引入

- $A(x) = \sum a_n x^n$
  $= \sum a(n) x^n$
  $\downarrow$ 用 t 代替 n
  $A(x) = \int_0^{+\infty} a(t)\, x^t\, dt$
  $= \int_0^{+\infty} a(t) \cdot (e^{\ln x})^t\, dt$
  $\downarrow$ $0<x<1$ $\quad \ln x < 0 \quad$ 用 $s = -\ln x$
  $= \int_0^{+\infty} f(t)\, e^{-st}\, dt$

基本公式

- $F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt \leftarrow$ Laplace 变换
  - 线性变换.
    $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$
    $\mathcal{L}(cg) = c\,\mathcal{L}(g)$.

例子 1

- $1 \rightarrow \mathcal{L}(1) = \frac{1}{s}$.
  $\int_0^{+\infty} e^{-st} dt = \lim\limits_{R\to\infty} \int_0^R e^{-st} dt = \lim\limits_{R\to\infty} \frac{1}{-s} e^{-st}\Big|_0^R$
  $= \lim\limits_{R\to\infty} \frac{e^{-Rt}-1}{-s}$
  $= \frac{1}{s} \qquad s>0$

例子 2

- $e^{at} f(t) \longrightarrow \mathcal{L}(\qquad) = F(s-a)$. $\qquad s>a$.
  指数移位函数.

例子 3

- $\cos at \rightsquigarrow \mathcal{L}(\qquad) = \frac{s}{s^2+a^2}$.
  $\cos at = \frac{1}{2}(e^{-iat} + e^{iat})$
  $\int_0^{+\infty} \cos at\, e^{-st}\, dt$
  $= \frac{1}{2} \int_0^{+\infty} (e^{-iat} + e^{iat})\, e^{-st}\, dt$.
  $= \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right)$
  $= \frac{1}{2} \frac{2s}{s^2+a^2}$

简化求解过程，将多个变量归于一个变量.

Laplace 应用于解线性方程.

- $Ay''+By'+C=0$

  ○ $f'(t) \rightsquigarrow \mathcal{L}(\quad) = sF(s) - f(0)$

  ○ $f''(t) = [\; f'(t) \;]'$

$$\mathcal{L} = s\,\mathcal{L}(f'(t)) - f'(0)$$
$$= s\,[sF(s) - f(0)] - f'(0)$$
$$= s^2F(s) - sf(0) - f'(0).$$

- $y'' - y = e^{-t} \qquad y(0)=1 \qquad y'(0)=0$

$$\downarrow \mathcal{L}$$

$$s^2F(s) - s - F(s) = \frac{1}{s+1} \qquad\qquad \frac{1}{s+1} + s = \frac{s^2+s+1}{s+1}$$

$$(s^2-1)F(s) = \frac{1}{s+1} + s$$

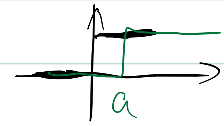
$$F(s) = \left(\frac{1}{s+1} + s\right)\frac{1}{s^2-1} = \frac{s^2+s+1}{(s+1)^2(s-1)}$$
$$= \frac{-1/2}{(s+1)^2} + \frac{1/4}{s+1} + \frac{3/4}{s-1}$$

$$\downarrow \mathcal{L}^{-1}$$

$$-\frac{1}{2}te^{-t} + \frac{1}{4}e^{-t} + \frac{3}{4}e^{t}$$

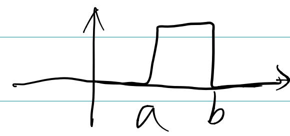
Laplace 对于有跳跃不连续点 function 有优势.

- 单位阶跃函数.

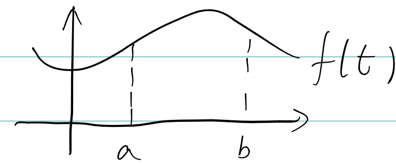


$u(t)$.

$u_a(t) = u(t-a)$ 平移



$$U_{ab}(t) = U_a(t) - U_b(t)$$
$$= u(t-a) - u(t-b)$$

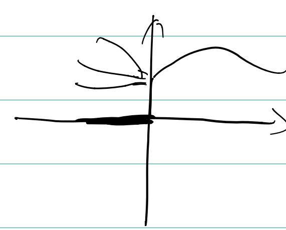


$u_{ab}(t) f(t)$.

$$\mathcal{L}(u(t)) = \int_0^{+\infty} e^{-st} u(t)\,dt = \frac{1}{s} \quad s>0.$$
$$\mathcal{L}(1) = \frac{1}{s}.$$

then

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) =$$



$F(s) \xrightarrow[\mathcal{L}^{-1}]{} u(t)f(t)$ 唯一

- $u(t-a)f(t-a) \xrightarrow{\mathcal{L}} e^{-as}F(s)$

- $E(x) = \int x \, dF(x)$ 期望的定义.
  $E(x^2) = \int x^2 dF(x)$
  $E(x^k) = \int x^k dF(x)$.

