The Feedback Dynamics of a Driven Cart Pendulum

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Abstract

The system consists of a pendulum under gravity, whose pivot is free to move in a horizontal line, and which can be driven by applying a force to said pivot. The Lagrangian of the system is found, and by employing the Euler-Lagrange equations, two coupled equations of motion are extracted. Different suggested integration routines are also covered.

1 Introduction

1.1 The pendulum

We will be investigating the following physical system:

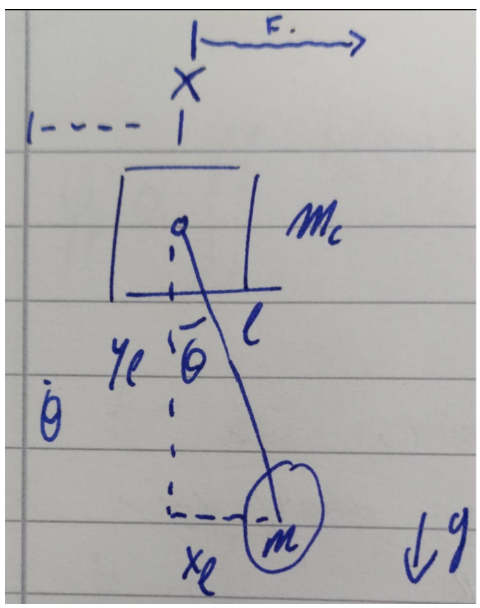

Figure 1: Diagram of a Cart Pendulum

A mass m is suspended below a cart of mass m_c by a mass-less rod, length l, restricted to swing only in one vertical plane. The angle θ subtended by the vertical and the rod is free to vary, the potential in this variable arising from the constant gravitational field strength g. The cart has freedom of motion on a horizontal axis in the plane of the pendulum, its coordinate x along this axis encodes the other dynamical variable of the system. The system has a 4-dimensional state space, when considering also the time derivatives $\dot{\theta}$ and $\dot{x}$. The system can be perturbed by applying a force F parallel to the x direction at the cart, which both accelerates it and provides a moment about m. For clarity, the coordinates relative to the cart y_l and x_l of the bob are also provided, as they will be used while evaluating the system's Lagrangian.

1.2 Implementation in a teaching lab

The following is based largely on previous work by Pietro's team, particularly Jeppe and Will Harpur-Davies, who designed the linear bearing fixtures, cart and belt-drive motor housing, and provided much documentation for driving and reading from the system with magnetic sensors. With their help, Lucy Ivy constructed a duplicate for our use, with the intent of providing a functional demonstration of normalised oscillation to a IA or IB physics practical class.

2 System Analysis

2.1 Evaluating the Lagrangian

Most of the following can be found in Bechhoefer's *Control Theory for Physicists* [Bec21], as well as here.

As should be familiar, the Lagrangian of a system is defined as follows:

$$L(q_1, q_2, \ldots, \dot{q_1}, \dot{q_2}; t) = T(q_1, q_2, \ldots, \dot{q_1}, \dot{q_2}; t) - V(q_1, q_2, \ldots, \dot{q_1}, \dot{q_2}; t)$$

The time dependence here being implicit in the generalised coordinates q_n. Let us first evaluate $V(q_1, q_2, \ldots, \dot{q_1}, \dot{q_2}; t)$. The system is affected by gravity. Due to the constraint on the motion, gravity can do no work on m_c, so the only term in the potential resulting from it is mgy_l. However, F can affect the entire system, and so the work done by it is xF. Expanding and writing V in terms of the natural coordinates of the system, we have:

$$\begin{aligned} V &= mgh - xF \\ V &= mg(l - y_l) - xF \\ V &= mg(l - lcos\theta) - xF \end{aligned}$$

The dependence of $V = V(\theta, x)$ can be clearly seen.

