引入

- $A(x) = \sum a_n x^n$

  $= \sum a(n) x^n$

  $\downarrow$ 用 t 代替 n

  $A(x) = \int_0^{+\infty} a(t) \, x^t \, dt$

  $= \int_0^{+\infty} a(t) \cdot (e^{\ln x})^t \, dt$

  $\downarrow$ $0 < x < 1$ $\ln x < 0$ 用 $S = -\ln x$

  $= \int_0^{+\infty} f(t) \, e^{-st} \, dt$

基本公式

- $F(s) = \int_0^{+\infty} f(t) \, e^{-st} \, dt \Leftarrow$ Laplace 变换

  · 线性变换.

  $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$

  $\mathcal{L}(cg) = c\,\mathcal{L}(g).$

例子1

- $1 \rightarrow \mathcal{L}(1) = \frac{1}{s}.$

  $\int_0^{+\infty} e^{-st} \, dt = \lim_{R \to \infty} \int_0^R e^{-st} \, dt = \lim_{R \to \infty} \frac{1}{-s} e^{-st} \Big|_0^R$

  $= \lim_{R \to \infty} \frac{e^{-Rt} - 1}{-s}$

  $= \frac{1}{s} \qquad s > 0$

例子2

- $e^{at} f(t) \rightarrow \mathcal{L}(\quad) = F(s-a).$ $s > a.$

  指数移位函数.

例子3

- $\cos at \rightsquigarrow \mathcal{L}(\quad) = \frac{s}{s^2 + a^2}.$

  $\cos at = \frac{1}{2}(e^{-iat} + e^{iat})$

  $\int_0^{+\infty} \cos at \, e^{-st} \, dt$

  $= \frac{1}{2} \int_0^{+\infty} (e^{-iat} + e^{iat}) e^{-st} \, dt.$

  $= \frac{1}{2} \left(\frac{1}{s-ia} + \frac{1}{s+ia} \right)$

  $= \frac{1}{2} \cdot \frac{2s}{s^2 + a^2}$

- $Ay'' + By' + C = 0$

  0 $f'(t) \rightsquigarrow \mathcal{L}(\ ) = sF(s) - f(0)$

  0 $f''(t) = [\ \ f'(t)\ \ ]'$

  $\quad \mathcal{L} = s\mathcal{L}(f'(t)) - f'(0)$

  $\quad = s[sF(s) - f(0)] - f'(0)$

  $\quad = s^2 F(s) - sf(0) - f'(0).$

- $y'' - y = e^{-t} \qquad y(0)=1 \qquad y'(0)=0$

  $\qquad \downarrow \mathcal{L}$

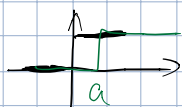 $s^2 F(s) - s - F(s) = \dfrac{1}{s+1} \qquad\qquad \dfrac{1}{s+1} + s = \dfrac{s^2+s+1}{s+1}$

  $(s^2-1)F(s) = \dfrac{1}{s+1} + s$

  $F(s) = \left(\dfrac{1}{s+1} + s\right)\dfrac{1}{s^2-1} = \dfrac{s^2+s+1}{(s+1)^2(s-1)}$

  $\quad = \dfrac{-1/2}{(s+1)^2} + \dfrac{1/4}{s+1} + \dfrac{3/4}{s-1}$

  $\qquad \downarrow \mathcal{L}^{-1}$

  $-\dfrac{1}{2}te^{-t} + \dfrac{1}{4}e^{-t} + \dfrac{3}{4}e^{t}$

- 单位阶跃函数.

$u(t)$.

$U_a(t) = U(t-a)$ 平移

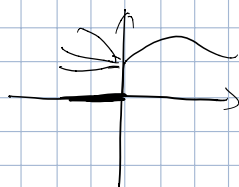$U_{ab}(t) = U_a(t) - U_b(t)$

$\qquad = U(t-a) - U(t-b)$

$f(t)$
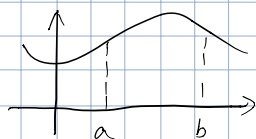
$U_{ab}(t) f(t)$.

$\mathcal{L}(u(t)) = \int_0^{+\infty} e^{-st} u(t) dt = \frac{1}{s} \quad s > 0.$

$\mathcal{L}(1) = \frac{1}{s}.$

then

$\mathcal{L}^{-1}(\frac{1}{s}) =$

$F(s) \xrightarrow[\mathcal{L}^{-1}]{} U(t) f(t)$ 唯一

- $U(t-a) f(t-a) \xrightarrow{\mathcal{L}} e^{-as} F(s)$

- $E(万) = \int 万 \, dF(万)$ 期望的定义.

  $E(万^2) = \int 万^2 \, dF(万)$

  $E(万^k) = \int \chi^k \, dF(万)$.

