Math 478 HW 3

Lucas Bouck

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1 Problem 2.7.1

Use the best approximation formula to find coefficients $\{a_k\}$ and $\{b_k\}$ for a Fourier series expansion for $f(x) = x - 1$ on the interval $(0, 1)$. That is,

$$f(x) \stackrel{L^2}{=} a_0 + \sum_{k=1}^{\infty} a_k \cos(2k\pi x) + b_k \sin(2k\pi x)$$

Answer:

The a_0 coefficient is the following:

$$a_0 = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^1 x - 1\, dx}{\int_0^1 1\, dx} = \frac{\frac{x^2}{2} - x\Big|_0^1}{1} = -\frac{1}{2}$$

The a_k coefficient for $k \in \mathbb{N}$ is the following:

$$\begin{aligned} a_k &= \frac{\langle f, \cos(2k\pi x)\rangle}{\langle \cos(2k\pi x), \cos(2k\pi x)\rangle} = \frac{\int_0^1 x\cos(2k\pi x) - \cos(2k\pi x)\, dx}{\int_0^1 \cos^2(2k\pi x)\, dx} \\ &= \frac{-\frac{\sin^2(k\pi)}{2k^2\pi^2}}{\frac{1}{8}\left(4 + \frac{\sin(4k\pi)}{k\pi}\right)} = 0 \end{aligned}$$

The b_k coefficient for $k \in \mathbb{N}$ is the following:

$$\begin{aligned} b_k &= \frac{\langle f, \sin(2k\pi x)\rangle}{\langle \sin(2k\pi x), \sin(2k\pi x)\rangle} = \frac{\int_0^1 x\sin(2k\pi x) - \sin(2k\pi x)\, dx}{\int_0^1 \sin^2(2k\pi x)\, dx} \\ &= \frac{\frac{-2k\pi + \sin(2k\pi)}{4k^2\pi^2}}{\frac{1}{2} - \frac{\sin(4k\pi)}{8k\pi}} = \frac{-4k\pi}{4k^2\pi^2} = -\frac{1}{k\pi} \end{aligned}$$

2 Problem 2.7.2

2.1 Part a)

Answer:

First we must find γ_k. If $k = 0$, then

$$\gamma_0 = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^1 3x^2 - 2x^3\, dx}{\int_0^1 1\, dx} = x^3 - \frac{x^4}{2}\Big|_0^1 = \frac{1}{2}$$

If $k \in \mathbb{N}$, then

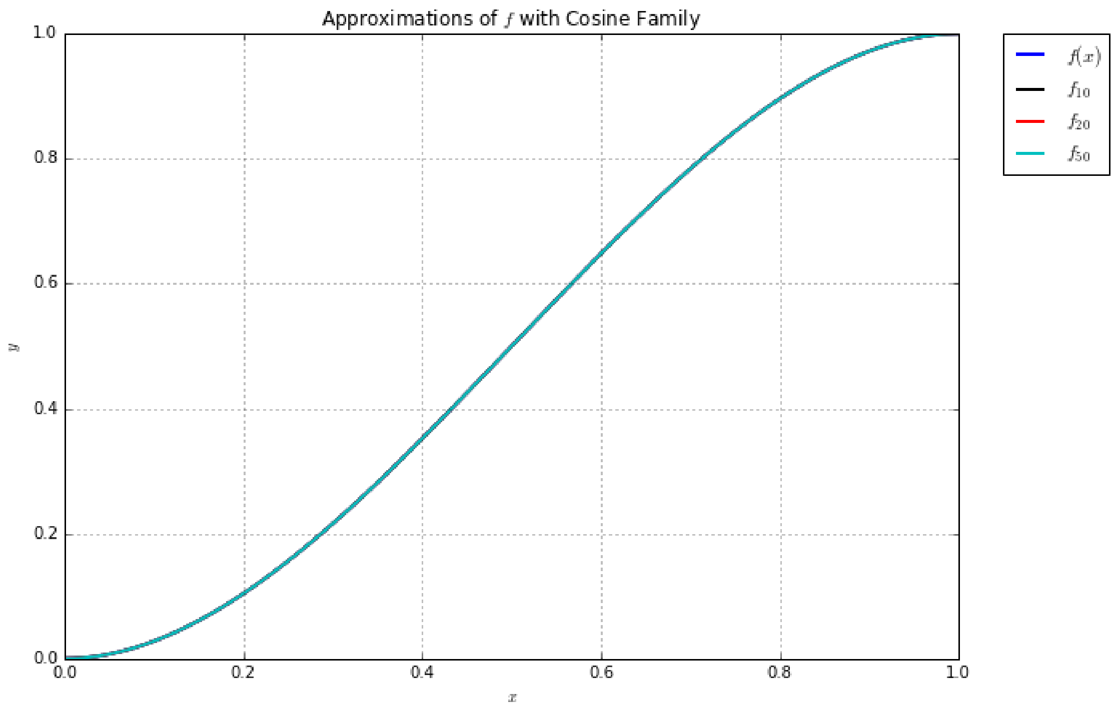
$$\gamma_k = \frac{\langle 3x^2 - 2x^3, \cos(k\pi x)\rangle}{\langle \cos(k\pi x), \cos(k\pi x)\rangle} = \frac{\int_0^1 \left(3x^2 - 2x^3\right)\cos(k\pi x)\,dx}{\int_0^1 \cos^2(k\pi x)\,dx}$$
$$= \frac{\frac{-12+12\cos(k\pi)+k\pi(6+k^2\pi^2)\sin(k\pi)}{k^4\pi^4}}{\frac{1}{4}\left(2 + \frac{\sin(2k\pi)}{k\pi}\right)} = \frac{-24+24\cos(k\pi)}{k^4\pi^4}$$
$$\gamma_k = \frac{-48}{k^4\pi^4} \text{ if } k \text{ is odd}$$
$$\gamma_k = 0 \text{ if } k \text{ is even}$$

