Written Assignment 10

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1 Problem 1a

Let the interval $[a, b]$ be given and let $c \in (a, b)$. Define f on $[a, b]$ by

$$f(x) = \begin{cases} 1 & \text{if } x = c \\ 0 & \text{if } x \neq c \end{cases}$$

Prove that f is Darboux integrable on $[a, b]$.

Proof:

Let $\epsilon > 0$. Let $P = \{x_0, ..., x_n\}$ be a partition of $[a, b]$ such that $||P|| < \frac{\epsilon}{2}$. Then without loss of generality there exists a $k \in \{1, ..., n\}$ such that $c \in [x_{k-1}, x_k]$. Then

$$U(f,P) - L(f,P) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = \sum_{i=1}^{n}(M_i)(x_i - x_{i-1})$$

There will be three cases.
Case 1: Suppose $c \in (x_{k-1}, x_k)$. Then $M_k = 1$, and

$$\sum_{i=1}^{n}(M_i)(x_i - x_{i-1}) = (x_k - x_{k-1}) < \frac{\epsilon}{2} < \epsilon$$

Case 2: Suppose $k \in \{2, ..., n\}$ and $c = x_{k-1}$. Then $M_k = M_{k-1} = 1$ and

$$\sum_{i=1}^{n}(M_i)(x_i - x_{i-1}) = (x_{k-1} - x_{k-2}) + (x_k - x_{k-1}) < \epsilon$$

Case 3: Suppose $k \in \{1, ..., n-1\}$ and $c = x_k$ Then $M_k = M_{k+1} = 1$ and

$$\sum_{i=1}^{n}(M_i)(x_i - x_{i-1}) = (x_{k+1} - x_k) + (x_k - x_{k-1}) < \epsilon$$

In all three cases, $U(f,P) - L(f,P) < \epsilon$, which is what we want.

2 Problem 1b

Prove that $\int_a^b f = 0$.

Proof:

We want to show that $\int_a^b f = 0$. Let $P = \{x_0, ..., x_n\}$ be a partition of $[a, b]$. Then

$$L(f, P) = \sum_{i=1}^{n} \inf\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) = \sum_{i=1}^{n} 0(x_i - x_{i-1}) = 0$$

Thus, for any partition, P, of $[a, b]$, $L(f, P) = 0$. Also for any partition, P, of $[a, b]$, $L(f, P) \leq 0$, so 0 is an upper bound of $S = \{L(f, P) : P \text{ is a partition of } [a, b]\}$. Suppose s is an upper bound of S. Then for any partition P, $s \geq L(f, P) = 0$, so $s \geq 0$. Therefore $\sup\{L(f, P) : P \text{ is a partition of } [a, b]\} = 0$, and since f is integrable, $\int_a^b f = 0$.

3 Problem 2a

Define f on $[-1, 1]$ by

$$f(x) = \begin{cases} x^2 \sin(1/x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that f' exists at every point of $[-1, 1]$.

Proof:

Let $x \in [-1, 1]$ there will be two cases.

Case 1: Suppose $x \neq 0$. Since t^2, $\sin(t)$, and $1/t^2$ are all differentiable when $x \neq 0$, then by the product and chain rule, $f'(x) = 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right)$.

Case 2: Suppose $x = 0$. I claim that $f'(x) = 0$. We must now show that $\lim_{h\to 0} \frac{f(0+h)-f(0)+0h}{h} = 0$. To simplify things, we have $\lim_{h\to 0} \frac{f(0+h)-f(0)+0h}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h^2}\right)$. Since $-1 \leq \sin\left(\frac{1}{h^2}\right) \leq 1$, for all $h \neq 0$, then $-|h| \leq h \sin\left(\frac{1}{h^2}\right) \leq |h|$. Let $\epsilon > 0$. Choose $\delta > 0$ such that $\delta < \epsilon$. If $0 < |h| < \delta$, then $||h| - 0| \leq |h| < \epsilon$ and $|-|h| - 0| \leq |h| < \epsilon$. Therefore, $\lim_{h\to 0} -|h| = \lim_{h\to 0} |h| = 0$. Since $-|h| \leq h \sin\left(\frac{1}{h^2}\right) \leq |h|$ for all $h \neq 0$, by the squeeze theorem, $\lim_{h\to 0} h \sin\left(\frac{1}{h^2}\right) = 0$. Therefore, $f'(0)$ exists and is equal to 0.

In both cases, f' exists, which means f' exists at every point of $[-1, 1]$.

