Written Assignment 4

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1 Problem 1

Let f be a function whose domain D_f is not bounded above. Prove that $\lim_{x\to\infty} f(x) = L$ if and only if for every sequence $x_n \in D_f$ with $x_n \to \infty$ as $n \to \infty$, $\lim_{n\to\infty} f(x_n) = L$.

Proof:
($\Rightarrow$)

Let f be a function whose domain D_f is not bounded above. Suppose $\lim_{x\to\infty} f(x) = L$. Let x_n be a sequence in D_f with $x_n \to \infty$ as $n \to \infty$. We want to show that $\lim_{n\to\infty} f(x_n) = L$.

Let $\epsilon > 0$. Then there exists an $M > 0$ such that if $x \geq M$ then $|f(x) - L| < \epsilon$. Choose $N \in \mathbb{N}$ such that if $n \geq N$ then $x_n \geq M$. We can do this because $x_n \to \infty$ as $n \to \infty$. If $n \geq N$, then $x_n \geq M$ and $|f(x_n) - L| < \epsilon$. Thus, we have shown that for all $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that if $n \geq N$ then $|f(x_n) - L| < \epsilon$. Therefore, $\lim_{n\to\infty} f(x_n) = L$.

($\Leftarrow$) (By Contrapositive)

Let f be a function whose domain D_f is not bounded above. Suppose $\lim_{x\to\infty} f(x) \neq L$. This means there exists an $\epsilon_0 > 0$ such that for all $M > 0$ there exists an $x \in D_f$ such that $x \geq M$ and $|f(x) - L| \geq \epsilon_0$. We want to show that there exists a sequence $x_n \in D_f$ such that $x_n \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} f(x_n) \neq L$.

Construct $x_n \in D_f$ in the following way. Choose x_1 such that $x_1 \geq 1$ and $|f(x_1) - L| \geq \epsilon_0$. Choose x_2 such that $x_2 \geq x_1 + 1$ and $|f(x_2) - L| \geq \epsilon_0$. Continue in the fashion and choose x_{n+1} such that $x_{n+1} \geq x_n + 1$ and $|f(x_{n+1}) - L| \geq \epsilon_0$. Since $x_{n+1} \geq x_n + 1$, $x_{n+1} > x_n$ for all n. For use later, it must be shown that $x_n \geq n$ for all $n \in \mathbb{N}$. Based on x_1 is defined, $x_1 \geq 1$, so the result holds for $n = 1$. Suppose the result is true for n. Then, $x_n \geq n$. We know that $x_{n+1} \geq x_n + 1 \geq n + 1$. By induction, $x_n \geq n$ for all $n \in \mathbb{N}$. We must show that x_n diverges to ∞ and $\lim_{n\to\infty} f(x_n) \neq L$.

Let $A > 0$ be a real number. We must show that there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $x_n > A$. By the Archimedean Principle, there exists an $N \in \mathbb{N}$ such that $N > A$. Choose that N. We know that $x_N \geq N > A$. Since x_n is monotonically increasing, if $n \geq N$ then $x_n > A$. Therefore $x_n \to \infty$.

We must now show that $\lim_{n\to\infty} f(x_n) \neq L$. Consider ϵ_0. Let N be a natural number. Based on how x_n is defined, $|f(x_N) - L| \geq \epsilon_0$. Thus, there exists an $\epsilon > 0$ such that for all natural N there exists an $n \geq N$ such that $|f(x_n) - L| \geq \epsilon_0$. Therefore, $\lim_{n\to\infty} f(x_n) \neq L$. We are done.

2 Problem 2

Let a be a cluster point of the domain D_f of a function f. Prove that $\lim_{x\to a} f(x) = \infty$ if and only if for every sequence $x_n \in D_f \setminus \{a\}$ with $x_n \to a$ as $n \to \infty$, $\lim_{n\to\infty} f(x_n) = \infty$.

Proof:
($\Rightarrow$)

Let a be a cluster point of the domain D_f of a function f. Suppose $\lim_{x\to a} f(x) = \infty$. Let $x_n \in D_f \setminus \{a\}$ be a sequence with $x_n \to a$ as $n \to \infty$. We want to show that $\lim_{n\to\infty} f(x_n) = \infty$.