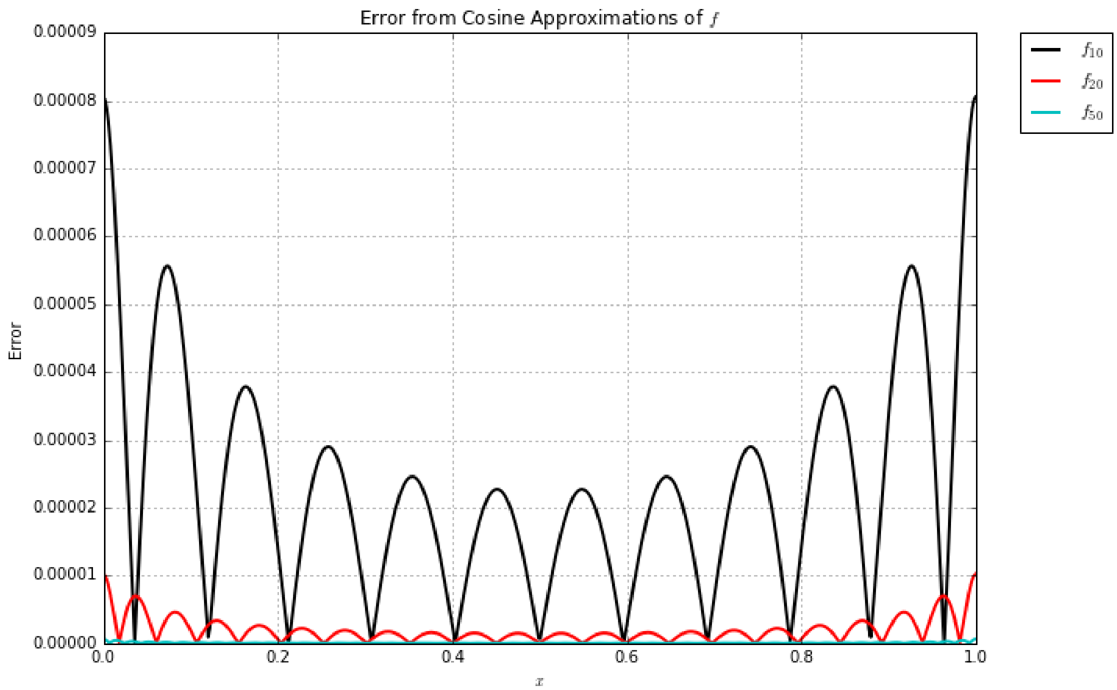
The next few bits below are the numerics for the problem, which involves graphing the partial sum approximations $f_N = \frac{1}{2} + \sum_{k=1}^{N} \gamma_k \cos(k\pi x)$ for $N = 10, 20, 50$ and graphing $|f_N - f|$ as a function of x. Please note that the plot of the Cosine approximations are all on top of the f we are trying to approximate, so it looks like there is only one function plotted.

