University of Mannheim - Department of Economics
Mehdi Hosseinkouchack Spring 2020

Topics in Time Series Analysis
Assignments

Useful inequalities:

a) $\Pr\{|X| > a\} \leq \frac{E(X^2)}{a^2}$ for $a > 0$.

b) **(Hölder's inequality)** $E|XY| \leq (E|X|^p)^{1/p}(E|Y|^q)^{1/q}$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$.

c) **(Mikowski's inequality)** $\sqrt[p]{E|X+Y|^p} \leq \sqrt[p]{E|X|^p} + \sqrt[p]{E|Y|^p}$ with $p \geq 1$.

Trigonometric identities:

a) $\sin^2 x + \cos^2 x = 1$

b) $e^{ix} = \cos x + i \sin x$

c) $2\cos x = e^{ix} + e^{-ix}$

d) $2i \sin x = e^{ix} - e^{-ix}$

e) $\cos 2x = \cos^2 x - \sin^2 x = \frac{1-\tan^2 x}{1+\tan^2 x}$

f) $\sin 2x = 2 \sin x \cos x = \frac{2\tan x}{1+\tan^2 x}$

Assignment 1 (due on 24.02.2020 at 13:45)

Question 1 *Show that if* $V_T - U_T \xrightarrow{p} 0$ *and* $U_T \xrightarrow{p} U$*, then* $V_T \xrightarrow{p} U$*.*

Question 2 *Let* $\{x_t\}$ *be an mds with respect to its own past, then show that* $E\left(x_t|x_{t-h}\right) = 0$ *and* $cov\left(x_t, x_{t-h}\right) = 0$*, for* $h > 0$*.*

Hint 1: Law of Iterated Expectations (LIE) says

$$E\left(Y\right) = E\left(E\left(Y|\Omega\right)\right),$$

where Ω is an information set.

Hint 2: LIE for conditional expectations says

$$E\left(Y|\Omega_1\right) = E\left(E\left(Y|\Omega_1, \Omega_2\right)|\Omega_1\right),$$

where Ω_1 and Ω_2 are two information sets.

Question 3 *Let* $x_t = \mu + \varepsilon_t + \frac{1}{2}\varepsilon_{t-1} + \frac{1}{3}\varepsilon_{t-2}$*, with* $\varepsilon_t \sim iid\left(0, \sigma^2\right)$*. For* $\{x_t\}_{t=1}^T$ *define* $\overline{X}_T = \frac{1}{T}\sum_{t=1}^T x_t$ *and show that* $\overline{X}_T - \mu \xrightarrow{p} 0$*.*

Assignment 2 (due on 02.03.2020 at 13:45)

Question 4 *Consider*

$$\begin{aligned} x_t &= \mu_t + \varepsilon_t,\ t = 1, 2, ..., T \\ \mu_t &= \begin{cases} \delta_1,\ t < \lfloor \tau T \rfloor \\ \\ \delta_2,\ t \geq \lfloor \tau T \rfloor \end{cases}, \end{aligned}$$

with $\varepsilon_t \sim iid\left(0, \sigma^2\right)$*,* $\tau \in (0, 1)$ *is a fixed and known constant and* $\delta_1 \neq \delta_2$*. Note that since* μ_t *is deterministic and* ε_t *is free of serial correlations,* $\{x_t\}$ *is free of serial correlations. Let*

$$\widehat{\rho}_x\left(1\right) = \frac{\sum_{t=2}^T\left(x_t - \overline{x}_T\right)\left(x_{t-1} - \overline{x}_T\right)}{\sum_{t=1}^T\left(x_t - \overline{x}_T\right)^2},\ \textit{with}\ \overline{x}_T = \frac{1}{T}\sum_{t=1}^T x_t.$$

- *Find the plim of* $\widehat{\rho}_x\left(1\right)$ *(as* $T \to \infty$*). Discuss why this estimator is not consistent.*
- *Provide a consistent estimator for the first autocorrelation of* $\{x_t\}$*.*

Question 5 *Consider* $\left(1 - aL\right)x_t = \left(1 + bL\right)\varepsilon_t$ *with* $|a| < 1$*. Find* c_j*'s in* $x_t = \sum_{j=0}^\infty c_j\varepsilon_{t-j}$*. (Note:* c_j*'s are a function of* a *and* b*).*

Assignment 3 (due on 09.03 at 13:45)

Question 6 *Using the seasonally adjusted German gdp series compute the quarterly gdp growth series as* $r_t = \log(gdp_t) - \log(gdp_{t-1})$ *and fit an* $ARMA(p,q)$ *model to* r_t *using Stata. To estimate the values of* p *and* q *use AIC. Submit your codes together with your choice of* p *and* q*. Are the residuals for your chosen model free of serial correlations?*

Hint 1: In Stata you may estimate an ARMA model using

```
arima depvar, arima(#p,0,#q)
```

Hint 2: for-loops can be coded in Stata using

```
forvalues lname=range {
Stata commands referring to`lname'
}
```

For "range" you may for example use `1(1)5` which implies that counter `lname` will assume values of 1, 2, 3, 4, 5.

