Written Assignment 2
Math 290, Dr. Walnut

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1 Problem 1

1.1 a)

Problem: Two positive real numbers a and b are said to be in the golden ratio if $a/b = (a+b)/a$. Prove that if a and b are in the golden ratio, then $a/b = b/(a-b)$.
Proof:
Let a and b both be real numbers. Also, let a and b be in the golden ratio. That means $a/b = (a+b)/a$. We want to show that if a and b are in the golden ratio, then $a/b = b/(a-b)$. By cross multiplying, we get $a^2 = ba + b^2$. By subtracting ba from both sides, we get $a^2 - ba = b^2$. We can factor out an a on the left side and get $a(a-b) = b^2$. By dividing both sides by b, we get $a(a-b)/b = b$. By dividing $(a-b)$ from both sides, we get $a/b = b/(a-b)$. Thus, if a and b are in the golden ratio, then $a/b = b/(a-b)$.

1.2 b)

Problem: Prove directly from the definition that if a and b are in the golden ratio then a/b is irrational.
Proof: (By Contradiction)
Let the positive real numbers a and b satisfy the golden ratio. This means $\frac{a}{b} = \frac{a+b}{a}$. Also, let a/b be rational. This means $a, b \in \mathbb{Z}$, a and b have no common divisors, and $b \neq 0$. By cross multiplying $\frac{a}{b} = \frac{a+b}{a}$, we get $a^2 = b(a+b)$. Since b is an integer, $(a+b)|a^2$. Since $a, b \in \mathbb{Z}$, $(a+b) \in \mathbb{Z}$. Since $(a+b) \in \mathbb{Z}$, $b|a^2$. There will be three cases.
Case 1: Let a=0. Then, $a^2 = 0$, and $b(a+b) = 0$. Since $a = 0$, $b^2 = 0$. Then, $b = 0$. Therefore, $b = 0$, and $b \neq 0$. We have a contradiction.
Case 2: Let a be even. Then a^2 is even. Since a^2 is even, either b or $a+b$ must be even. Since a is even, in order for a+b to be even, b must be even. Thus, b is even. Since both a and b are even, 2 divides both a and b. Therefore, a and b have a common divisor, but a and b do not have a common divisor. We have a contradiction.
Case 3: Let a be odd. Then $a = 2k+1$ where $k \in \mathbb{Z}$. Then $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 =$

$2(2k^2+2k)+1$. $2k^2+2k$ is an integer, so a^2 is odd. Since $b(a+b)=a^2$, b and $a+b$ must be odd. Then, $b=2m+1$, where $m \in \mathbb{Z}$, and $a+b=2n+1$, where $n \in \mathbb{Z}$. Then, $a+b=a+2m+1=2n+1$. Thus, $a=2n-2m=2(n-m)$. Since $n-m \in \mathbb{Z}$, a is even, but a is odd. We have a contradiction.

Since in all cases of a, if we have a/b being rational then we have a contradiction. By contradiction, this means a/b is irrational if a/b satisfies the golden ratio.

1.3 Problem 2

Problem: Prove that a natural number p is prime if and only if $p>1$ and there exists no natural number n with $1<n\leq\sqrt{p}$ such that $n|p$.
Proof:
($\Rightarrow$) (By contrapositve) There will be two cases for this direction.
Case 1: Let $p \in \mathbb{N}$, and $p \not> 1$. Then $p=1$. Since 1 cannot be a prime number, p is not prime.
Case 2: Let $p>1$. Let there be a natural n such that $n \in (1, \sqrt{p}]$ and $n|p$. Since $n \in (1, \sqrt{p}]$, $n \neq 1$. Since $n|p$, $nk=p$ for some integer k. Since $n \leq \sqrt{p}$, $n^2 \leq (\sqrt{p})^2 = p$. We do not need absolute value on the p because p is always positive. Since $n>1$, $n<n^2\leq p$. Thus, $n<p$. Thus, $n \neq p$. Since $n \neq 1$, $n \neq p$, and $n|p$, there exists a natural n that divides p that is not equal to 1 or p. Therefore, p is not prime.

For both cases, by contrapositve, if p is prime, then p>1 and there is no natural n with $1<n\leq\sqrt{p}$ such that $n|p$.

($\Leftarrow$) (By contrapositive) There will be 3 cases for this direction.
Say p is not prime and $p>1$. Then there exists a natural m such that $m|p$, $m \neq 1$, and $m \neq p$. Since m is natural and not equal to 1, $m>1$. Since $m|p$, $mk=p$ for some integer k. Because m and p are both positive, k must be positive. Since m and p are not zero, k must not be 0. Since both p and m are not equal to one, k is not equal to one. Thus, $k \in \mathbb{Z}$ and k is positive and nonzero. Thus, k is a natural number. Since $k \in \mathbb{N}$, and $k \neq 1$, $k>1$. From $mk=p$, we get $m=p/k$. Here are the 3 cases.
Case 1: Let $k=\sqrt{p}$. Then, $m=p/k=p/\sqrt{p}=\sqrt{p}$. Thus, $1<m\leq\sqrt{p}$.
Case 2: Let $k>\sqrt{p}$. Then, $m=p/k<p/\sqrt{p}=\sqrt{p}$. Thus, $m<\sqrt{p}$. Thus, $1<m\leq\sqrt{p}$.
Case 3: Let $k<\sqrt{p}$. Since $nk=p$, where m is a natural number, $k|p$. Since $k \in \mathbb{N}$, $k>1$, and $k|p$, there exists a natural number k such that $k|p$ and $1<k\leq\sqrt{p}$.

In the first two cases, let $m=n$. In the third case, let $k=n$. In all three cases, there exists a natural n such that $n|p$ and $1<n<p$. We have shown that if p is not prime and $p>1$, then there exists a natural n such that $n|p$ and $1<n\leq\sqrt{p}$. By contrapositve, if there does not exist a natural n such that $n|p$ and $1<n\leq\sqrt{p}$, then p is prime or $p\leq 1$. This statement is equivalent to if there does not exist a natural n such that $n|p$, and $1<n\leq\sqrt{p}$ and $p>1$, then the natural number p is prime. We are done.

We have proven that if p is prime, then p>1 and there is no natural n with $1<n\leq\sqrt{p}$

such that $n|p$ and that if there does not exist a natural n such that $n|p$, and $1 < n \leq \sqrt{p}$ and $p > 1$, then the natural number p is prime. Therefore, a natural number p is prime if and only if $p > 1$ and there exists no natural number n with $1 < n \leq \sqrt{p}$ such that $n|p$.