In [39]:
```
import numpy as np
from matplotlib import rcParams as rcParams
from matplotlib import pyplot as plt
rcParams.update({'font.size': 48})
%matplotlib inline
xvals=np.linspace(0,1,500)
def truef(x):
    return 3*(x**2)-2*(x**3)
def cos_approx(x,N):
    answer=x-x
    for k in range(0,N+1):
        if k==0:
            answer=answer+.5
        elif k%2==1:
            answer=answer-48*np.cos(k*np.pi*x)/((k**4)*(np.pi**4))
    return answer
plt.figure(figsize=(10,7))
plt.plot(xvals,truef(xvals),linewidth=2.0,label='$f(x)$')
plt.plot(xvals,cos_approx(xvals,10),'k',linewidth=2.0,label='$f_{10}$')
plt.plot(xvals,cos_approx(xvals,20),'r',linewidth=2.0,label='$f_{20}$')
plt.plot(xvals,cos_approx(xvals,50),'c',linewidth=2.0,label='$f_{50}$')
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.title('Approximations of $f$ with Cosine Family')
plt.grid()
plt.legend(bbox_to_anchor=(1.05, 1), loc=2, borderaxespad=0.)
plt.show()

plt.figure(figsize=(10,7))
plt.plot(xvals,np.abs(truef(xvals)-cos_approx(xvals,10)),'k',
         linewidth=2.0,label='$f_{10}$')
plt.plot(xvals,np.abs(truef(xvals)-cos_approx(xvals,20)),'r',
```

```
         linewidth=2.0,label='$f_{20}$')
plt.plot(xvals,np.abs(truef(xvals)-cos_approx(xvals,50)),'c',
         linewidth=2.0,label='$f_{50}$')
plt.xlabel('$x$')
plt.ylabel('Error')
plt.title('Error from Cosine Approximations of $f$')
plt.grid()
plt.legend(bbox_to_anchor=(1.05, 1), loc=2, borderaxespad=0.)
plt.show()
```





The series does seem to converge uniformly. Below is a proof that f_N converges uniformly on the interval $[0, 1]$.

Proof:

Denote $f_N = \frac{1}{2} + \sum_{k=1}^{N} \gamma_k \cos(k\pi x)$. Since $|\cos(k\pi x)| \leq 1$, $|\gamma_k \cos(k\pi x)| \leq |\gamma_k|$ for all $k \in \mathbb{N}$. Also we know that $\gamma_k = \frac{-48}{k^4\pi^4}$ if k is odd, and $\gamma_k = 0$ if k is even. Thus, $|\gamma_k \cos(k\pi x)| \leq |\gamma_k| \leq \frac{50}{k^4}$ for all $k \in \mathbb{N}$.

Let $M_k = \frac{50}{k^4}$. The infinite sum $\sum_{k=1}^{\infty} M_k = 50 \sum_{k=1}^{\infty} k^{-4}$ converges, and $M_k = \frac{50}{k^4} \geq |\gamma_k \cos(k\pi x)|$ for all $k \in \mathbb{N}$. By the Weierstraß M-test, the series $\sum_{k=1}^{N} \gamma_k \cos(k\pi x)$ converges uniformly, which implies $f_N = \frac{1}{2} + \sum_{k=1}^{N} \gamma_k \cos(k\pi x)$ converges uniformly.

2.2 Part b)

We now want to do the same problem but with the sine family instead of the cosine family. First we must find γ_k.

$$\gamma_k = \frac{\langle 3x^2 - 2x^3, \sin(k\pi x)\rangle}{\langle \sin(k\pi x), \sin(k\pi x)\rangle} = \frac{\int_0^1 (3x^2 - 2x^3)\sin(k\pi x)\,dx}{\int_0^1 \sin^2(k\pi x)\,dx}$$

$$= \frac{\frac{-6k\pi - k\pi(6+k^2\pi^2)\cos(k\pi) + 12\sin(k\pi)}{k^4\pi^4}}{\frac{1}{2} - \frac{\sin(2k\pi)}{4k\pi}} = \frac{-12k\pi - 2k\pi(6 + k^2\pi^2)\cos(k\pi)}{k^4\pi^4} = \frac{-12k\pi - (12k\pi + 2k^3\pi^3)(-1)^k}{k^4\pi^4}$$