Hint 3: Conditional statements can be coded in Stata using

```
if some_condition {
Commads to be executed if the condition is true
}
```

Hint 4: Using `ac var` you may plot the autocorrelations for `var`.

Hint 5: Consider $p_{\max} = q_{\max} = 3$.

Assignment 4 (due on 16.03.2020 at 13:45)

Question 7 *Consider* $x_t = \mu + a_1 x_{t-1} + \varepsilon_t$ *with* $\varepsilon_t \sim iid(0, \sigma^2)$, $E\varepsilon_t^4 < \infty$ *and* $|a_1| < 1$. *Find the asymptotic distribution of the OLS estimator for* a_1 *using* $\{x_t\}_{t=1}^T$.

Question 8 *Let* $S_T = T^{-\alpha} \sum_{t=1}^T t\varepsilon_t$ *with* $\varepsilon_t \sim iid(0, \sigma^2)$ *and* $E\varepsilon_t^4 < \infty$. *Is there a value for* α *such* $S_T \xrightarrow{d} \mathcal{N}(0, \lambda)$*? Provide a complete asymptotic analysis and find* λ *(as function of* σ^2*).*

Assignment 5 (due on 02.04.2020 at 13:00)

Question 9 *For the seasonally adjusted german GDP series and define* r_t *as in Assignment 3. Consider the* $AR(p)$ *model for* $p = 0, 1, 2$ *and* 3 *as three alternative models to* *provide a one step ahead forecast for r_t. Which is of these models do you prefer? (Hint: use the last 6 observations to calculate and report the corresponding $RMSE$ for all of these models; using this information you may be able to recommend one).*

Question 10 *Say, $\{\varepsilon_t\}$ is such that we have $\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor rT\rfloor}\varepsilon_t \Rightarrow \sigma W(r)$ with σ^2 being the variance of ε_t and $W(r)$ is a Wiener process. Show that $T^{-5/2}\sum_{t=1}^{T} t^2\varepsilon_t \xrightarrow{d} \sigma W(1) - 2\sigma\int_0^1 rW(r)\,dr$. Find the variance of this limit.*

Assignment 6 (due on 24.04.2020 at 13:00)

Question 11 *The goal of this assignment is to see how the approach of Phillips-Perron helps with providing a reliable test for the null of a unit root in the presence of short-run dynamics for a model with a constant, i.e. see whether the empirical size of the PP is close the nominal level. Consider the following model*

$$\begin{aligned} x_t &= \beta_0 + u_t, & (1)\\ u_t &= a u_{t-1} + e_t, & (2)\\ e_t &= \varepsilon_t - \frac{5}{6}\varepsilon_{t-1} + \frac{1}{6}\varepsilon_{t-2}, & (3)\end{aligned}$$

where we are interested in testing $H_0 : a = 1$ versus $H_1 : a < 1$. Using Stata (or MatLab or other) simulate $\{x_t\}_{t=1}^{T}$ with $T = 1000$, for $S = 1000$ replications setting $a = 1$ and $\beta_0 = 1$ and assuming $\varepsilon_t \overset{iid}{\sim} \mathcal{N}(0,1)$. For each replication consider a type I error of $\alpha = 5\%$ and the following three strategies when testing for H_0:

a) *strategy 1: ignore the short-run dynamics in e_t;*

b) *strategy 2: consider the correction proposed by Phillips-Perron for which you estimate the long-run variance using slide 76 with $q = \lfloor T^{4/5}\rfloor$;*

c) *strategy 3: as strategy 2 except that you set $q = \lfloor T^{1/3}\rfloor$.*

To calculate the empirical size you should compare the t-stat with the corresponding critical value (-2.86) for the DF test for the model given by (1) and (3), that is to calculate

$$P_i = \frac{1}{S}\sum_{s=1}^{S} I\left\{t_{(s)}^i < -2.86\right\},$$

where $I\left\{t_{(s)}^i < -2.86\right\}$ takes a value of 1 if $t_{(s)}^i < -2.86$ and a value of zero otherwise, and $t_{(s)}^i$ is the statistic for the null of $a = 1$ for replication number s for $i = 1$ (strategy

1), $i = 2$ (strategy 2) and $i = 3$ (strategy 3), respectively. In your answer report P_1, P_2 and P_3. Further, send me the programs you used for this exercise. Which strategy do you prefer?

