WRITTEN ASSIGNMENT 6
MATH 290, DR. WALNUT

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1. Problem 1a

Use induction to prove that for all natural numbers n, and all real numbers $r \neq 1$,

$$\sum_{j=0}^{n-1} r^j = \frac{r^n - 1}{r - 1}$$

.

Proof:

Let $n = 1$. Then $\frac{r^n-1}{r-1} = \frac{r-1}{r-1} = 1$. Also, $\sum\limits_{j=0}^{n-1} r^j = \sum\limits_{j=0}^{0} r^j = r^0 = 1$. Thus, the result holds for $n = 1$.

Let n be a natural number. Assume $\sum\limits_{j=0}^{n-1} r^j = \frac{r^n-1}{r-1}$ is true. We want to show that $\sum\limits_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$. Since $r = r$, then $r^n = r^n$. By adding $\sum\limits_{j=0}^{n-1} r^j$ to both sides, we get $r^n + \sum\limits_{j=0}^{n-1} r^j = r^n + \sum\limits_{j=0}^{n-1} r^j$. By the induction hypothesis, $r^n + \sum\limits_{j=0}^{n-1} r^j = r^n + \frac{r^n-1}{r-1}$. Then, $r^n + \sum\limits_{j=0}^{n-1} r^j = \sum\limits_{j=0}^{n} r^j = r^n + \frac{r^n-1}{r-1} = \frac{r^n(r-1)}{r-1} + \frac{r^n-1}{r-1} = \frac{r^{n+1}-r^n}{r-1} + \frac{r^n-1}{r-1} = \frac{r^{n+1}-r^n+r^n-1}{r-1} = \frac{r^{n+1}-1}{r-1}$
Thus, $\sum\limits_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$, which is what we want. Therefore, for all natural numbers n and all real numbers $r \neq 1$, $\sum\limits_{j=0}^{n-1} r^j = \frac{r^n-1}{r-1}$.

2. Problem 1b

Prove that if $2^p - 1$ is prime, then p is prime.

Proof: (By contrapositive)

Let p be a natural number. Assume p is not prime. This means that there exist two natural numbers m and k such that $mk = p$, $m, k \neq 1$ and $m, k \neq p$. Since $m, k \in \mathbb{N}$ and $m, k \neq 1$, $m, k > 1$. Also, let $k \leq m$ without loss of generality. Since $mk = p$, we can write

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$2^p-1=2^{mk}-1$. From the previous result proved and because 2 is a real number and not equal to 1, $\frac{2^{mk}-1}{2-1}=2^{mk}-1=\sum\limits_{j=0}^{mk-1}2^j$. We can rewrite this as

$$\sum_{j=0}^{mk-1}2^j=\sum_{j=0}^{k-1}2^j+...+\sum_{j=(m-1)k}^{mk-1}2^j=\sum_{i=1}^{m}\left(\sum_{j=(i-1)k}^{ik-1}2^j\right)$$

Since $k>1$, we know the first term will be greater than 1. Also, from the first sum, we know there are mk iterations in the sum. Each subsum is k iterations long, so we know that there will be m subsums of k length. Then, we will factor out a $2^{(i-1)k}$ from each subsum. Then, our sum is,

$$2^0\sum_{j=0}^{k-1}2^{j-0}+...+2^{(m-1)k}\sum_{j=(m-1)k}^{mk-1}2^{j-(m-1)k}=\sum_{i=1}^{m}\left(2^{(i-1)k}\sum_{j=(i-1)k}^{ik-1}2^{j-(i-1)k}\right)$$

We can then adjust each index and exponent of 2 in each subsum. The sum is now equal to,

$$2^0\sum_{j=0}^{k-1}2^j+...+2^{(m-1)k}\sum_{j=0}^{k-1}2^j=\sum_{i=1}^{m}\left(2^{(i-1)k}\sum_{j=0}^{k-1}2^j\right)$$

Since k is a natural number, i is a natural number, and $i\geq 1$, $2^{(i-1)k}$ is an integer. Since each term of the total sum is $\sum\limits_{j=0}^{k-1}2^j$ multiplied by some integer, $\sum\limits_{j=0}^{k-1}2^j$ is a divisor of each term. Let $a=\sum\limits_{j=0}^{k-1}2^j$. Since $2^0=1$ and $2^j\in\mathbb{N}$ where $j\geq 1$, $a\in\mathbb{N}$. Since a divides each term and 2^p-1 is equal to the sum of all the terms, based on a previous result proved, $a|2^p-1$. Since $k>1$, $a=\sum\limits_{j=0}^{k-1}2^j>1$. Thus, $a=\sum\limits_{j=0}^{k-1}2^j\neq 1$. Since $k\in\mathbb{N}$ and $2\in\mathbb{R}$, $a=\sum\limits_{j=0}^{k-1}2^j=2^k-1$. Since $k<p$, $2^k<2^p$, and $2^k-1<2^p-1$. Thus, $a=\sum\limits_{j=0}^{k-1}2^j\neq 2^p-1$. Therefore there exists a natural number a such that $a|\,(2^p-1)$, $a\neq 1$, and $a\neq 2^p-1$. Therefore, 2^p-1 is not prime. By contrapositive, if 2^p-1 is prime, then p is prime.

3. Problem 2

Define the relation S on $\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})$ by $(m,n)S(p,q)$ if and only if $mq=np$. Prove S is an equivalence relation on $\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})$.

Proof:

Let S be a relation on $\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})$. Define the relation S by $(m,n)S(p,q)$ if and only if $mq=np$. We want to show that S is an equivalence relation.

We first want to show that S is reflexive. Let $(m,n)\in\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})$. Since $mn=nm$, $(m,n)S(m,n)$. Thus, S is reflexive.

Next, we want to show that S is symmetric. Let $(m,n),(p,q) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, and let $(m,n)S(p,q)$. Then, $mq = np$. Then, $pn = qm$. Therefore, $(p,q)S(m,n)$. Therefore, S is symmetric.

The last thing we want to show is that S is transitive. Let $(m,n),(p,q),(k,l) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, and let $(m,n)S(p,q)$ and $(p,q)S(k,l)$. Then, $mq = np$, and $pl = qk$. By multiplying n to both sides of $pl = qk$, we get $npl = nqk$. Since $mq = np$, $mql = nqk$. Then, $ml = nk$. Therefore, $(m,n)S(k,l)$. Therefore, S is transitive.

Since S is reflexive, symmetric, and transitive, S is an equivalence relation.