Written Assignment 7

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1 Problem 1a

Prove that if f is Lipschitz continuous on an interval I, then it is uniformly continuous on I.

Proof:

Let f be Lipschitz continuous on an interval I. We want to show it is uniformly continuous on I. Let $\epsilon > 0$. Because f is Lipschitz continuous on I, there exists an M such that for all $x, y \in I$, $|f(x) - f(y)| \leq M|x - y|$. Choose M such that $M > 0$ and such that for all $x, y \in I$, $|f(x) - f(y)| \leq M|x - y|$. Choose $\delta > 0$ such that $\delta = \frac{\epsilon}{M}$. Let $x, y \in I$. If $|x - y| < \delta$, then $|f(x) - f(y)| \leq M|x - y| < M\delta = M\frac{\epsilon}{M} = \epsilon$. Thus, $|f(x) - f(y)| < \epsilon$, and f is uniformly continuous on I.

2 Problem 1b

Prove that the function $f(x) = \sqrt{x}$ is continuous on $[0, 1]$. Explain why $f(x) = \sqrt{x}$ is also uniformly continuous on $[0, 1]$.

Proof:

Let $f(x) = \sqrt{x}$ be a function with the domain $[0, \infty)$ and the real numbers as the codomain. We want to show that f is continuous on $[0, 1]$. Let $\epsilon > 0$ and let $a \in [0, 1]$. Choose $\delta > 0$ such that $\delta < \epsilon$. For all $x \in [0, 1]$ if $|x - a| < \delta$, then $|x - a| < \epsilon$. Then $|\sqrt{x} - \sqrt{a}||\sqrt{x} + \sqrt{a}| < \epsilon$, which means $|\sqrt{x} - \sqrt{a}| = |f(x) - f(a)| < \epsilon$. Therefore, f is continuous on $[0, 1]$.

The reason f is uniformly continuous on $[0, 1]$ is that when we choose δ, our δ is only a function of ϵ, which is the intuitive understanding of uniformly continuity.

3 Problem 1c

Prove that $f(x) = \sqrt{x}$ is *not* Lipschitz continuous on $[0, 1]$.

Proof:

Let $f(x) = \sqrt{x}$ be a function with the domain $[0, \infty)$ and the real numbers as the codomain. We want to show that f is not Lipschitz continuous on $[0, 1]$, which means for all M there exist $x, y \in [0, 1]$ such that $|f(x) - f(y)| > M|x - y|$. There will be two cases. Let $M \leq 0$. Choose $x = 1$ and $y = 0$. Then, $|\sqrt{1} - \sqrt{0}| > 0 \geq M|1 - 0|$. For the next case let $M > 0$. Choose $y = 0$ and choose $x = \frac{1}{n}$ where $n \in \mathbb{N}$ and $n > M^2$. Then,

$$\begin{aligned}
n^2 &> M^2 n \\
\frac{1}{n} &> \frac{M^2}{n^2} \\
x &> M^2 x^2 = (Mx)^2 \\
\sqrt{x} &> |Mx| = M|x| \\
|\sqrt{x} - 0| &> M|x - 0| \\
|\sqrt{x} - \sqrt{y}| &> M|x - y|
\end{aligned}$$

Since $n \in \mathbb{N}$ and $x = \frac{1}{n}$, $x \in [0, 1]$. Then there exist $x, y \in [0, 1]$ such that $|f(x) - f(y)| > M|x - y|$. Therefore, f is not Lipschitz continuous on $[0, 1]$.

4 Problem 2a

Suppose f is not bounded on $[a, b]$ and show that there is a sequence $x_n \in [a, b]$ such that for all $n \in \mathbb{N}, |f(x_n)| > n$.

