Written Assignment 6

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1 Problem 1

Show that f is continuous on D_f if and only if for every sequence $x_n \in D_f$ that converges to an element of D_f the sequence $f(x_n)$ converges in $\mathbb{R}$.

Proof:
$(\Rightarrow)$

Let f be a function that is continuous on D_f. Let $x_n \in D_f$ be a sequence that converges to $a \in D_f$. We want to show that $f(x_n)$ converges in $\mathbb{R}$. Since f is continuous at a and $x_n \to a$, $f(x_n) \to f(a)$. Therefore, $f(x_n)$ converges in $\mathbb{R}$.

$(\Leftarrow)$ (By Contrapositive)

Let f be a function and suppose f is not continuous at $a \in D_f$. This means there exists an $\epsilon_0 > 0$ such that for all $\delta > 0$ there exists an $x \in D_f$ such that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon_0$. We want to show that there exists a sequence $x_n \in D_f$ such that $x_n \to a$ and $f(x_n)$ diverges.

We shall now construct x_n. Let $x_n = a$ if n is odd. If n is even, choose $x_n \in D_f$ so that it satisfies $|x_n - a| < \frac{1}{n}$ and $|f(x_n) - f(a)| \geq \epsilon_0$. We can do this because f is not continuous at a. We must now show that $x_n \to a$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. If $n \geq N$, then $\frac{1}{n} < \epsilon$. If n is even, then $|x_n - a| < 1/n < \epsilon$. If n is odd, $|x_n - a| = 0 < \epsilon$. Thus, we have found a $N \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - a| < \epsilon$. Therefore, $x_n \to a$.

We must finally show that $f(x_n)$ is divergent. Define n_k to be the sequence of even n. Then the subsequence $f(x_{n_k}) = f(a)$ for all n_k, which means $f(x_{n_k}) \to f(a)$. Consider the subsequence $f(x_{n_j})$ where n_j is the sequence of odd n. Based on how the sequence x_n is defined, we know that there exists an $\epsilon_0 > 0$ such that $|f(x_{n_j}) - f(a)| \geq \epsilon_0$ for all n_j. This means $f(x_{n_j})$ does not converge to $f(a)$. There will be two cases. Suppose there exists an $L \in \mathbb{R}$ such that $f(x_{n_j}) \to L$ and $L \neq f(a)$. Then $f(x_n)$ would contain 2 different subsequences that converge to two different real numbers, and by Corollary 1.5.1 from the book, this would mean that $f(x_n)$ is divergent. Suppose $f(x_{n_j})$ is divergent. Then for all $x \in \mathbb{R}$, $f(x_{n_j})$ does not converge to x. By the contrapositive of Theorem 1.5.1 from the book, this means $f(x_n)$ does not converge to x. Thus, $f(x_n)$ is divergent. In both cases, $f(x_n)$ is divergent.

2 Problem 2

Prove that if f is bounded on D_f if and only if for every sequence $x_n \in D_f$, the sequence $f(x_n)$ has a convergent subsequence.

Proof:
($\Rightarrow$)
Let f be a function with domain D_f. Suppose f is bounded on D_f, which means there is an $M > 0$ such that $|f(x)| \leq M$ for all $x \in D_f$. Let $x_n \in D_f$ be a sequence. We want to show that $f(x_n)$ has a convergent subsequence. Since $x_n \in D_f$, $|f(x_n)| \leq M$ for all $n \in \mathbb{N}$. By the Bolzano-Weierstrass Theorem, $f(x_n)$ has a convergent subsequence.

($\Leftarrow$) (By Contrapositive)
Let f be a function with domain D_f. Suppose f is not bounded on D_f, which means for all $M > 0$ there exists a $x \in D_f$ such that $|f(x)| > M$. We must show that there exists a sequence $x_n \in D_f$ such that $f(x_n)$ has no convergent subsequence.

We shall now construct x_n. Choose $x_1 \in D_f$ such that $|f(x_1)| > 1$. Choose $x_2 \in D_f$ such that $|f(x_2)| > |f(x_1)| + 1$. We continue in this pattern and choose $x_n \in D_f$ such that $|f(x_n)| > |f(x_{n-1})| + 1$.

For future use, we must show that if $n > m$, then $|f(x_n)| > |f(x_m)| + 1$. We will do this by induction on $n - m$. Suppose $n - m = 1$, then by how the sequence is defined $|f(x_n)| > |f(x_m)| + 1$. Now suppose the result is true for $n - m = l$. We want to show that it is true for when $n - m = l + 1$. Suppose $n - m = l + 1$. We know that $|f(x_n)| > |f(x_{n-1})| + 1$ and that $n - 1 - m = l$. By the inductive hypothesis, $|f(x_{n-1})| > |f(x_m)| + 1$. Then, $|f(x_n)| > |f(x_{n-1})| + 1 > |f(x_m)| + 2 > |f(x_m)| + 1$. Thus, if $n > m$, then $|f(x_n)| > |f(x_m)| + 1$

We must now show that $f(x_n)$ has no convergent subsequences. We will do this by contradiction. Suppose $f(x_n)$ has a convergent subsequence denoted $f(x_{n_k})$ and $f(x_{n_k}) \to L$, where $L \in \mathbb{R}$. Consider $\epsilon = \frac{1}{2}$. Then there exists a $K \in \mathbb{N}$ such that if $k \geq K$ then $|f(x_{n_k}) - L| < \frac{1}{2}$. Let $k \in \mathbb{N}$ with $k > K$. Then we have $|f(x_{n_k}) - L| < \frac{1}{2}$ and $|f(x_{n_K}) - L| = |L - f(x_{n_k})| < \frac{1}{2}$. Adding the two inequalities yields $|f(x_{n_k}) - L| + |L - f(x_{n_K})| < 1$. By the triangle inequality, $|f(x_{n_k}) - L + L - f(x_{n_K})| = |f(x_{n_k}) - f(x_{n_K})| \leq |f(x_{n_k}) - L| + |L - f(x_{n_K})| < 1$. Then, $||f(x_{n_k})| - |f(x_{n_K})|| \leq |f(x_{n_k}) - f(x_{n_K})| < 1$, and $|f(x_{n_k})| - |f(x_{n_K})| < 1$. This means $|f(x_{n_k})| < 1 + |f(x_{n_K})|$. But since $k > K$, $n_k > n_K$ and $|f(x_{n_k})| > |f(x_{n_K})| + 1$. This means $|f(x_{n_k})| > |f(x_{n_K})| + 1$ and $|f(x_{n_k})| < 1 + |f(x_{n_K})|$, which is a contradiction. Therefore, $f(x_n)$ has no convergent subsequences.