Time Series Assignment 5

Lucas Cruz Fernandez

Q9

We first read in the data and again calculate the growth rate r_t as in Assignment 3. The code for this is omitted for brevity.

We save the last 6 observations for calculating the Root Men Squared Error at the end and define some general parameters that are of use later.

```
no_models = 4 #number of models we want to estimate
n <- length(growth_rate)  #helpful when "reducing" data sets later
valid <- growth_rate[c((n-5):n)] #contains last 6 elements of growth_rate
```

The below function creates a one-step ahead prediction using the supplied *data* and binds all predictions together in one list, which is returned.

```
forecast_onestep <- function(data, no_models){
  prediction_list <- vector()  #list of predictions to be returned
  for (i in seq(from=0, to=(no_models - 1))){     #loop over the models we are interested in
    model <- arima(data, order = c(i, 0, 0),
                   optim.control = list(maxit = 1000))    #estimate the model
    prediction <- predict(model, n.ahead = 1, se.fit=FALSE) #make prediction based on model
    prediction_list <- c(prediction_list, prediction)   #add prediction to prediction list
  }
  return(prediction_list)
}
```

We will use this function now to forecast the last 6 values of the dataset. First, we create an empty 4×6 matrix, where the columns represent the period we forecast and each row represents a different model. The forecasting process in the *for*-loop then works as follows: to predict the last period we use all but the last observation, thus, we remove it from the dataset. We use this "reduced" dataset to make the forecast using our before defined function. The result of this function - the list of predictions from each model - is then added as the $i^t h$ column of the matrix. The second to last period is then predicted using all but the last two observations and so on. In the end, the matrix contains all predictions of all models for the periods of interest (e.g. element [2, 2] of the matrix is the prediction of the second to last period using an AR(1) model).

```
#apply function to different datasets to get different predictions
n <- length(growth_rate)
predictions_mat <- matrix(data = NA, nrow = 4, ncol = 6)
for(i in 1:6){
  data <- growth_rate[c(1:(n-i))]
  predictions_mat[, i] <- forecast_onestep(data = data, no_models = no_models)
}
predictions_mat <- Rev(predictions_mat, 2) #predictions returned are in "wrong" order;
#we want the first column to represent the eraliest forecasted period (T - 6), not the last
```

Finally, to calculate the RMSE we use a predefined function from the Metrics package. The errors vector is filled with the corresponding RMSE of each model.

```
errors <- vector(length = no_models)
for(i in 1:dim(predictions_mat)[1]){
  errors[i] <- rmse(valid, predictions_mat[i, ])
}
```

In this list of errors the second one is the smallest. The second model we used to create the one-step ahead forecasts is an $AR(1)$ model. Thus, based on the RMSE this is the best model to forecast the GDP growth rate.

```
which(errors == min(errors))
```

```
## [1] 2
```

Q10

First, we will derive the given limit of the process described in the task.
We omit the factor $T^{-\frac{5}{2}}$ for ease of notation. It will later be added when we focus on the limit with respect to $T \to \infty$. Now, we can show that:

$$\begin{aligned}\sum_{t=1}^{T} t^2\epsilon_t &= \sum_{t=1}^{T}\sum_{s=1}^{t} t\epsilon_t \\ &= \sum_{s=1}^{T}\left(\sum_{t=1}^{T} t\epsilon_t - \sum_{t=1}^{s-1} t\epsilon_t\right) \\ &= T\sum_{t=1}^{T} t\epsilon_t - \sum_{s=1}^{T}\sum_{t=1}^{s-1} t\epsilon_t\end{aligned}$$

We know from class that the first term $T\sum_{t=1}^{T} t\epsilon_t$ can be rewritten as:

$$T(T\sum_{t=1}^{T} t\epsilon_t - \sum_{t=1}^{T} x_t)$$

We now focus on the latter term $\sum_{s=1}^{T}\sum_{t=1}^{s-1} t\epsilon_t$, which we can write as:

$$\begin{aligned}&\sum_{s=1}^{T}\sum_{t=1}^{s-1}\sum_{p=1}^{t}\epsilon_t \\ =&\sum_{s=1}^{T}\sum_{p=1}^{s-1}\sum_{t=p}^{s-1}\epsilon_t \\ =&\sum_{s=1}^{T}\sum_{p=1}^{s-1}\left(\sum_{t=1}^{s-1}\epsilon_t - \sum_{t=1}^{p-1}\epsilon_t\right) \\ =&\sum_{s=1}^{T}\sum_{p=1}^{s-1} x_{s-1} - \sum_{s=1}^{T}\sum_{p=1}^{s-1} x_{p-1}\end{aligned}$$

where x_t is defined as $x_t = \sum_{t=1}^{t} \epsilon_t$, i.e. the shock accumulator described in class.
We can rewrite the above term even further:

$$\begin{aligned}
&\sum_{s=1}^{T}(s-1)x_{s-1} - \sum_{s=1}^{T}\left(\sum_{p=1}^{T} x_{p-1} - \sum_{p=s}^{T} x_{p-1}\right) \\
&= \sum_{s=1}^{T} s x_{s-1} - \sum_{s=1}^{T} x_{s-1} + T\sum_{s=1}^{T} x_{s-1} - \sum_{s=1}^{T}\sum_{p=1}^{T} x_{p-1} \\
&= \sum_{s=1}^{T} s x_{s-1} - \sum_{s=1}^{T} x_{s-1} + T\sum_{s=1}^{T} x_{s-1} - \sum_{s=1}^{T} s x_{s-1}
\end{aligned}$$

