- Jeffery 先验是一种信息先验分布，计算方法是根号下的 Fisher information 矩阵的行列式.

  $P(\theta) \propto \sqrt{\det I(\theta)}$.

- 具体的好处就是，不受到原分布的参数选择差异的影响，所以说是 non-informative 的，这一性质对于 Beyesian 方法中很重要，因为先验的选择往往会影响后验参数的估计，

- 减少参数量.

- Fish information

  $I(\theta) = -E\left[\frac{d^2 \log f}{d\theta^2}\right]$ ← 似然

  $$\frac{d^2 \log f}{d\theta^2} = \frac{d}{d\theta}\left(\frac{\frac{df}{d\theta}}{f}\right)$$

- Jeffery 先验

$f(t|\theta)$ 为似然函数.

Fish information. $I(\theta) = E\left[\left(\frac{d\log f(t,\theta)}{d\theta}\right)^2\right]$

$$= \int \left(\frac{d\log f(t,\theta)}{d\theta}\right)^2 f(t,\theta)\,dt.$$

$p(\theta) \propto \sqrt{I(\theta)}$. 为 Jeffery 先验.

- 若 $\varphi(\theta)$ 是一一映射

$$p(\varphi) = p(\theta)\left|\frac{d\theta}{d\varphi}\right|$$

$$\propto \sqrt{I(\theta)\cdot\left(\frac{d\theta}{d\varphi}\right)^2}$$

$$= \sqrt{E\left(\frac{d\log f}{d\theta}\right)^2\left(\frac{d\theta}{d\varphi}\right)^2}$$

$$= \sqrt{E\left(\left(\frac{d\log f}{d\theta}\cdot\frac{d\theta}{d\varphi}\right)^2\right)}$$

$$= \sqrt{E\left(\frac{d\log f}{d\varphi}\right)^2}$$

$$= \sqrt{I(\varphi)}$$

先验不变性，并且没有新的参数.

- $x \sim N(\mu, \sigma^2)$

① μ 不定, σ 固定.

不合适先验

$\int_{-\infty}^{+\infty} 1\,dt = \infty$

$$\log f = -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

定值. $I(\mu) \propto 1$.

$$\sqrt{I(\mu)} = \sqrt{E\left(\frac{(x-\mu)^2}{\sigma^4}\right)} = \sqrt{\frac{1}{\sigma^2}}$$

没有信息的先验.

② μ 固定，求 δ.

令 $\tau = \frac{1}{\delta^2}$

$$f(\tau) = \frac{1}{\sqrt{2\pi}\delta} \exp\left(\frac{-(x-\mu)^2}{2\delta^2}\right) = \frac{\tau^{\frac{1}{2}}}{\sqrt{2\pi}} \exp\left(-\frac{\tau(x-\mu)^2}{2}\right)$$

$$\ln f = \frac{1}{2}\ln\tau - \frac{\tau}{2}(x-\mu)^2$$

$$\frac{d\ln f}{d\tau} = \frac{1}{2\tau} - \frac{1}{2}(x-\mu)^2$$

$$E\left[\left(\frac{d\ln f}{d\tau}\right)^2\right] = E\left[\frac{1}{4\tau^2} - \frac{(x-\mu)^2}{2\tau} + \frac{(x-\mu)^4}{4}\right]$$

$$= \frac{1}{4\tau^2} - \frac{1}{2\tau^2} + \frac{3}{4\tau^2}$$

$$= \frac{1}{2\tau^2}$$

$$= \sqrt{I(\tau)}$$ 这里假设了 τ 为 Jeffrey 先验

- 泊松分布.

$$f(x|\lambda) = e^{-\lambda}\frac{\lambda^n}{n!}$$ $\pi(\lambda)$ = λ 的 Jeffrey 先验.

$$\pi(\lambda) = E\left[\left(\frac{n}{\lambda} - 1\right)^2\right]$$

$$= E\left(1 + \frac{n^2}{\lambda^2} - \frac{2n}{\lambda}\right)$$

$$= \sqrt{\frac{1}{\lambda}}.$$

- 优点：不引入任何新参数.

- $x = \theta + \varepsilon \qquad \varepsilon \sim N(0, \tau^{-\frac{1}{2}})$

① τ 固定，

$p(\theta) = N(\theta | 0, \lambda^{\frac{1}{2}})$

$p(x|\theta)p(\theta) = \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{(x-\theta)^2}{2\tau}\right) \frac{1}{\sqrt{2\pi\lambda}} \exp\left(-\frac{\theta^2}{2\lambda}\right).$

$= \frac{1}{\sqrt{2\pi\tau}} \exp($

$\hat{\theta} = \frac{\lambda x}{\lambda + \tau}$ （峰值）

② τ 不固定.

第一种做法：$p(\theta, \tau) = p(\theta)p(\tau).$

$p(\theta, \tau | x) \propto p(x|\theta,\tau)p(\theta,\tau)$

$= p(x|\theta,\tau)p(\theta)p(\tau)$

$p(\theta) \sim N(0, \lambda^{\frac{1}{2}})$

$p(\tau) = Ga(\alpha, \frac{1}{2}\beta)$ $\qquad \tau^{\frac{\alpha}{2}-1}$

$\propto \frac{1}{\sqrt{2\pi}} \tau^{\frac{1}{2}} \exp\left(-\frac{\tau(x-\theta)^2}{2}\right) \frac{\lambda^{\frac{1}{2}}}{\sqrt{2\pi}} \exp\left(-\frac{\lambda\theta^2}{2}\right) \cdot \left(\frac{\beta}{2}\right)^{\alpha/2} \exp\left(\frac{\beta\tau}{2}\right)$

$L = \tau^{\frac{\alpha+1}{2}-1} \exp\left(-\frac{\tau}{2}((x-\theta)^2 + \beta)\right) \exp\left(-\frac{\lambda}{2}\theta^2\right) \lambda^{\frac{1}{2}}.$

$\ln L = \left(\frac{\alpha+1}{2}-1\right)\ln\tau - \frac{\tau}{2}\left[(x-\theta)^2 + \beta\right] + \frac{1}{2}\ln\lambda - \frac{\lambda}{2}\theta^2$

$Q = 2\ln L = (\alpha - 1)\ln\tau + \tau\left[(x-\theta)^2 + \beta\right] - \ln\lambda + \lambda\theta^2$

极大似然解法.

$$\begin{cases} \frac{\partial Q}{\partial \tau} = \frac{1}{\tau}(\alpha - 1) + (x-\theta)^2 + \beta = 0 \\ \frac{\partial Q}{\partial \theta} = 2\tau(x-\theta) + \lambda\theta = 0. \end{cases}$$

无法解

第二种做法 $P(\theta,\tau)=P(\theta|\tau)P(\tau)$.

$P(\theta|\tau)\sim N(0,(\lambda\tau)^{\frac{1}{2}})$ τ为Gam分布.

代入原方程有

极大似然

$$\begin{cases}\frac{\partial Q}{\partial\theta}=(-(x-\theta)+\lambda\theta)=0\\ \frac{\partial Q}{\partial\tau}=-\frac{\alpha}{2}\frac{1}{\tau}+\frac{1}{2}((x-\theta)^2+\beta+\lambda\theta^2)=0\end{cases}$$

习题.

上面(θ,τ)共轭. 但太多超参数.

第三种做法: $P(\theta,\tau)=P(\theta|\tau)P(\tau)$

$P(\tau)\propto 1$ 满足Jeffery分布.

$P(\tau)\propto\pi(\tau)=\frac{1}{\tau^2}$.

$L=\exp(\frac{\tau}{2}x^2)$ 指数分布.