ASSET PRICING INDEX

POTENTIAL RISKS AND CONTRACTS

2 companies: C & H (Swedish) SEK — ACME (American) \$

Agreement: ACME delivers 1000 computer games for $\underbrace{\text{1000\$ per game}}^{\text{Tot. 1 Mln \$}}$ to C & H in 6 months = T (maturity).

Today currency rate 8,00 SEK/\$ => Different scenarios at time T
- rate is = => 8 Mln SEK
- rate is ↑ => 8.5 Mln SEK
- rate is ↓ => 7.5 Mln SEK

We want to cover this risk

Strategy 1

C & H can buy 1 Mln \$ at 8.000.000 SEK TODAY

But... ① Keep money in bank account and not using
② C & H may not have access to 8 Mln SEK today
} Drawback

Strategy 2

C&H can enter a forward contract => agrees today to buy at time T 1 Mln \$ at a rate K (fixed today). Note that this strategy costs nothing today

e.g. K = 8,2 Possible Scenarios at time T
- -> K = 8,5 SEK/\$ => C & H buys 1 Mln \$ at 8.2 Mln SEK (potential profit)
- -> K = 7,9 SEK/\$ => " " " (potential loss)

Strategy 3

C&H buy an European call option (right to buy at fixed date at a fixed strike price)

Different from forward since it has a price in time 0

Goal for this course:
- ① What is the fair price of the contract? => PRICING (buyer)
- ② How does the seller of an option (any contract) covers for the potential losses? => SELLER (hedging)

Market
- efficency => absence of arbitrage
- mature -> hedge any claim in the market
- price that market attributes to the risk

BINOMIAL MARKET MODEL (1 Period case)

t = running time, we'll have in this case only 2 periods => t: 0 (Today) —— T=1 (Tomorrow)

T = time horizon, Duration

2 traded assets
- ZCB with constant interest rate
  - price today: $B_0 = b$ (we'll assume $b=1$)
  - price at T: $B_T = b \cdot (1+r)$
- stock with stochastic price
  - price today $S_0 = s$
  - price at T:

$$S_T = \begin{cases} s \cdot u & \text{with probability } P_u \text{ (up)} \\ s \cdot d & \text{with prob. } P_d \text{ (down)} \end{cases} \qquad u,d>0,\ s>0$$

We can represent this situation in 2 different ways:
- Tree
- Mathematical

Tree representation:

BOND: $b \longrightarrow b \cdot (1+r)$, with $t=0$ and $t=1=T$

STOCK: $s \xrightarrow{P_u} s \cdot u$, $s \xrightarrow{P_d} s \cdot d$

We'll assume
- $u > d$
- $P_u + P_d = 1$
- $P_d = 1 - P_u$

Mathematical representation:

$S_T = s \cdot Z$ → Z is a binomial R.V. with $P(Z=u) = P_u$, $P(Z=d) = P_d$

Note that
- Bond Value at Maturity (VaM) is DETERMINISTIC => Riskless
- Stock VaM is a RANDOM VARIABLE => Risky

1-PERIOD BINOMIAL MARKET MODELS EXAMPLES

Example 1 ↴

Stock whose value today is 100, it's estimated that tomorrow's value increases to 110 with prob. 1/3 or to 105.
Compute u, d and provide a representation of the stock price in terms of Z.

Tree Representation

100 — $P_u = 1.1$ → $110 = 100 \cdot 1.1$

100 — $P_d = 1.05$ → $105 = 100 \cdot 1.05$

Z Binomial R.V.

$$Z = \begin{cases} 1.1 & p_u = 1/3 \\ 1.05 & p_d = 2/3 \end{cases}$$

$110 = s \cdot u \Rightarrow u = 1.1$ $\left(\frac{110}{s} = u \right)$

$105 = s \cdot d \Rightarrow d = 1.05$

Example 2 ↴

The value of the stock tomorrow depends on the decision of the ECB that has 4 possible decisions with same probability and independent ↴

①	②	③	④
w_1 -> stock's price ↑ 5%	w_2 -> stock's price ↓ 5%	w_3 -> stock's price ↓ 5%	w_4 -> stock's price ↓ 5%

Describe the stock price dynamics given that stock price today is s

$(\Omega, \mathcal{F}, P)$

space state $\Omega = \{w_1, w_2, w_3, w_4\}$

$\mathcal{F}$ → set of all possible events

$P(w_1) = 1/4 = P(w_2) = 1/4 = P(w_3) = 1/4 = P(w_4) = 1/4$

Tree representation ↴

$S_0 = s$

$$S_T = \begin{cases} s \cdot (1.05) & \text{if } w_1 \rightarrow P_u = 1/4 \\ s \cdot (0.95) & \text{if } w_2, w_3, w_4 \rightarrow P_d = 3/4 \end{cases}$$

(branches labelled P_u and P_d)

PORTFOLIO

Def. "portfolio"

Portf is a pair $h=(x,y)$ chosen at $t=0$ and to be valid until $t=T$

n bonds   n stocks

$h(x,y)$

0 ——— T

x and y can be positive (Long (Buy)) or negative (Short (Sell)). Value of the portfolio

$$\begin{cases} t=0,T \\ V_t^h = x \cdot B_t + y \cdot S_t \end{cases}$$

it's stochastic! (completely known at time 0, R.V. at time t)

Example: $h=(3,-1)$

Long on 3 bonds; short on 1 stock

Assumptions (Non realistic in real market)

1) $x, y \in \mathbb{R}$ (short position and fractional holdings are allowed and we can buy/short sell ∞ number of stocks)

2) No bid-ask spread (selling price = buying price)

3) No transaction costs

4) The Market is completely liquid

Review

Bond $\rightarrow B_0 = b \rightarrow B_1 = b\cdot(1+r)$ (1) by default; r risk free rate

Stock $\rightarrow S_0 = s \rightarrow S_1 = \begin{cases} s\cdot u & P_u \\ s\cdot d & P_d = 1-P_u \end{cases}$

Portfolio $\rightarrow h=(x,y)$ x and y decided at $t=0$ to be valid until $t=1$

$V_0^h = x\cdot b + y\cdot s$ → Value of the portf h at time 0

$$V_1^h = x\cdot b\cdot(1+r) + y\cdot S_1 = \begin{cases} x\cdot b\cdot(1+r) + y\cdot Su & P_u \\ x\cdot b\cdot(1+r)+y\cdot Sd & P_d = 1-P_u \end{cases}$$

ARBITRAGE STRATEGY

An arbitrage portfolio is a portfolio $h=(x,y)$ s.t.:
→ Please note that it arises in case of mispricing

ARBITRAGE CONDITIONS (Homework 1, Exercise 2)

1) $V_0^h = 0$ ← the prtf is costless

2) $V_1^h \geq 0$ ← the strategy is risk free (The value of the prtf tomorrow is ≥ 0 with probability 1)

$P(V_1^h < 0) = 1 - P(V_1^h \geq 0) = 0$

3) $P(V_1^h > 0) = \alpha > 0$ ← potential profit at future time

Exercise

$b=1 \quad r=0.2 \quad s=100$

At time $t=1$ S may rise to 110 with $p_u=0.4$ or to 105.

ⓐ Show that the market is not arbitrage free

Bond: $1 \longrightarrow 1.2$

Stock: $100 \xrightarrow{0.4} 110 \Rightarrow u = 1.1$

$100 \xrightarrow{0.6} 105 \Rightarrow d = 1.05$

$Z = \begin{cases} 1.1 & P_u = 0.4 \\ 1.05 & 1-P_u = 0.6 \end{cases}$ $\quad P(V_1^h \geq 0) = 1 \quad P(V_1^h > 0) = 1$

ⓑ Build an arbitrage structure

$h=(x,y) \quad V_0^h = x \cdot b + y \cdot s = 100 \cdot 1 - 1 \cdot 100 = 0$ ← ① zero-cost strategy

$V_1^h = 100 \cdot 1.2 - 1 \cdot 100 \cdot Z = \begin{cases} 120-110=10 & P=0.4 \\ 120-105=15 & P=0.6 \end{cases}$ ② ③

What if $r=0.1$? We still have an arbitrage

What if $r=0.08$? We don't have an arbitrage anymore

ABSENCE OF ARBITRAGE THEOREM

The BMM is arbitrage free IFF $d < 1+r < u$

Potential returns of the stock: d, u — Potential return of the bond: $1+r$

Interpretation

The market is arbitrage free IFF the stocks' return does not dominate the return of the bond $(1+r \leq d < u)$ nor is dominated by the return of the bond $(d < u \leq 1+r)$

Consequence

LAW OF ONE PRICE

If the BMM is arbitrage free, any two prtfs $h=(x,y)$, $h'=(x',y')$ with the same value tomorrow $V_1^h = V_1^{h'}$ (P-as) (Probability almost sure) must have the same value today $V_0^h = V_0^{h'}$

THEOREM PROOF (For the exam)

If the market is arbitrage free, then $d < 1+r < u$.

or equivalently

If $d < 1+r < u$ does not hold, then an arbitrage exists $\Rightarrow \begin{cases} \text{CASE 1} \\ 1+r \leq d < u \\ \text{CASE 2} \\ d < u \leq 1+r \end{cases}$

CASE 1 ⤵ $1+r \leq d < u$

Arbitrage strategy: $\left[\begin{array}{l} \text{Shortsell } s \text{ units of bond } (+s€) \text{ (that are dominated by the stock's return)} \\ \text{Buy the stock } (-s€) \end{array}\right.$

$h=(x,y) \quad x=-s \quad y=1 \Rightarrow V_0^h = -s \cdot 1 + 1 \cdot s = 0$ ① ✓

$$V_1^h = +s\cdot(1+r) - 1\cdot s\cdot Z = \begin{cases} -s\cdot(1+r)+s\cdot u \\ -s\cdot(1+r)+s\cdot d \end{cases} = \begin{cases} s(-(1+r)+u) > 0 & P \\ s(-(1+r)+d) \geq 0 & 1-P>0 \end{cases}$$ ② ✓ ③ ✓

CASE 2 ⤵ $d < u \leq 1+r$

Arbitrage strategy: $\left[\begin{array}{l} \text{Shortsell the stock } (+s€) \\ \text{buy the bond } (-s€) \end{array}\right.$

$h=(x,y)$ $\quad V_0^h = s\cdot 1 - 1\cdot s = 0$ ① ✓

$x = s$

$y=-1$

$$V_1^h = -s\cdot 1\cdot Z + s(1+r) = \begin{cases} s((1+r)-u) \geq 0 & P \\ s((1+r)-d) > 0 & 1-P_u > 0 \end{cases}$$ ② ✓ ③ ✓

THEOREM PROOF BIS

If $d < 1+r < u$, then the market is arbitrage free.

Let $h=(x,y)$ be a prtf s.t. $V_0^h = 0 \Rightarrow x\cdot b + y\cdot s = 0 \Rightarrow x = -y\cdot s$

$$V_1^h = -y\cdot s\cdot(1+r) + y\cdot s\cdot Z = \begin{cases} y\cdot s\overbrace{(-(1+r)+u)}^{>0} & P \\ y\cdot s\underbrace{(-(1+r)+d)}_{<0} & 1-P \end{cases}$$ No arbitrage

If $d < 1+r < u$, $V_1^h(u)$ and $V_1^h(d)$ have necessarily different signs. Therefore ② and ③ do not hold $\Rightarrow$ Any prtf with 0 value at time 0 is not an arbitrage.

RISK NEUTRAL PROBABILITY

In a binomial market model with 1 bond and 1 stock the market is arbitrage free IFF $\begin{cases} d < 1+r < u \\ \text{law of one price holds} \end{cases}$ → We can therefore say that $(1+r)$ is a convex combination of d and u

$d \quad 1+r \quad u$

We can so say that ↓

$$* \; \underbrace{1+r = q \cdot u + (1-q) \cdot d}_{A} \quad \text{with } q \in (0,1)$$ (this is a weighted average of u and d)

In our market model we have:

$B: 1 \longrightarrow 1+r$

$S: s \xrightarrow{P_u} S \cdot u$, $\; s \xrightarrow{P_d} S \cdot d$

$S_1 = S \cdot Z$ with $Z = \begin{cases} u & P_u \\ d & 1-P_u \end{cases}$ and $\underbrace{E[Z] = u \cdot P_u + (1-P_u) \cdot d}_{B}$

They are similar but with different weights. We can still interpret A as $E[Z]$. Since $q \in (0,1)$ we can interpret it as probabilities and we can say that in a fictiscious world we have:

"Fictiscious world" Market (The stock has the same return of the bond)

$$Z = \begin{cases} u & q \\ d & 1-q \end{cases} \qquad \begin{aligned} Q(Z=u) &= q \\ Q(Z=d) &= 1-q \end{aligned}$$

$$\overset{Q}{E}[S_1] = s \cdot \underline{u \cdot q} + \underline{s \cdot d \cdot (1-q)} = s \cdot (1 \overset{*}{+} r)$$

If we wanna isolate S (the price) we get: $\boxed{S = \frac{1}{1+r} \cdot \overset{Q}{E}[S_1]}$ → Risk neutral probability

Today's price of the stock = Discounted Expected Future value of a Stock in a specific world Q where $\begin{cases} Q(Z=u) = q \\ Q(Z=d) = 1-q \end{cases}$

"Risk Neutral Valuation Formula"
"Risk adjusted probability"
"Equivalent Martingale measure"

Discount Factor Review

$\frac{C}{1+r} \quad \underset{0}{|} \longleftarrow \overset{C}{\underset{1}{|}}$

Proposition ↓

For the BMM under non-arbitrage condition $(d < 1+r < u)$ the risk neutral probability Q is given by $Q(Z=u) = q$ with $\boxed{q = \frac{(1+r) - d}{u - d}}$

Proof ↓

$$\begin{aligned} \text{Non arbitrage} &\iff 1+r = qu + (1-q) \cdot d \quad (A) \\ &\iff 1+r = q(u-d) + d \\ &\iff q = \frac{(1+r) - d}{u - d} \end{aligned}$$

Q is independent of real world probabilities P and does not depends on the level of risk aversion of an agent in the market but only depends on r, u, d which are subjective features of the market.

AGENT IN THE MARKET

Characteristics:

- Profit maximiser
- Risk Averse → (Utility function describes level of risk aversion in the market)

expected payoff $\rightarrow Y_1 \rightsquigarrow U(Y_1) \leftarrow$ random, so we take the expectation $E[U(Y_1)]$

Certainty equivalent: $CE(Y_1) = U^{-1}(E[U(Y_1)])$

this is the fixed amount that makes an agent indifferent between investing in an "asset" with random pay-off Y_1 (and taking the risk) and taking the money at time 1 and do nothing.

If the utility function $U(Y_1)$ is linear $U(Y_1) = Y_1$ the agent is risk neutral

$$\Downarrow$$

$CE(Y_1) = E[Y_1]$ Value at time 1

What if I want the value at time 0?

$\dfrac{CE(Y_1)}{1+r}$ (Value that an agent assigns today to a random payoff)

$$Y_0 = \frac{CE(Y_1)}{1+r} = \frac{E[Y_1]}{1+r}$$

→ the same of the risk neutral valuation formula

→ expected discounted value of the future pay-off

for a risk neutral agent

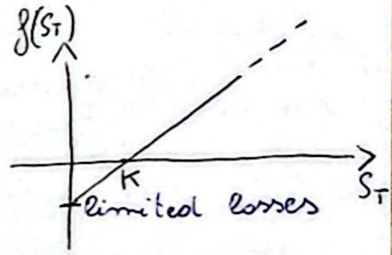
Securities
- Basic Securities
  - Fixed Income
    - Bond
    - Bank Account
    - ...
  - Equities
    - Stocks
- Derivatives
  - Options
    - Plain Vanilla
      - Only characterized by strike and maturity
    - Exotic
  - Swap
  - Forward Futures

Forward / Futures
(OTC) (REGULATED)

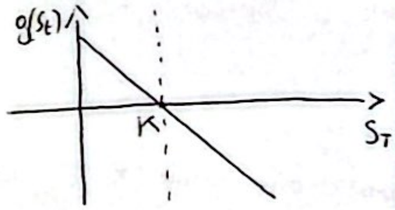
- $t=0$
- $t=T$

One part agrees to buy (is an obligation, main difference with options that are rights) an asset at T, price K ($F(t,T)$). The value of the contract at $t=0$ is ZERO.

