sub 高斯

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- $\phi_X(\lambda) \le \frac{\lambda^2 \delta}{2}$ for all $\lambda \in \mathbb{R}$.

  $\log E[e^{\lambda x}] = \phi_X(\lambda)$. $N(0, \delta^{\frac{1}{2}})$.

小性质

- 若 $X \in G(\nu)$

  (1) $E[X]=0$, (2) $Var[X] \le \nu$

  证明:

  利用指数的泰勒展开,

  $\sum \frac{\lambda^n x^n}{n!} = e^{\lambda x}$

  $E[\sum_{n=0}^{\infty} \frac{\lambda^n x^n}{n!}] = E[e^{\lambda x}] \le \frac{\lambda^2 \nu}{2} = \sum_{n=0}^{\infty} \frac{\lambda^{2n} \nu^n}{2^n n!}$.

  $\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} E[X^n] \le \frac{\lambda^2 \nu}{2}$

  $\lambda E[X] + \frac{\lambda^2}{2} E[X^2] \le \frac{\lambda^2 \nu}{2} + o(\lambda^2)$.

  $$\begin{cases} E[X] \le 0 & \lambda \to 0^+ \\ E[X] \ge 0 & \lambda \to 0^- \end{cases} \Rightarrow E[X]=0$$

  则: $\frac{\lambda^2}{2} E[X^2] \le \frac{\lambda^2}{2}\nu + o(\lambda^2)$

举例.

- $P_X = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$

  $E[e^{\lambda x}] = \frac{1}{2}e^1 + \frac{1}{2}e^{-1}$.

  $\log(\frac{1}{2}e + \frac{1}{2}e^{-1}) = 1 + \log\frac{1}{2}(1+e^{-2})$,

  $= \log(\frac{1}{2}e^{\lambda} + \frac{1}{2}e^{-\lambda}) \le \frac{\lambda^2}{2}$. $= g(1)$

- $G(\nu)$

  "$a \vee b$" $\triangleq \max\{a,b\}$:

  $P(X>t) \vee P(-X<t) \le e^{-\frac{t^2}{2\nu}}$ 高斯.

应用.

定理. • X 为随. a.v. $E[X]=0$.

$$P(X>x)\vee P(-X>x)\le e^{-\frac{x^2}{2v}}.$$

then for 任何 $p>0$

$$E[|X|^p]\le p\cdot(2v)^{\frac{p}{2}}\cdot\Gamma(\frac{p}{2})$$

$$E[|X|^p]=\int_0^{+\infty}P(|X|^p>t)dt.$$

$$=\int_0^{+\infty}p\,t^{p-1}\,P(|X|>t)dt.$$

$$\le\int_0^{+\infty}p\,t^{p-1}\cdot 2\exp(-\frac{t^2}{2v})dt.$$

$u=\frac{t^2}{2v}$

$$=p(2v)^{\frac{p}{2}}\underbrace{\int_0^{+\infty}\exp(-u)\cdot u^{\frac{p}{2}-1}du}_{\Gamma(\frac{p}{2})}.$$

反过来 • 若 $E[X^{2q}]\le q!\cdot C^q$.

then $X\in\mathcal{G}(4C)$ $(v=4C)$

证明: $\hat{X}$ 是 X 的 copy.

$$E[e^{\lambda X}]E[e^{-\lambda\hat{X}}]=E[e^{\lambda(X-\hat{X})}]$$ 奇次项为0.

$$=\sum_{q=0}^{\infty}\frac{\lambda^{2q}E[(X-\hat{X})^{2q}]}{(2q)!}$$

由. Jessen不等式有. $\because E[(X-\hat{X})^{2q}]\le 2^{2q-1}(E[X^{2q}]+E[\hat{X}^{2q}])$

$$=2^{2q}E[X^{2q}].$$

原式为: $E[e^{\lambda X}]E[e^{-\lambda\hat{X}}]=\le\sum\frac{\lambda^{2q}\cdot 2^{2q}\cdot C^q\cdot q!}{(2q)!}$ $\underline{E[e^{-\lambda X}]\ge 1}$

而 $\frac{(2q)!}{q!}=\prod(q+1)!\ge\prod 2^q\cdot q!$

则 $E[e^{\lambda X}]\le\sum\frac{\lambda^{2q}\cdot 2^q\cdot C^q}{q!}=e^{2\lambda^2C}=e^{\frac{\lambda^2 4C}{2}}=\mathcal{G}(4C).$