The expression for $T(q_1, q_2, \ldots, \dot{q_1}, \dot{q_2}; t)$ is slightly more cumbersome. We first solve for all kinetic energies in terms of the absolute and relative coordinates x, x_l and y_l

$$\begin{aligned} T &= \frac{1}{2}m_l\dot{x}^2 + \frac{1}{2}m[(\dot{x_l} + \dot{x})^2 + \dot{y_l}^2] \\ &= \frac{1}{2}m_l\dot{x}^2 + \frac{1}{2}m(\dot{x_l}^2 + 2\dot{x_l}\dot{x} + \dot{x}^2 + \dot{y_l}^2) \\ &= \frac{1}{2}\dot{x}^2(m_c + m) + \frac{1}{2}m(\dot{x_l}^2 + \dot{y_l}^2) + m\dot{x_l}x \\ &= \frac{1}{2}\dot{x}^2(m_c + m) + \frac{1}{2}m[(lsin\theta\dot{\theta})^2 + (lcos\theta\dot{\theta})^2] + mlsin\theta\dot{\theta}x \\ &= \frac{1}{2}\dot{x}^2(m_c + m) + \frac{1}{2}ml^2\dot{\theta}^2 + mlsin\theta\dot{\theta}x \end{aligned}$$

Again the dependence of T on the coordinates in the state space is clear.

The Lagrangian therefore evaluates to:

$$L(\theta, x, \dot{\theta}, \dot{x}) = \frac{1}{2}\dot{x}^2(m_c + m) + \frac{1}{2}ml^2\dot{\theta}^2 + mlsin\theta\dot{\theta}x + xF + mg(lcos\theta - l)$$

2.2 Dimensionless Equations of Motion

Substituting this into the Euler-Lagrange equations yields the following coupled ordinary differential equations:

$$(m_c + m)\ddot{x} + ml(\ddot{\theta}cos\theta - \dot{\theta}^2 sin\theta) = F \tag{1}$$

$$\ddot{x}lcos\theta + \ddot{\theta}l^2 + glsin\theta = 0 \tag{2}$$

These can be integrated straight away, or can be developed a bit further to fit the control theory approach to solving this problem. This will be our preferred method, since we are looking to apply a feedback loop through the time-dependent forcing $F(t)$, which will indirectly also depend on the angular position and velocity of the pendulum. This will enable us to use a technique called "normalised oscillation" to characterise the oscillatory properties of the system.

In general, the first step in control theory is to move away from an expression of the equations rooted in the physical system, and towards one that more succinctly expresses the relationships between coordinates, i.e. one which contains dimensionless variables and parameters [Bec21]. One may achieve this in equations 1 and 2 by choosing an energy scale $l/g(m_c + m)$ and a gravitational potential scale lg respectively:

$$\ddot{x_d} = \frac{\ddot{x}}{g}, \qquad \ddot{x} := \ddot{x_d} \tag{3}$$

$$\theta = \theta \tag{4}$$

where 3 is a re-scaling of $\ddot{x}$ to dimensionlessness, and a relabeling of this dimensionless variable x_d to x for the sake of clear notation, since we will not be transforming back to dimensions. In 4, θ need not be updated since it is already dimensionless. All that remains is to divide 1 by the remaining factor in the force scale we chose, and 2 by l:

$$\ddot{x} + \gamma(\ddot{\theta}cos\theta - \dot{\theta}^2 sin\theta) = \mu(t) \tag{5}$$

$$\ddot{x}cos\theta + \ddot{\theta} + sin\theta = 0 \tag{6}$$

where γ is a dimensionless parameter $= \frac{m}{m_c+m}$ and $\mu(t) = \frac{F(t)}{m+m_c}$. This enables easier processing of data, as the entire behaviour of the system unique to a specific set of dimensions is encoded by these variables. We can find, if a system contains n variables and parameters, and we have access to m basic scales such as mass, length and time, it is possible to find a system of equations with n-m dimensionless parameters (c.f. Buckingham Pi Theorem) [Bec21]. We are now at a point where we can implement an integration routine to find explicit solutions for x and θ, and can use this to accurately perform motor driven oscillations of the cart pendulum.

References

[Bec21] John Bechhoefer. *Control Theory for Physicists*, page 29. Cambridge University Press, 2021.