Assignment 4
Math 290, Dr. Walnut

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1 Problem 1

Prove that for all sets A and B, $A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$.
Proof:

($\Rightarrow$) Let A and B be sets and let $A = B$. This means $A \subseteq B$ and $B \subseteq A$. We want to show that $\mathcal{P}(A) = \mathcal{P}(B)$.

Let X be a set and let $X \in \mathcal{P}(A)$. By the definition of a power set, $X \subseteq A$. This means if k is an element of X, then it is an element of A. Since $A \subseteq B$, then k is an element of B. Thus, $X \subseteq B$. Then, $X \in \mathcal{P}(B)$. Therefore, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Let X be a set and let $X \in \mathcal{P}(B)$. By the definition of a power set, $X \subseteq B$. This means if k is an element of X, then it is an element of B. Since $B \subseteq A$, then k is an element of A. Thus, $X \subseteq A$. Then, $X \in \mathcal{P}(A)$. Therefore, $\mathcal{P}(B) \subseteq \mathcal{P}(A)$. Therefore, $\mathcal{P}(A) = \mathcal{P}(B)$.

($\Leftarrow$) Let A and B be sets and let $\mathcal{P}(A)$ and $\mathcal{P}(B)$ be the power sets of A and B. Assume $\mathcal{P}(A) = \mathcal{P}(B)$. We want to show that $A = B$.

Let x be an element of A. Let S be a set and let $S = \{x\}$. Since x is the only element of S and $x \in A$, $S \subseteq A$. Since $S \subseteq A$, $S \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, $S \in \mathcal{P}(B)$. Since $S \in \mathcal{P}(B)$, $S \subseteq B$. Since $x \in S$, then $x \in B$. Thus, $A \subseteq B$.

Let x be an element of B. Let S be a set and let $S = \{x\}$. Since x is the only element of S and $x \in B$, $S \subseteq B$. Since $S \subseteq B$, $S \in \mathcal{P}(B)$. Since $\mathcal{P}(B) \subseteq \mathcal{P}(A)$, $S \in \mathcal{P}(A)$. Since $S \in \mathcal{P}(A)$, $S \subseteq A$. Since $x \in S$, then $x \in A$. Thus, $B \subseteq A$. Therefore $A = B$.

2 Problem 2a

For a natural number k, let $k\mathbb{Z}$ denote the set of all integer multiples of k. Prove that for all $a, b \in \mathbb{N}$, $a = b$ if and only if $a\mathbb{Z} = b\mathbb{Z}$.
Proof:
Let $a, b \in \mathbb{N}$.

($\Rightarrow$) Let $a = b$. We want to show that $a\mathbb{Z} = b\mathbb{Z}$.

Let $x \in a\mathbb{Z}$. This means x is an integer multiple of a. This means there exists an integer z such that $az = x$. Since $a = b$, then $az = bz$. Then, $bz = x$. Thus, x is an integer multiple of b. Thus, $x \in b\mathbb{Z}$. Therefore, $a\mathbb{Z} \subseteq b\mathbb{Z}$.

Let $x \in b\mathbb{Z}$. This means x is an integer multiple of b. This means there exists an integer z such that $bz = x$. Since $a = b$, then $az = bz$. Then, $az = x$. Thus, x is an integer multiple of a. Thus, $x \in a\mathbb{Z}$. Therefore, $b\mathbb{Z} \subseteq a\mathbb{Z}$. Therefore, $a\mathbb{Z} = b\mathbb{Z}$.

($\Leftarrow$) (By Contrapositive) Assume $a \neq b$. We want to show that $a\mathbb{Z} \neq b\mathbb{Z}$. There will be two cases.

Case 1: Let $a < b$. Let $b\mathbb{Z} \subseteq a\mathbb{Z}$. We want to show that $a\mathbb{Z} \nsubseteq b\mathbb{Z}$. We want to show that there exists an integer x such that $x \in a\mathbb{Z}$ and $x \notin b\mathbb{Z}$. Let $x = a$. Then, $a * 1 = x$. Since $x = a < b$, then $b \not| x$. Then, for all integers k, $bk \neq x$. Then, x is not an integer multiple of b. Thus, $x \in a\mathbb{Z}$ and $x \notin b\mathbb{Z}$.Thus, $a\mathbb{Z} \nsubseteq b\mathbb{Z}$.

