Definition (Metric Space)

A metric space is an ordered pair (M, d) where M is a non-empty set and d is a metric on M, i.e., a function $d : M \times M \to \mathbf{R}$ such that for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in M$, the following holds:

1. $d(\mathbf{x}, \mathbf{y}) \geq 0$,
2. $d(\mathbf{x}, \mathbf{y}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{y}$,
3. $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$,
4. $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ (triangle inequality).

Definition (Cauchy Sequences in a Metric Space)

Given a metric space (M, d), a sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is Cauchy, if for every positive real number $\epsilon > 0$, there is a positive integer N such that for all natural numbers $m, n > N$, the distance

$$d(\mathbf{x}_m - \mathbf{x}_n) \leq \epsilon.$$

Roughly speaking, the terms of the sequence are getting closer and closer together in a way that suggests that the sequence ought to have a limit in M. Nonetheless, such a limit does not always exist within M.