Long Payoff
$S_T - K = f(S_T)$



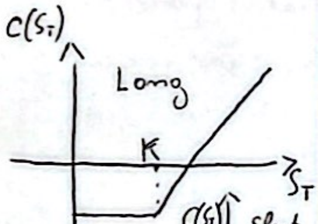
Short Payoff
$K - S_T = g(S_T)$

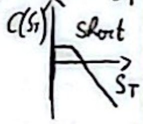


Call options (European, Long)

Gives to the buyer the right (and not the obligation) to buy the underlying at maturity T at price K (strike)

Payoff $C(S_T) = \begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{otherwise} \end{cases}$

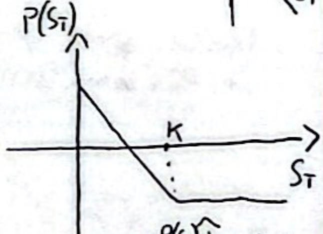


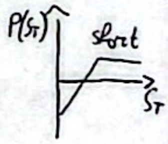


Put options (European, Long)

Gives to the buyer the right to sell the underlying at maturity T at price K (strike)

Payoff $P(S_T) = \begin{cases} K - S_T & \text{if } S_T < K \\ 0 & \text{otherwise} \end{cases}$





European vs American

European → they can be exercised only at expiration

American → they can be exercised at any time prior to expiration

Bermuda → they can be exercised on predetermined dates

Terminology:

	Call	Put
• in the money	$S_T > K$	$S_T < K$
• at the money	$S_T = K$	$S_T = K$
• out of the money	$S_T < K$	$S_T > K$

$\boxed{\text{Put-Call PARITY}}$

Conditions: Long on a call, Short on a Put with some underlying S, Strike K and maturity T

The payoff of the contract is:

$$\underbrace{C(S_T) - P(S_T)}_{\text{Contract 1}} = \underbrace{S_T - K}_{\text{Contract 2}} \qquad \begin{cases} \text{If } S_T > K & (S_T - K) - 0 \\ \text{If } S_T \leq K & 0 - (K - S_T) \end{cases} = S_T - K$$

Contract 1
- Long Call option
- Short Put option

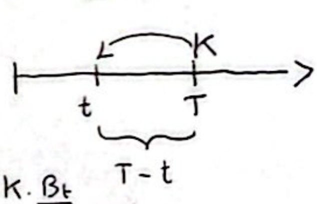
Contract 2

S_T: 1 unit of stock (Long)

K: K units of bonds (Short)

If the market is arbitrage free, the law of one price holds:

$$\boxed{\Pi(t,C) - \Pi(t,P) = S_T - K \cdot (1+r)^{-(T-t)}}$$



Put-Call Parity

Price at time t of the Call option

also know as

$$\Pi(t,C) = \Pi(t,P) + S_T - K \cdot \frac{B_t}{B_t}$$

In the case of dividend distribution:

$$\boxed{\Pi(t,C) - \Pi(t,P) = S_t - D(t,T) - K \cdot \frac{B_t}{B_t}}$$

These are the prices that we'll compute in the next section

expected discounted value of dividend to be distributed from t to T

PRICING AND HEDGING

Consider the 1 period binomial MM with $\begin{cases} \text{1 Bond } B \\ \text{1 Stock } S \end{cases}$. Consider a financial derivative with random payoff X at $t=1$.

In general:

$$X = \phi(S_1) = \begin{cases} \phi(s \cdot u) & P_u \\ \phi(s \cdot d) & 1 - P_u \end{cases}$$ is the payoff

Denote $\pi(t, X)$ the price at t of a derivative with payoff X. For the 1 period model we have

$\pi(1, X) = X$

$\pi(0, X)$

Definition:

We say that a payoff is achievable (or replicable) if there is a strategy $h = (x, y)$ s.t. $V_1^h = X$

Value at time 1 of the strategy h = payoff

replicating strategy or hedging strategy

COMPLETNESS OF THE MARKET

Def.:

If all derivatives can be replicated we say that the market is complete

or

Completness means to be able to cover from the risk using market instruments.

The market that we have now is:

$$\begin{cases} B & 1 \longrightarrow 1+r \\ S & s \xrightarrow{P_u} S \cdot u, \quad s \xrightarrow{P_d} S \cdot d \end{cases} \qquad S_1 = s \cdot Z$$

source of risk

When a derivative can be replicated?

If $X = \phi(S_1) \leftarrow$ payoff is a function of the stock price only

If $X = \phi(S_1, Y) \leftarrow$ with Y R.V. $\neq Z$, investing in the market let us cover S_1, but no chance to cover $Y \longrightarrow$ e.g. political decision

$\rightarrow$ e.g. insurance with mortality probab.

Proposition:

The binomial market is complete

Proof:

Let X be the pay-off of a derivative. In the binomial market $X = \phi(S_1)$.

Let $h = (x, y)$ be a replicating strategy:

$$\text{(A)} \quad V_1^h = X = \phi(S_1) = \begin{cases} \phi(Su) & P_u \\ \phi(Sd) & 1 - P_u = P_d \end{cases}$$

$$\parallel$$

$$\text{(B)} \quad x(1+r) + y \cdot S_1 = \begin{cases} x(1+r) + y \cdot Su & P_u \\ x(1+r) + y \cdot Sd & P_d = 1 - P_u \end{cases}$$

(A) and (B) have to be the same:

$$\begin{cases} x(1+r) + y \cdot Su = \phi(Su) \\ x(1+r) + y \cdot Sd = \phi(Sd) \end{cases} \longrightarrow \begin{cases} x = \dfrac{1}{1+r} \cdot \dfrac{u \cdot \phi(Sd) - d \cdot \phi(Su)}{u - d} \\ y = \dfrac{1}{S} \cdot \dfrac{\phi(Su) - \phi(Sd)}{u - d} \end{cases} \quad (h_x^o, h_y^o)$$

Suppose now that:

$d < 1+r < u \Rightarrow 1 - P$ BMM $\begin{cases} \text{arbitrage free} \\ \text{complete} \end{cases}$

Take a derivative $X = \phi(S_1)$

For the law of one price, we have that $V_1^h = X$

$$\downarrow \qquad \downarrow$$

$$V_0^h = \Pi(0, X)$$

This means that in the BMM, pricing and hedging have the same answer

MARKET 1-PERIOD BINOMIAL

$B_0 = 1 \longrightarrow B_1 = 1 + r$

$$S_0 = s \longrightarrow S_1 = \begin{cases} s \cdot u & p_u \\ s \cdot d & p_d \end{cases}$$

Assumptions

1) $d < 1+r < u \iff$ market is arbitrage free $\Rightarrow$ law of one price holds

1) $\iff Q$ risk neutral measure

$$Q(Z=u) = q \qquad q = \frac{(1+r)-d}{u-d} \qquad s = \frac{1}{1+r} \cdot E^Q[S_1]$$

2) Market is complete, $X = \phi(S)$

2) $\iff$ a hedging strategy $h = (x, y)$
$$\begin{cases} x = \frac{1}{1+r} \cdot \frac{u\phi(sd) - d\phi(su)}{u-d} \\ y = \frac{1}{s} \cdot \frac{\phi(Su) - \phi(Sd)}{u-d} \end{cases}$$

If market is $\begin{cases} \text{arbitrage free} \\ \text{complete} \end{cases}$ $\Rightarrow$

$V_1^h = X$ (completness) every derivative can be replicated

$V_0^h = \Pi(0, X)$ (absence of arbitrage) costless strategy

$$V_0^h = x + y \cdot s$$

$$= \frac{1}{1+r} \cdot \frac{u\phi(Sd) - d\phi(Su)}{u-d} + s \cdot \frac{1}{s} \cdot \frac{\phi(Su) - \phi(Sd)}{u-d} =$$

$$= \frac{1}{1+r} \cdot \left(\frac{u\phi(Sd) - d\phi(Su) + (1+r)\phi(Su) - (1+r)\phi(Sd)}{u-d} \right) =$$

$$= \frac{1}{1+r} \cdot \left(\phi(Su) \cdot \overbrace{\frac{(1+r)-d}{u-d}}^{q} + \phi(Sd) \cdot \overbrace{\frac{u-(1+r)}{u-d}}^{1-q} \right) = \frac{1}{1+r} \cdot E^Q[X] = V_0^h = \Pi(0, X)$$

Exam exercise

$S = 100$ $\quad b = 1$ $\quad u = 1.2$ $\quad d = 0.8$ $\quad r = 0$ $\quad P(Z=u) = P_u = 0.6$

① Verify that the market is arbitrage free

$$d < 1+r < u \;\rightarrow\; 0.8 < 1 < 1.2$$

② Compute the price dynamics

$t = 0$ $\qquad$ $t = 1$

$$B_0 = 1 \longrightarrow B_1 = 1+r = 1$$

$$S_0 = 100 \longrightarrow S_1 = \begin{cases} 120 & P_u = 0.6 \\ 80 & P_d = 0.4 \end{cases}$$

③ Compute the risk neutral probabilities

$$q = \frac{(1+r)-d}{u-d} = \frac{1+0-0.8}{1.2-0.8} = \frac{1}{2} = Q(Z=u)$$

$$1-q = \frac{1}{2} = Q(Z=d)$$

④ Let X be the pay-off of a call option with underlying S_T, $K = 110$, $T = 1$
Compute the price of the derivative

$$X = (S_1 - K)^+ = \begin{cases} (120-110)^+ = 10 & P_u \\ (80-110)^+ = 0 & 1-P_u \end{cases}$$

$$\Pi(0,X) = \frac{1}{1+r} \cdot E^Q[X]$$

$$\Pi(0,X) = \frac{1}{1+0} \cdot \left(10 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2}\right) = 5$$

⑤ Compute the replicating strategy and verify that: • $V_0^h = \Pi(0,X)$
• $V_1^h = \Pi(1,X) = X$

$h = (x, y)$

$$x = \frac{1}{1+r} \cdot \frac{u \cdot \phi(Sd) - d \cdot \phi(Su)}{u-d} \qquad y = \frac{1}{S} \cdot \frac{\phi(Su) - \phi(Sd)}{u-d}$$

$$= \frac{1}{1} \cdot \frac{1.2 \cdot 0 - 0.8 \cdot 10}{1.2 - 0.8} \qquad = \frac{1}{100} \cdot \frac{10-0}{1.2-0.8}$$

$= -20$ → short sell 20 units of bonds $\qquad$ $= \frac{1}{4}$ → long position on 0.25 stock

We can now check

$$V_0^h = x + y \cdot S = -20 + 100 \cdot 0.25 = 5 = \Pi(0,X)$$

$$V_1^h = x(1+r) + y \cdot S_1 = \left.\begin{cases} x(1+r) + y \cdot Su = -20+30 = 10 \\ x(1+r) + y \cdot Sd = -20+20 = 0 \end{cases}\right\} X$$

MULTIPERIOD BINOMIAL MARKET MODEL

We'll have T trading dates, 1 trading period

$t = 0, 1, \dots, T$

$0 \quad 1 \quad 2 \quad 3 \quad \dots \quad T$

r = periodic interest rate

R = annual interest rate

Monthly interest rate (compounded)

$1+R = (1+r)^{12}$

$r = (1+R)^{1/12} - 1$

Price dynamics:

$B_0 = 1 \qquad B_1 = (1+r) \qquad B_2 = (1+r) \cdot B_1 = (1+r)^2 \qquad B_T = (1+r)^T$

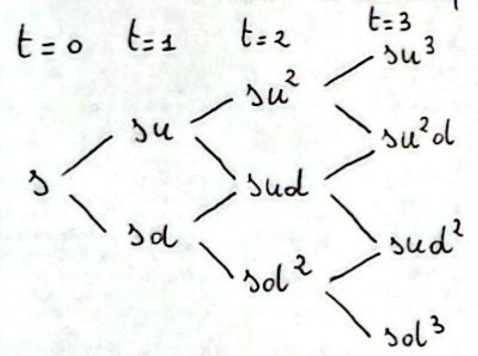
Stock: u, d constant $\quad d < u$

$$S_0 = s \qquad S_1 = \begin{cases} Su & p_u \\ Sd & 1-p_u \end{cases} \qquad \begin{cases} (Su)u \\ (Su)d \end{cases} \quad \begin{cases} (Sd)u \\ (Sd)d \end{cases}$$

or in short

$S_0 = s \qquad S_1 = S_0 \cdot Z_0 \qquad S_2 = S_1 \cdot Z_1 \quad \dots \quad S_{n+1} = S_n \cdot Z_n \quad (Z_n)_{n \geq 0}$ are i.i.d. binomial R.V. with $Z_n \begin{cases} u & p_u \\ d & 1-p_u \end{cases}$

or tree representation → can be too long



We'll collect information to filter the tree

Def. "Information"

Information is a sequence $(\mathcal{F}_n)_{n \geq 0}$ of σ algebras that at each time n tells us what are plausible state of nature (i.e. tells us which part of the tree we can erase)

From a mathematical point of view:

$\mathcal{F}_n = \sigma(S_0, S_1, S_2, \dots, S_n)$

$\mathcal{F}_0 = \sigma(S_0) \leftarrow$ at time 0 we have the trivial σ algebra since we have no information at all.

Def. "Filtration"

The information set $(\mathcal{F}_n)_{n \geq 0}$ is a sequence of σ algebra (contains all the plausible information) that have some characteristics:

$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ ($\mathcal{F}_2$ is more informative than $\mathcal{F}_1, \mathcal{F}_0 \dots$)

At each point in time $\mathcal{F}_n$ contains all the information from time 0 to time n:

$$\mathcal{F}_n = \sigma(S_0, S_1, \dots, S_n)$$

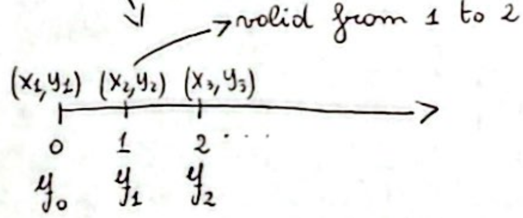
PORTFOLIO

Def

A prtf is a stochastic process $\{h_t = (x_t, y_t),\ t = 1, \dots, T\}$ s.t.

- h_t is $\mathcal{F}_{t-1}$-measurable
- x_t and y_t are respicterely the amount invested in bond (or in the bank account) and the number of shares/stock decided at $t-1$ to hold until time t (with $(x_0, y_0) = (x_1, y_1)$).

What does it means?

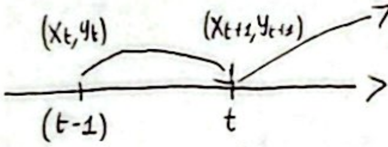


Value of our prtf

$V_0^h = x_1 + y_1 \cdot S$

$V_1^h = x_1 \cdot (1+r) + y_1 \cdot S_1$

$V_2^h = x_2 \cdot (1+r) + y_2 \cdot S_2$



At each point in time you decide how much do you wanna invest until the next period.

$x_t(1+r) + y_t \cdot S_t = x_{t+1} + y_{t+1} \cdot S_t$

Def. Self-financing strategy

A Self-financing strategy is a prtf strategy with satisfies the budget equation

$$\underbrace{x_t(1+r) + y_t \cdot S_t}_{\text{Value of prtf 1 second before rebalancing } (V_t^h)} = \underbrace{x_{t+1} + y_{t+1} \cdot S_t}_{\text{Value of the prtf 1 second after rebalancing } (V_{t+1}^h)}$$

Def. Arbitrage

Conditions to be an arbitrage $\begin{cases} V_0^h = 0 \\ V_1^h \geq 0 \text{ with prob. } 1 \\ P(V_1^h > 0) > 0 \end{cases}$

Please note: If it starts costless, it remain costless

An arbitrage is a self-financing prtf s.t. $\begin{cases} V_0^h = 0 \\ V_t^h \geq 0 \\ P(V_t^h > 0) > 0 \end{cases}$

Theorem

The Multiperiod BMM is arbitrage free iff $d < 1+r < u$

equivalent to

This condition is equivalent to the existence of a probability measure Q s.t. $Q(z_i = u) = q$, $q = \frac{(1+r)-d}{u-d}$

Risk neutral pricing formula

$$S = \frac{1}{1+r} \cdot E^Q[S_1]$$

RISK NEUTRAL PROBABILITY Q

The risk neutral probability Q is a probability measure s.t.

