Written Assignment 8
Math 290, Dr. Walnut

Lucas Bouck

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1 Problem 1a

Let R be a relation on the set A. Suppose that R is a partial order. Prove that if every nonempty subset of A has a smallest element, then R is a well ordering.
Proof:
Let R be a partial order on A. Assume that every nonempty subset of A has a smallest element. We want to show that R is a well ordering. Let x and y be distinct elements in A. Consider the set $B = \{x, y\}$. Since $x, y \in A$, then $B \subseteq A$. Since $x, y \in B$, B is nonempty. From our first assumption, it follows that B has a smallest element. This means there is an element in B such that this element is the greatest lower bound of B. There are now two cases to consider. If x is the smallest element of B, then x relates by R to every element in B, which means xRy. If y is the smallest element of B, then y relates by R to every element in B, which means yRx. In both cases, either xRy or yRx. Because R is a partial order on A, and for any two distinct elements, $x, y \in A$ either xRy or yRx, R is a linear order of A. Since R is a linear order of A and every nonempty subset of A has a smallest element, R is a well ordering.

2 Problem 1b

Let R be a relation on A. Suppose that R is a partial order on A. Prove that if every nonempty subset B of A contains a unique element that is related by R to every element of B, then R is a well ordering.
Proof:
Let R be a relation on the set A. Assume that every nonempty subset B of A contains a unique element that is related by R to every element of B. We want to show that R is a well ordering on A.

We need to show that R is a partial order. First, we will show that R is reflexive. Let $x \in A$. Consider the set $B = \{x\}$. Since $x \in A$, $B \subseteq A$. Also, since $x \in B$, B is nonempty.

Then, there exists an unique element in B such that this element relates by R to every element in B. Since x is the only element in B, xRx. Thus, R is reflexive.

Next, we must show that R is antisymmetric. We will do this by contradiction. Suppose that R is not antisymmetric. This means that there exists $x, y \in A$ such that xRy, yRx, and $x \neq y$. Next consider the set $C = \{x, y\}$. Since R is reflexive, xRx, and yRy. Then, xRy, xRx, yRx and yRy. This means that x relates by R to all the elements in C, y relates to all the elements in C. Since $x, y \in C$, $C \subseteq A$. Additionally, since $x, y \in C$, C is nonempty. Then, there exists a unique element in C such that this element relates by R to every element in C. The elements x and y relate to every element in C, so this contradicts the fact that C has a unique element such that this element relates by R to every element in C. Therefore, by contradiction, R is symmetric.

Finally we must show that R is transitive. Let x, y, z be distinct elements in A, and let xRy and yRz. We want to show that xRz. Consider the set $D = \{x, y, z\}$. Since $x, y, z \in A$, $D \subseteq A$. Also, since $x, y, z \in D$, D is nonempty. Therefore, D contains a unique element such that this element relates by R to every element in D. Since xRy and R is antisymmetric, y does not relate by R to x, and y cannot be the unique element that relates by R to all the elements in D. Since yRz and R is antisymmetric, z does not relate by R to y, and z cannot be the unique element in D that relates by R to all the elements in D. Since y, z do not relate to all elements in D and there must be a unique element in D such that the unique element relates by R to all elements in D, x is the unique element in D such that x relates by R to all elements in D. Since $z \in D$, xRz. Thus, R is transitive. Since R is reflexive, antisymmetric, and transitive, R is a partial order.

We need to show that every nonempty subset of A has a smallest element. Let B be a nonempty subset of A. From our assumptions, there exists a unique element $x \in B$ such that xRb for all $b \in B$. We want to show that B has a smallest element. Since $x \in B$ and $B \subseteq A$, $x \in A$. Since xRb for all $b \in B$, x is a lower bound of B. Consider the lower bound of B called $y \in A$. By the definition of lower bound, yRb for all $b \in B$. Since $x \in B$, yRx. Therefore, if $y \in A$ is a lower bound of B, yRx. Thus, x is the greatest lower bound of B. Since $x \in B$ and x is the greatest lower bound of B, x is the smallest element of B. Therefore, if every nonempty subset B of A contains a unique element that is related by R to every element of B, then every nonempty subset of A has a smallest element.

Since every nonempty subset of A has a smallest element and R is a partial order on A, by the last result that we proved, R is a well ordering on A.

3 Problem 2a

Let A, B, and C be sets. Suppose that $f : A \to B$ and $g : B \to C$. Prove that if $g \circ f$ is injective, then f is injective.

Proof:

Let A, B, and C be sets. Suppose that $f : A \to B$ and $g : B \to C$. Assume that $g \circ f$

is injective. We want to show that f is injective, which means we want to show that if $f(x) = f(y)$, then $x = y$. Let $x, y \in A$ and assume $f(x) = f(y)$. Since $f(x) = f(y)$ and g is a function, $g(f(x)) = g(f(y))$. Since $g \circ f$ is injective and $g(f(x)) = g(f(y))$, then $x = y$. Therefore, f is injective.

4 Problem 2b

Let A, B, and C be sets. Suppose that $f : A \to B$ and $g : B \to C$. Prove that if $g \circ f$ is surjective, then g is surjective.

Proof:

Let A, B, and C be sets. Suppose that $f : A \to B$ and $g : B \to C$. Assume that $g \circ f$ is surjective. We want to show that f is surjective, which means we want to show that if $c \in C$, then there exists a $b \in B$ such that $g(b) = c$. Let $c \in C$. Since $g \circ f$ is surjective, there exists an $a \in A$ such that $g(f(a)) = c$. Let $b = f(a)$. Since f's codomain is B, $f(a) \in B$. Then, $b \in B$. Since $g(f(a)) = c$, then $g(b) = c$. Therefore, for all $c \in C$, there exists a $b \in B$ such that $g(b) = c$. Therefore, g is surjective.