4 Problem 2b

Prove that f' is not bounded, and thus not continuous on $[-1, 1]$.

Proof:

Let $M > 0$. Choose $n \in \mathbb{N}$ such that $n > \frac{M}{2\sqrt{\pi}}$. Let $x = \frac{1}{n\sqrt{\pi}}$. Since n is a natural number, $0 < x < 1$. Then, $|f'(x)| = |\frac{2}{n\sqrt{\pi}} \sin\left(n^2\pi\right) - 2n\sqrt{\pi}\cos\left(n^2\pi\right)|$. Then, $|f'(x)| = |2n\sqrt{\pi}|$. Since $n > \frac{M}{2\sqrt{\pi}}$, we know that $2n\sqrt{\pi} > M$ and $|f'(x)| = |2n\sqrt{\pi}| > M$. Therefore, for all $M > 0$, there exists an $x \in [-1, 1]$ such that $|f'(x)| \geq M$, and f' is not bounded on $[-1, 1]$.

5 Problem 3a

Define f on $[-1, 1]$ by

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that f' exists at every point of $[-1, 1]$.

Proof:

Let $x \in [-1, 1]$ there will be two cases.

Case 1: Suppose $x \neq 0$. Since t^2, $\sin(t)$, and $1/t$ are all differentiable when $x \neq 0$, then by the product and chain rule, $f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$.

Case 2: Suppose $x = 0$. I claim that $f'(x) = 0$. We must now show that $\lim_{h\to 0} \frac{f(0+h)-f(0)+0h}{h} = 0$. To simplify things, we have $\lim_{h\to 0} \frac{f(0+h)-f(0)+0h}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right)$. Since $-1 \leq \sin\left(\frac{1}{h}\right) \leq 1$, for all $h \neq 0$, then $-|h| \leq h\sin\left(\frac{1}{h}\right) \leq |h|$. Let $\epsilon > 0$. Choose $\delta > 0$ such that $\delta < \epsilon$. If $0 < |h| < \delta$, then $||h| - 0| \leq |h| < \epsilon$ and $|-|h| - 0| \leq |h| < \epsilon$. Therefore, $\lim_{h\to 0} -|h| = \lim_{h\to 0} |h| = 0$. Since $-|h| \leq h\sin\left(\frac{1}{h}\right) \leq |h|$ for all $h \neq 0$, by the squeeze theorem, $\lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$. Therefore, $f'(0)$ exists and is equal to 0.

In both cases, f' exists, which means f' exists at every point of $[-1, 1]$.

6 Problem 3b

Prove that f' is bounded on $[-1, 1]$.

Proof:

Choose $M = 3$. Let $x \in [-1, 1]$. If $x = 0$, then $|f'(x)| = 0 < 3$. If $x \neq 0$, then $f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$. Since $|\sin\left(\frac{1}{x}\right)| \leq 1$ and $|\cos\left(\frac{1}{x}\right)| \leq 1$ for all $x \neq 0$, then $-2x - 1 \leq 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \leq 2x + 1$. Since $x \in [-1, 1]$, $-2x - 1 \geq -3$ and $2x + 1 \leq 3$. Therefore, $|f'(x)| = |2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)| \leq |2x + 1| \leq 3$ for all $x \in [-1, 1]$, and f' is bounded on $[-1, 1]$.

7 Problem 3c

Prove that f' is not continuous on $x = 0$.

Proof:

We need to show that f' is not continuous at $x = 0$, which means we need to construct a sequence x_n such that $x_n \to 0$ but $f'(x_n)$ does not converge to $f'(0) = 0$. We shall now construct x_n. For all $n \in \mathbb{N}$, choose $x_n = \frac{1}{2n\pi}$. We must first show that $x_n \to 0$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{2\pi\epsilon}$. If $n \geq N$, then $n > \frac{1}{2\pi\epsilon}$. If $n \geq N$, then $x_n = \frac{1}{2n\pi} < \epsilon$. Therefore, $x_n \to 0$. We must now show that $f'(x_n)$ does not converge to 0. For all $n \in \mathbb{N}$, $f'(x_n) = \frac{2}{2n\pi} \sin(2n\pi) - \cos(2n\pi) = 1$. Since $f'(x_n) = 1$ for all n, $f'(x_n) \to 1$, which means $f'(x_n)$ does not converge to 0. We have found a $x_n \in [-1, 1]$ such that $x_n \to 0$ but $f'(x_n)$ does not converge to $f'(0)$. Therefore, $f'(x)$ is not continuous at $x = 0$.