1) Q is equivalent to P

2) $S_t = \frac{1}{1+r} \cdot E^Q[S_{t+1} \mid \mathcal{F}_t]$ -> Today's price = Expected discounted value of tomorrow's price

$$S_t(1+r) = E^Q[S_{t+1} \mid \mathcal{F}_t] \qquad S_t = E^Q\left[\frac{S_{t+1}}{(1+r)} \middle| \mathcal{F}_t\right]$$

-> being a constant, we can put it wherever we want

We can now divide both sides for $\frac{1}{(1+r)^t}$ nothing else that $\frac{1}{B_t}$

$$\frac{S_t}{(1+r)^t} = \frac{1}{(1+r)^t} \cdot E^Q\left[\frac{S_{t+1}}{1+r} \middle| \mathcal{F}_t\right]$$

$$\tilde{S}_t = E^Q\left[\frac{S_{t+1}}{(1+r)^{t+1}} \middle| \mathcal{F}_t\right]$$

$$\tilde{S}_t = E^Q\left[\tilde{S}_{t+1} \mid \mathcal{F}_t\right]$$

-> "Equivalent Martingale Measure"

Called in this way because the normalized price is a Martingale

$\tilde{S}_t = \frac{S_t}{B_t}$

-> $(\tilde{S}_t)_{t=0,1,\dots,T}$ is a Martingale under Q (but not under P)

Def. -> REPLICABLE DERIVATIVE

A derivative with pay-off X and maturity T is achievable if there exists a self-financing strategy whose value at the maturity is equals to the pay-off ($V_t^h = X = \phi(S_T)$)

Def. -> COMPLETE MARKET

If all the derivative are replicable/achievable/hedgeable, the market is complete.

Theorem ->

The Multiperiod BMM is complete => $X = \phi(S_t)$ -> for an European derivative the pay-off is a function of the stock price only

We know that for the law of one price ->

$$V_T^h = X$$

$$V_t^h = \Pi(t, X)$$

with $d < 1+r < u$

We now want to compute prices via the Binomial Algorithm

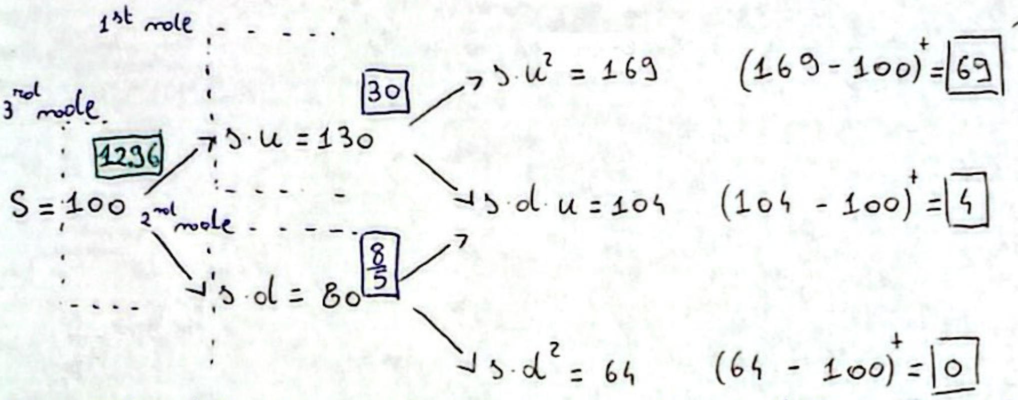
Exam exercise Binomial Market Model (Multiperiod)

$B_0 = 1 \quad S_0 = 100 \quad u = 1.3 \quad d = 0.8 \quad r = 0 \quad P_u = 0.7 \quad T = 2$

⓪ Is the market arbitrage free? $d < 1+r < u \quad 0.8 < 1 < 1.3$

① Compute the market dynamics and risk neutral probability

$B_0 = 1 \longrightarrow B_1 = 1+r = 1 \longrightarrow B_2 = (1+r)^2 = 1$



→ numbers in the box correspond to the value of the derivative

$q = \frac{(1+r)-d}{u-d} = \frac{1-0.8}{0.5} = \frac{2}{5}$

$1-q = \frac{3}{5}$

② Compute the price of European call option, $T=2$, $K=100$ $\quad \phi(S_2) = (S_2-K)^+ = \boxed{}$

③ Compute the hedging strategy

We'll answer at 2 and 3 at the same time

1st node

$$\pi(1,X) = \frac{1}{1+r} \cdot E^Q[\phi(S_2) \mid S_1 = 130] = 1 \cdot \left(\frac{2}{5} \cdot 69 + \frac{3}{4} \cdot 4\right) = \boxed{30}$$

$$\begin{cases} X_2 = 1 \cdot \dfrac{1.3 \cdot 4 - 0.8 \cdot 69}{0.5} = -100 \\ Y_2 = \dfrac{1}{130} \cdot \dfrac{69-4}{0.5} = \underline{1} \end{cases}$$

If $S_1 = 130$ ↴

$V_1^h = -100 + 1 \cdot 130 = 30$ ✓ $= \pi(1,X)$ $\qquad V_2^h = -100 \cdot 1 + 1 \cdot \begin{cases} 169 = 69 \\ 104 = 4 \end{cases} = \phi(S_2)$ ✓ } Strategy self financing

2nd node

$$\pi(1,X) = \frac{1}{1+r} \cdot E^Q[\phi(S_2) \mid S_1 = 80] = 1 \cdot \left(4 \cdot \frac{2}{5} + 0 \cdot \frac{3}{5}\right) = \boxed{\frac{8}{5}} = (1.6)$$

$$\begin{cases} X_2 = 1 \cdot \dfrac{1.3 \cdot 0 - 0.8 \cdot 4}{0.5} = -6.4 \\ Y_2 = \dfrac{1}{80} \cdot \dfrac{4-0}{0.5} = 0.1 \end{cases}$$

If $S_1 = 80$ ↴

$V_1^h = -6.4 + 0.1 \cdot 80 = 1.6 = \pi(1,X)$ ✓ $\qquad V_2^h = -6.4 + 0.1 \begin{cases} 104 = 4 \\ 64 = 0 \end{cases} = \phi(S_2)$ ✓ } Strategy self financing

3rd node

$$\pi(0,X) = 1 \cdot \left(\frac{2}{5} \cdot 30 + \frac{3}{5} \cdot 1.6\right) = \boxed{12.96}$$

$$\begin{cases} X_1 = 1 \cdot \dfrac{1.3 \cdot 1.6 - 0.8 \cdot 30}{0.5} = -43.84 \\ Y_1 = \dfrac{1}{100} \cdot \dfrac{30-1.6}{1.2-0.8} = 0.568 \end{cases}$$

If $S_0 = 100$ ↴

$V_1^h = -43.84 + 0.568 \cdot 100 = 12.96 = \pi(1,X)$ ✓ $\qquad V_2^h = -43.84 + 0.568 \cdot \begin{cases} 130 = 30 \\ 80 = 1.6 \end{cases} = \phi(S_1)$

Theorem ↴

Let X be an European derivative, with market complete and arbitrage free:

$$\pi(t,X) = \frac{1}{(1+r)^{T-t}} \cdot E^Q[X \mid \mathcal{F}_t]$$

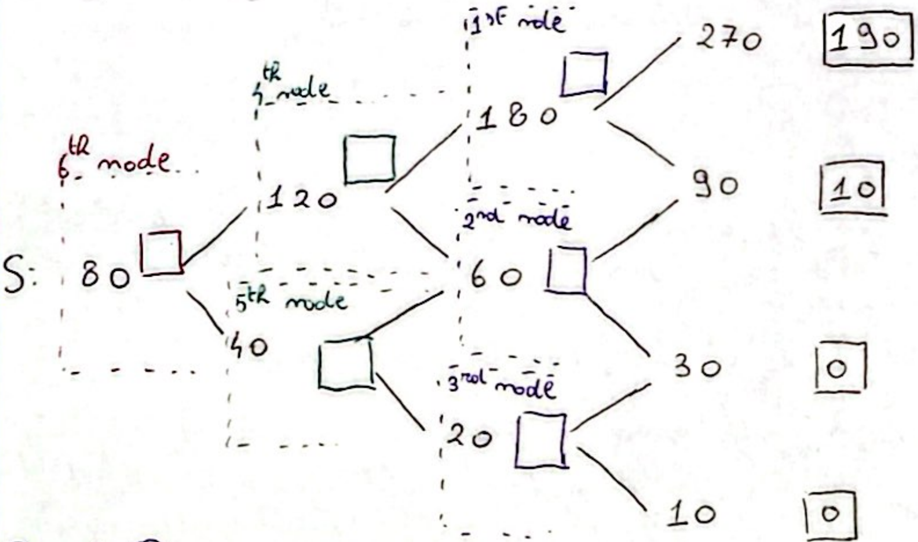
Exercise Multiperiod BMM

$T=3 \quad S_0=80 \quad u=1.5, \; d=0.5, \; r=0, \; P_u=0.6$ → Useless

⓪ Arbitrage free → $d < 1+r < u \qquad 0.5 < 1 < 1.5$ ✓

① Price dynamics, Risk neutral probability

B: $\underset{1}{t_0} \longrightarrow \underset{1}{t_1} \longrightarrow \underset{1}{t_2} \longrightarrow \underset{1}{t_3}$



$q = \frac{(1+r)-d}{u-d} = \frac{1-0.5}{1} = \frac{1}{2}$

$1-q = \frac{1}{2}$

$\phi(S_3) = (S_3 - K)^+ = \square$

② and ③

Price of European call option, $T=3$, $K=80$ and Hedging Strategy

$\Pi(2,X) = \frac{1}{1+r} \cdot E^q\left[\phi(S_3) | S_2 = 180\right]$

$$\begin{cases} X_3 = \frac{1}{1+r} \cdot \frac{u \cdot \phi(Sd) - d \cdot \phi(Su)}{u-d} \\ y_3 = \frac{1}{S_2(180)} \cdot \frac{\phi(Su) - \phi(Sd)}{u-d} \end{cases}$$

MARTINGALE

Def. "Martingale"

Let $(M_t)_{t=0,1,\dots,T}$ be a stochastic process. $(M_t)_{t=0,1,\dots,T}$ is a Martingale on $(\Omega, \mathcal{F}, P)$ if

1. $(M_t)_t$ is F-adapted $\forall t$ (M_t is $\mathcal{F}_t$ measurable)
2. $E[M_t \mid \mathcal{F}_{t-1}] = M_{t-1}$ Best prediction of today's value is yesterday's value

Properties of a Martingale (We have 3 σ algebra $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_u$)

I) $E[M_u \mid \mathcal{F}_t] = M_t$

II) $E\left[E[M_u \mid \mathcal{F}_t] \mid \mathcal{F}_s\right] = M_s$ Tower Rule with $t < s$

As a consequence of $E[M_t] = E[M_0]\ \forall t \Rightarrow$ expectation of a Martingale is constant and does not depend on time

Properties of Conditional Expectation

($\mathcal{F} \subset \mathcal{G}$ are 2 σ algebras, Y R.V., $\mathcal{F}_0 = \{\Omega, \phi\}$ the trivial σ algebra

I) $E[Y \mid \mathcal{F}] = X$ (The R.V. that minimizes the distance between X and Y)

II) $E[Y \mid \mathcal{F}] = Y$ if Y is $\mathcal{F}$-measurable

III) $E[Y \mid \mathcal{F}_0] = E[Y]$ when we condition on the trivial σ algebra

NORMALISED OR DISCOUNTED MARKET

Normalized Market $\left(\cdot \frac{1}{B_t}\right)$

$B_0 = 1 \longrightarrow B_{t+1} = B_t(1+r) = (1+r)^{t+1} \longrightarrow \bar{B}_t = \frac{B_t}{B_t} = 1$

$S_0 = s \longrightarrow S_{t+1} = S_t \cdot Z_t \longrightarrow \bar{S}_t = \frac{S_t}{B_t}$

Let X be a T-derivative (maturity=T) $\longrightarrow \bar{X} = \frac{X}{B_t}$

Def.
- equivalent martingale measure
- risk neutral probability
- risk adjusted probability

It is a probability measure Q

- Q is equivalent to P
- $\bar{S}_t = E^Q[\bar{S}_{t+1} \mid \mathcal{F}_t] \Rightarrow ((\bar{S}_t)_t$ is a Martingale under Q)

1st FUNDAMENTAL THEOREM OF ASSET PRICING

1st FTAP

A market model (whatever market model) is arbitrage free iff there exists at least one equivalent martingale measure

2nd FTAP

If a Market admits a unique martingale measure, it is arbitrage free and complete

Corollary (of 1st FTAP and 2nd FTAP)

Assume that $d < 1+r < u$, then the multiperiod BMM is arbitrage free and complete

Proof

$q = \frac{(1+r)-d}{u-d}$ $\quad 1-q = \frac{u-(1+r)}{u-d}$ $\quad q \in (0,1) \Rightarrow$ we can interprete it as a probability measure

$$\begin{cases} Q(S_t = u \cdot S_{t-1} \mid \mathcal{F}_{t-1}) = q \to \text{probability to jump up} \\ Q(S_t = d \cdot S_{t-1} \mid \mathcal{F}_{t-1}) = 1-q \to \text{p to jump down} \end{cases}$$

- $Q \sim P$
- $\bar{S}_t = E^Q[\bar{S}_{t+1} \mid \mathcal{F}_t]$
- Q is unique

Note that in a Multiperiod BMM

If $(h_t)_t = (x_t, y_t)_t$ is a self-fin. strategy, then $\bar{V}_t^h = \frac{V_t^h}{B_t}$

and

$\bar{V}_t^h = E^Q[\bar{V}_{t+1}^h \mid \mathcal{F}_t] \sim V_t^h = \frac{1}{1+r} \cdot E^Q[V_{t+1}^h \mid \mathcal{F}_t]$

equivalent to

Let X be the pay-off of a T-derivative

$\Pi(t,X) = V_t^h$

$\bar{\Pi}(t,X) = \frac{\Pi(t,X)}{B_t}$ is a martingale under Q

$\Pi(t,X) = \frac{1}{1+r} \cdot E^Q[\Pi(t+1,X) \mid \mathcal{F}_t]$

AMERICAN OPTIONS

We now are in a Multiperiod BMM. $d < 1+r < u$, $t = 0, 1, \dots, T$

Consider an American option with maturity T.

The Pay-off (intrinsic value) of an American option is a stochastic process $(X_t)_{t=1,\dots,T}$

American Call option $\begin{cases} X_t = (S_t - K)^+ \\ X_t = 5 \longrightarrow \text{Stopping Value} \end{cases}$

$=$

$\frac{1}{1+r} \cdot E^Q[\Pi(t+1) \mid \mathcal{F}_t] \to$ Continuation value

we have to compare them

Binomial Algorithm for American options

$t = T \quad \to \quad \Pi(T, (X_t)_t) = X_t$

$t = T-1 \quad \to \quad X_{T-1}$

$\frac{1}{1+r} \cdot E^Q[\Pi(T, (X_t)_t) \mid \mathcal{F}_{t-1}]$

A

$$\Pi(T-1, (X_t)_t) = \max\left\{X_{T-1}, \frac{1}{1+r} \cdot E^Q[\Pi(T, (X_t)_t) \mid \mathcal{F}_{t-1}]\right\}$$

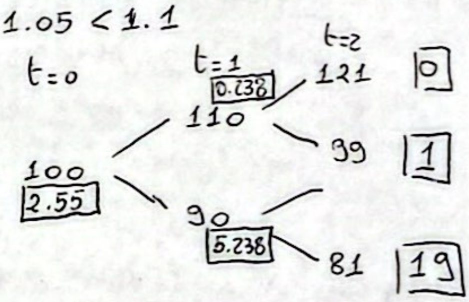
$t = T-n$ We'll repeat these steps until we arrive

Exercise

$S_0 = 100 \quad B_0 = 1 \quad T = 2 \quad d = 0.9 \quad u = 1.1 \quad r = 5\% = 0.05$

Compute the price of an American Put option $K = 100$ and $T = 2$

1) Arbitrage free? $d < 1+r < u \to 0.9 < 1.05 < 1.1$



2) Price dynamics

$t=0 \qquad t=1 \qquad t=2$

$\underline{B}: 1 \longrightarrow 1.05 \longrightarrow (1.05)^2$

3) Payoff

$X_t = (K - S_t)^+$

4) Risk neutral probabilities $\to q = \frac{(1+r) - d}{u - d} = \frac{1.05 - 0.9}{1.1 - 0.9} = 0.75 \Rightarrow q = 0.25$

At $t = 1$

If $S_1 = 110 \to X_1 = 0$ $\left((100-110)^+\right)$ vs $\frac{1}{1+r} \cdot E^Q[\Pi(T, (X_t)_t) \mid \mathcal{F}_{t-1}] = \frac{1}{1.05} \cdot (0 \cdot 0.75 + 1 \cdot 0.25) = \frac{0.25}{1.05} \sim 0.238$

stopping value vs Continuation value

$\Pi(1, (X_t)) = 0.238 \to$ The highest between the two A $\to (0.238 > 0)$

At $t = 1$

If $S_1 = 90 \to X_1 = (100-90)^+ = 10$ vs $\frac{1}{1.05} \cdot (1 \cdot 0.75 + 19 \cdot 0.25) = 5.238$

chance to get much less

$\Pi(1, (X_t) = 10$

At $t = 0$

$X_0 = (100 - 100)^+ = 0$ vs $\frac{1}{1.05} \cdot (0.238 \cdot 0.75 + 10 \cdot 0.25) \sim 2.55$

$\Pi(0, (X_t)) = 2.55$

AMERICAN DERIVATIVES

Consider a Multiperiod BMM, 1 Bond, 1 Stock

Def. ↴

An american Derivative is a financial contract whose payoff is a process $(X_t)_t$.
The exercise time of an American derivative is a stopping time τ.