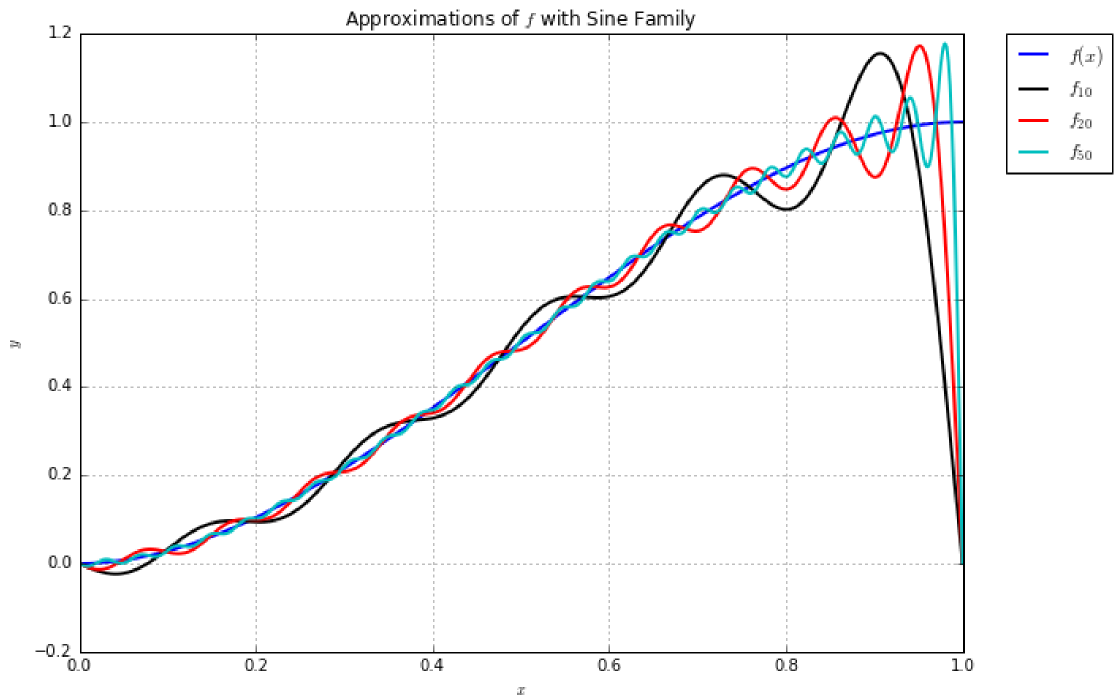
$$\gamma_k = \frac{2}{k\pi} \text{ if } k \text{ is odd}$$

$$\gamma_k = \frac{-24 - 2k^2\pi^2}{k^3\pi^3} \text{ if } k \text{ is even}$$

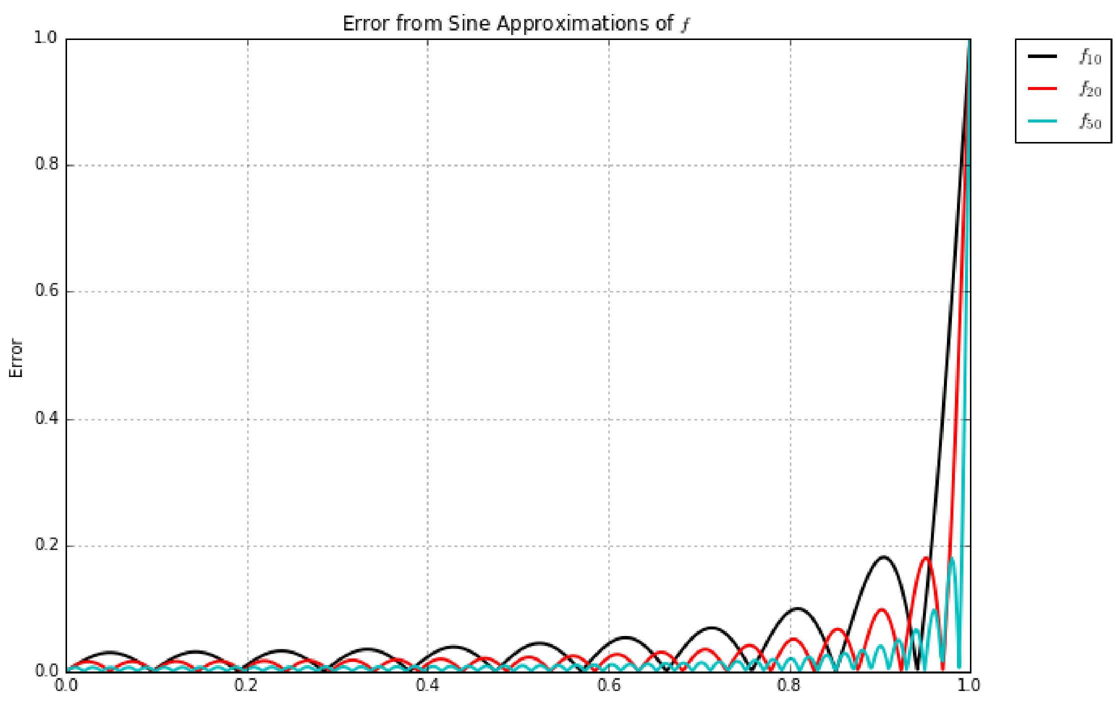
The next few bits below are the numerics for the problem, which involves graphing the partial sum approximations $f_N = \sum_{k=1}^{N} \gamma_k \sin(k\pi x)$ for $N = 10, 20, 50$ and graphing $|f_N - f|$ as a function of x.

