已知边缘和条件概率求联合分布

- 已知 $p(x) = N(x \mid \mu, \Lambda^{-1})$

  $p(y \mid x) = N(y \mid Ax+b, L^{-1})$

求：① $p(y)$，② $p(x \mid y)$

解：① $y = Ax + b + \varepsilon$

$\varepsilon \sim N(0, L^{-1})$

$$
\begin{aligned}
E[y] &= E[Ax+b+\varepsilon] \\
&= E(Ax+b) + E[\varepsilon] \\
&= A\mu + b + 0 \\
&= A\mu + b
\end{aligned}
$$

$$
\begin{aligned}
var[y] &= var[Ax+b+\varepsilon] \\
&= var[Ax+b] + var[\varepsilon] \\
&= A\Lambda^{-1}A^T + L^{-1}
\end{aligned}
$$

$y \sim N(A\mu+b,\ A\Lambda^{-1}A^T + L^{-1})$

② $z = \begin{pmatrix} x \\ y \end{pmatrix} \sim N\left(\begin{bmatrix} \mu \\ A\mu+b \end{bmatrix}, \begin{bmatrix} \Lambda^{-1} & \Delta \\ 0 & L^{-1}+A\Lambda^{-1}A^T \end{bmatrix} \right)$

$$
\begin{aligned}
\Delta &= cov(x, y) \\
&= E[(x-Ex)(y-E(y))^T] \\
&= E[(x-\mu)(y-A\mu-b)^T] \\
&= E[(x-\mu)(Ax+b+\varepsilon-A\mu-b)^T] \\
&= E[(x-\mu)(Ax-A\mu+\varepsilon)^T] \\
&= E[(x-\mu)(x-\mu)^T A^T + (x-\mu)\varepsilon^T] \\
&= E[(x-\mu)(x-\mu)^T A^T] + \underbrace{E[(x-\mu)\varepsilon^T]}_{=0} \\
&= E[(x-\mu)(x-\mu)^T]A^T \\
&= var[x]A^T = \Lambda^{-1}A^T
\end{aligned}
$$

$p(x \mid y) = N(\_\_,\ \_\_\_\_)$