Def. ↴

A stopping time τ is a R.V. that can assumes values $0, 1, \dots, T$ s.t. $\{\tau \leq t\}$ is $\mathcal{F}_t$-measurable

Theorem ↴ "PRICING FORMULA FOR AN AMERICAN OPTION"

In an arbitrage free market ($d < 1+r < u$), the price of an American derivative with payoff $(X_t)_t$ is ↴

$$\pi(0, (X_t)_t) = E^Q\left[\frac{X_\tau}{B_\tau}\right] \Longleftrightarrow \pi(t, (X_t)_t) = \max\left\{X_t, \frac{1}{1+r} \cdot E^Q\left[\pi(t+1, (X_t)_t) \mid \mathcal{F}_t\right]\right\}$$

Intuitive proof ↴

The seller would ask to the buyer at least an amount that allows to buy a self-financing strategy that replicates at least the intrinsic value (payoff).

$(h_t)_t = (X_t, Y_t)_t$ ← self-financing strategy that you can buy with money comes from the buyer of the option

$V_t^h \geq X_t$ equivalent to $\underbrace{\bar{V}_t^h = \frac{V_t^h}{B_t} \geq \frac{X_t}{B_t}}_{*1}$ → we discount B_t (always ≥ 0) because $\bar{V}_t^h$ becomes a Martingale under the risk neutral measure (with nice properties ① ②)

① Optional Stopping theorem ↴

$$E^Q[\bar{V}_\tau^h] = E^Q[\bar{V}_0^h]$$

since it's deterministic

$$= E^Q\left[\frac{V_0^h}{B_0}\right] = E^Q[V_0^h] = V_0^h \implies E^Q[\bar{V}_\tau^h] = V_0^h$$ ← this equality holds for every stopping time

② ↴

If τ^* is the optimal exercise time $V_{\tau^*}^h = X_{\tau^*} \implies V_0^h = E^Q[\bar{V}_{\tau^*}^h] = E^Q[\bar{X}_{\tau^*}]$

$V_0^h = \underbrace{E^Q[\bar{V}_\tau^h] \geq E^Q[\bar{X}_\tau]}_{*1}$ at every stopping time

↓

equivalent to

↑

$V_0^h \geq \underbrace{\max E^Q[\bar{X}_\tau]}$ for every stopping time. we decide to stop when this quantity ($E^Q[\bar{X}_\tau]$) is as much as largest possible

This should be an equality due to $E^Q(\bar{X}_{\tau^*}) = V_0^h$ at exercise time

$E^Q(\bar{X}_{\tau^*})$

$$\pi(0, (X_t)_t) = \max_\tau E^Q\left[\frac{X_t}{B_t}\right]$$

Generalised Market

$t_0 \qquad t_1$

$B_0 = 1 \rightarrow B_1 = (1+r)^1$

$S \rightarrow S_u, \dots, S_u$

We now have a stocks that can takes more than just 2 values → it can instead takes a set of values:

$\Omega = \{w_1, \dots, w_K\} \quad P(\{w_i\}) > 0 \quad \forall i = 1, \dots, K$

1 Bond: $\quad B_0 = 1 \rightarrow B_t = (1+r)^t$

d Risky Assets $\begin{pmatrix} S_t^1 \\ \vdots \\ S_t^d \end{pmatrix}$ $\quad S_0^i = S_i \rightarrow S_{t+1}^i = S_t^i \cdot Z_t^i$ with $Z_t^i = \begin{cases} \alpha_1^i(t) & P_1 \\ \alpha_2^i(t) & P_2 \\ \vdots & \vdots \\ \alpha_K^i(t) & P_K \end{cases}$

Time independent

Normalised / Discounted Market

$\bar{B}_t = \frac{B_t}{B_t} = 1 \quad \forall t \qquad \bar{S}_t^i = \frac{S_t^i}{B_t}$

$(B_t, S_t^1, \dots, S_t^d)$ original market

$(1, \bar{S}_t^1, \dots, \bar{S}_t^d)$ Normalised/Discounted Market

We now would like to find q to compute the price

$\Pi(0, X) = E^{\mathbb{Q}}\left[\frac{X}{B_T}\right]$

We'll see that the only problem, whatever the market is, is to compute the risk neutral prob. measure

Def. PRTF

A prtf. is a process $(h_t)_t = (h_t^0, h_t^1, \dots, h_t^d)$ where h_t^i is # of units/shares of assets in the prtf decided at time t, to be valid until t+1. The value is

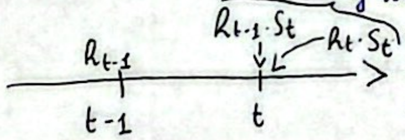
$$V_t^h = \underbrace{h_{t-1}^0 \cdot B_t}_{X_{t-1}} + h_{t-1}^1 \cdot S_t^1 + \dots + h_{t-1}^d \cdot S_t^d = h_{t-1} \odot S_t$$

inner product

Def. SELF FINANCING STRATEGY

A self financing strategy is a prtf that satisfies the budget equation

$$h_{t-1} \cdot S_t = h_t \cdot S_t$$



if we do not withdraw and do not put liquidity these values are the same

For a self-financing strategy, the P&L between two consecutive trading dates is only due to changes in asset prices

$$\begin{cases} V_t^h = h_{t-1} \cdot S_t = h_t \cdot S_t \\ V_{t-1}^h = h_{t-2} \cdot S_{t-1} = h_{t-1} \cdot S_{t-1} \end{cases} \Rightarrow V_t^h - V_{t-1}^h = h_{t-1} \cdot S_t - h_{t-1} \cdot S_{t-1} = h_{t-1} \cdot (S_t - S_{t-1})$$

Def. ARBITRAGE

CONDITIONS

An arbitrage is a self-financing strategy s.t.
1. $V_0^h = 0$
2. $P(V_T^h \geq 0) = 1$
3. $P(V_T^h > 0) > 0$

Def. ARBITRAGE FREE MARKET

$h_0^0 + h_0^1 \cdot s_1 + h_0^2 \cdot s_2 + \dots + h_0^d \cdot s_3 = 0$ But how do we choose h_0^i?

There are infinitely many ways that we must test

Solution: 1^{st} FTAP (To check the arbitrage free conditions)

The market is arbitrage free IFF there is at least one risk neutral probabilit[y] Q (one equivalent martingale measure)

Def. EQUIVALENT MARTINGALE MEASURE

An EMM Q is a probability s.t.

1) $Q \sim P$ (They are equivalent since they have the same null set and sure set)

2) $\boxed{\bar{S}_t^i = E^Q\left[\bar{S}_{t+1}^i \mid \mathcal{Y}_t\right]}$ $\rightarrow S_t^i = \frac{1}{1+r} \cdot E^Q \cdot \left[S_{t+1}^i \mid \mathcal{Y}_t\right]$

$$\begin{cases} q_1, q_2, \dots, q_K \in (0,1) \\ q_1 + q_2 + \dots + q_K = 1 \end{cases}$$

Normalized price of any asset is a Martingale under Q but not under P

Example 1 period case

$$S_0^i = \frac{1}{1+r} \cdot E^Q\left[S_1^i \mid \mathcal{Y}_0\right] = \frac{1}{1+r} \cdot E^Q\left[S_1^i\right] \quad \text{with } S_1^i = \begin{cases} \alpha_1^i & p_1 & q_1 \\ \alpha_2^i & p_2 & q_2 \\ \vdots & \vdots & \vdots \\ \alpha_K^i & p_K & q_K \end{cases}$$

SYSTEM TO BE SOLVED

$$\boxed{S_i = \frac{1}{1+r} \cdot \left(\alpha_1^i \cdot q_1 + \alpha_2^i \cdot q_2 + \dots + \alpha_K^i \cdot q_K\right) \quad \begin{matrix} q_1, q_2, \dots, q_K \in (0,1) \\ q_1 + \dots + q_K = 1 \end{matrix} \quad Q \sim P}$$

⚠

1^{st} FTAP (In generalized discrete time market)

The market is arbitrage free IFF there exists at least one EMM

2^{nd} FTAP

An arbitrage free market is complete IFF the EMM is unique

Pricing Formula

If the market is arbitrage free => any replicable derivative with payoff X has price

$$\Pi(0,X) = E^Q\left[\frac{X}{B_T}\right]$$

$$\Pi(t,X) = B^t \cdot E^Q\left[\frac{X}{B_t} \mid \mathcal{Y}_t\right] \text{ equivalent to } \Pi(t,X) = \frac{1}{(1+r)^{T-t}} \cdot E^Q\left[X \mid \mathcal{Y}_t\right]$$

TRINOMIAL MARKET: EXAMPLES

Example 1

$B_0 = 1 \quad \rightarrow \quad B_t = (1+r)^t \qquad r = 0$

$$S_0 = s = 100 \rightarrow S_1 = \begin{cases} 120 & p = 1/3 \\ 100 & p = 1/3 \\ 80 & p = 1/3 \end{cases}$$

We now look for $\rightarrow q_1, q_2, q_3 \in (0,1)$

$$\begin{cases} q_1 + q_2 + q_3 = 1 \\ S_i = \frac{1}{1+r} \cdot (\alpha_1^i q_1 + \alpha_2^i \cdot q_2 + \ldots + \alpha_K^i q_K) = \frac{1}{1} \cdot (120 \cdot q_1 + 100 \cdot q_2 + 80 \cdot q_3) = 100 \end{cases}$$

System can have
- unsolvable (0 solutions)
- unique solution
- ∞ many solutions

$$\begin{cases} q_1 + q_2 + q_3 = 1 \\ 100 = (120 \cdot q_1 + 100 q_2 + 80 \cdot q_3) \end{cases} \rightarrow \begin{cases} q_1 = 1 - q_2 - q_3 \\ 1 = 1.2(1 - q_2 - q_3) + q_2 + 0.8 q_3 \end{cases} \quad \begin{cases} q_1 = 1 - q_2 - q_3 \\ 0.2 - 0.2 q_2 - 0.4 q_3 \end{cases} \quad \begin{cases} '' \\ q_2 = \frac{0.2 - 0.4 q_3}{0.2} \end{cases}$$

$$\begin{cases} q_1 = q_3 \\ q_2 = 1 - 2q_3 \\ q_3 = q_1 \rightarrow \text{free variable, arbitrary called } \bar{q} \end{cases} \quad \begin{cases} q_1 = \bar{q} \\ q_2 = 1 - 2q_3 \\ q_3 = \bar{q} \end{cases}$$

∞ many solutions parametrised by $\bar{q} \in (0,1)$ but $1 - 2\bar{q} \in (0,1) \Rightarrow$
$\Rightarrow q \in (0, 1/2)$

∞ many EMM $\Rightarrow$ market is arbitrage free (1st FTAP) but not complete (2nd FTAP)

Example 2

The same for bond and r, but...

$$S_0^1 = s_1 = 100 \qquad S_1^1 = \begin{cases} 110 & p_1 = 1/3 \\ 100 & p_2 = 1/3 \\ 90 & p_3 = 1/3 \end{cases}$$

$$S_0^2 = s_2 = 100 \qquad S_2^2 = \begin{cases} 105 & p_1 = 1/3 \\ 100 & p_2 = '' \\ 90 & p_3 = '' \end{cases}$$

Is the market arbitrage free?

$$\begin{cases} q_1 + q_2 + q_3 = 1 \\ 100 = 110 q_1 + 100 q_2 + 90 q_3 \\ 100 = 105 q_1 + 100 q_2 + 90 q_3 \end{cases} \quad \begin{cases} q_1 = \bar{q} \\ q_2 = 1 - 2\bar{q} \\ q_3 = \bar{q} \end{cases}$$

∞ many solutions $\quad q \in (0, 1/2)$

Example 3

The same for bond and r, but...

$$S_0^1 = s_1 = 100 \qquad S_1^1 = \begin{cases} 130 & p_1 = 1/3 \\ 90 & '' \\ 85 & '' \end{cases}$$

$$S_0^2 = s_2 = 100 \qquad S_2^2 = \begin{cases} 105 & p_1 = 1/3 \\ 100 & '' \\ 95 & '' \end{cases}$$

Is the market arbitrage free?

$$\begin{cases} q_1 + q_2 + q_3 = 1 \\ 100 = 130 q_1 + 90 q_2 + 85 q_3 \\ 100 = 105 q_1 + 100 q_2 + 95 q_3 \end{cases} \quad \begin{cases} q_1 = 2/7 \\ q_2 = 3/7 \\ q_3 = 2/7 \end{cases} \quad q \in (0, 1/2)$$

$\exists!$ solution

Example 4

The same for bond and r, but...

$$S_0^1 = s_1 = 100 \qquad S_1^1 = \begin{cases} 130 & p_1 = 1/3 \\ 90 & '' \\ 85 & '' \end{cases}$$

$$S_0^2 = s_2 = 100 \qquad S_2^2 = \begin{cases} 110 & p_1 = 1/3 \\ 105 & '' \\ 97 & '' \end{cases}$$

Is the market arbitrage free?

$$\begin{cases} q_1 + q_2 + q_3 = 1 \\ 100 = 130 q_1 + 90 q_2 + 85 q_3 \\ 100 = 110 q_1 + 105 q_2 + 97 q_3 \end{cases} \quad \begin{cases} q_1 = 21/59 \\ q_2 = 50/59 \\ q_3 = -12/59 \end{cases} \quad q \in (0, 1/2)$$

$\Rightarrow$ market is not arbitrage free

ARBITRAGE STRATEGY?

MULTIPERIOD TRINOMIAL MARKET EXERCISE

$B_0 = 1 \qquad r = 0$

$S_0^1 = 1 \qquad S_{t+1}^1 = S_t \cdot Z_t^1 \qquad Z_t^1 = \begin{cases} 2 = \alpha^1 & p_1 \\ 1 = \alpha^2 & p_2 \\ 1/2 = \alpha^3 & p_3 \end{cases}$ $\qquad \underline{X = (S_2^2 - S_2^1)^+} \to$ Payoff

$S_0^2 = 1 \qquad S_{t+1}^2 = S_t \cdot Z_t^2 \qquad Z_t^2 = \begin{cases} 8/3 = \beta^1 & p_1 \\ 8/9 = \beta^2 & p_2 \\ 1/3 = \beta^3 & p_3 \end{cases}$

Skipping the 1^{st} part (linear system), it turns out that the unique EMM Q is:

$Q(Z_t^1 = \alpha^1 \mid \cancel{\mathcal{F}_{t-1}}) = Q(Z_t = \beta_1 \mid \cancel{\mathcal{F}_{t-1}}) = 1/6 = q_1$ → we could drop it because α and β are time independent

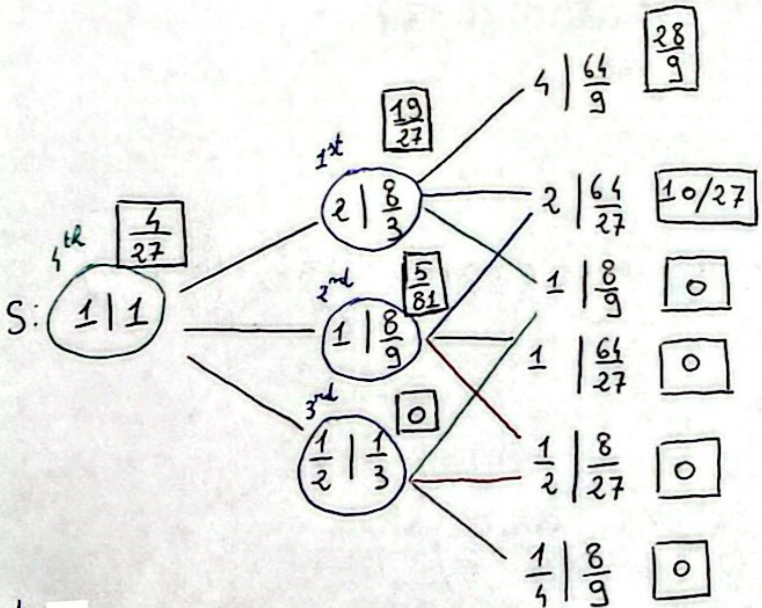
$Q(Z_t^1 = \alpha^2 \mid \cancel{\mathcal{F}_{t-1}}) = Q(Z_t = \beta_2 \mid \cancel{\mathcal{F}_{t-1}}) = \frac{1}{2} = q_2$