Let $M > 0$. We want to show that there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $f(x_n) > M$. Since $\lim_{x\to a} f(x) = \infty$, there is a $\delta > 0$ such that for all $x \in D_f$, if $0 < |x - a| < \delta$ then $f(x) > M$. Since $x_n \to a$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|x_n - a| < \delta$. Choose that N. Since $x_n \in D_f \setminus \{a\}$, $x_n \neq a$ for all n. Thus, if $n \geq N$ then $0 < |x_n - a| < \delta$. Then, if $n \geq N$, $f(x_n) > M$. Therefore, there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $f(x_n) > M$, and $\lim_{n\to\infty} f(x_n) = \infty$.

($\Leftarrow$) (By Contrapositive)

Let a be a cluster point of the domain D_f of a function f. Suppose $\lim_{x\to a} f(x) \neq \infty$. This means there exists an $M > 0$ such that for all $\delta > 0$ there exists an $x \in D_f$ such that $0 < |x - a| < \delta$ and $f(x) \leq M$. We will refer to this real number as M_0. We want to show that there exists a sequence $x_n \in D_f \setminus \{a\}$ with $x_n \to a$ as $n \to \infty$ such that $\lim_{n\to\infty} f(x_n) \neq \infty$.

We shall construct the sequence x_n. Choose x_n such that $x_n \in D_f \setminus \{a\}$ and $0 < |x_n - a| < \frac{1}{n}$ and $f(x_n) \leq M_0$. We can do this because for all $\delta > 0$ there exists an $x \in D_f$ such that $0 < |x - a| < \delta$ and $f(x) \leq M_0$. We shall show that this sequence converges to a. Let $\epsilon > 0$. Choose N such that $\frac{1}{N} < \epsilon$. We can do this because of the Archimedean Principle. If $n \geq N$, then $\frac{1}{n} < \epsilon$. Then $|x_n - a| < \frac{1}{n} < \epsilon$. Thus, there exists an N such that if $n \geq N$ then $|x_n - a| < \epsilon$. Therefore, $x_n \to a$ as $n \to \infty$.

We must now show that $\lim_{n\to\infty} f(x_n) \neq \infty$. Consider M_0. Let N be a natural number. Based on how x_n was constructed, $f(x_N) \leq M_0$. Thus, there exists an $M > 0$ such that for all natural numbers N, there exists an $n \geq N$ such that $f(x_n) \leq M_0$. Therefore, $\lim_{n\to\infty} f(x_n) \neq \infty$. We are done.

3 Problem 3

Suppose that f and g are functions continuous on $\mathbb{R}$ and that $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Proof:

Suppose that f and g are functions continuous on $\mathbb{R}$ and that $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Let $a \in \mathbb{R}$. We want to show that $f(a) = g(a)$. There will be two cases. If a is a rational number, then $f(a) = g(a)$. Now suppose a is irrational. Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Due to the density if $\mathbb{Q}$ in $\mathbb{R}$, for all $\delta > 0$, there exists a rational number y in the open interval $(a, a+\delta)$, which means there exists a rational number y that satisfies $0 < |y - a| < \delta$. Since $y \in \mathbb{R}$, this means a is a cluster point of $\mathbb{R}$. Since a is a cluster point of $\mathbb{R}$ and f and g are continuous on $\mathbb{R}$, $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$.

Since the rational numbers are dense in the real numbers, there exists a sequence $r_n \in \mathbb{Q}$ such that r_n converges to a as $n \to \infty$. Based on the sequential characterization of limits of functions and the facts that $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$, and $r_n \to a$, $f(r_n) \to L$ and $g(r_n) \to M$. Since $f(r) = g(r)$ for all $r \in \mathbb{Q}$, $f(r_n) = g(r_n)$ for all $n \in \mathbb{N}$. Thus, $f(r_n) \to L$ and $f(r_n) \to M$. Therefore, $L = M$ and $f(a) = g(a)$. We are done.