In [40]:
```
import numpy as np
from matplotlib import pyplot as plt
%matplotlib inline
xvals=np.linspace(0,1,500)
def truef(x):
    return 3*(x**2)-2*(x**3)
def sin_approx(x,N):
    answer=x-x
    for k in range(1,N+1):
        if k%2==1:
            answer=answer+2*np.sin(k*np.pi*x)/(k*np.pi)
        elif k%2==0:
            answer=answer-(24+2*(k**2)*(np.pi**2))*np.sin(k*np.pi*x)/\
            ((k**3)*(np.pi**3))
    return answer
plt.figure(figsize=(10,7))
plt.plot(xvals,truef(xvals),linewidth=2.0,label='$f(x)$')
plt.plot(xvals,sin_approx(xvals,10),'k',linewidth=2.0,label='$f_{10}$')
plt.plot(xvals,sin_approx(xvals,20),'r',linewidth=2.0,label='$f_{20}$')
plt.plot(xvals,sin_approx(xvals,50),'c',linewidth=2.0,label='$f_{50}$')
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.title('Approximations of $f$ with Sine Family')
plt.grid()
plt.legend(bbox_to_anchor=(1.05, 1), loc=2, borderaxespad=0.)
plt.show()

plt.figure(figsize=(10,7))
plt.plot(xvals,np.abs(truef(xvals)-sin_approx(xvals,10)),'k',
         linewidth=2.0,label='$f_{10}$')
plt.plot(xvals,np.abs(truef(xvals)-sin_approx(xvals,20)),'r',
         linewidth=2.0,label='$f_{20}$')
plt.plot(xvals,np.abs(truef(xvals)-sin_approx(xvals,50)),'c',
         linewidth=2.0,label='$f_{50}$')
plt.xlabel('$x$')
plt.ylabel('Error')
plt.title('Error from Cosine Approximations of $f$')
plt.grid()
plt.legend(bbox_to_anchor=(1.05, 1), loc=2, borderaxespad=0.)
plt.show()
```



Approximations of f with Sine Family
$f(x)$
f_{10}
f_{20}
f_{50}
y
x



Error from Sine Approximations of f
f_{10}
f_{20}
f_{50}
Error
x

It is clear from the plots that the Sine Family Approximation will not converge uniformly to the function f on the interval . This is due to $\sin(k\pi 1) = 0$, while $f(1) = 1$. Thus, $|f_N(1) - f(1)| = 1$ for all N and f_N doesn't converge uniformly on the interval.

3 Problem 2.7.3

Consider the BVP on $[0, 1]$ given by $u'' + \lambda u = 0$ with $u(0) = u'(0)$ and $u(1) = u'(1)$. For which values of λ can find nontrivial solutions? What solutions do you obtain?

Answer:

For the first case, let's consider $\lambda = 0$. Then, we have $u'' = 0$, which means the general solution will be $u = c_1 x + c_2$ and $u' = c_1$. By bringing in the boundary conditions, we get

$$\begin{aligned} u(0) &= u'(0) & u(1) &= u'(1) \\ c_2 &= c_1 & c_1 + c_2 &= c_1 \\ c_2 &= c_1 & c_2 &= 0 \end{aligned}$$

So $c_1 = c_2 = 0$ and we get the trivial solution.

