Topics in Time Series Analysis: Assignment 7

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24^{th} of April 2020

Q12

We consider a process as described in the assignment. We know that the spectral density function can be written as:

$$\begin{aligned}
f(\lambda) &= |\psi(\lambda)|^2 \frac{\sigma^2}{2\pi} \\
&= \sum_{j=0}^{q} b_j e^{ij\lambda} \sum_{j=0}^{q} b_j e^{-ij\lambda} \frac{\sigma^2}{2\pi}
\end{aligned}$$

We can further rewrite this as:

$$\begin{aligned}
f(\lambda) &= \left(\sum_{j=0}^{q} b_j e^{ij\lambda}) (\sum_{j=0}^{q} b_j e^{-ij\lambda} \right) \frac{\sigma^2}{2\pi} \\
&= \frac{\sigma^2}{2\pi} \left(\sum_{j=0}^{q} b_j^2 + 2 \sum_{h=1}^{q} \sum_{j=0}^{q} b_j b_{j+h} cos(\lambda h) \right) \\
&= \frac{\sigma^2}{2\pi} \left(1 + \sum_{j=1}^{q} b_j^2 + 2 \sum_{h=1}^{q} \sum_{j=0}^{q} b_j b_{j+h} cos(\lambda h) \right)
\end{aligned}$$

We set $q = 3$ to demonstrate how the re-arrangement is justified:

$$\begin{aligned}
& \left(\sum_{j=3}^{q} b_j e^{ij\lambda} \right) \left(\sum_{j=0}^{3} b_j e^{-ij\lambda} \right) \\
\Leftrightarrow \quad & (b_0 + b_1 e^{-i\lambda} + b_2 e^{-2i\lambda} + b_3 e^{-3i\lambda})(b_0 + b_1 e^{-i\lambda} + b_2 e^{-2i\lambda} + b_3 e^{-3i\lambda}) \\
\Leftrightarrow \quad & (b_0^2 + b_1^2 + b_2^2 + b_3^2 + b_0 b_1 e^{-i\lambda} + b_0 b_1 e^{i\lambda} + b_1 b_2 e^{-i\lambda} + ... + b_0 b_2 e^{-2i\lambda} + ... \\
\Leftrightarrow \quad & \sum_{j=0}^{3} b_j^2 + 2 \sum_{h=1}^{3} \sum_{j=0}^{3} b_j b_{j+h} cos(\lambda h)
\end{aligned}$$

We can further show that:

$$
\begin{aligned}
\gamma(h) &= \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda h} f(\lambda) d\lambda \\
&= \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda h}\left(1+\sum_{j=1}^{q} b_j^2 + 2\sum_{h=1}^{q}\sum_{j=0}^{q} b_j b_{j+h} cos(\lambda h)\right) d\lambda \\
&= \frac{\sigma^2}{2\pi}\left[\left(1+\sum_{j=0}^{q} b_j^2\right)\int_{-\pi}^{\pi} e^{i\lambda h} d\lambda + \int_{-\pi}^{\pi}\sum_{h=1}^{q}\sum_{j=0}^{q} b_j b_{j+h}(e^{-i\lambda h}+e^{i\lambda h})e^{i\lambda h} d\lambda\right] \\
&= \frac{\sigma^2}{2\pi}\left[\left(1+\sum_{j=0}^{q} b_j^2\right)\int_{-\pi}^{\pi} cos(\lambda h) + isin(\lambda h) d\lambda + \int_{-\pi}^{\pi}\sum_{j=0}^{q} b_j b_{j+h}(1+e^{2i\lambda h}) d\lambda\right] \\
&= \frac{\sigma^2}{2\pi}\left[\left(1+\sum_{j=0}^{q} b_j^2\right)\underbrace{[sin(\lambda h) - icos(\lambda h)]_{-\pi}^{\pi}}_{=0} + \int_{-\pi}^{\pi}\sum_{j=0}^{q} b_j b_{j+h} d\lambda + \int_{-\pi}^{\pi}\underbrace{cos(2\lambda h) + isin(2\lambda h)}_{=0} d\lambda\right] \\
&= \frac{\sigma^2}{2\pi}\left(\sum_{j=0}^{q} b_j b_{j+h}\pi - (-\pi)\sum_{j=0}^{q} b_j b_{j+h}\right) \\
&= \sigma^2\sum_{j=0}^{q} b_j b_{j+h}
\end{aligned}
$$