$Q(Z_t^1 = \alpha^3 \mid \cancel{\mathcal{F}_{t-1}}) = Q(Z_t = \beta_3 \mid \cancel{\mathcal{F}_{t-1}}) = \frac{1}{3} = q_3$

1) Compute the prices

$$B: \underset{t_0}{1} \longrightarrow \underset{t_1}{1} \longrightarrow \underset{t_2}{1}$$

$\to = \left(\frac{64}{9} - 4\right)^+ = \left(\frac{64-36}{9}\right)^+ = \left(\frac{28}{9}\right)$



$t=1$

At $t=1$, if $(S_1^1, S_1^2) = (2, \frac{8}{3}) \Rightarrow \Pi(1,X) = \frac{1}{1+r} \cdot E^Q\left[\Pi(2,X) \mid (S_1^1, S_1^2) = (2, \frac{8}{3})\right] =$

$= \frac{28}{9} \cdot \frac{1}{6} + \frac{10}{27} \cdot \frac{1}{2} + \cancel{0 \cdot \frac{1}{3}} = \frac{19}{27}$

If $(S_1^1, S_1^2) = (1, \frac{8}{9}) \Rightarrow \Pi(1,X) = \frac{10}{27} \cdot \frac{1}{6} + \cancel{0 \cdot \frac{1}{2}} + \cancel{0 \cdot \frac{1}{3}} = \frac{5}{81}$

If $(S_1^1 = \frac{1}{2}, S_1^2 = \frac{1}{3}) \Rightarrow \Pi(1,X) = 0$ → which is the value for something that in future will give to me 0? 0

$t=0$

At $t=0$, if $(S_0^1, S_0^2) = (1,1)$

$\Pi(0,X) = \frac{1}{1+r} \cdot E^Q\left[\Pi(1,X) \mid (S_0^1, S_0^2) = (1,1)\right] = \frac{19}{27} \cdot \frac{1}{6} + \frac{5}{81} \cdot \frac{1}{2} + \cancel{0 \cdot \frac{1}{3}} = \frac{4}{27}$

2) Hedging strategy

$t=1 \quad (h_2^0, h_2^1, h_2^2)$ replication

$$\begin{cases} h_2^0 \cdot B_2 + h_2^1 \cdot S_2^1 + h_2^2 \cdot S_2^2 = \Pi(2,X) \\ h_2^0 \cdot B_1 + h_2^1 \cdot S_1^1 + h_2^2 \cdot S_1^2 = \Pi(1,X) \end{cases}$$

self-financing

$t=0 \quad (h_1^0, h_1^1, h_1^2)$ replication

$$\begin{cases} h_1^0 \cdot B_1 + h_1^1 \cdot S_1^1 + h_1^2 \cdot S_1^2 = \Pi(1,X) \\ h_1^0 \cdot B_0 + h_1^1 \cdot S_0^1 + h_1^2 \cdot S_0^2 = \Pi(0,X) \end{cases}$$

Self-financing

At $t=1$, if $(2, 8/3)$

$$\begin{cases} h_2^0 \cdot 1 + h_2^1 \cdot 4 + h_2^2 \cdot \frac{64}{9} = \frac{28}{9} \\ h_2^0 \cdot 1 + h_2^1 \cdot 2 + h_2^2 \cdot \frac{64}{27} = \frac{10}{27} \\ h_2^0 \cdot 1 + h_2^1 \cdot 1 + h_2^2 \cdot \frac{8}{9} = 0 \\ \boxed{h_2^0 \cdot 1 + h_2^1 \cdot 2 + h_2^2 \cdot \frac{8}{3} = \frac{19}{27}} \end{cases}$$

$(h_2^0, h_2^1, h_2^2) = \left(\frac{8}{27}, -\frac{35}{27}, \frac{9}{8}\right)$

1^{st} node

$\to \frac{8}{27} - \frac{35}{27} \cdot 2 + \frac{9}{8} \cdot \frac{8}{3} = \frac{8-70+81}{27} = \frac{19}{27}$ ✓

This equation is redundant, we'll use it to check if the solutions are correct

At $t=1$, if $(1, 8/9)$

$$\begin{cases} h_2^0 \cdot 1 + h_2^1 \cdot 2 + h_2^2 \cdot \frac{64}{27} = \frac{10}{27} \\ h_2^0 \cdot 1 + h_2^1 \cdot 1 + h_2^2 \cdot \frac{64}{81} = 0 \\ h_2^0 \cdot 1 + h_2^1 \cdot 1/2 + h_2^2 \cdot \frac{8}{27} = 0 \\ \boxed{h_2^0 \cdot 1 + h_2^1 \cdot 1 + h_2^2 \cdot \frac{8}{9} = \frac{5}{81}} \end{cases}$$

$(h_2^0, h_2^1, h_2^2) = \left(\frac{10}{81}, -\frac{50}{81}, \frac{5}{8}\right)$

2^{nd} node

At $t=1$, if $\left(\frac{1}{2}, \frac{1}{3}\right)$ -> Today's price 0, future's price 0 => $(h_2^0, h_2^1, h_2^2) = (0, 0, 0)$

3^{rd} node

At $t=0$, (We have no if)

$$\begin{cases} h_1^0 \cdot 1 + h_1^1 \cdot 2 + h_1^2 \cdot \frac{8}{3} = \frac{19}{27} \\ h_1^0 \cdot 1 + h_1^1 \cdot 1 + h_1^2 \cdot \frac{8}{9} = \frac{5}{81} \\ h_1^0 \cdot 1 + h_1^1 \cdot \frac{1}{2} + h_1^2 \cdot \frac{1}{3} = 0 \\ \boxed{h_1^0 \cdot 1 + h_1^1 \cdot 1 + h_1^2 \cdot 1 = \frac{4}{27}} \end{cases}$$

$(h_1^0, h_1^1, h_1^2) = \left(\frac{1}{9}, -\frac{20}{27}, \frac{7}{9}\right)$

4^{th} node

CONTINUOUS TIME MARKET MODELS: BLACK & SCHOLES - MERTON MODEL

1 Bond (or Bank account) $B_0 = 1$ $B_t = e^{rt}$ with r = instantaneous risk free rate

1 Stock $S_0 = s$ $S_t = s \cdot e^{(u - \frac{1}{2}\sigma^2)t + \sigma W_t}$

Correction term

$(W_t)_{t \geq 0}$ Continuous time stochastic process called "Brownian Motion"

Volatility of the stock (how much it deviates from it's expected return) => measure of riskiness

$u \in \mathbb{R}$ (even if it should be positive). It's the expected rate of return of the stock

$E[S_t] = s \cdot e^{ut}$ (P)

Reminder: Risk Neutral Pricing Formula

$\Pi(0, X) = E^Q\left[\frac{X}{B_T}\right]$

Under Q, the stock will have the same rate of return of the bond

Def. CONTINUOUS TIME STOCHASTIC PROCESS

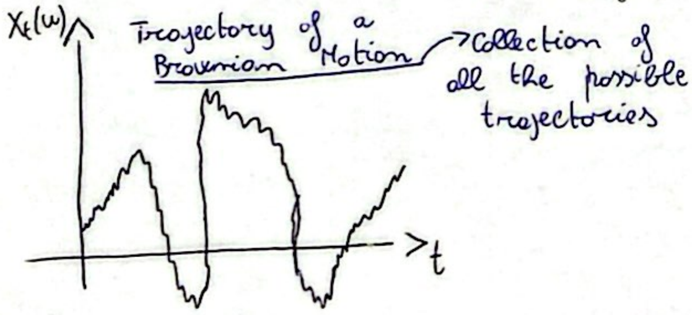
A CTSP $(X_t)_{t \geq 0}$ is a family of R.V.s, indexed by time. At each time t, X_t is a R.V. Fix $(\Omega, \mathcal{G}, P)$, Ω contains infinitely many states ω

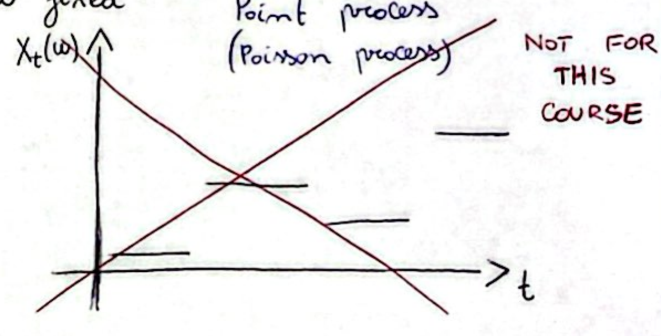
$F = (\mathcal{G}_t)_{t \geq 0}$ filtration. This is a family of σ-algebras indexed by time, s.t $\mathcal{G}_s \subset \mathcal{G}_t$

Def. TRAJECTORY

A trajectory of a Stochastic process is a deterministic function

$t \longrightarrow X_t(\omega)$ for ω fixed





MARTINGALE IN CONTINUOUS TIME

A Martingale in continuous time is a stochastic process $(M_t)_{t\geq 0}$ such that:

1) M_t is $\mathcal{F}$-measurable -> measurability means that at time t we know exactly what M_t is

2) $E[M_t | \mathcal{F}_t] = M_s \quad \forall s<t$ -> today's value is the best prediction for the future -> Pay attention to that

By definition of Martingale we get:

I) $E[M_t] = E[M_0] \quad \forall t$ -> (A Martingale has constant expectation)

II) $E[E[M_t|\mathcal{F}_s]|\mathcal{F}_u] = M_u$ -> (Tower Rule)

MARKOV PROCESS

A Markov Process $(X_t)_{t\geq 0}$ is a stochastic process with the property:

$$E[f(X_t)|\mathcal{F}_s] = E[f(X_t)|X_t] = G(s, X_t)$$

Usually they are different *1

?

i.e. If we wanna predict the future distribution of the process conditional to the whole past history, the only information that matters is the today's value of the process and not the history.

Please pay attention
We don't know if the future distribution will be the today's distribution (generally no) *1

BROWNIAN MOTION -> (It has a Gaussian distribution with $\mu = 0$, $\sigma^2 = T$)

Def. ↴

A BM is a stochastic process $(W_t)_{t \geq 0}$ adapted to $\mathcal{F}$ (W_t is $\mathcal{F}$-measurable) with the following features:

1) $W_0 = 0$

2) $W_t - W_s \overset{*1}{\sim} N(0, t-s) \quad \forall s < t$

3) $0 < t_1 < t_2 < t_3 \ldots$

$W_{t_2} - W_{t_1}$, $W_{t_3} - W_{t_2} \ldots$ are independent and independent of $\mathcal{F}_{t_{i-1}}$

4) $(W_t)_{t \geq 0}$ has continuous trajectory

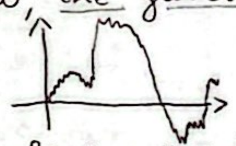
2)+3) mean that a BM has independent and stationary increments $(W_t - W_s)$

Important facts ↴

Cond (1) + Cond (2) => $W_t = W_t - W_0$

Cond (3) => If W_t drives the price of a stock, then future changes of the price are independent of today's value

Cond (4) => For each fixed state of nature ω, the function (the trajectory) $t \to W_t(\omega)$ is $(0, +\infty) \to \mathbb{R}$ continuous however it is irregular



so that it's not differentiable anywhere w.r.t. time

Properties ↴

A) A Brownian Motion is a Martingale. This means that it should be:

I) Adapted to some measure: W_t is $\mathcal{F}$-measurable ✓

II) $E[W_t \mid \mathcal{F}_s] = W_s \quad \forall s < t$ ↴

$$E[W_t \mid \mathcal{F}_s] = E[W_t \overbrace{- W_s + W_s}^{+\text{and}-} \mid \mathcal{F}_s] = \underbrace{E[W_t - W_s \mid \mathcal{F}_s]}_{(W_t - W_s) \text{ is independent from } \mathcal{F}_s} + \overbrace{E[W_s \mid \mathcal{F}_s]}^{\text{measurable quantity always come out from the expectation}}$$

$$= E[W_t - W_s] + W_s = W_s \quad \forall s < t \quad ✓$$

$*1$

B) A Brownian Motion is a Markov process.

$$E[f(X_t) \mid \mathcal{F}_s] = E[f(X_t) \mid W_s]$$

BLACK & SCHOLES - MERTON MODEL

1 Bond (Bank account) $\quad B_t = e^{rt}$

1 Stock $\quad \boxed{S_t = s\, e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}}$ → Alone it's not sufficient to describe stock's price

"DIFFUSION PROCESS"

• LOCAL DYNAMICS → $\boxed{dS_t = S_t \mu\, dt + S_t \sigma\, dW_t}$ → describes the evolution of stock's price in an infinitesimal time

expected rate of return of the stock

$$dS_t \sim \boxed{S_{t+dt} - S_t \cong S_t \mu\, dt + S_t \sigma (W_{t+dt} - W_t)}$$

• INTEGRAL REPRESENTATION → We simply apply $\int_0^t$ on both sides of local dynamics

$$\int_0^t dS_u = \int_0^t S_u \mu\, du + \int_0^t S_u \sigma\, dW_t$$

We use u instead of t since t is an extreme of integration

$$\boxed{S_t - S_0 = \int_0^t S_u \mu\, du + \int_0^t S_u \sigma\, dW_t}$$

But we now have a problem: W_t is not differentiable

So what is dW_t? We'll give a meaning to $\int_0^t S_u \sigma\, dW_t$

CONSTRUCTION OF STOCHASTIC INTEGRAL ("ITÔ INTEGRAL")

Goal: $\int_0^t Y_u\, dW_t$

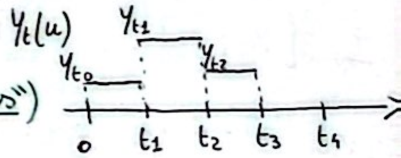
Step 1

$(Y_t)_{t \geq 0}$ that is such that
- I) Y_t is $\mathcal{F}_t$ measurable $\forall t$
- II) $E\left[\int_0^t Y_u^2\, du\right] < +\infty$

A process $(Y_t)_{t \geq 0}$ with these 2 characteristics is an L^2-process $\quad E[\ldots^2] < \infty$

Step 2

Suppose that $(Y_t)_{t \geq 0}$ is piecewise constant (a.k.a. "simple process")



$$\int_0^t Y_u\, dW_u = \int_{t_0}^{t_1} Y_u\, dW_u + \int_{t_1}^{t_2} Y_u\, dW_u + \ldots + \int_{t_{m-1}}^{t_m} Y_u\, dW_u = \int_{t_0}^{t_1} Y_{t_0}\, dW_u + \int_{t_1}^{t_2} Y_{t_1}\, dW_u + \ldots$$

$$= \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} Y_u\, dW_u = \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} Y_{t_i}\, dW_t = \sum_{i=0}^{m-1} Y_{t_i} \cdot \int_{t_i}^{t_{i+1}} dW_u$$

it's constant

→ fundamental theorem of calculus only requires the continuity of the trajectory ⇒ we can apply it

$$= \sum_{i=0}^{m-1} Y_{t_i} \cdot \left(W_{t_{i+1}} - W_{t_i}\right)$$

So, in short: $(Y_t)_{t \geq 0}$ piecewise constant $\Rightarrow \int_0^t Y_u\, dW_u = \sum_{i=0}^{m-1} Y_{t_i} \cdot \left(W_{t_{i+1}} - W_{t_i}\right)$

Step 3

$(Y_t)_{t \geq 0}$ is "general" (non-piecewise constant)

Approximate $(Y_t)_{t \geq 0}$ with a sequence of "simple process" $Y_t \sim Y_t^m$ (where $(Y_t^m)_{t \geq 0}$ is a simple process)

$$\int_0^t Y_u^m\, dW_u = \sum_{i=0}^{m-1} Y_{t_i}^m \left(W_{t_{i+1}} - W_{t_i}\right)$$

We would like to have a small error so we take the lim

$$= \int_0^t Y_u\, dW_u \leftarrow \lim_{m \to \infty} \int_0^t Y_u^m\, dW_u = \lim_{m \to \infty} \sum_{i=0}^{m-1} Y_{t_i}^m \left(W_{t_{i+1}} - W_{t_i}\right)$$

ITÔ INTEGRAL

Let $(Y_t)_{t\geq 0}$ be an L^2- process -> $I_t = \int_0^t Y_u \, dW_u$

$(I_t)_{t\geq 0}$ is a stochastic process

I) I_t is $\mathcal{F}_t$- measurable

II) $I_0 = 0$

III) $(I_t)_{t\geq 0}$ is a martingale → constant expection $= 0$

IV) $E[I_t^2] = E[\int_0^t Y_u^2 \, du] < +\infty$ → finite since it's an L^2- process

or in extended form $\underbrace{E[(\int_0^t Y_u \, dW_u)^2] = E[\int_0^t Y_u^2 \, du]}_{\text{ITÔ ISOMETRY}}$

Back to our B&S-M model, we have

LOCAL DYNAMICS (A) -> $dS_t = S_t u \, dt + S_t \sigma \, dW_t$

$S_t - S_0 = \underbrace{\int_0^t S_u \mu \, du}_{\text{RIEMANN-STIELTJES INTEGRAL}} + \underbrace{\int_0^t S_u \sigma \, dW_u}_{\text{ITÔ INTEGRAL}}$ ← INTEGRAL REPRESENTATION (B)

But how (A) and (B) can be equal to $S = s \cdot e^{(u - \frac{1}{2}\sigma^2)t + \sigma W_t}$?