- 矩 生成函数.

  $\psi_万(t) = E(e^{t万}) = \int e^{t万} \, dF_X(万) = \int e^{t万} f_X(万) \, d万$

  $\psi'_万(0) = \int \chi e^{t万} \, dF_X(万) = \int 万 \, dF_X(万)$

  $\psi^{(k)}_万(0) = \int 万^k \, dF_X(万) = E(X^k)$.

- $L(t) = E[e^{-tx}] = \int e^{t万} \, dF_X(万)$ 拉普拉斯 function.

- $L(\mu, t) = \int exp(-t万) \, \mu(d万)$

- 测度 def.

  $g(0, \omega) \to R$ 无穷阶可导,

  $(-1)^n g^{(n)}(\lambda) \geq 0$ $n \in N \cup \{0\}$ 且 $\lambda > 0$.

  $\begin{cases} g(万) \geq 0 \\ g'(万) \leq 0 \\ g''(万) \geq 0 \end{cases}$

- 定理 (Bernstein): 若 $g: (0, +\omega) \to R$ 是 完全单调函数,

  口唯一测度 μ, 对于 $[0, \omega)$ 为 Laplace 数据.

  $g(\lambda) = \int_0^\omega exp(-\lambda t) \mu(d万) = L(\mu, \lambda)$

- 推论: $g(0+)=1$ $\quad g(+\infty)=0$

$$\mu(dt) = F(dt)$$
$$g(\lambda) = \int \exp(-\lambda t) \, dF(t).$$

证明:

$$g(\lambda) = \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!}(\lambda-a)^k + \int_a^\lambda \frac{g^{(n)}(s)}{(n-1)!}(\lambda-s)^{n-1} ds$$

$$= \sum (-1)^k \frac{g^{(k)}(a)}{k!}(a-\lambda)^k + \int_\lambda^a \frac{(-1)^n g^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1} ds.$$

① $\lim\limits_{a\to\infty} \int_\lambda^a \frac{(-1)^n g^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1} ds$

$$= \int_\lambda^\infty \frac{(-1)^n g^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1} ds.$$

$$\le \phi(\infty)$$

② $P_k(\lambda) = \lim\limits_{a\to\infty} \frac{(-1)^k g^{(k)}(a)}{k!}(a-\lambda)^k$ 不依赖于λ

$$P_k(v) = \lim\limits_{a\to\infty} \frac{(-1)^k g^{(k)}(a)}{k!}(a-v)^k$$

$$= \lim\limits_{a\to\infty} \frac{(-1)^k g^{(k)}(a)}{k!} \frac{(a-v)^k}{(a-\lambda)^k}(a-\lambda)^k$$

$$= P_k(\lambda)$$

$$g(\lambda) = \sum P_k + \int_\lambda^\infty \frac{(-1)^n g^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1} ds.$$

$$g(\infty) = 0 \implies P_k = 0$$

$$g(\lambda) = \int_\lambda^\infty \frac{(-1)^n g^{(n)}(s)}{(n-1)!}(s-\lambda)^{n-1} ds.$$

$\color{red}{\int_0^{+\infty} ?}$ $\iff$ $g(\lambda) = \int_0^\infty \left(1-\frac{\lambda}{s}\right)_+^{n-1} \cdot \frac{(-1)^n g^{(n)}(s)}{(n-1)!} s^{n-1} ds$ $\quad$ $(a)_+ = \begin{cases} a & a>0 \\ 0 & a<0 \end{cases}$

$$S = \frac{n}{t} \color{red}{\leftarrow 这里可以这样吗?} \color{blue}{可以吧}$$

$$g(\lambda) = \int_0^\infty \left(1-\frac{\lambda t}{n}\right)_+^{n-1} \cdot \frac{(-1)^n g^{(n)}\left(\frac{n}{t}\right)\left(\frac{n}{t}\right)^{n-1}}{n!} dt$$

$$= \int_0^{+\infty} e^{-\lambda t} \cdot f(t) dt$$

- 如果 $g(t)$ 关于原点对称, 并二阶可导, 凸函数, $g(+\infty)=0$ $\quad$ $g(0+)=1$

那么μ.

$$g(t) = \int_0^\infty \frac{1}{s}\left(1-\frac{t}{s}\right)_+ s g''(s) ds \quad t>0$$

(mixture Berntem-Ferj.)

- Example. $\exp(-|\beta|)^{\alpha}$ $0 < \alpha \le 1$.

- Thm: 若函数 可表的成高斯 scale mixture, 则对每 $P(\sqrt{v})$ 是完全单调面,

- Thm: 计 $f(b) > 0$, then $e^{ug(b)}$ 是完全单调面 计 $f'(b)$是完全单调