Written Assignment 7
Math 290, Dr. Walnut

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1 Problem 1A

Let R be a relation on a nonempty set A. Prove that if R is symmetric and transitive and $\text{Dom}(R) = A$, then R is reflexive.
Proof:

Let R be a relation on a nonempty set A. Assume R is symmetric and transitive and $\text{Dom}(R) = A$. We want to show that R is reflexive. Let $x \in A$. Since $\text{Dom}(R) = A$, there exists a $y \in A$ such that $(x, y) \in R$. Thus, xRy. Since R is symmetric, yRx. Since R is transitive and xRy and yRx, xRx. Since for all $x \in A$, then xRx, then R is reflexive.

2 Problem 1B

Suppose that R is reflexive and transitive and define the relation L on A by xLy if and only if xRy and yRx. Prove that L is an equivalence relation.
Proof:

Let R and L be relations on A. Assume R is reflexive and transitive and define the relation L on A by xLy if and only if xRy and yRx. We want to show that L is an equivalence relation.

First, we want to show that L is reflexive. Let $x \in A$. Since R is reflexive, xRx and xRx. Since xRx and xRx, xLx. Therefore, L is reflexive.

Next, we want to show that L is symmetric. Let xLy. Then, xRy and yRx. Then, yRx and xRy. Therefore, yLx, and L is symmetric.

Finally, we want to show that L is transitive. Let xLy and yLz. Then, xRy, yRx, yRz, and zRy. Since R is transitive, xRy, and yRz, then xRz. Also, since R is transitive, zRy, and yRx, then zRx. Since xRz and zRx, then xLz. Therefore, L is transitive.

Since L is reflexive, symmetric, and transitive, L is an equivalence relation.

3 Problem 1C

Suppose that S is a symmetric relation on A such that $R \subseteq S$. Prove that $R \cup R^{-1} \subseteq S$.
Proof:

Let S be a symmetric relation on A. Let R be a relation on A and assume $R \subseteq S$. We want to show that $R \cup R^{-1} \subseteq S$. Let $(x, y) \in R \cup R^{-1}$. This means $(x, y) \in R$ or $(x, y) \in R^{-1}$. There will be two cases. Let $(x, y) \in R$. Since $R \subseteq S$, then $(x, y) \in S$. In this case, $R \cup R^{-1} \subseteq S$. Next, suppose $(x, y) \in R^{-1}$. Then, $(y, x) \in R$. Since $R \subseteq S$, $(y, x) \in S$ and ySx. Since S is symmetric, xSy. Then, $(x, y) \in S$. In this case, $R \cup R^{-1} \subseteq S$. In both cases, $R \cup R^{-1} \subseteq S$. We are done.

4 Problem 2

Let $m \in \mathbb{N}$ and for $x \in \mathbb{Z}$ define $\overline{x}^m$ to be the equivalence class of x in $\mathbb{Z}_m$. Prove that for any integers a, b, c, and d, if $\overline{a}^m = \overline{c}^m$ and $\overline{b}^m = \overline{d}^m$ then $\overline{ab}^m = \overline{cd}^m$.
Proof:

Let a, b, c, and d be integers, and let $m \in \mathbb{N}$. Assume $\overline{a}^m = \overline{c}^m$ and $\overline{b}^m = \overline{d}^m$. We want to show that $\overline{ab}^m = \overline{cd}^m$. Since $\overline{a}^m = \overline{c}^m$, then $m|(c-a)$. This means $mk = c - a$ for some integer k. Then, $c = mk + a$. Since $\overline{b}^m = \overline{d}^m$, $m|(d-b)$. This means $ml = d - b$ for some integer l. Then, $d = ml + b$. Then, $cd = (mk + a)(ml + b) = m^2kl + mla + mkb + ab$. Then, $cd - ab = m^2kl + mla + mkb = m(mkl + la + kb)$. Let $mkl + la + kb = z$. Since $m, k, l, a, b \in \mathbb{Z}$, then $z \in \mathbb{Z}$. Thus, there exists an integer z such that $mz = cd - ab$. Therefore, $m|(cd - ab)$, and $\overline{ab}^m = \overline{cd}^m$.