Noising an image by adding noise sampled by a Gaussian distribution to achieve desired SNR

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Abstract

This document is not a complete treaty of the problem and its purpose is simply to give a justification of the final result. Given the lack of rigour i would like to state that this works well enough in practice.

1 Assumptions

The only assumption needed are that the original noise is normally distributed, that the original signal value is the expected value, and that the only form of variance of the original signal comes from said noise.

2 Derivation

Let the signal to noise ratio be $SNR = \frac{S^2}{\sigma^2}$ where S is the signal's expected value and σ^2 is its variance. Let $S^{'} = \frac{S}{n}$ be the new signal and $SNR^{'} = \frac{S^{'2}}{\sigma^2}$ its signal to noise ratio (note that σ did not change): it is clear that $SNR^{'} = \frac{S^2}{n^2\sigma^2}$. We want to achieve this SNR by adding to the initial signal extractions from a Gaussian distribution with mean $\mu = 0$ and standard deviation $\sigma^{'}$ to be determined.
The new expected value will be:

$$E(x + x') = E(x) + E(x') = E(x) + 0 = S$$

So, as predictable, no problems arise with S.
The new variance will be:

$$\begin{gathered} E((x + x' - E(x + x'))^2) = E((x + x')^2) - \mu^2 = \\ = E(x^2 + x'^2 + 2xx') - \mu^2 = E(x^2) - \mu^2 + E(x'^2) + 2E(x)E(x') = \\ = \sigma^2 + \mu^2 - \mu^2 + \sigma'^2 + \mu'^2 + 2E(x)E(x') \end{gathered}$$

The two random variables are uncorrelated so $E(x)E(x') = 0$ and, remembering that $\mu' = 0$, by setting this equal to $n^2\sigma^2$:

$$\begin{gathered} \sigma^2 + \sigma'^2 = n^2\sigma^2 \\ \sigma'^2 = (n^2 - 1)\sigma^2 \end{gathered}$$

giving us the variance of the gaussian distribution we wish to add. On the algorithm which is in the Preprocessing.py file (in the same repository where this document should have been found) the achieved SNR ratio is equal to four up to the fourth significant digit.