Wishart 分布

what

高斯分布的平方.

- $\underset{n\times p}{X}$ $\quad \underset{p\times p}{X^TX}$ ← Wishart 分布

$n \geq p$

- S 正定, 自由度 r, Σ 正定, 计

pdf 为 $p(S) = \dfrac{|S|^{\frac{r+p-1}{2}} \operatorname{etr}(-\frac{1}{2}\Sigma^{-1}S)}{\pi^{\frac{p(p-1)}{2}} |\Sigma|^{r/2} \prod_{i=1}^{r} \Gamma(\frac{r+1-i}{2})}$ $\quad r \geq p$.

$S = X^TX$ ← 随机矩阵给一个名字.

$\Updownarrow$

$X \sim N(0, I_n \otimes \Sigma)$ $\qquad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$.

where

性质

有了这些性质就可以直接使用

- Thm. $S \sim W_p(\Sigma, r)$ $\qquad S_{11.2} = S_{11} - S_{12} \cdot S_{22}^{-1} S_{21}$

$\qquad \Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{11}^{-1}\Sigma_{21}$

Then.

① $S_{22} \sim W_{p-q}(\Sigma_{22}, r)$

② $S_{11.2} \sim W_q(\Sigma_{11.2}, r-p+q)$.

③ $S_{11.2} \sim (S_{12}, S_{22})$ 是独立的.

④ $S_{12} | S_{22} \sim N(\Sigma_{12}\Sigma_{22}^{-1}S_{22}, S_{11.2} \otimes S_{22})$.

作一个变换

$\begin{cases} S_{11.2} = S_{11} - S_{12}S_{22}^{-1}S_{21} \\ B_{12} = S_{12} \\ B_{22} = S_{22} \end{cases}$ $\qquad$ 利用三线性 + 外积.

① ∧ ② ∧ ③ 得到 Jacobian.

$d(S_{11}, S_{12}, S_{22}) = d(S_{11.2}, B_{12}, B_{22})$.

$p(S_{11}, S_{12}, S_{22}) = p(S_{11.2}, B_{12}, B_{22})$.

$|S| = |S_{11.2}|\,|S_{22}| = |S_{11.2}|\,|B_{22}|$

how
如何计算 S 矩阵

得到 S.

从这里抽出 T

$$tr(\Sigma^{-1}S) = tr\left(\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\right)$$

$$\doteq tr(C_{11}S_{11}) + 2\,tr(C_{21}C_{12}) + tr(C_{22}S_{22}).$$

- $\Sigma^{-1} = AA^T \Rightarrow A\Sigma A^T = I$

$W = ASA^T \qquad W \sim W_p(I, r)$

$dW =$

let $S \sim W_p(I, r)$ 且 $S = T^T T$ T为上三角 $t_{ii} > 0$

then t_{ij} $(1 \le j < i \le p)$ 是相互独立的.

$t_{ii}^2 \sim \chi^2_{r-i+1}$ $1 \le i \le p$ 且

$t_{ij} \sim N(0,1)$ $(1 \le j < i \le p)$

- Wishart 的单变量分布是 χ^2 分布.