STOCHASTIC CALCULUS AND ITÔ FORMULA

Review local dynamics
$dS_t = S_t u \, dt + S_t \sigma \, dW_t$
General case
$dX_t = a(t, X_t) dt + b(t, X_t) dW_t$

Motivation

$X = \phi(S_T)$ T- Derivative $\Pi(t, X) = E^Q\left[\frac{X}{e^{r(T-t)}} \middle| \mathcal{F}_t\right]$

It turns out that the B&S-M market is Markovian => S is Markovian too.

$\Pi(t,X) = e^{-r(T-t)} \cdot E^Q[\phi(S_T) | \mathcal{F}_t] = e^{-r(T-t)} \cdot \tilde{G}(t, S_t) = G(t, S_t)$ → Which is this function ?

Remember that $\boxed{dX_t = a(t, X_t) dt + b(t, X_t) dW_t}$ -> LOCAL DYNAMICS OF A DIFFUSION

Let $(X_t)_{t\geq 0}$ be a stochastic process with local dynamics

Assume that: $E[\int_0^T b^2(u, X_u) du] < +\infty$ and $E[\int_0^T |a(u, X_u)| du] < +\infty$

Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a function of two variables: $f(t,x)$

Assume that: $f(t,x)$ is twice differentiable $\left(\frac{\partial f}{\partial t}(t,x), \frac{\partial f}{\partial x}(t,x), \frac{\partial^2 f}{\partial t^2}(t,x), \frac{\partial^2 f(t,x)}{\partial x^2}\right)$

Let $Z_t = f(t, X_t)$ $\boxed{\text{FIND THE DYNAMICS OF } Z_t}$ $(dZ_t = \ldots)$

Remark

$W_{t+\Delta t} - W_t \sim N(0, \Delta t) \Rightarrow \begin{cases} E[W_{t+\Delta t} - W_t] = 0 \\ Var[W_{t+\Delta t} - W_t] = \Delta t = E[(W_{t+\Delta t} - W_t)^2] \end{cases}$

$\Gamma_t = (W_{t+\Delta t} - W_t)^2 \Rightarrow E[\Gamma_t] = \Delta t$

$Var(\Gamma_t) = E[\Gamma^2] - (E[\Gamma])^2 = \underbrace{E[(W_{t+\Delta t} - W_t)^4]}_{4^{th} \text{ moment of a Normal} = 3\sigma^2} - (\Delta t)^2 = 3(\Delta t)^2 - (\Delta t)^2 = 2(\Delta t)^2$

We now have this R.V.

$$(W_{t+\Delta t} - W_t)^2 - \begin{cases} E\left[(W_{t+\Delta t} - W_t)^2\right] = \Delta t \\ Var\left[(W_{t+\Delta t} - W_t)^2\right] = 2(\Delta t)^2 \end{cases}$$

$\rightarrow$ this term is negligible w.r.t. Δt — so small

If Δt is small, $(W_{t+\Delta t} - W_t)$ is almost deterministic (it doesn't have a variation) and is almost equal to it's expectation

Can we make it equals?

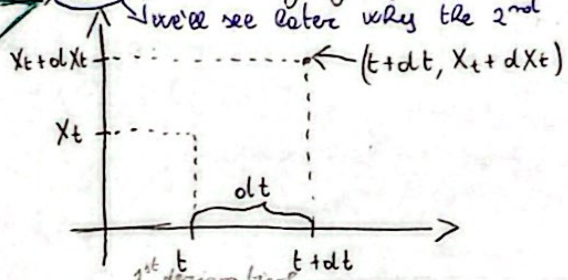
if $\Delta t \rightsquigarrow dt$

$(W_{t+dt} - W_t)^2 = \boxed{(dW_t)^2 = dt}$ and $\boxed{(dt\, dW_t) = 0}$ and $\boxed{(dt)^2 = 0}$

Back to our goal, find the dynamics of Z_t: $dZ_t = \dots$

Take Taylor approximation 2nd order of $f(b, x)$ at the point (t, X_t) with increments dt, dX_t

we'll see later why the 2nd



$$f(t+dt, X_t+dX_t) = \overbrace{f(t,X_t)}^{\text{function}} + \overbrace{\frac{\partial f}{\partial t}(t,X_t)\cdot dt + \frac{\partial f}{\partial X}(t,X_t)\cdot dX_t}^{\text{1st derivative}} +$$

$$\underbrace{+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(t,X_t)\cdot(dt)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(t,X_t)\cdot(dX_t)^2 + \frac{\partial^2 f}{\partial x\,\partial t}(t,X_t)(dt\,dX_t)}_{\text{2nd order approximation}}$$

We now want to substitute the dynamics of X $(dX_t = \overbrace{a(t,X_t)}^{S_{tu}}dt + \overbrace{b(t,X_t)}^{S_{t\sigma}}dW_t)$

$$= \underline{f(t,X_t)} + \underline{\frac{\partial f}{\partial t}(t,X_t)dt} + \underline{\frac{\partial f}{\partial X}(t,X_t)\cdot\left(a(t,X_t)dt + b(t,X_t)dW_t\right)} + \cancel{\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(t,X_t)\cdot(dt)^2} +$$

$$+\underline{\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)}\cdot\left(\cancel{a^2(t,X_t)(dt)^2} + \underline{b^2(t,X_t)\underbrace{(dW_t)^2}_{=dt}} + \cancel{2a(t,X_t)b(t,X_t)(dt\,dW_t)}\right) +$$

$$\underbrace{+\frac{\partial^2 f}{\partial t\,\partial x}(t,X_t)\left(\cancel{a(t,X_t)(dt)^2} + \cancel{b(t,X_t)(dt\,dW_t)}\right)}_{0}$$

We know that $(dW_t)^2 = dt$ and $(dt)^2 = 0$ and $(dt\,dW_t) = 0$ so we can start erasing and we remain with

$$= \overbrace{f(t,X_t)}^{\text{function}} + \overbrace{\frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial X}(t,X_t)\cdot\left(a(t,X_t)dt + b(t,X_t)dW_t\right)}^{\text{1st derivative}} + \frac{1}{2}\frac{\partial^2 f}{\partial^2 x^2}\cdot b^2(t,X_t)\cdot dt$$

We now understand the reason for which we stop at the 2nd order

Further terms will be $\Delta t^n = 0$

ITÔ FORMULA (CONCLUSION)

$dX_t = a(t,X_t)dt + b(t,X_t)dW_t$

We can finally write

ITÔ FORMULA (Local dynamics)

$$\boxed{\underbrace{f(t+dt, X_t+dX_t) - f(t,X_t)}_{df(t,X_t)} = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)b^2(t,X_t)dt}$$

or alternatively

ITÔ FORMULA

$$\boxed{\underset{A}{f(t,X_t)} - \underset{B}{f(0,X_0)} = \underset{C}{\int_0^t \left[\frac{\partial f}{\partial t}(u,X_u) + \frac{\partial f}{\partial x}(u,X_u)a(u,X_u) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(u,X_u)b^2(u,X_u)\right]du} + \underset{D}{\int_0^t \frac{\partial f}{\partial x}(u,X_u)b(u,X_u)dW_u}}$$

(Integral representation)

What are the conditions under which this guy is a Martingale?

Constant Expectation

Let's compute it

$$\underset{A}{E[f(t,X_t)]} = \underset{B}{f(0,X_0)} + \underset{C}{E\left[\int_0^t \dots du\right]} + \underset{D}{E\left[\cancel{\int_0^t \dots dW_u}\right]}$$

If $I_0 = 0$ and I is a Martingale => Constant expectation

-> This one cancels out since it's an ITÔ integral

If $\frac{\partial f}{\partial x}(t,X_t) \cdot b(t,X_t)$ is an L^2-process

$\Rightarrow E\left[\int_0^t \dots dW_u\right] = 0$

So we have

Reimann-integral

$$E[f(t,X_t)] = f(0,X_0) + E\left[\int_0^t \underbrace{\frac{\partial f}{\partial t}(u,X_u) + \frac{\partial f}{\partial x}(u,X_u)a(u,X_u) - \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(u,X_u)b^2(u,X_u)}_{I}\right]du$$

Under which condition this expectation is constant?

The only possibility to get the integral constant is to have the integrand (I) equals to 0

In conclusion

If the integrand is 0 => $f(t,X_t)$ has constant expectation $= \forall t$ and it's a Martingale

Exercise

$dX_t = \left(u - \frac{1}{2}\sigma^2\right)dt + \sigma\, dW_t \qquad f(t,X) = e^X \qquad S_t = e^{X_t}$ → Remember that it requires: $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial x} = e^{X_t}$, $\frac{\partial^2 f}{\partial x^2} = e^{X_t}$

• Find the local dynamics of S_t using the ITÔ formula

$$dS_t = d f(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)\,dX_t + \frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial x^2}(t,X_t)}_{e^{X_t}}\underbrace{b^2(t, X_t)}_{\sigma^2}dt =$$

$$= \cancel{0\, dt} + e^{X_t}\left[\left(u - \frac{1}{2}\sigma^2\right)dt + \sigma\, dW_t\right] + \frac{1}{2}e^{X_t}\sigma^2\, dt =$$

$$= e^{X_t}\left(u - \cancel{\frac{1}{2}\sigma^2} + \cancel{\frac{1}{2}\sigma^2}\right)dt + e^{X_t}\sigma\, dW_t$$

$$= e^{X_t}\cdot u \cdot dt + e^{X_t}\sigma\, dW_t = \boxed{S_t\, u \cdot dt + S_t\, \sigma\, dW_t}$$

→ Local dynamics of the Stock in the B-S-M model

or equivalently

$$X_t = X_0 + \int_0^t \left(u - \frac{1}{2}\sigma^2\right)du + \int_0^t \sigma\, dW_t = X_0 + u - \frac{1}{2}\sigma^2 + \sigma(W_t - \cancel{W_0})$$

$$\Rightarrow S_t = e^{X_0}\cdot e^{\left(u - \frac{1}{2}\sigma^2\right) + \sigma W_t}$$

$\left(X_t = X_0 + \left(u - \frac{1}{2}\sigma^2\right) + \sigma W_t \Rightarrow e^{X_t} = e^{X_0}\cdot e^{\left(u - \frac{1}{2}\sigma^2\right) + \sigma W_t}\right.$

$\Downarrow$

$S_t = e^{X_0}\cdot e^{\left(u - \frac{1}{2}\sigma^2\right) + \sigma W_t}$

PRODUCT RULE

$dX_t = a(t, X_t)dt + b(t, X_t)dW_t \qquad dY_t = \alpha(t, X_t)dt + \beta(t, X_t)dU_t$

U_t, W_t Brownian Motion with correlation equals to ρ

• Compute

Covariance between X and Y

$$\boxed{dZ_t = d(X_t, Y_t) = X_t\, dY_t + Y_t\, dX_t + \rho \cdot b(t, X_t)\cdot \beta(t, Y_t)\, dt} \quad \text{with } Z_t = X_t \cdot Y_t$$

$dS_t = S_t\, u\, dt + S_t\, \sigma W_t$

$B_t = e^{rt}$ or local dynamics: $dB_t = B_t\, r\, dt$ (not apply ITÔ formula since it's deterministic) *1

$B_t^{-1} = e^{-rt}$ or local dynamics: $dB_t^{-1} = -B_t^{-1} r\, dt$

Review Normalised/Discounted Market

$\tilde{S}_t = \frac{S_t}{B_t} = S_t \cdot B_t^{-1}$

$\tilde{B}_t = \frac{B_t}{B_t} = 1$

applying the product rule

$d\tilde{S}_t = S_t\, dB_t^{-1} + B_t^{-1} dS_t + 0$ → Covariance between deterministic and Stochastic is 0

$$= S_t \cdot B_t^{-1}(-r)dt + B_t^{-1}\cdot S_t\, \mu\, dt + B_t^{-1}\cdot S_t\, \sigma\, dW_t$$

$$= \tilde{S}_t(u - r)dt + \tilde{S}_t\, \sigma\, dW_t$$

$d\tilde{B}_t = 0$ → (since it's a constant, the local variation is zero)

LOCAL DYNAMICS EXERCISE

Local dynamics of the asset $dX_u = \mu X_u du + \sigma X_u dW_u$ $\begin{cases} X_0 = 40 \\ \mu = 0.1/\text{year} \\ \sigma = 0.4/\text{year} \end{cases}$

(μ and σ constant)) local dynamics of the B-S model

1) Is it more likely to exercise an European Call or European Put on $(X_t)_{t\geq 0}$ with $K = 24$, $T = 2$?

$C(X_2) = (X_2 - 24)^+$ We definitely exercise the call if $X_2 > 24$

$P(X_2) = (24 - X_2)^+$ We definitely exercise the put if $X_2 < 24$

Compute the probabilities

$P(\text{Call is exercised}) = P(X_2 \geq 24) = \dots = P(Z > -0.974) = 0.835$ *1 — obtained at the end of the exercise

$P(\text{Put is exercised}) = P(X_2 < 24) = 1 - P(X_2 \geq 24) = \dots = 0.165$

But we don't know X_2 => we need to compute it

1st possibility: $X_2 - X_0 = \int_0^2 X_u \mu du + \int_0^2 X_u \sigma dW_u$ -> But we don't know X_u so this is not the right way

2nd possibility: $X_t = X_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ ← $X_t = \dots$ since dX_u is exactly the local dynamics of the B&S model

But are we sure?

Apply the logarithm to X and see if I get the exponential

$f(t, X) = \log(X)$

$$d(\log X_t) = \frac{1}{X_t} \cdot dX_t + \frac{1}{2} \cdot \left(-\frac{1}{X_t^2}\right) \cdot \sigma^2 \cdot X_t^2 dt$$

$$= \frac{1}{\cancel{X_t}} \cdot \cancel{X_t} \mu dt + \frac{1}{\cancel{X_t}} \cdot \cancel{X_t} \sigma dW_t - \frac{1}{2}\sigma^2 \cdot \frac{1}{\cancel{X_t^2}} \cdot \cancel{X_t^2} dt$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$$

$$\log X_t - \log X_0 = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) du + \int_0^t \sigma dW_u$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma(W_t - W_0)$$ ✓

Apply the ITÔ formula to the exponential and see if I get X_t

$f(t, X_t) = e^{X_t}$ PROOF AT PAGE 18

$$e^{\log X_t} = e^{\log X_0} \cdot e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

$$X_t = X_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

So we want to compute $X_2 = X_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)\cdot 2 + \sigma W_2}$ and $P(X_2 > 24)$

$$P\left(X_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)\cdot 2 + \sigma \cdot W_2} > 24\right) \quad \text{with } W_2 \sim (0, 2)$$

We can take the logarithm

$$P\left(\log X_0 + \left(\mu - \frac{1}{2}\sigma^2\right)\cdot 2 + \sigma \cdot W_2 > \log 24\right)$$

$$=$$

$$P\left(W_2 > \frac{\log(24) - \log(X_0) - \left(\mu - \frac{1}{2}\sigma^2\right)\cdot 2}{\sigma}\right)$$

Since we would like to arrive to a Standard Normal we divide for $\sqrt{\sigma^2}$

$$P\left(\frac{W_2}{\sqrt{2}} > \frac{\log(24) - \log(40) - \left(0.1 - \frac{1}{2}0.4^2\right)\cdot 2}{\sigma \cdot \sqrt{2}}\right) = P(Z > -0.974) = 0.835 \text{ *1}$$

2) Consider an European call option with strike K^*. What is the value of K so that the payoff is at least 6 with a probability of at least 20%?