Proof:

Let f be a function and suppose that f is not bounded on $[a, b]$. This means for all $M > 0$ there exists a $x \in [a, b]$ such that $|f(x)| > M$. We want to show that there is a sequence $x_n \in [a, b]$ such that for all $n \in \mathbb{N}, |f(x_n)| > n$. We shall now construct x_n. Choose $x_1 \in [a, b]$ such that $|f(x_1)| > 1$. We can do this because f is not bounded on $[a, b]$. If $n > 1$, choose $x_n \in [a, b]$ such that $|f(x_n)| > |f(x_{n-1})| + 1$.

We must now show that for all $n \in \mathbb{N}, |f(x_n)| > n$. We will do this by induction on n. Let $n = 1$. Based on how the sequence is defined, $|f(x_1)| > 1$. Now let $n \in \mathbb{N}$ and suppose the result is true for n. Then $|f(x_{n+1})| > |f(x_n)| + 1 > n + 1$, so the result holds for $n + 1$. By the principle of mathematical induction, $|f(x_n)| > n$ for all $n \in \mathbb{N}$. Therefore, there exists a sequence $x_n \in [a, b]$ such that for all $n \in \mathbb{N}, |f(x_n)| > n$.

5 Problem 2b

Use part a and the Bolzano- Weierstrass Theorem to show there exists a $x_0 \in [a, b]$ such that f is not continuous at x_0.

Proof:

Let f be a function and suppose that f is not bounded on $[a, b]$. By part a, there is a sequence $x_n \in [a, b]$ such that for all $n \in \mathbb{N}, |f(x_n)| > n$. We will use such sequence and refer to it as x_n. Since $x_n \in [a, b]$, by Bolzano- Weierstrass Theorem, x_n has a convergent subsequence, which will be called x_{n_k}. Since $[a, b]$ is a closed interval, x_{n_k} converges to a real number that is in $[a, b]$. Let $x_0 \in [a, b]$ be a real number such that $x_{n_k} \to x_0$. We shall now show that $f(x_{n_k})$ does not converge to $f(x_0)$.

Let $\epsilon = 1$. We must show for all $K \in \mathbb{N}$ that there exists a $k \geq K$ such that $|f(x_{n_k}) - f(x_0)| \geq \epsilon$. Let m be a natural number such that $|f(x_0)| < m$. There will be two cases for K. First let K be a natural number such that $n_K < m + 1$. Then since n_k is an increasing sequence of natural numbers, there is a $k \geq K$ such that $n_k > m + 1$. Choose such n_k. Then, $|f(x_{n_k})| > n_k > m+1 > |f(x_0)|+1$. Then also $|f(x_{n_k})|-|f(x_0)| > 1$. By the triangle inequality, $|f(x_{n_k})-f(x_0)| \geq ||f(x_{n_k})|-f(x_0)|| > |f(x_{n_k})|-|f(x_0)| > 1$. Thus, there exists a $k \geq K$ such that $|f(x_{n_k}) - f(x_0)| \geq \epsilon$. For the second case, let K be a natural number such that $n_K \geq m + 1$. Since $n_K \geq m + 1$, $|f(x_{n_K})| > n_K > m + 1 > |f(x_0)| + 1$. Then, $|f(x_{n_K})| - |f(x_0)| > 1$. By the triangle inequality, $|f(x_{n_K}) - f(x_0)| \geq ||f(x_{n_K})| - f(x_0)|| > |f(x_{n_K})| - |f(x_0)| > 1$. Thus, there exists a $k \geq K$ such that $|f(x_{n_k}) - f(x_0)| \geq \epsilon$. In both cases, there exists a $k \geq K$ such that $|f(x_{n_k}) - f(x_0)| \geq \epsilon$.

We have shown that there exists a sequence $x_{n_k} \in [a, b]$, such that $x_{n_k} \to x_0$ but $f(x_{n_k})$ does not converge to $f(x_0)$. By the sequential characterization of limits, f is not continuous at x_0. Therefore, there exists a $x_0 \in [a, b]$ such that f is not continuous at x_0.