This holds for all $\gamma(h)$ where $h > 0$. Moreover:

$$
\begin{aligned}
\gamma(0) &= \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi}\left(1+\sum_{j=1}^{q} b_j^2 + 2\sum_{h=1}^{q}\sum_{j=0}^{q} b_j b_{j+h} cos(\lambda h)\right) d\lambda \\
&= \frac{\sigma^2}{2\pi}\left[\int_{-\pi}^{\pi}\left(1+\sum_{j=1}^{q} b_j^2\right) d\lambda + 2\sum_{h=1}^{q}\sum_{j=0}^{q} b_j b_{j+h}\int_{-\pi}^{\pi} cos(\lambda h) d\lambda\right] \\
&= \sigma^2\left(1+\sum_{j=1}^{q} b_j^2\right)
\end{aligned}
$$

It's important to note that in our derivation of $\gamma(h)$ the multiplication of $(e^{-i\lambda h}+e^{i\lambda h})e^{i\lambda h}$ our inside the summation over h only becomes $(1 + e^{2i\lambda h})$ when the h in the summation is the same as the h we are calculating $\gamma(h)$ for.

For example, consider $\gamma(1)$:

$$\begin{aligned}
\gamma(1) &= \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda} f(\lambda) d\lambda \\
&= \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda h}\left(1+\sum_{j=1}^{q} b_j^2 + 2\sum_{h=1}^{q}\sum_{j=0}^{q} b_j b_{j+h} cos(\lambda h)\right) d\lambda \\
&= \frac{\sigma^2}{2\pi}\left[\left(1+\sum_{j=0}^{q} b_j^2\right)\int_{-\pi}^{\pi} e^{i\lambda} d\lambda + \int_{-\pi}^{\pi} e^{i\lambda}\sum_{h=1}^{q}\sum_{j=0}^{q} b_j b_{j+h}(e^{-i\lambda h}+e^{i\lambda h}) d\lambda\right] \\
&= \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda}\sum_{h=1}^{q}\sum_{j=0}^{q} b_j b_{j+h}(e^{-i\lambda h}+e^{i\lambda h}) d\lambda \\
&= \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi}\sum_{j=0}^{q} b_j b_{j+1}(e^{-i\lambda}+e^{i\lambda})e^{i\lambda} + \sum_{j=0}^{q} b_j b_{j+1}(e^{-2i\lambda}+e^{2i\lambda})e^{i\lambda} + ... + \sum_{j=0}^{q} b_j b_{j+1}(e^{-qi\lambda}+e^{qi\lambda})e^{i\lambda} d\lambda \\
&= \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi}\sum_{j=0}^{q} b_j b_{j+1}(e^{-i\lambda}+e^{i\lambda})e^{i\lambda} d\lambda \\
&= \frac{\sigma^2}{2\pi}\left(\pi\sum_{j=0}^{q} b_j b_{j+1} - (-\pi)\sum_{j=0}^{q} b_j b_{j+1}\right) \\
&= \sigma^2\sum_{j=0}^{q} b_j b_{j+1}
\end{aligned}$$

Except for the first term all parts will go to zero.
Putting all derivations together we can see that $f(\lambda)$ can be also written as:

$$f(\lambda) = \frac{1}{2\pi}\left(\gamma(0) + 2\sum_{h=1}^{q}\gamma(h) cos(\lambda h)\right)$$