We now change all indices to t for ease of notation and as in the lecture rewrite x_{t-1} to x_t:

$$\begin{aligned}
\sum_{t=1}^{T} x_{t-1} &= \sum_{t=1}^{T} x_t - T(x_T + x_0) \\
&= \sum_{t=1}^{T} x_t - o_p(1)
\end{aligned}$$

Note that the latter term is $o_p(1)$ when we add the factor $T^{-\frac{5}{2}}$, which is omitted here for brevity and ease of notation.
Now combining all terms that we have derived so far we get:

$$T^2 \sum_{t=1}^{T} \epsilon_t - 2\sum_{t=1}^{T} t x_t + \sum_{t=1}^{T} x_t$$

Adding the factor we omitted in the beginning we have:

$$T^{-\frac{1}{2}} \sum_{t=1}^{T} \epsilon_t - 2T^{-\frac{3}{2}} \sum_{t=1}^{T} \frac{t}{T} x_t + T^{-\frac{5}{2}} \sum_{t=1}^{T} x_t$$

The last term is $o_p(1)$ when $T \to \infty$, for the other two terms we know from class that:

1. $T^{-\frac{1}{2}} \sum_{t=1}^{T} \epsilon_t \xrightarrow{d} \sigma W(1)$
2. $2T^{-\frac{3}{2}} \sum_{t=1}^{T} \frac{t}{T} x_t \xrightarrow{d} 2\sigma \int_0^1 rW(r)dr$

Note that $\frac{t}{T}$ is essentially r. Combining these two terms we therefore have:

$$T^{-\frac{1}{2}} \sum_{t=1}^{T} \epsilon_t - 2T^{-\frac{3}{2}} \sum_{t=1}^{T} \frac{t}{T} x_t \xrightarrow{d} \sigma W(1) - 2\sigma \int_0^1 rW(r)dr$$

We now turn to the analysis of this limit, namely finding its variance. Again, we know from class that the variance of the first term is $\frac{\sigma^2}{3}$. The variance of the second term can be found by doing the following derivations:
First, we rewrite the integral:

$$\int_0^1 rW(r)dr = \int_0^1 (1-r^2)dW(r)$$

We plug this into the term and find its variance by following the same steps as in the lecture:

$$\begin{aligned}
Var(2\sigma \int_0^1 rW(r)dr) &= 4\sigma^2 Var(\int_0^1 (1-r^2)dW(r)) \\
&= 4\sigma^2 \int_0^1 (1-r^2)^2 dr \\
&= 4\sigma^2 \int_0^1 (1-2r^2+r^4)dr \\
&= 4\sigma^2 \left[r - \frac{2}{3}r^3 + \frac{1}{5}r^5 \right]_0^1 \\
&= 4\sigma^2 \frac{8}{15}
\end{aligned}$$

To find the variance of the whole limit we know that $Var(a-b) = Var(a) + Var(b) - 2Cov(a,b)$, where $a = \sigma W(1)$ and $b = 2\sigma \int_0^1 rW(r)dr$. We therefore also need to find the covariance. Note that $E[W(r)] = 0$.

$$\begin{aligned}
Cov\left(\sigma W(1), 2\sigma \int_0^1 rW(r)dr\right) &= 2\sigma^2 \left(E\left[W(1) \int_0^1 rW(r)dr \right] - E\left[W(1)\right] E\left[\int_0^1 rW(r)dr \right] \right) \\
&= 2\sigma^2 E\left[W(1) \int_0^1 (1-r^2)dW(r) \right] \\
&= 2\sigma^2 E\left[W(1) \left[r - \frac{1}{3}r^3 \right]_0^1 \right] \\
&= 2\sigma^2 E\left[W(1) \left[r - \frac{1}{3}r^3 \right] \right] \\
&= 2\sigma^2 E\left[W(1) \left[r - \frac{1}{3}r^3 \right] \right] \\
&= 2\sigma^2 E\left[W(1)\right] \left[r - \frac{1}{3}r^3 \right] & = 0
\end{aligned}$$

Thus, the variance of the whole limit is just:

$$Var(\sigma W(1)) + Var\left(\int_0^1 rW(r)dr \right) = \frac{\sigma^2}{3} \frac{32}{15}\sigma^2 \tag{1}$$

$$= \frac{37}{15}\sigma^2 \tag{2}$$