \right)$$

$$+ \frac{\beta_\ell}{2} \sin\left(\ell\pi x\right) e^{-t/2} \left(\sqrt{4\ell^2\pi^2-1} \cos\left(\frac{\sqrt{4\ell^2\pi^2-1}}{2} t \right) - \sin\left(\frac{\sqrt{4\ell^2\pi^2-1}}{2} t \right) \right)$$

Thus,

$$\begin{aligned}
u(0,x) &= \sum_{\ell=1}^{\infty} \alpha_\ell \sin\left(\ell\pi x\right) = \sin(2\pi x) \\
u_t(0,x) &= \sum_{\ell=1}^{\infty} \frac{-\alpha_\ell}{2} \sin\left(\ell\pi x\right) + \frac{\beta_\ell}{2} \sqrt{4\ell^2\pi^2-1} \sin\left(\ell\pi x\right) = \sin(\pi x) + 5\sin(3\pi x) \\
\alpha_2 &= \frac{\langle \sin\left(2\pi x\right), \sin(2\pi x) \rangle}{\langle \sin\left(2\pi x\right), \sin\left(2\pi x\right) \rangle} = 1 \\
\alpha_\ell &= 0, \text{ otherwise} \\
\beta_2 &= \frac{2}{\sqrt{16\pi^2-1}} \frac{\langle \sin\left(2\pi x\right), \frac{1}{2}\sin(2\pi x) + \sin(\pi x) + 5\sin(3\pi x) \rangle}{\langle \sin\left(2\pi x\right), \sin\left(2\pi x\right) \rangle} = \frac{1}{\sqrt{16\pi^2-1}} \\
\beta_\ell &= \frac{2}{\sqrt{4\ell^2\pi^2-1}} \frac{\langle \sin\left(\ell\pi x\right), \sin(\pi x) + 5\sin(3\pi x) \rangle}{\langle \sin\left(\ell\pi x\right), \sin\left(\ell\pi x\right) \rangle}, \text{ otherwise} \\
\beta_1 &= \frac{2}{\sqrt{4\pi^2-1}} \\
\beta_3 &= \frac{10}{\sqrt{36\pi^2-1}} \\
\beta_\ell &= 0, \text{ otherwise}
\end{aligned}$$

Our solution is now

$$\begin{aligned}
u(t,x) = \sin\left(\pi x\right) e^{-t/2} \frac{2}{\sqrt{4\pi^2-1}} \sin\left(\frac{\sqrt{4\pi^2-1}}{2} t \right) \\
+ \sin\left(2\pi x\right) e^{-t/2} \left(\cos\left(\frac{\sqrt{16\pi^2-1}}{2} t \right) + \frac{1}{\sqrt{16\pi^2-1}} \sin\left(\frac{\sqrt{16\pi^2-1}}{2} t \right) \right) \\
+ \sin\left(3\pi x\right) e^{-t/2} \frac{10}{\sqrt{36\pi^2-1}} \sin\left(\frac{\sqrt{36\pi^2-1}}{2} t \right)
\end{aligned}$$

which satisfies the initial conditions, the PDE, and boundary conditions.