单变量指数分布

Def • $E_f(f, g, h, \phi, c)$:

通式 $$p(x|\theta) = f(x)g(\theta)\exp(c\cdot\phi(\theta)\cdot h(x))$$

• the family called reglar if $\mathcal{X}$ 不依赖于 θ

E_f 的充分统计量 • 如果 $x_1, \cdots, x_n \in \mathcal{X}$，是充分可交换的

$$p(x_1, \cdots, x_n) = \int_\theta \prod_{i=1}^{n} E_f(x_i|f, g, h, \phi, c)\, dF(\theta)$$

则:

$$t_n = t_n(x_1, \cdots, x_n)$$
$$= [n,\ h(x_1) + \cdots + h(x_n)]$$ 是充分统计量.

伯努利分布 • $$p(x|\theta) = \theta^x (1-\theta)^{1-x} \quad x \in \{0, 1\},$$
$$= (1-\theta)\left(\frac{\theta}{1-\theta}\right)^x$$
$$= (1-\theta)\cdot\exp\left(x\log\frac{\theta}{1-\theta}\right)$$ ← 指数族分布形式

$f(x) = 1 \quad g(\theta) = (1-\theta) \quad c = 1$

$h(x) = 1 \quad \phi(\theta) = \log\frac{\theta}{1-\theta}$.

指数分布 • $F_X(x|\theta) = \theta\exp(-\theta x)$

高斯分布 • $$p(x|\theta) = N(x|0, s^2) = \left(\frac{1}{2\pi s^2}\right)^{\frac{1}{2}} \exp\left(-\frac{x^2}{2s^2}\right).$$
$$= \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \frac{1}{\theta^{\frac{1}{2}}}\cdot\exp\left(-\frac{1}{2}\frac{x^2}{\theta}\right).$$ 指数分布形式

$f(x) = (2\pi)^{-\frac{1}{2}} \quad g(\theta) = \theta^{\frac{1}{2}} \quad c = -\frac{1}{2} \quad h(x) = x^2 \quad \phi(\theta) = \theta^{-1}$.

均匀分布 • $$p(x|\theta) = U(x|[0, \theta]) = \frac{1}{\theta} \quad x \in [0, \theta].$$

$f(x) = 1 \quad h(x) = 0 \quad g(\theta) = \theta^{-1} \quad \phi(\theta) = 0$

$$p(x_1, \cdots, x_n|\theta) = \prod_{i=1}^{n} p(x_i|\theta)$$
$$= \theta^{-n} I_{[0,\theta]}(\max\{x_i\}).$$ 指数族分布

$t_n = [n, \max\{x_i\}]$

k参数的指数族

- $p(x|\theta) = E_f[x|f, g, h, \phi, \theta, c]$　　θ有k维.

$$= f(x)g(\theta)\exp\left(\sum_j c_j\phi_j(\theta)h_j(x)\right)$$

充分统计量为
- $p(x_1, \cdots, x_n|\theta) = \prod_{i=1}^{n} E_{fk}(x_i|f, g, h, \phi, \theta, c)$

$$t_n = [n, \sum_{i=1}^{n} h_1(x_i), \cdots, \sum_{i=1}^{n} h_k(x_i)].$$

正态
- $p(x|\theta) = N(x|\mu, \lambda^{-1})$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{\lambda}{2}(x-\mu)^2\right) \qquad \boxed{\theta = (\mu, \lambda)}.$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\lambda^{\frac{1}{2}}\exp\left(-\frac{\lambda}{2}(x^2 - 2x\mu + \mu^2)\right)$$

指数族分布.

$$= \left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\lambda^{\frac{1}{2}}\exp\left(-\frac{\lambda}{2}\mu^2\right)\exp\left(-\frac{\lambda}{2}(x^2 - 2x\mu)\right)$$

$f(x) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}}$　　$g(\theta) = \lambda^{\frac{1}{2}}\exp\left(-\frac{\lambda}{2}\mu^2\right)$

$c_1 = 1$　　$c_2 = -\frac{1}{2}$

$\phi_1(\theta) = \lambda\mu$　　$\phi_2(\theta) = x^2$.

