引入

$$S = \sum (X_i - E[X_i])$$

$$\phi_S(\lambda) = \sum \log E[e^{\lambda(X_i - E[X_i])}].$$

$$= \sum \log E[e^{\lambda X_i} e^{-\lambda E[X_i]}]$$

$$= \sum [\log E[e^{\lambda X_i}] - \lambda E[X_i]] \quad \leftarrow \log u \le u-1 \quad u>0.$$

$$\le \sum (E[e^{\lambda X_i}] - 1 - \lambda E[X_i])$$

$$= \sum E[e^{\lambda X_i} - \lambda X_i - 1].$$

bennett不等式

X_i 为独立随机变量, $X_i \le b$. for all $b>0$

$$v = \sum E[X_i^2]$$

定义: $\phi(u) = e^u - u - 1$ 对所有 $u \in \mathbb{R}$.

则有 $$\log E[e^{\lambda X}] \le n \log\left(1 + \frac{v}{nb^2}\phi(b\lambda)\right) \le \frac{v}{b^2}\phi(b\lambda)$$

对任意 $t>0$.

对应泊松的chernoff不等式

$$P(S \ge t) \le \exp\left\{-\frac{v}{b^2} h\left(\frac{bt}{v}\right)\right\}.$$

$$h(u) = (1+u)\log(1+u) - u. \quad u>0$$

相当2值

bernstein's 不等式

- $h(x) = (1+x)\log(1+x) - x$

$h(x) \geq \frac{x^2}{2(1+\frac{x}{3})} = g(x)$

$P(S \geq t) \leq \exp\left\{-\frac{t^2}{2(\nu + \frac{bt}{3})}\right\}$

$h(0) = 0$

记:

$h'(x) = \log(1+x)$ $\quad g(0) = 0$

$h'(0) = 0$ $\quad g'(x) = \frac{x}{1+\frac{x}{3}} - \frac{x^2 \cdot \frac{1}{3}}{2(1+\frac{x}{3})^2}$

$g'(0) = 0$ $\quad = \frac{x}{1+\frac{x}{3}} - \frac{x^2}{6(1+\frac{x}{3})^2}$.

$h''(x) = \frac{1}{1+x}$ $\quad g''(x) = \frac{27}{(x+3)^3}$.

$\Downarrow$

$h^{(n)}(0) \geq g^{(n)}(0)$

$\Downarrow$ 泰勒展开.

$h(x) \geq g(x)$

新 bernstein 不等式

- $X_1, \cdots, X_n$ 是独立实值随机变量，$v>0$. 有 $\sum E[X_i^2] \le v$.
若 $\sum E[(X_i)_+^q] \le \frac{c^{q-2} v q!}{2}$ $(q \ge 3)$
如果 $S = \sum (X_i - E[X_i])$
那么对于所有 $\lambda \in (0, \frac{1}{c})$，$\psi_S(\lambda) \le \frac{v\lambda^2}{2(1-c\lambda)}$，$\psi_S^*(t) \ge \frac{v}{c^2} h_1(\frac{ct}{v})$，
$t>0$，$h_1(u) = 1+u-\sqrt{1+2u}$ $(u>0)$
有 $P(S \ge \sqrt{2vt} + ct) \le e^{-t}$.

证明：

$$\phi(u) = e^u - u - 1 \quad u \le 0 \quad \phi(u) \le \frac{u^2}{2}.$$

对 $\lambda>0$ $\quad \phi(\lambda X_i) \le \frac{\lambda^2 X_i^2}{2} + \sum_{q=3}^{\infty} \frac{\lambda^q (X_i)_+^q}{q!}$.

$$E[\phi(\lambda X_i)] \le E[\frac{\lambda^2 X_i^2}{2}] + \sum_{q=3}^{\infty} \frac{\lambda^q E[X_{i+}]^q}{q!} \le \frac{v}{2} \sum_{q=2}^{\infty} \lambda^q c^{q-2}.$$

$(\lambda c) < 1 \Rightarrow$ 收敛. $\quad = \frac{v\lambda^2}{2} \sum_{q=0}^{\infty} (\lambda c)^q$

$$\psi_S(\lambda) \le \sum_{i=1}^{n} E[e^{\lambda X_i} - \lambda X_i - 1] = \sum_{i=1}^{n} E[\phi(\lambda X_i)]$$

$$\lambda \in (0, \frac{1}{c}) \quad \psi_S(\lambda) \le \sum_{i=1}^{n} \phi(\lambda X_i) \le \frac{v}{2} \frac{\lambda^2}{1-c\lambda}.$$

$$\psi_S^*(\lambda) = \sup_{\lambda \in (0,\frac{1}{c})} (t\lambda - \psi_S(\lambda)) \ge \sup_{\lambda \in (0,\frac{1}{c})} (t\lambda - \frac{\lambda^2 v}{2(1-c\lambda)}).$$

$$= \frac{v}{c^2} h_1(\frac{ct}{v})$$

h_1 是增的，$h^{-1}(u) = u + \sqrt{2u}$.

$$\psi^*(t) = \frac{v}{c^2} h_1(\frac{ct}{v}).$$

$$\psi^{*-1}(t) = \frac{v}{c}(\frac{c^2}{v} u + \sqrt{2\frac{c^2}{v} u}) = cu + \sqrt{2vu}.$$

$$P(S \ge t) \le \exp\{-\frac{v}{c^2} h_1 \frac{ct}{v}\} = \exp\{-\psi^*(t)\}.$$

$$P(S \ge \psi^{*-1}(t)) \le \exp\{-t\}$$

$$P(S \ge ct + \sqrt{2vt}) \le \exp\{-t\}$$

推论.
- 令 $X_1 \ldots X_n$ 是独立随机变量，$S = \sum_{i=1}^{\infty} (X_i - E[X_i])$

$$P(S \ge t) \le \exp\{-\frac{t^2}{2(v+ct)}\}$$

$$h_1(u) \ge \frac{u^2}{2(1+u)}$$