For the next possible case, let's consider $\lambda = -k^2 < 0$ for some $k \in \mathbb{R}$. Then the ODE is $u'' - k^2 u = 0$ and by solving the characteristic equation, we get eigenvalues $r = \pm k$. Hence the general solution is $u = c_1 e^{kx} + c_2 e^{-kx}$ and $u' = c_1 k e^{kx} - c_2 k e^{-kx}$. From the boundary conditions, we get

$$\begin{aligned} u(0) &= u'(0) & u(1) &= u'(1) \\ c_1 + c_2 &= c_1 k - c_2 k & c_1 e^k + c_2 e^{-k} &= c_1 k e^k - c_2 k e^{-k} \\ (1-k)c_1 + (1+k)c_2 &= 0 & (1-k)e^k c_1 + (1+k)e^{-k} c_2 &= 0 \end{aligned}$$

In matrix form, this is

$$\begin{pmatrix} 1-k & 1+k \\ (1-k)e^k & (1+k)e^{-k} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A\boldsymbol{c} = \mathbf{0}$$

To get a nontrivial solution, we want $\det(A) = (1-k)(1+k)e^{-k} - (1+k)(1-k)e^k = 0$, so

$$\begin{aligned} (1-k)(1+k)e^{-k} - (1+k)(1-k)e^k = (1-k^2)(e^{-k} - e^k) &= (1+\lambda)(e^{-k} - e^k) = 0 \\ (1+\lambda) &= 0 \\ \lambda &= -1 \end{aligned}$$

For $\lambda = 1$, $k = 1$ or $k = -1$. The answer will be the same no matter which k we choose, so choose $k = -1$. Then,

$$A\boldsymbol{c} = \begin{pmatrix} 2 & 0 \\ 2e^k & 0 \end{pmatrix} \boldsymbol{c} = 0$$

Then, $c_1 = 0$ and c_2 is free. Thus, the solution will be $u(x) = 0e^{-1x} + c_2 e^x = \gamma e^x$ for some $\gamma \in \mathbb{R}$.

For the final case that we need to cover, let $\lambda = k^2 > 0$. Then, we have the ODE $u'' + k^2 = 0$. Solving the characteristic equation, we have eigenvalues $r = \pm ki$ and we get the general solution $u =$

$c_1 \cos(kx) + c_2 \sin(kx)$ and $u' = -c_1 k \sin(kx) + c_2 k \sin(kx)$. Considering the boundary conditions, we have

$$\begin{aligned} u(0) &= u'(0) & u(1) &= u'(1) \\ c_1 &= c_2 k & c_1 \cos(k) + c_2 \sin(k) &= -c_1 k \sin(k) + c_2 k \cos(k) \\ c_1 - kc_2 &= 0 & c_1(\cos(k) + k\sin(k)) + c_2(\sin(k) - k\cos(k)) &= 0 \end{aligned}$$

In matrix form, this is

$$\begin{pmatrix} 1 & -k \\ \cos(k) + k\sin(k) & \sin(k) - k\cos(k) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A\boldsymbol{c} = \mathbf{0}.$$

For the solution to be non-trivial, we want $\det(A) = \sin(k) - k\cos(k) + k(\cos(k) + k\sin(k)) = 0$. By simplifying, we get $\sin(k)(1 + k^2) = 0$. Since $1 + k^2 \neq 0$, we need $\sin(k) = 0$, so $k = \ell\pi$ for some $\ell \in \mathbb{N}$. We'll exclude $k = 0$ because we assumed $k^2 > 0$. Thus, $\lambda = \ell^2\pi^2$. Back in matrix form, we get

$$\begin{pmatrix} 1 & -\ell\pi \\ (-1)^\ell & -\ell\pi(-1)^\ell \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{0}$$

Thus, $c_1 = \ell\pi c_2$ so the general solution looks like $u(x) = \alpha(\ell\pi\cos(kx) + \sin(kx))$ for some $\alpha \in \mathbb{R}$.