正则指数族
- $pdf = p(\vec{y}|\vec{\varphi}) = cef(\vec{y}|a, b, \vec{\varphi})$

$$= a(\vec{y})\exp\left(y^T\vec{\varphi} - b(\vec{\varphi})\right)$$

$\vec{y} = (y_1, \cdots, y_k)$　　$\vec{\varphi} = (\varphi_1, \cdots, \varphi_k)$

$y_i = h_i(x)$　　$\varphi_i = c_i\phi_i(\theta)$

一阶统计量为: $E(y|\varphi) = \int \vec{y}\, a(\vec{y})\exp[y^T\vec{\varphi} - b(\vec{\varphi})]\, dy$

Proof:

$$\int a(\vec{y})\exp[y^T\vec{\varphi} - b(\vec{\varphi})]\, dy = 1$$

两边对 $\vec{\varphi}$ 求导

$$\int a(\vec{y})\exp[y^T\vec{\varphi} - b(\vec{\varphi})]\cdot[y^T - \nabla b(\vec{\varphi})]\, dy = 0$$

$$E[y|\varphi] = \nabla b(\vec{\varphi})$$

泊松
- $e^{-\lambda}\frac{\lambda^x}{x!}$

$$= \frac{1}{x!}\exp[x\log\lambda - \lambda]$$

$$= \frac{1}{x!}\exp[x\theta - e^\theta]$$

则 一阶统计量为 $(e^\theta)' = e^\theta$.

指数族应用

共轭先验 统计推理的重要技巧 ↑ 指数族分布的优势

- 如果 $X=(X_1,\dots,X_n)$ 是随机样本 从 regular 指数族分布,

  $$p(x|\theta) = \prod_{i=1}^{n} f(x_i)\,[g(\theta)]^n \exp\left[\sum_{j=1}^{k} c_j \phi_j(\theta)\left(\sum_{i=1}^{n} h_j(x_i)\right)\right]$$

  则共轭先验为. 对于 θ 形式为

  $$p(\theta|\tau) = [K(\tau)]^{-1}[g(\theta)]^{\tau_0}\exp\left[\sum_{j=1}^{k} c_j \phi_j(\theta)\tau_j\right]$$

  $$K(\tau) = \int_\Theta [g(\theta)]^{\tau_0}\exp\left[\sum_{j=1}^{k} c_j \phi_j(\theta)\tau_j\right] d\theta.$$

伯努利似然 || Beta的先验.

- $$p(x|\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}$$

  $$= (1-\theta)^n \exp\left[\left(\log\frac{\theta}{1-\theta}\right)\sum_{i=1}^{n} x_i\right]$$

  $$p(\theta|\tau) = (1-\theta)^{\tau_0}\exp\left(\log\frac{\theta}{1-\theta}\,\tau_1\right)$$

  $$= (1-\theta)^{\tau_0}\left(\frac{\theta}{1-\theta}\right)^{\tau_1}$$

  $$= \theta^{\tau_1}(1-\theta)^{\tau_0-\tau_1}$$

  $$= \theta^{\alpha-1}(1-\theta)^{\beta-1}$$ 先验分布.

泊松分布 || Gamma 先验

- $$p(x|\theta) = \prod \theta^{x_i}\exp[-\theta]/x_i!$$

  $$= \left(\prod x_i!\right)^{-1}\exp(-n\theta)\exp\left[\log\theta\sum_{i=1}^{n} x_i\right]$$

  $$p(\theta|\tau) \propto \exp(-\tau_0\theta)\exp(\log\theta\,\tau_1)$$

  $$= \theta^{\tau_1}\exp(-\tau_0\theta)$$

正态分布.

$(2\pi)^{-\frac{n}{2}}[\lambda^{\frac{1}{2}}\exp(-\frac{\lambda}{2}\mu^2)]^n\exp(\mu\lambda\sum x_i - \frac{\lambda}{2}\sum x_i^2)$

- $$p(x|\mu,\lambda) = \prod\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}}\exp\left[-\frac{\lambda}{2}(x_i-\mu)^2\right]$$

  $$\theta=(\mu,\lambda)$$

  $$p(\mu,\lambda|\tau) \propto \left[\lambda^{\frac{1}{2}}\exp\left(-\frac{\lambda}{2}\mu^2\right)\right]^{\tau_0}\exp\left[\mu\lambda\tau_1-\frac{\lambda}{2}\tau_2\right]$$

  $$= \lambda^{\frac{\tau_0}{2}}\exp\left(-\frac{\tau_0\lambda}{2}\mu^2\right)\exp\left[\mu\lambda\tau_1-\frac{\lambda}{2}\tau_2\right].$$