Case 2: Let $b < a$. Let $a\mathbb{Z} \subseteq b\mathbb{Z}$. We want to show that $b\mathbb{Z} \nsubseteq a\mathbb{Z}$. We want to show that there exists an integer y such that $y \in b\mathbb{Z}$ and $y \notin a\mathbb{Z}$. Let $y = b$. Then, $b * 1 = y$. Since $y = b < a$, then $a \not| y$. Then, for all integers l, $al \neq y$. Then, y is not an integer multiple of a. Thus, $y \in b\mathbb{Z}$ and $y \notin a\mathbb{Z}$. Thus, $b\mathbb{Z} \nsubseteq a\mathbb{Z}$.

In both cases, $a\mathbb{Z} \neq b\mathbb{Z}$.

3 Problem 2b

Prove for all natural numbers a and b, $a|b$ if and only if $b\mathbb{Z} \subseteq a\mathbb{Z}$.

Proof:

($\Rightarrow$) Let a and b be natural numbers and let $a|b$. Then, there exists an integer k such that $ak = b$. We want to show that $b\mathbb{Z} \subseteq a\mathbb{Z}$. Let x be an integer in the set $b\mathbb{Z}$. This means there exists an integer m such that $bm = x$. Since $ak = b$, $akm = x$. Since $k, m \in \mathbb{Z}$, $km \in \mathbb{Z}$. Thus, x is an integer multiple of a. Thus, $x \in a\mathbb{Z}$. Therefore, $b\mathbb{Z} \subseteq a\mathbb{Z}$.

($\Leftarrow$) (By Contrapositive) Let a and b be natural numbers. Assume $a \not| b$. We want to show that $b\mathbb{Z} \nsubseteq a\mathbb{Z}$. This means we want to show that there exists an integer $x \in b\mathbb{Z}$ such that $x \notin a\mathbb{Z}$. Let $x = b$. Since $b * 1 = x$, x is an integer multiple of b and $x \in b\mathbb{Z}$. Since $x = b$ and $a \not| b$, $a \not| x$. Thus, for all integers k, $ak \neq x$. Thus, x is not an integer multiple of a, and $x \notin a\mathbb{Z}$. Therefore, $b\mathbb{Z} \nsubseteq a\mathbb{Z}$.

4 Problem 3

Given sets A and B. Define the symmetric difference of A and B, denoted $A \triangle B$, by $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Prove that $A \triangle B = (A \cup B) \setminus (A \cap B)$.

Proof:

Let A and B be sets. We want to show that $A \triangle B = (A \cup B) \setminus (A \cap B)$, which means want to show that $A \triangle B \subseteq (A \cup B) \setminus (A \cap B)$ and $(A \cup B) \setminus (A \cap B)$.

Let $x \in A \triangle B$. Then, $x \in (A \setminus B) \cup (B \setminus A)$. There will be two cases.

Case 1: Let $x \in A$ and $x \notin B$. Since $x \in A$, $x \in (A \cup B)$. Since $x \notin B$, $x \notin (A \cap B)$. Since $x \in (A \cup B)$ and $x \notin (A \cap B)$, $x \in (A \cup B) \setminus (A \cap B)$. Therefore, $A \triangle B \subseteq (A \cup B) \setminus (A \cap B)$.

Case 2: Let $x \in B$ and $x \notin A$. Since $x \in B$, $x \in (A \cup B)$. Since $x \notin A$, $x \notin (A \cap B)$. Since $x \in (A \cup B)$ and $x \notin (A \cap B)$, $x \in (A \cup B) \setminus (A \cap B)$. Therefore, $A \triangle B \subseteq (A \cup B) \setminus (A \cap B)$.

Let $x \in (A \cup B) \setminus (A \cap B)$. Then, $x \in (A \cup B)$ and $x \notin (A \cap B)$. There will be two cases.

Case 1: Let $x \in A$ and $x \notin B$. Then, $x \in A \setminus B$. Then, $x \in (A \setminus B) \cup (B \setminus A)$. Thus, $x \in A \triangle B$. Therefore, $(A \cup B) \setminus (A \cap B) \subseteq A \triangle B$.

Case 2: Let $x \in B$ and $x \notin A$. Then, $x \in B \setminus A$. Then, $x \in (A \setminus B) \cup (B \setminus A)$. Thus, $x \in A \triangle B$. Therefore, $(A \cup B) \setminus (A \cap B) \subseteq A \triangle B$.

Since $A \triangle B \subseteq (A \cup B) \setminus (A \cap B)$ and $(A \cup B) \setminus (A \cap B) \subseteq A \triangle B$, then $A \triangle B = (A \cup B) \setminus (A \cap B)$.