Math 478 HW 2

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1 Problem 1.5.8

1.1 Part a)

Answer:

Suppose u, v solve $u_{tt} = c^2 u_{xx}$ with homogenous Neumann boundary conditions and with the initial conditions $u(0,x) = v(0,x) = f(x)$ and $u_t(0,x) = v_t(0,x) = g(x)$. Let $w(t,x) = u(t,x) - v(t,x)$. Then,

$$\begin{aligned} w_{tt} &= u_{tt} - v_{tt} \\ &= c^2 u_{xx} - c^2 v_{xx} \\ &= c^2(u_{xx} - v_{xx}) = c^2 w_{xx} \end{aligned}$$

For the boundary conditions, since $w \equiv u - v$, which means $w_x \equiv u_x - v_x$, and $u_x = v_x = 0$ on the boundary, $w_x = 0$ on the boundary and w satisfies the wave equation with homogenous Neumann boundary conditions. The initial conditions for w are below.

$$\begin{aligned} w(0,x) &= u(0,x) - v(0,x) = f(x) - f(x) = 0 \\ w_t(0,x) &= u_t(0,x) - v_t(0,x) = g(x) - g(x) = 0 \end{aligned}$$

Thus, the initial conditions, $w(0,x)$ and $w_t(0,x)$, are the zero function.

1.2 Part b)

Answer:

We must show that the a solution u to the wave equation $u_{tt} = c^2 u_{xx}$ with homogenous Neumann boundary conditions and initial conditions $u(0,x) = f(x)$ and $u_t(0,x) = g(x)$ is unique.

Suppose there exists another solution v to the wave equation with the same boundary conditions and initial conditions. Consider the function $w(t,x) = u(t,x) - v(t,x)$. From the result in Part a, $w(t,x)$ solves $w_{tt} = c^2 w_{xx}$ with homogenous Neumann boundary conditions. Also from part a, $w(0,x) = 0$ and $w_t(0,x) = 0$. By the result in 1.5.7b, $w(t,x) \equiv 0$. This means $u(t,x) - v(t,x) \equiv 0$. Therefore, $u(t,x) = v(t,x)$, and the solution to the wave equation $u_{tt} = c^2 u_{xx}$ with homogenous Neumann boundary conditions and initial conditions $u(0,x) = f(x)$ and $u_t(0,x) = g(x)$ is unique.

2 Problem 1.5.10

Answer:

Let $\Omega \subset \mathbb{R}^d$ and assume that u satisfies $u_t = k\Delta u$ on Ω subject to homogenous Dirichlet boundary conditions. Define $E(t) = \int_\Omega u(t,x)^2\,dx$. Show that $dE(t)/dt \le 0$ for all $t \ge 0$.

First, we'll take the derivative of $E(t)$ with respect to time.

$$\frac{dE(t)}{dt} = \frac{d}{dt}\int_\Omega u(t,x)^2\,dx = \int_\Omega 2u(t,x)u_t(t,x)\,dx.$$

Since $u_t = k\Delta u$ on Ω,

$$\int_\Omega 2u(t,x)u_t(t,x)\,dx = 2k\int_\Omega u(t,x)\Delta u(t,x)\,dx = 2k\int_\Omega u(t,x)\mathrm{div}(\nabla u(t,x))\,dx.$$

From Theorem 1.37, we have

$$\int_\Omega \nabla u\cdot\nabla v\,dx = \int_{\partial\Omega} v\nabla u\cdot n(s)\,ds - \int_\Omega \Delta uv\,dx.$$

Rearranging this, we get

$$\int_\Omega \Delta uv\,dx = \int_{\partial\Omega} v\nabla u\cdot n(s)\,ds - \int_\Omega \nabla u\cdot\nabla v\,dx.$$

In our case, just let $v = u$ and we get

$$2k\int_\Omega \Delta uu\,dx = 2k\int_{\partial\Omega} u\nabla u\cdot n(s)\,ds - 2k\int_\Omega \nabla u\cdot\nabla u\,dx.$$

Our solution u satisfies homogenous Dirichlet boundary conditions so $2k\int_{\partial\Omega} u\nabla u\cdot n(s)\,ds = 2k\int_{\partial\Omega} 0(\nabla u\cdot n(s))\,ds = 0$. Then,

$$\begin{aligned}2k\int_\Omega \Delta uu\,dx &= -2k\int_\Omega \nabla u\cdot\nabla u\,dx\\ &= -2k\int_\Omega \sum_{j=1}^d \left(\frac{\partial u}{\partial x_j}\right)^2\,dx\end{aligned}$$

We know $\left(\frac{\partial u}{\partial x_j}\right)^2 \geq 0$, so $\sum_{j=1}^d \left(\frac{\partial u}{\partial x_j}\right)^2 \geq 0$. Hence, $\int_\Omega \sum_{j=1}^d \left(\frac{\partial u}{\partial x_j}\right)^2 dx \geq 0$, which implies

$$\frac{dE(t)}{dt} = -2k \int_\Omega \sum_{j=1}^d \left(\frac{\partial u}{\partial x_j}\right)^2 dx \leq 0.$$

Therefore $\frac{dE(t)}{dt} \leq 0$, which is what we want.