To answer the original question, $u'' + \lambda u = 0$ has nontrivial solutions when $\lambda = -1$ and $\lambda = \ell^2\pi^2$ for any $\ell \in \mathbb{N}$. The corresponding solutions are then

$$\begin{aligned} &u(x) = \gamma e^x, \gamma \in \mathbb{R} \text{ when } \lambda = -1 \\ &u(x) = \alpha(\ell\pi\cos(\ell\pi x) + \sin(\ell\pi x)), \alpha \in \mathbb{R} \text{ when } \lambda = \ell^2\pi^2 \text{ and } \ell \in \mathbb{N} \end{aligned}$$

4 Problem 2.7.5

For the domain $I = [0, L]$, find all eigenvalues and eigenfunctions of the Sturm-Liouville problem $u'' + \lambda u = 0$ with $u(0) = u'(L) = 0$.

Answer:

Let's consider the different cases. First let $\lambda = 0$. Then the general solution to $u'' = 0$ is $u = c_1 x + c_2$ and $u' = c_1$. Since $u(0) = c_2 = u'(L) = c_1 = 0$, u is the trivial solution.

In the next case, let $\lambda = -k^2 < 0$. Then from the characteristic polynomial of $u'' - k^2 u = 0$, we get eigenvalues $r = \pm k$ and get a the general solution $u = c_1 e^{-kx} + c_2 e^{kx}$ with $u' = -c_1 k e^{-kx} + c_2 k e^{kx}$. With the boundary conditions, we get

$$\begin{aligned} u(0) = c_1 + c_2 &= 0 & u'(L) = -c_1 k e^{-Lk} + c_2 k e^{kL} &= 0 \\ -c_1 &= c_2 & c_2 k e^{-Lk} + c_2 k e^{kL} &= 0 \\ c_2 k(e^{-kL} + e^{kL}) &= 0 & & \\ c_2 &= 0 & c_1 &= 0 \end{aligned}$$

Therefore, u is the trivial solution.

The last case is $\lambda = k^2 > 0$. The eigenvalues from the characteristic equation are $r = \pm k$, and the solution is $u = c_1 \cos(kx) + c_2 \sin(kx)$ with $u' = -c_1 k \sin(kx) + c_2 k \cos(kx)$. With the boundary conditions, we get

$$\begin{aligned}
u(0) = c_1 &= 0 & u'(L) = c_2 k \cos(kL) = 0 \\
kL &= \frac{(2\ell - 1)\pi}{2}, \ell \in \mathbb{N} \\
k &= \frac{(2\ell - 1)\pi}{2L}, \ell \in \mathbb{N} \\
\lambda &= \frac{(2\ell - 1)^2\pi^2}{4L^2}, \ell \in \mathbb{N}
\end{aligned}$$

Given $\lambda = \frac{(2\ell-1)^2\pi^2}{4L^2}$, the solution to $u'' + \lambda u = 0$ is $u(x) = \sin(\frac{(2\ell-1)\pi}{2L}x)$.

The eigenvalues of the Sturm-Liouville problem are $\lambda_\ell = \frac{(2\ell-1)^2\pi^2}{4L^2}, \ell \in \mathbb{N}$ with eigenfunctions $\varphi_\ell(x) = \sin(\frac{(2\ell-1)\pi}{2L}x)$. To prove that $\{\varphi_\ell\}$ is a complete set, we can use Corollary 2.38. We only need to show that the ℓth member $\varphi_\ell(x)$ has $\ell - 1$ roots in the open interval $(0, L)$. Let $\ell \in \mathbb{N}$. Then $\varphi_\ell(x) = \sin(\frac{(2\ell-1)\pi}{2L}x)$. The roots of φ_ℓ occur when $\frac{(2\ell-1)\pi}{2L}x = n\pi$ for $n \in \mathbb{N}$, so the roots are $x = \frac{2Ln}{(2\ell-1)}$ for $n \in \mathbb{N}$. Since $2\ell - 2 < 2\ell - 1$, then

$$\begin{aligned}
L(2\ell - 2) &< L(2\ell - 1) \\
\frac{2L(\ell - 1)}{2\ell - 1} &< L \\
0 < \frac{2L(1)}{2\ell - 1} < \frac{2L(2)}{2\ell - 1} < ... < \frac{2L(\ell - 1)}{2\ell - 1} &< L
\end{aligned}$$

The fact $\frac{2L\ell}{2\ell-1} > L$ follows from the fact that $2\ell > 2\ell - 1$. Therefore, the only roots of $\varphi_\ell(x)$ in $(0, L)$ are $x = \frac{2Ln}{2\ell-1}$ where $n = 1, 2, ..., \ell - 1$. This means $\varphi_\ell(x)$ has $\ell - 1$ roots in $(0, L)$, and $\{\varphi_\ell\}$ is a complete set.