- 矩生成函数.
  Laplace 变换.
  $\varphi_x(t) = E(e^{tx}) = \int e^{tx} dF_X(x) = \int e^{tx} f_X(x) dx$
  $\varphi'_x(0) = \int x e^{tx} dF_X(x) = \int x dF_X(x) = E(X)$
  $\varphi_X^{(k)}(0) = \int x^k dF_X(x) = E(X^k)$.

- $\mathcal{L}(t) = E[e^{-tx}] = \int e^{tx} dF_X(x)$ 拉普拉斯function.

- $\mathcal{L}(\mu, t) = \int \exp(-tx)\mu(dx)$ 函数L完全单调, n阶可导, $L(0)=1$

- 测度 def.
  $g(0, \infty) \rightarrow R$ 无穷阶可导,
  $(-1)^n g^{(n)}(\lambda) \geq 0$ $n \in N \cup \{0\}$ 且 $\lambda > 0$.

$$\begin{cases} g(x) \geq 0 \\ g'(x) \leq 0 \\ g''(x) \geq 0 \end{cases}$$

- 定理 (Bernstem): 若 $g: (0, +\infty) \rightarrow R$ 是完全单调函数,
  ∃唯一测度 μ, 对于 $[0, \infty)$ 为 Laplace 变换.
  $g(\lambda) = \int_0^\infty \exp(-\lambda t)\mu(dt) = \mathcal{L}(\mu, \lambda)$

- 推论：$g(0+)=1 \quad g(+\infty)=0$

$$\mu(dt)=F(dt)$$

$$g(\lambda)=\int \exp(-\lambda t)\,dF(t).$$

证明：

$$g(\lambda)=\sum_{k=0}^{n-1}\frac{g^{(k)}(a)}{k!}(\lambda-a)^k+\int_a^\lambda \frac{g^{(n)}(s)}{(n-1)!}(\lambda-s)^{n-1}ds$$

$$=\sum(-1)^k\frac{g^{(k)}(a)}{k!}(a-\lambda)^k+\int_\lambda^a\frac{(-1)^n g^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1}ds.$$

① $\lim_{a\to\infty}\int_\lambda^a \frac{(-1)^n g^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1}ds$

$$=\int_\lambda^\infty \frac{(-1)^n g^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1}ds.$$

$$\le \phi(t)$$

② $P_k(\lambda)=\lim_{a\to\infty}\frac{(-1)^k g^{(k)}(a)}{k!}(a-\lambda)^k$ 不依赖于λ.

$$P_k(\upsilon)=\lim_{a\to\infty}\frac{(-1)^k g^{(k)}(a)}{k!}(a-\upsilon)^k$$

$$=\lim_{a\to\infty}\frac{(-1)^k g^{(k)}(a)}{k!}\frac{(a-\upsilon)^k}{(a-\lambda)^k}(a-\lambda)^k$$

$$=P_k(\lambda)$$

$$g(\lambda)=\sum P_k+\int_\lambda^\infty\frac{(-1)^n g^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1}ds.$$

$$g(\infty)=0\Rightarrow P_k=0$$

$$g(\lambda)=\int_\lambda^\infty\frac{(-1)^n g^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1}ds.$$

$\int_0^{+\infty}$?

$$\Leftrightarrow\quad g(\lambda)=\int_0^\infty\left(1-\frac{\lambda}{s}\right)_+^{n-1}\cdot\frac{(-1)^n g^{(n)}(s)}{(n-1)!}s^{n-1}ds \qquad (a)_+=\begin{cases}a & a>0\\ 0 & a<0\end{cases}$$

$s=\frac{n}{t}$ ← 这里可以这样吗? 可以吧

$$g(\lambda)=\int_0^\infty\left(1-\frac{\lambda t}{n}\right)_+^{n-1}\cdot\frac{(-1)^n g^{(n)}(\frac{n}{t})(\frac{n}{t})^{n+1}}{n!}dt.$$

$$=\int_0^{+\infty}e^{-\lambda t}\cdot f(t)\,dt$$

- 如果 $g(t)$ 关于原点对称，有二阶导，凸函，$g(+\infty)=0 \quad g(0+)=1$

那么 $g(t)=\int_0^\infty \frac{1}{s}\left(1-\frac{t}{s}\right)_+ s\,g''(s)\,ds \quad t>0$

(Mixture Berntem-Ferj.)