3 Problem 1.5.11

Answer:

We want to find a solution $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ of $u_t + 2u_x = 0$, with the initial condition $u(0,x) = e^{-x^2}$. Note that the transport equation can be rewritten as $\nabla u \cdot (1,2) = 0$. This means the vector $(1,2)$ is parallel to the level curves of u, which means $u(0,x_0) = u(t, x_0 + 2t)$.

Taking $x = x_0 + 2t$ as a level curve, then $x_0 = x - t2$ and

$$u(t,x) = u(0,x_0) = u(0, x - 2t).$$

Given the initial condition $u(0,x) = e^{-x^2}$, our solution $u(t,x)$ is

$$u(t,x) = u(0, x-2t) = e^{-(x-2t)^2}$$

4 Problem 1.5.14

Answer:

We want to find a solution u on the domain $x > 0$ and $t > 0$ to $u_t + x^2 u_{xx} = 0$ with the initial condition $u(0,x) = \cos(x)$ and boundary condition $u(t,0) = 1$

Another way of representing of $u_t + x^2 u_x = 0$ is $\nabla u \cdot (1, x^2) = 0$, which means the level curves of u are parallel to $(1, x^2)$. Thus, if we find a $\varphi(t)$ that satisfies $dx/dt = x^2$ with initial values (t_0, x_0), then the solution will be of the form $f(\varphi(0))$, where f is the initial condition. Suppose we have the IVP $\varphi(t_0) = x_0$, $dx/dt = x^2$. Then,

$$\begin{aligned} \frac{dx}{dt} &= x^2 \\ \int \frac{1}{x^2} dx &= \int dt \\ -\frac{1}{x} &= t + C \\ \varphi(t) &= \frac{-1}{t+C} \end{aligned}$$

Since $\varphi(t_0) = x_0$, then $x_0 = \frac{-1}{t_0+C}$, and $C = \frac{1}{-x_0} - t_0$. Then, $\varphi(t) = \frac{-1}{t+(\frac{1}{-x_0}-t_0)}$. The level curve that goes through (t_0, x_0) is $\varphi(t)$, so $u(t_0, x_0) = u(0, \varphi(0)) = \cos(\varphi(0)) = \cos\left(\frac{1}{\frac{1}{x_0}+t_0}\right) = \cos\left(\frac{x_0}{1+x_0t_0}\right)$. Thus the solution to $u_t + x^2u_x = 0$ on the domain $x > 0$ and $t > 0$ with the initial condition $u(0, x) = \cos(x)$ and boundary condition $u(t, 0) = 1$ is

$$u(x,t) = \cos\left(\frac{x}{1+tx}\right) \tag{1}$$

Please note that the original $\cos(\varphi(0)) = \cos\left(\frac{1}{\frac{1}{x}+t}\right)$ is undefined at the boundary $x = 0$. The limit of $\cos\left(\frac{1}{\frac{1}{x_0}+t_0}\right)$ as $x \to 0$ is 1 and we can define our solution as a piecewise function where $u(x,t) = \cos\left(\frac{1}{\frac{1}{x_0}+t_0}\right)$, when $x > 0$ and $u(x,t) = 1$, when $x = 0$. This piecewise function is continuous at $x = 0$, so the piecewise function can be a solution to our PDE. The solution we state in (1), which is $u(x,t) = \cos\left(\frac{x}{1+tx}\right)$, is defined at $x = 0$ and $\cos\left(\frac{0}{1+t0}\right) = 1$, which is equal to the piecewise function at $x = 0$. Also, $\cos\left(\frac{x}{1+tx}\right) = \cos\left(\frac{1}{\frac{1}{x}+t}\right)$, when $x > 0$. Therefore, I will use equation (1) as the solution to the stated PDE with the stated initial and boundary conditions. In summary,

$$\begin{aligned} u(x,t) &= \cos\left(\frac{x}{1+tx}\right) \\ u(0,x) &= \cos\left(\frac{x}{1+0x}\right) = \cos(x) \\ u(t,0) &= \cos\left(\frac{0}{1+t0}\right) = 1 \end{aligned}$$

$$\text{and } u \text{ satisfies } u_t + x^2u_x = 0$$

Therefore, u is a solution on the domain $x > 0$ and $t > 0$ to $u_t + x^2u_x = 0$ with the initial condition $u(0, x) = \cos(x)$ and boundary condition $u(t, 0) = 1$.

Let's look at what the solution does as $t \to \infty$.

$$\lim_{t\to\infty} \cos\left(\frac{x}{1+tx}\right) = 1$$

so the asymptotic behavior of u as $t \to \infty$ is $u(t, x) \to 1$.

Let's look at the problem for all $x \in \mathbb{R}$. If we look at the level set that would go through the point $(t_0, x_0) = (t_0, -1/t_0)$, using the characteristics from before, $\varphi(t) = 1/t$ and $u(t_0, x_0) = \cos(\varphi(0))$ would be undefined. Thus, our solution u at $(t, -1/t)$ will be undefined. Therefore, our solution cannot exist for all $x \in \mathbb{R}$.