# Topics in Time Series Analysis: Assignment 2

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## Q7

We consider $x_t = \mu + a_1 x_{t-1} + \epsilon_t$ and the given assumptions. The OLS estimate of the coefficient is then given by:

$$\hat{a}_1 = \frac{C\hat{o}v(x_t, x_{t-1})}{V\hat{a}r(x_{t-1})}$$

We can rearrange this term to show that this is equal to

$$\hat{a}_1 = a_1 + \frac{C\hat{o}v(x_{t-1}, \epsilon_t)}{V\hat{a}r(x_{t-1})}$$

To make sure that the term does not converge to a "abnormal" distribution we center and scale the OLS coefficient $\hat{a}_1$:

If we are able to show that one part of this fraction converges in probability to a constant and the other converges in distribution we could apply Slutsky's Theorem to find their joint convergence in distribution.

We first look at the denominator. As we have shown in class we are able to apply LLN1 if $x_t$ is a linear process that is absolutely summable and has bounded fourth moments. To show this we can rewrite $x_t$ as:

$$x_t = \frac{\mu}{1-a} + \sum_{j=0}^{\infty} a^j \epsilon_{t-j}$$

We have shown in class that the exact same process is linear, absolutely summable and has finite fourth moments. Thus, we omit these derivations here. With all conditions fulfilled we are able to apply LLN1 and show that the estimated variance converges to its true value which is

$$E[x_{t-1}^2] - E[x_{t-1}^2] = E[(\frac{\mu}{1-a} + \sum_{j=0}^{\infty} a^j \epsilon_{t-j})^2] - \big(\frac{\mu}{1-a}\big)^2 \tag{1}$$

$$= E[\frac{\mu^2}{(1-a)^2} + 2\mu \sum_{j=0}^{\infty} a^j \epsilon_{t-j} + \sum_{j=0}^{\infty} a^{2j} \epsilon_{t-j}^2] - \frac{\mu^2}{(1-a)^2} \tag{2}$$

$$= \frac{\sigma^2}{1-a^2} \tag{3}$$

This is indeed a constant.

We now turn to the numerator of the fraction in (1). For convergence in distribution we want to apply the first CLT from class. We will show now that all the condition for its application are indeed fulfilled. Note that $Z_t \coloneqq x_{t-1}\epsilon_t$.
For the following derivations we need to know: align* E[x$_{t-1}^2$] $= E[(\frac{\mu}{1-a} + \sum_{j=0}^{\infty} a^j \epsilon_{t-j})^2]$
$= E[\frac{\mu^2}{(1-a)^2} + 2\mu \sum_{j=0}^{\infty} a^j \epsilon_{t-j} + \sum_{j=0}^{\infty} a^{2j} \epsilon_{t-j}^2]$
$= \frac{\mu^2}{(1-a)^2} + \frac{\sigma^2}{1-a^2}$

**1.**

$\frac{1}{\sqrt{T}} \sum_{t=2}^{T} Z_t$ is a mds:

$$E[Z_t|I_{t-1}] = x_{t-1} E[\epsilon_t|I_{t-1}] = 0$$

**2.**

$Z_t$ has finite fourth moments.

$$E[Z_t^4] = \overbrace{E[x_{t-1}^4]}^{<\infty \text{ since absolute summability implies 4 summability}} \underbrace{E[\epsilon_t^4]}_{<\infty \text{ as given}}$$

**3.**

Is $E[Z_t^2] > 0$

$$E[Z_t^2] = E[\epsilon_t^2]E[x_{t-1}^2]$$
$$= \frac{\mu^2\sigma^2}{(1-a)^2} + \frac{\sigma^4}{1-a^2} \quad > 0$$

**4.**

What's $\frac{1}{T}\sum_{t=1}^{T} E[Z_t^2]$?

$$\begin{aligned}\frac{1}{T}\sum_{t=0}^{T} E[Z_t^2] &= \frac{1}{T}\sum_{t=0}^{T} \frac{\mu^2\sigma^2}{(1-a)^2} + \frac{\sigma^4}{1-a^2} \\ &= \frac{\mu^2\sigma^2}{(1-a)^2} + \frac{\sigma^4}{1-a^2} \qquad > 0\end{aligned}$$

**5.**

The variance of $Z_t$ must converge towards its true value in probability.

$$\frac{1}{T}\sum_{t\geq 1} \underbrace{Z_t^2}_{\text{no mds}} \xrightarrow{p} \frac{\mu^2\sigma^2}{(1-a)^2} + \frac{\sigma^4}{1-a^2}$$

