高斯推导公式

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

这里有很多假设，

Σ_{11} 对称 $\Sigma_{12}^T = \Sigma_{21}$.

把一个高斯用两个相互独立的高斯替代.

这里是将 $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ 的空间到 $\begin{pmatrix} x_1 \\ x_{22\cdot1} \end{pmatrix}$ 空间的线性变换.

Jacobian 矩阵为 B.

目前需要讨论的问题:

记 $x_{21} = x_2 - \Sigma_{21}\Sigma_{11}^{-1} x_1$

- $$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \sim \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22\cdot1} \end{pmatrix}. \quad \Sigma = AQA^{-1}.$$

$$= \begin{pmatrix} I_p & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I_q \end{pmatrix}\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\begin{pmatrix} I_p & -\Sigma_{11}^{-1}\Sigma_{12} \\ 0 & I_q \end{pmatrix}$$ AQA^{-1} 形式.

$$= \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22\cdot1} \end{pmatrix}$$ 块对角化. 化简得.

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}^{-1} = \begin{pmatrix} I_p & -\Sigma_{11}^{-1}\Sigma_{12} \\ 0 & I_q \end{pmatrix}\begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22\cdot1}^{-1} \end{pmatrix}\begin{pmatrix} I_p & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I_q \end{pmatrix}$$

$$\underset{n\times n}{(A} - \underset{n\times1}{B}\,\underset{1\times1}{D^{-1}}\,\underset{1\times n}{C})^{-1} = A^{-1} + A^{-1}B\underbrace{(D - CA^{-1}B)^{-1}}_{1\times1}CA^{-1}$$ 简化计算.

线性变换 Jacobian 矩阵

$$z = \begin{pmatrix} x_1 \\ x_{22\cdot1} \end{pmatrix} = \underbrace{\begin{pmatrix} I & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I \end{pmatrix}}_{B}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow dz = (\det B)\,dx$$

$$\downarrow \quad \Sigma(x-\mu) = B^{-1}z \longrightarrow (B^{-1}z)^T\Sigma(B^{-1}z) = z(B\Sigma B^T)^{-1}z$$

$$\downarrow \quad z^T(B\Sigma B^T)^{-1}z = (z_1, z_2)\begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22\cdot1} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

$\downarrow$

将一个高斯分为 $z_1^T\Sigma_{11}z_1$ 和 $z_2^T\Sigma_{22\cdot1}z_2$ 的两个独立高斯.

- $(B\Sigma B^T)^{-1} = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22\cdot1} \end{pmatrix}$

- 条件概率分布.

$$X_2|X_1 \sim N(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(X_1-\mu_1), \Sigma_{22\cdot1})$$
$$p(X_1, X_2) = P(X_1)\,P(X_2|X_1)$$

$$X_2 = X_{2\cdot1} + \Sigma_{21}\Sigma_{11}^{-1}X_1$$
$$E(X_2) = \mu_{ba} + \Sigma_{21}\Sigma_{11}^{-1}X_1$$