$C(X) = (X_2 - K^*) \geqslant 6 \qquad X_2 = X_0 \cdot e^{(u - \frac{1}{2}\sigma^2)\cdot 2 + \sigma \cdot W_2} \qquad P(X_2 \geqslant 6 + K^*) = 0.2$

$P(X_2 \geqslant 6 + K^*)$

$= P(\log X_0 + (u - \frac{1}{2}\sigma^2)\cdot 2 + \sigma \cdot W_2 \geqslant \log(6 + K^*))$

$$= P\left(\frac{W_2}{\sqrt{2}} \geqslant \frac{\log(6+K^*) - \log(X_0) - (u - \frac{1}{2}\sigma^2)\cdot 2}{\sigma \cdot \sqrt{2}}\right) = 0.2$$

$$= P\left(Z \geqslant \frac{\log(6+K^*) - \log(40) - (0.1 - \frac{1}{2}\cdot(0.4^2))\cdot 2}{\sqrt{2}\cdot 0.4}\right) = 0.2$$

$$= P\left(Z \geqslant \underbrace{\frac{\log(6+K^*) - \log(40) - 0.04}{0.5657}}_{\star}\right) = 0.2$$

$P(Z \geqslant \star) = 0.2$ corresponds to 0.84

$$\frac{(\log(6+K^*) - \log(40) - 0.04)}{0.5657} = 0.84$$

$\log(6+K^*) = (0.84 \cdot 0.5657) + \log(40) + 0.04$

$\log(6+K^*) = 4.2041$

$6 + K^* = e^{4.2041}$

$K^* = e^{4.2041} - 6 = 60.95$

Since it's "at least", we should compute also for value >0.2

$P(Z \geqslant \star) = 0.3$ corresponds to 0.53

$$\frac{\log(6+K^*) - \log(40) - 0.04}{0.5657} = 0.53 \Rightarrow \log(6+K^*) = (0.53 \cdot 0.5657) + \log(40) + 0.04$$

$\log(6+K^*) = 4.028$

$K^* = e^{4.028} - 6 = 50.187$

K^* should be $\leqslant 60.95$

FEYNMAN-KAČ FORMULA

$(X_t)_{t\geq 0}$ $\quad dX_u = \mu(u, X_u)\,du + \sigma(u, X_u)\,dW_u \qquad X_t = X$

Compute $F(t, X_u) = Z_t$

1) Apply the ITÔ formula to get the local dynamics of Z_t $\quad \frac{\partial F}{\partial x_t}(t, X_t)\,dX_t$

$$dZ_t = dF(t,X_t) = \frac{\partial F}{\partial t}(t,X_t)\,dt + \frac{\partial F}{\partial x}(t,X_t)\,\mu(t,X_t)\,dt + \frac{\partial F}{\partial x}(t,X_t)\,\sigma(t,X_t)\,dW_t +$$

$$+ \boxed{\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,X_t)\,\sigma^2(t,X_t)\,dt}$$

2) We want to get conditions to guarantee that Z_t is a Martingale

If $\frac{\partial F}{\partial x}(t,X_t)\cdot\sigma(t,X_t)$ is an L^2-process then the boxed term identifies a Martingale.

$$F(t,X_t) - F(0,X_0) = \boxed{\int_0^t \dots\, du} + \underbrace{\int_0^t \frac{\partial F}{\partial x}(u,X_u)\,\sigma(u,X_u)\,dW_u}_{\in L^2 \Rightarrow \text{ is a Martingale}}$$

This term should be zero

(it should be a constant but that is impossible since it's w.r.t. time)

So, Z_t is a Martingale if $\frac{\partial F}{\partial t}(t,X_t) + \frac{\partial F}{\partial x}(t,X_t)\mu(t,X_t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,X_t)\sigma^2(t,X_t) = 0$

Suppose now that $F(T, X_t) = \phi(X_T)$

$$\Downarrow$$

$$F(t,X) = E[\phi(X_t)\,|\,X_t = X]$$

Why? Demonstrate it

$$dF(u,X_u) = \left\{\frac{\partial F}{\partial t}\cdot(u,X_u) + \frac{\partial F}{\partial x}(u,X_u)\mu(u,X_u) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(u,X_u)\sigma^2(u,X_u)\right\}du + \frac{\partial F}{\partial x}(u,X_u)\sigma(u,X_u)\,dW_u$$

We now take the integral on both sides

$$\int_t^T dF(u,X_u) = \underbrace{\int_t^T\left\{\frac{\partial F}{\partial t}\cdot(u,X_u) + \frac{\partial F}{\partial x}(u,X_u)\mu(u,X_u) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(u,X_u)\sigma^2(u,X_u)\right\}du}_{0} + \int_t^T\frac{\partial F}{\partial x}(u,X_u)\sigma(u,X_u)\,dW_u$$

$0 \rightarrow$ Why this is 0? Because if Z_t is a Martingale, then this term should be 0

$$F(T,X_t) - F(t,X) = \int_t^T \frac{\partial F}{\partial x}(u,X_u)\,\sigma(u,X_u)\,dW_u$$

3) Compute now the conditional expectation given $X_t = X$

$$E[F(T,X_T) - F(t,X)\,|\,X_t = x] = \underbrace{E\left[\int_t \cancel{\frac{\partial F}{\partial x}(u,X_u)}\sigma(u,X_u)\,dW_u\,\middle|\,X_t = X\right]}_{=0}$$

Why we can take it out?

$$E[\underbrace{F(T,X_T)}_{\phi(X_T)}\,|\,X_T = x] - F(t,X) = 0$$

$$F(t,X) = E[\phi(X_t)\,|\,X_t = X] \qquad \blacksquare$$

Summary (1)

Let $F(t,x)$ be the solution of the partial differential equation (PDE)

$$\begin{cases} \frac{\partial F}{\partial t}(t,x) + \frac{\partial F}{\partial x}(t,x)\mu(t,x) + \frac{1}{2}\frac{\partial^2 F}{\partial x}(t,x)\sigma^2(t,x) = 0 \\ F(t,x) = \phi(x) \end{cases}$$

Drift of the process

Volatility of the process

Then $F(t,x) = E[\phi(X_T) \mid X_T = x]$

where $(X_t)_{t \geq 0}$ is the process with local dynamics $dX_u = \mu(u, X_u)du + \sigma(u, X_u)dW_u$

$X_t = x$

Summary (2)

Let $F(t,X)$ be the solution of the PDE

Why do we consider it?

$$\begin{cases} \frac{\partial F}{\partial t}(t,x) + \frac{\partial F}{\partial x}(t,x)\mu(t,x) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,x)\sigma^2(t,X) - r F(t,x) = 0 \\ F(t,X) = \phi(x) \end{cases}$$

Then $F(t,X) = e^{-r(T-t)} \cdot E[\phi(X_T) \mid X_T = X]$ where $dX_u = \mu(u, X_u)du + \sigma(u, X_u)dW_u$

$X_t = X$

FEYNMAN-KAČ EXERCISE

Suppose that the price of a derivative is $\Pi(t) = F(t,X)$ where $F(t,X)$ is the solution of $\rightarrow$ $\frac{\partial F}{\partial t} + \frac{\partial F}{\partial X} \cdot \mu X + \frac{1}{2} \cdot \frac{\partial^2 F}{\partial X^2} \cdot \sigma^2 \cdot X^2 - rF = 0$

1) Write the pay-off

$\phi(X_T) = \ln(X_T^2) \longrightarrow$

2) Write the local dynamics of the underlying asset

$dX_u = aX_u du + bX_u dW_u \qquad X_t = X$

3) Find $F(t,X)$

$F(t,X) = e^{-r(T-t)} \cdot E[\ln(X_T^2) \mid X_T = x]$ stochastic representation of the process F

4) Compute the conditional expectation explicitily

Hint: We know from B&S that $X_t = X_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)t + bW_t}$

$F(t,X) = e^{-r(T-t)} \cdot E[2\ln(X_T) \mid X_T = x]$

$X_t = X \cdot e^{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}$

$\ln(X_t) = \ln(x) + (\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)$

$$E[2\ln(X_T) \mid X_T = x] = E\left\{2\left[\ln(x) + (\mu - \tfrac{1}{2}\sigma^2)(T-t) + \sigma\overbrace{(W_T - W_t)}^{N(0,\,T-t)}\right]\right\}$$

$$= 2\left[\ln(x) + (\mu - \tfrac{1}{2}\sigma^2)(T-t)\right]$$

$$F(t,x) = e^{-r(T-t)} \cdot 2\left[\ln(x) + (\mu - \tfrac{1}{2}\sigma^2)(T-t)\right]$$

DEFINITIONS

$(\Omega, \mathcal{Y}, P)$ $\quad F=(\mathcal{Y}_t)_{t\geq 0}$ $\quad \mathcal{Y}_t=\sigma(S_u, u<t)$

Market

Consider a market with 1 riskless asset (Bond/Bank account) and 1 risky asset (stock)

$dB_t = B_t \cdot r \cdot dt \qquad B_0 = 1 \longrightarrow B_t = e^{rt}$

$dS_t = S_t \cdot \alpha(t, S_t)dt + S_t \cdot \sigma(t, S_t)dW_t, \quad S_0 = s_0$

This is the B&S market iff α and β are constant

Portfolio

A prtf is a process $(h_t)_{t\geq 0} = (h_t^0, h_t^1) \geq 0$. (h_t^0, h_t^1) are $\mathcal{Y}$-measurable and represen[t] units of the bond and shares of stock chosen at t^- to hold until time t

t^- $\quad t$ — potentially could be 1 millisecond

Value

The value of the prtf is $V_t^h = h_t^0 \cdot B_t + h_t^1 \cdot S_t$ ← it holds for any prtf

Self-financing

A prtf is self-financing if it satisfies the BUDGET EQUATION

$$\underbrace{dV_t^h}_{V_t^h - V_{t-1}^h} = \underbrace{h_t^0 dB_t + h_t^1 dS_t}_{h_t^0(B_t - B_{t-1}) + h_t^1(S_t - S_{t-1})}$$

← it holds only for self-financing prtf

Arbitrage

An arbitrage is a self-financing prtf s.t.

$V_0^h = 0$ (costless)

$P(V_T^h \geq 0) = 1$ (riskless)

$P(V_T^h > 0) > 0$ (profit with a positive probability)

1^{st} FTAP

The market is arbitrage free IFF there is a risk neutral probability measure (EMM)

Replicability

A T-derivative with pay-off $X = \phi(S_T)$ is replicable if there exists a self financing strategy s.t. $X = V_T^h$

Completeness

A market is complete iff all T-derivatives are replicable

2^{nd} FTAP

An arbitrage free market is complete IFF the EMM is unique

identifying Q means identifies the stock price distribution

BJORK META-THEOREM

Consider a Market with N-risky assets and D Brownian Motions (Sources of Risk)

1) If $N \leq D$ the market is arbitrage free → many sources of risk and few assets => no mispricing

2) If $N \geq D$ the market is complete → there are too many assets and few sources of risk ⇒ possibility of mispricing

$$dS_t = S_t a\,dt + S_t \sigma_2\, dW_t^1$$
$$dP_t = P_t a\,dt + P_t \sigma_2\, dW_t^1$$

Since they are driven by the same BM, one of the two must be overpriced/underpriced

3) If $N = D$ the market is complete and Arbitrage free

Law of one price →

If the market is arbitrage-free and X is achievable, then →

$$\Pi(t, X) = V_t^h$$ → Value at time t of the replicating prtf

Relative prtf →

A relative prtf (u_t^0, u_t^1) expresses a prtf (h_t^0, h_t^1) in fractions of the wealth

$$u_t^0 = \frac{h_t^0 \cdot B_t}{V_t^h} \qquad u_t^1 = \frac{h_t^1 \cdot S_t}{V_t^h}$$

$$V_t^h = h_t^0 \cdot B_t + h_t^1 \cdot S_t = V_t^h u_t^0 + V_t^h \cdot u_t^1 \Rightarrow u_t^0 + u_t^1 = 1$$

If a prtf (h_t^0, h_t^1) is self-financing

$$dV_t^h = h_t^0 \cdot dB_t + h_t^1 \cdot dS_t$$
$$= u_t^0 \cdot V_t^h \cdot \frac{dB_t}{B_t} + u_t^1 \cdot V_t^h \cdot \frac{dS_t}{S_t}$$
$$= V_t^h \left(u_t^0 \cdot \frac{dB_t}{B_t} + u_t^1 \cdot \frac{dS_t}{S_t} \right)$$

ROOF: $F(t,S_t)$ (2-

y the Bjork Meta-Theorem this market is complete and arbitrage free

$\boxed{\text{Market}}$

ssumptions:

$*^1\; dB_t = B_t\, r\, dt \qquad B_t = e^{rt}$

$*^2\; dS_t = S_t \cdot \alpha(t,S_t)dt + S_t\sigma(t,S_t)\,dW_t$

) The market is perfectly liquid

i) There are no transaction costs

et $X = \phi(S_T)$ be a T-derivative (Contingent claim or T-Claim)

ind $\Pi(t,X)$

Assumption:

$\Pi(t,X) = F(t,S_t) \Rightarrow$ Finding $\Pi(t,X)$ means to find $F(t,s)$

Step 1: Find the local dynamics

$dS_t = S_t\, \alpha(t,S_t)dt + S_t\, \sigma(t,S_t)\,dW_t$

$$
\begin{aligned}
d\Pi(t,X) &= dF(t,S_t)\\
&= \frac{\partial F}{\partial t}(t,S_t)dt + \frac{\partial F}{\partial S}(t,S_t)dS + \frac{1}{2}\cdot\frac{\partial^2 F}{\partial S^2}(t,S_t)\cdot S_t^2\sigma^2(t,S_t)dt\\
&= \frac{\partial F}{\partial t}(t,S_t)dt + \frac{\partial F}{\partial S}(t,S_t)\cdot S_t\alpha(t,S_t)dt + \frac{\partial F}{\partial S}(t,S_t)\cdot S_t\sigma(t,S_t)\,dW_t +\\
&\quad + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(t,S_t)S_t^2\sigma^2(t,S_t)dt\\
&= \left(\frac{\partial F}{\partial t}(t,S_t) + \frac{\partial F}{\partial S}(t,S_t)\cdot S_t\alpha(t,S_t) + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}S_t^2\sigma^2(t,S_t)\right)dt +\\
&\quad + \frac{\partial F}{\partial S}\cdot S_t\,\sigma(t,S_t)\,dW_t
\end{aligned}
$$

Step 1 bis: Mathematical Step (Multiply and Divide times $F(t,S_t)$)

$$
= F(t,S_t)\cdot\underbrace{\left(\frac{\frac{\partial F}{\partial t} + \frac{\partial F}{\partial S}\cdot S_t\alpha(t,S_t) + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}S_t^2\sigma^2(t,S_t)}{F(t,S_t)}\right)}_{\alpha_\Pi(t,S_T)}dt + F(t,S_t)\underbrace{\left(\frac{\frac{\partial F}{\partial S}\cdot S_t\cdot\sigma(t,S_t)}{F(t,S_t)}\right)}_{\sigma_\Pi(t,S_t)}dW_t
$$

$d\Pi(t,X) = dF(t,S_t) = F(t,S_t)\cdot\alpha_\Pi(t,S_T)dt + F(t,S_t)\sigma_\Pi(t,S_t)\,dW_t \leftarrow$ local dynamics of the price

Step 2: Local dynamics of the replicating ptf

(u_t^0, u_t^1) relative ptf with $\quad u_t^0 = \frac{h_t^0 B_t}{V_t^h} \qquad u_t^1 = \frac{h_t^1 S_t}{V_t^h}$

We would like the value of the ptf to satisfy the budget equation

$$
\begin{aligned}
dV_t^h &= V_t^h\cdot u_t^0\cdot\frac{\overset{*1}{dB_t}}{B_t} + V_t^h\cdot u_t^1\cdot\frac{\overset{*2}{dS_t}}{S_t} = V_t^h\cdot u_t^0\, r\,dt + V_t^h u_t^1\left(\alpha(t,S_t)dt + \sigma(t,S_t)\,dW_t\right)\\
&= V_t^h\cdot\left(u_t^0 r + u_t^1\alpha(t,S_t)\right)dt + V_t^h u_t^1\,\sigma(t,S_t)\,dW_t
\end{aligned}
$$

Step 3) Matching   Remember that $F(t,S_t) = \pi(t,S_t) = V_t$ → Matching

$$d\pi(t,S_t) = dF(t,S_t) = \underline{F(t,S_t)} \cdot \alpha_\pi(t,S_T)\,dt + \underline{F(t,S_t)}\,\sigma_\pi(t,S_t)\,dW_t$$

$$dV_t^h = \underline{\underline{V_t^h}} \cdot \left(u_t^0 r + u_t^1 \cdot \alpha(t,S_t)\right) dt + \underline{\underline{V_t^h}}\, u_t^1 \cdot \sigma(t,S_t)\,dW_t$$