We can rewrite:

$$\frac{1}{T}\sum_{t\geq 1} Z_t^2 = \frac{1}{T}\sum_{t\geq 1} \underbrace{x_{t-1}^2(\epsilon_t^2 - \sigma 2)}_{:=Y_t} + \sigma^2 x_{t-1}^2)$$

$Y_t$ is a mds: $E[Y_t|I_{t-1}] = E[x_{t-1}\epsilon_t^2|I_{t-1}] - E[x_{t-1}\sigma^2|I_{t-1}] = x_{t-1}\sigma^2 - x_{t-1}\sigma^2 = 0$
This allows us to show that the above term converges as follows

$$= \underbrace{\frac{1}{T}\sum_{t\geq 1} Y_t}_{\xrightarrow{p} 0 \text{ by LLN1}} + \sigma^2 \underbrace{\frac{1}{T}\sum_{t\geq 1} x_{t-1}^2}_{\xrightarrow{p} \frac{\sigma^2\mu^2}{(1-a)^2} + \frac{\sigma^4}{1-a^2} \text{ by LLN1}}$$

If we now put all our above findings together we can show that:

$$\begin{aligned}&\Rightarrow \frac{1}{\sqrt{T}}\sum_{t=2}^{T} x_{t-1}\epsilon_t \xrightarrow{d} N\left(0, \frac{\sigma^2\mu^2}{(1-a)^2} + \frac{\sigma^4}{1-a^2}\right) \\ &\Rightarrow \frac{1}{T}\sum_{t=2}^{T} V\hat{a}r(x_{t-1}) \xrightarrow{p} \frac{\sigma^2}{1-a^2}\end{aligned}$$

Therefore by Slutsky's Theorem:

$$\Rightarrow \sqrt{T}(\hat{a}_1 - a_1) \xrightarrow{d} N\big(0, \frac{(1-a^2)^2\mu^2}{(1-a)^2\sigma^2} + 1 - a^2\big)$$

# Q8

First, we rearrange such that

$$S_T = \frac{1}{T^\alpha}\sum_{t=1}^{T} t\epsilon_t = \frac{1}{T^{\alpha-1}}\sum_{t=1}^{T}\frac{t}{T}\epsilon_t = \frac{1}{T^{\alpha-1}}\sum_{t=1}^{T} Z_t$$

Now we check whether the conditions for use of CLT1 are fulfilled for $Z_t$:

**1.**

Is $Z_t$ a mds?

$$E|\frac{t}{T}\epsilon_t| = \frac{t}{T}E|\epsilon_t| = 0$$
$$I_t = \sigma(\epsilon_s, s \leq t)$$
$$E[Z_t|I_t] = \frac{t}{T}E[\epsilon_t|I_{t-1}] = 0$$

**2.**

Does $Z_t$ have bounded fourth moments?

$$E[Z_t^4] = E[(\frac{t}{T})^4\epsilon_t^4] = \underbrace{(\frac{t}{T})^4}_{\in(0,1]} E[\epsilon_t^4] < \infty$$

since it's given that $E[\epsilon_t^4] < \infty$.

**3.**

Is $E[Z_t^2] > 0$?

$$E[Z_t^2] = E[(\frac{t}{T})^2\epsilon_t^2] = \underbrace{(\frac{t}{T})^2}_{\in(0,1]} E[\epsilon_t^2] = \frac{t^2}{T^2}\sigma^2 > 0$$

**4.**

What's $\frac{1}{T}\sum_{t=1}^{T} E[Z_t^2]$?

$$
\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} E[Z_t^2] &= \frac{1}{T}\sum_{t=1}^{T} \frac{t^2}{T^2}\sigma^2 \\
&= \frac{1}{T^3}\sigma^2 \sum_{t=1}^{T} t^2 \\
&= \frac{1}{T^3}\sigma^2 \frac{T(T+1)(2T+1)}{6} \\
&= \frac{1}{T^3}\sigma^2 \frac{2T^3+3T^2+T}{6} \\
&= \frac{1}{3}\sigma^2 + o_p(1) > 0
\end{aligned}
$$

**5.**

To what does $\frac{1}{T}\sum_{t=1}^{T} Z_t^2$ converge?

$$
\frac{1}{T}\sum_{t=1}^{T} Z_t^2 = \frac{1}{T}\sum_{t=1}^{T} \frac{t^2}{T^2}\sigma^2 \xrightarrow{p} \frac{1}{3}\sigma^2
$$

By setting $\alpha = \frac{3}{2}$ we get

$$
S_T = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} \frac{t}{T}\epsilon_t \quad \xrightarrow{d} \quad \mathcal{N}(0, \frac{1}{3}\sigma^2)
$$