基本公式

- $S = \sum (x_i - E[X_i])$

$$\phi_S(\lambda) = \sum \log E[e^{\lambda(X_i - E[X_i])}].$$
$$= \sum \log E[e^{\lambda X_i} e^{-\lambda E[X_i]}]$$
$$= \sum [\log E[e^{\lambda x_i}] - \lambda E[X_i]] \quad \leftarrow \log u \le u-1 \quad u>0.$$
$$\le \sum (E[e^{\lambda x_i}] - 1 - \lambda E[X_i])$$
$$= \sum E[e^{\lambda x_i} - \lambda X_i - 1].$$

bennett不等式

- X_i 为独立随机变量, $X_i \le b$. for all $b>0$

$$V = \sum E[X_i^2]$$

定义: $\phi(u) = e^u - u - 1$ 对所有 $u \in \mathbb{R}$.

则有 $\log E[e^{\lambda x}] \le n \log\left(1 + \frac{V}{nb^2} \phi(b\lambda)\right) \le \frac{V}{b^2} \phi(b\lambda)$

对任意 $t>0$.

对应的相应chernoff不等式

$$P(S \ge t) \le \exp\left\{ -\frac{V}{b^2} h\left(\frac{bt}{V}\right) \right\}.$$

$$h(u) = (1+u)\log(1+u) - u. \quad u>0$$

证明:

- 令 $b=1$, $u^{-2}(e^{u}-u-1)$ 非递减.

$$x_i^{-2}(e^{\lambda x_i}-\lambda x_i-1)\le e^{\lambda}-\lambda-1$$

$$e^{\lambda x_i}-\lambda x_i-1\le\phi(\lambda)\,x_i^{2}$$

$$E[e^{\lambda x_i}-\lambda x_i-1]\le E[\phi(\lambda)x_i^{2}]\Leftrightarrow E[e^{\lambda x_i}]\le E[\lambda x_i]+1+E[x_i^{2}]\,\phi(\lambda)$$

$$\phi_S(\lambda)=\sum\left(\log E[e^{\lambda x_i}]-\lambda E[x_i]\right)$$

$$\le n\sum\frac{1}{n}\left[\log\left(1+\lambda x_i+\phi(\lambda)E[x_i^{2}]\right)-\lambda E[x_i]\right]$$

$$\le n\log\left(1+\lambda\frac{1}{n}\sum x_i+\phi(\lambda)\frac{1}{n}\cdot v\right)-\sum\lambda E[x_i]$$

$$\le n\left(\sum\frac{\lambda}{n}Ex_i\right)+\frac{1}{n}\phi(\lambda)v-\sum\lambda E[x_i]$$

$$=\frac{1}{n}\phi(\lambda)v$$