For the law of one price $\left(\pi(t,x) = V_t^h = F(t,S_t)\right)$, we so need to equate the 2 coeff

$$\begin{cases} \alpha_\pi(t,S_t) = u_t^0 \cdot r + u_t^1 \alpha(t,S_t) \\ \sigma_\pi(t,S_t) = u_t^1 \cdot \sigma(t,S_t) \\ u_t^0 + u_t^1 = 1 \end{cases} \qquad \begin{cases} u_t^1 = \dfrac{\sigma_\pi(t,S_t)}{\sigma(t,S_t)} \text{ — Computed before} \\ u_t^0 = 1 - u_t^1 = 1 - \dfrac{\sigma_\pi(t,S_t)}{\sigma(t,S_t)} \end{cases}$$

$$\begin{cases} \overbrace{\dfrac{\dfrac{\partial F}{\partial t}(t,S_t) + \dfrac{\partial F}{\partial S}(t,S_t) \cdot S_t\,\alpha(t,S_t) + \dfrac{1}{2}\dfrac{\partial^2 F}{\partial S^2} S_t^2 \sigma^2(t,S_t)}{F(t,S_t)}}^{\alpha_\pi(t,S_t)} = r - \underbrace{r \cdot \dfrac{\frac{\partial F}{\partial S}(t,S_t) \cdot S_t \cdot \cancel{\sigma(t,S_t)}}{F(t,S_t) \cdot \cancel{\sigma(t,S_t)}}}_{r \cdot \left(1 - \frac{\sigma_\pi(t,S_t)}{\sigma(t,S_t)}\right)} + \underbrace{\alpha(t,S_t) \cdot \dfrac{\frac{\partial F}{\partial S}(t,S_t) S_t \cancel{\sigma(t,S_t)}}{F(t,S_t) \cdot \cancel{\sigma(t,S_t)}}}_{\frac{\sigma_\pi(t,S_t)}{\sigma(t,S_t)} \cdot \alpha(t,S_t)} \\ = \\ = \end{cases}$$

We can now multiply both sides times $F(t,S_t)$ (assuming that is $\neq 0$)

$$\underbrace{\frac{\partial F}{\partial t}(t,S_t)}_{1} + \cancel{\frac{\partial F}{\partial S}(t,S_t) \cdot S_t\,\alpha(t,S_t)} + \underbrace{\frac{1}{2}\frac{\partial^2 F}{\partial S^2} \cdot S_t^2 \sigma^2(t,S_t)}_{2} = \underbrace{r \cdot F(t,S_t)}_{3} - \underbrace{r \cdot \frac{\partial F}{\partial S}(t,S_t) S_t}_{4} + \cancel{\frac{\partial F}{\partial S}(t,S_t) \cdot S_t\,\alpha(t,S_t)}$$

$F(t,S_t)$ is the price of the derivative $F(t,S_t) = \pi(t,x)$

PDE

F is the solution of
$$\begin{cases} \overset{1}{\dfrac{\partial F}{\partial t}(t,S)} + \overset{4}{\dfrac{\partial F}{\partial S}(t,S) \cdot r \cdot S} + \overset{2}{\dfrac{1}{2} \cdot \dfrac{\partial^2 F}{\partial S^2}(t,S) \cdot S^2 \cdot \sigma^2(t,S)} - \overset{3}{rF(t,S)} = 0 \\ F(T,S) = \phi(S) \end{cases}$$

"Cauchy problem"

"Kolmogorov Backward equation"

This metodology, not only tells us the price, but also the hedging strategy

$$\begin{cases} u_t^1 = \dfrac{\frac{\partial F}{\partial S}(t,S_t) S_t}{F(t,S_t)} \\ u_t^0 = 1 - u_t^1 \end{cases} \quad \text{or} \quad \begin{cases} h_t^0 = \dfrac{u_t^0 \cdot V_t^h}{B_t} = \dfrac{u_t^0 \cdot F(t,S_t)}{B_t} \\ h_t^1 = \dfrac{u_t^1 \cdot V_t^h}{S_t} = \dfrac{u_t^1 \cdot F(t,S_t)}{S_t} = \dfrac{\partial F}{\partial S}(t,S_t) \end{cases}$$

DELTA OF THE PRICE

This represents the sensitivity of the derivative price w.r.t. e change of the stock price:

Delta ⊕ => When the Stock's price ↑ => Derivative's price ↑

Delta ⊖ => When the Stock's price ↑ => Derivative's price ↓

But it also represents the exposure on the stock:

Delta ⊕ => Long on the Stock

Delta ⊖ => Short on the Stock

BLACK AND SCHOLES MODEL

$dB_t = B_t r \, dt$ —> Bond dynamic

$dS_t = S_t \mu \, dt + S_t \sigma \, dW_t$ —> Stock dynamic

$X = \phi(S_T)$ $\Pi(t,X) = F(t,S_t)$ where $F(t,S_t)$ is the solution of:

$$\left\{\begin{array}{l} \frac{\partial F}{\partial t}(t,S) + \frac{\partial F}{\partial S} \cdot S \cdot r + \frac{\partial^2 F}{\partial S^2}(t,S) \cdot S^2 \cdot \sigma^2 - rF(t,S) = 0 \\ F(T,S) = \phi(S) \end{array}\right]$$

BLACK AND SCHOLES FORMULA

$$\left\{\begin{array}{l} h_t^1 = \frac{\partial F}{\partial S}(t,S_t) \\ h_t^0 = \frac{\mu_t^0 \cdot F(t,S_t)}{B_t} \end{array}\right]$$

Replicating Strategy

(Feynman - Kac)

For the F-K formula we have that:

$$F(t,S) = e^{-r(T-t)} \cdot E[\phi(X_T) \mid X_T = s] \text{ if } dX_t = X_t r \, dt + X_t \sigma(t,X_t) \, dU_t$$

general BM

But we have a problem, we cannot apply this formula since → $r \neq \mu$

we have a different dynamics w.r.t. the one of the F-K formula ($dY_t = Y_t a \, dt + Y_t \sigma \, dW_t$)

But How can we apply FK to our model?

$dB_t = B_t r \, dt$

$dS_t = S_t \mu \, dt + S_t \sigma \, dW_t$

But we want to have $dS_t = S_t r \, dt + S_t \sigma(t,S_t) \, d\tilde{W}_t$ with:

$$\boxed{d\tilde{W}_t = dW_t + \frac{\mu(t,S_t) - r}{\sigma(t,S_t)} \, dt}$$

Market price of risk

"New Brownian Motion"

The only way to obtain it is to modify W_t that is no more a BM on $(\Omega, \mathcal{F}, P)$ but is a BM on $(\Omega, \mathcal{F}, Q)$

Under the risk neutral probability measure

the one under which $\tilde{W}_t$ is a Brownian motion

it changes the distribution of the Brownian motion

We can then apply F-K theorem and:

Let $F(t,y)$ be the solution of:

$$\left\{\begin{array}{l} \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} \cdot r \cdot y + \frac{1}{2} \frac{\partial^2 F}{\partial y^2} \cdot \sigma^2 \cdot y^2 - rF = 0 \\ F(T,y) = \phi(y) \end{array}\right.$$

"RISK NEUTRAL VALUATION FORMULA"

Then $\boxed{F(t,y) = \underbrace{e^{-r(T-t)} \cdot E^Q[\phi(S_T) \mid S_T = s]}_{}}$ where $dS_t = S_t r \, dt + S_t \sigma \, d\tilde{W}_t$

expected discounted value of the pay-off under the probability measure Q

DYNAMICS OF THE MARKET

REAL	VS	RISK NEUTRAL
$dB_t = B_t \cdot r \cdot dt$		$dB_t = B_t r dt$
W_t is a BM		$(\tilde{W}_t)_{t\geq 0}$ is a BM under Q $\quad \tilde{W}_t \sim N(0,t)$
$dS_t = S_t \alpha(t,S_t)dt + S_t\sigma(t,S_t)dW_t$		$dS_t = S_t r dt + S_t \sigma(t,S_t) d\tilde{W}_t$

But the two worlds must be equivalent. the relation is given by

$$d\tilde{W}_t = dW_t + \frac{\alpha(t,S_t) - r}{\sigma(t,S_t)}$$

$$S_t\alpha(t,S_t)dt + S_t\sigma(t,S_t)dW_t = S_t r dt + S_t\sigma(t,S_t)d\tilde{W}_t \quad (\text{isolate } d\tilde{W}_t)$$

Exercise

Derivative with this payoff $X = \phi(S_T)$

Find Price and Hedging Strategy

$\Pi(t,X) = F(t,S_t)$ where $F(t,S)$ is the solution of (the Kolmogorov Backward equation)

$$\begin{cases} \frac{\partial F}{\partial t}(t,s) + \frac{\partial F}{\partial S}(t,s)\cdot S\cdot r + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(t,s)\cdot S^2\cdot\sigma^2(t,s) - rF(t,s) = 0 \\ F(T,s) = \phi(s) \end{cases}$$

$$F(t,S) = e^{-r(T-t)}\cdot E^Q[\phi(S_T) \mid S_t = s]$$

When it's not there we assume that $r=0$

We'll use the dynamics in the Q world

Find the hedging strategy

$$\begin{cases} h_t^1 = \frac{\partial F}{\partial S}(t,S_t) \leftarrow \text{DELTA} \\ h_t^0 = \frac{u_t^0 \cdot F(t,S_t)}{B_t} \end{cases}$$

REAL

$dB_t = B_t\, r\, dt$

W_t is a BM

$dS_t = S_t \cdot \mu \cdot dt + S_t \cdot \sigma \cdot dW_t$

they are constant in the B&S Market

Q

$dB_t = B_t\, r\, dt$

$(\tilde{W}_t)_{t\geq 0}$ BM under Q $\tilde{W}_t \sim N(0,t)$

$dS_t = S_t\, r\, dt + S_t\, \sigma\, d\tilde{W}_t$

$d\tilde{W}_t = dW_t + \frac{\mu - r}{\sigma}\, dt$ → Value that market attributes to risk

We also have to modify the Kolmogorov Backward equation

$X = \phi(S_T)$

$\Pi(t,X) = F(t,S_t)$ where $F(t,S_t)$ is the solution of

$$\begin{cases} \frac{\partial F}{\partial t}(t,S_t) + \frac{\partial F}{\partial S}(t,S_t)\cdot S \cdot r + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(t,S)\cdot S^2 \cdot \sigma^2 - rF(t,s) = 0 \\ F(T,s) = \phi(s) \end{cases}$$

B&S equation

with $F(t,s) = e^{-r(T-t)} \cdot E^Q\left[\phi(S_T) \mid S_t = s\right]$

I) $\tilde{S}_t = \frac{S_t}{B_t}$ is a Martingale under Q

II) $d\tilde{S}_t = \tilde{S}_t \cdot \sigma(t,S_t)\, d\tilde{W}_t$

III) $\tilde{\Pi}(t,X) = \frac{\Pi(t,X)}{B_t} = \frac{F(t,S_t)}{B_t}$ is a Q-Martingale

S_t, $\Pi(t,X)$ under Q have a rate of return $= r$ (the same of the riskless asset

In the risk neutral world any traded financial instrument must have the same rate of return (r)

PRICING OF CALL AND PUT OPTIONS IN THE B&S MARKET

European Call option

Maturity T, Strike K, $X = \phi(S_T) = (S_T - K)^+ = \begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{otherwise} \end{cases}$

Find the price of this option

CALL

$$\boxed{C(t,s) = F(t,s) = sN(d_1(t,s)) - Ke^{-r(T-t)} N(d_2(t,s))}$$

with

I) $N(\cdot)$ is the cumulative distribution of a standard Normal R.V.

$$d_1(t,S) = \frac{1}{\sigma\sqrt{T-t}} \cdot \left[\left(\ln\left(\frac{S}{K}\right)\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right]$$

This represents a standardized measure of how far the current stock price is from the strike price, accounting for time to expiration and the volatility of the stock

$$d_2(t,S) = d_1(t,s) - \sigma\sqrt{T-t}$$

Proof (IDEA) (Not for the exam)

$$F(t,S_t) = e^{-r(T-t)} \cdot E^Q\left[(S_T-K)^+ \mid S_T = s\right]$$

$$dS_t = S_t r\,dt + S_t \sigma\, d\hat{W}_t$$

$$S_t = S \cdot e^{\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(\hat{W}_T - \hat{W}_t\right)\right]}$$

$$Y = \ln(S) + \left(r - \tfrac{1}{2}\sigma^2\right)(T-t) + \sigma\left(\hat{W}_T - \hat{W}_t\right)$$

$$= \ln(S) + \left(r - \tfrac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\cdot Z \qquad Z \sim N(0,1)$$

once we know the distribution, we can plug it into expectation

$$F(t,S_t) = e^{-r(T-t)} \cdot E^Q\left[(S_T - K)\underbrace{I_{S_T>K}}_{\text{indicator function}} \mid S_t = s\right]$$

$$= e^{-r(T-t)} \cdot \int_{\frac{\ln(K/s) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}}^{+\infty} \left(s e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\cdot z} - K\right) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz$$

European put option

PUT

$$\boxed{P(t,s) = K\cdot e^{-r(T-t)} N(-d_2(t,s)) - sN(-d_1(t,S))}$$

Put-Call Parity

$$\boxed{P(t,s) = K \cdot \frac{B_t}{B_t} + C(t,S_t) - S_t}$$

B&S MARKET: EXERCISE

$\mu = 0.1 \quad \sigma = 0.4 \quad K = 18 \quad S_0 = 16 \quad r = 0.04 \quad T = 1$

1) Price at $t=0$, hedging strategy of European call

$$C_0 = S_0 N(d_1) - K e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{16}{18}\right) + \left(0.04 + \frac{1}{2}\cdot(0.4)^2\right)\cdot 1}{0.4 \cdot 1} = 0.0055$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = 0.0055 - 0.4\cdot\sqrt{1} = -0.3945$$

We can finally plug the numbers in

$$C(0, S_0) = 16\, N(0.0055) - e^{-0.04\cdot 1}\cdot 18 N(-0.3945) \sim 2.04\ €$$

2) Price of an European Put with B&S formula and check with Put-Call Parity

$$P_t = K\cdot e^{-r(T-t)}\cdot N(-d_2(t,S)) - S N(-d_1(t,S)) =$$

$$= 18\cdot e^{-0.04\cdot 1}\cdot N(0.3945) - 16 N(-0.0055) \sim 3.34\ €$$

Put-Call Parity $\rightarrow P(0, S_0) = 18\cdot e^{-0.04} + 2.04 - 16 = 3.34\ €$

3) Suppose that in 6 months, the price of the underlying asset is 16.4. Would have it been more convenient to buy the Call option after 6 months (and invest the initial amount in the bond) or at time 0?

Call price in 6 months $\rightarrow$ $C(6\text{ months}, 6\text{ months}) \quad \overset{?}{\bigcirc} \quad C(0, S_0)\cdot e^{r\cdot\frac{1}{2}}$

$$1.356 \quad < \quad 2.04\cdot e^{0.04\cdot\frac{1}{2}}$$

$$C\left(\tfrac{1}{2}, S_{1/2}\right) = S_{1/2}\cdot N\left(d_1\left(\tfrac{1}{2}, S_{1/2}\right)\right) - K e^{-r\cdot\frac{1}{2}} N\left(d_2\left(\tfrac{1}{2}, S_{\frac{1}{2}}\right)\right) =$$ We first have to compute d_1 and d_2

$$= 16.4\cdot N(-0.117) - 18\cdot e^{-0.04\cdot\frac{1}{2}} N(-0.4) \sim 1.356$$

$$d_1\left(\tfrac{1}{2}, S_{\frac{1}{2}}\right) = \frac{\ln\left(\frac{S_{\frac{1}{2}}}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\left(1 - \frac{1}{2}\right)}{\sigma\sqrt{1 - 1/2}} = \frac{\ln\left(\frac{16.4}{18}\right) + \left(0.04 + \frac{1}{2}(0.4)^2\right)\cdot\frac{1}{2}}{0.4\cdot\sqrt{1/2}} = -0.117$$

$$d_2\left(\tfrac{1}{2}, S_{\frac{1}{2}}\right) = d_1\left(\tfrac{1}{2}, S_{1/2}\right) - 0.4\cdot\sqrt{\tfrac{1}{2}} = -0.4$$

So I can invest today in Bonds and in 6 months I can buy the Call option and save/earn the difference.