Note: empirical size of a test is the estimated probability of rejecting a correct null hypothesis. Ideally we would like to have a test that has an empirical size close to the nominal type I error, α. A test is undersized if the empirical rejection frequencies are smaller than α and is over-sized if the empirical rejection frequencies are above α. An under-sized test is usually called conservative, since it rejects less than it is supposed to; so to speak it accepts less risk than we are willing to take. An over-sized test rejects a correct null hypothesis more than it should, implying meaningless power against the alternative hypothesis.

Assignment 7 (due on 27.04.2020 at 13:45)

Question 12 *Consider an MA(q) process $x_t = \mu + \sum_{j=0}^{q} b_j \varepsilon_{t-j}$, with $b_0 = 1$ and the transfer function*

$$\Psi(\lambda) = \sum_{j=0}^{q} b_j e^{-ij\lambda}.$$

Show that the spectral density of x_t can be written as

$$f_x(\lambda) = \frac{1}{2\pi}\left(\gamma_x(0) + 2\sum_{h=1}^{q} \gamma_x(h)\cos(h\lambda)\right).$$

Hint 1: start with the definition of spectral density given on slide 128 and carefully expand the power transfer function to reach at the desired result.

Hint 2: you may wish to take a look at the trigonometric identities provided at the cover page.

Assignment 8 (due on 04.05.2020 at 13:45)

Question 13 *An ideal band-pass filter, $F(L, \lambda_l, \lambda_h) = \sum_{j=-\infty}^{\infty} c_j L^j$, would have a transfer function of the form*

$$\Psi(\lambda) = \begin{cases} 1, \text{ if } \lambda_l \le |\lambda| \le \lambda_h \ (\lambda_l \text{ and } \lambda_h \text{ are in } [0,\pi]), \\ 0, \text{ otherwise.} \end{cases}$$

Find the coefficients of this filter, c_j, as a function of λ_l and λ_h.

Hint: $c_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ij\lambda}\Psi(\lambda)\, d\lambda$.

Question 14 *Consider a simple moving average process:*

$$x_t = \frac{1}{2q+1}\sum_{|j|\le q}\varepsilon_{t-j},\ \varepsilon_t \sim iid\,(0,1)\,.$$

Let $f_x(\lambda;q)$ denote the spectral density of $\{x_t\}$. Find $f_x(\lambda;q)$ and plot it on $\lambda \in [0,\pi]$ for $q \in \{1,3,10\}$. Calculate $s(\lambda_0;q) = 2\int_0^{\lambda_0} f_x(\lambda)\, d\lambda$ for $\lambda_0 \in [0,\pi]$ and provide a plot of $VR(\lambda_0;q) = \frac{s(\lambda_0;q)}{s(\pi;q)}$ for $q \in \{1,3,10\}$.

Hint 1: $VR(\lambda_0;q)$ is the variance share of x_t which is due to those of its components that happen with frequencies below λ_0.

Hint 2: To calculate integrals in Stata use `integ yvar xvar`, where `yvar` is a numerical evaluation of a function at `xvar`.

Hint 3: You may also find $\int_0^{\lambda_0} f_x(\lambda)\, d\lambda$ analytically.

Assignment 9 (due on 13.05.2020 at 13:45)

Question 15 *In this assignment you are to see the behavior of the Whittle estimation approach for a simple MA(1) model using a simulation study.*

Consider an MA(1) model $x_t = \varepsilon_t + b\varepsilon_{t-1}$. Using Stata (or MatLab or else) simulate $\{x_t\}_{t=1}^{T}$ with $T = 1000$, for $S = 1000$ replications setting $b = 0.25$ for (i) $\varepsilon_t \overset{iid}{\sim} N(0,1)$, (ii) $\varepsilon_t \overset{iid}{\sim} t_{(5)}$ (t distribution with 5 degrees of freedom) and (iii) $\varepsilon_t \overset{iid}{\sim} unif\,(0,1)$ (uniformly distributed on the unit interval). Let $\tau = i, ii$ and iii denote the specifications for the error term. Let $b_s^{(\tau)}$ be the estimate for b from replication #s and let $\sigma_s^{2,(\tau)}$ be the corresponding estimate for $\sigma_\varepsilon^{2,(\tau)}$ from replication #s. Report the estimation biases: $\left(\frac{1}{S}\sum_{s=1}^{S} b_s^{(\tau)} - b\right)\times T$ and $\left(\frac{1}{S}\sum_{s=1}^{S}\sigma_s^{2,(\tau)} - \sigma_\varepsilon^{2,(\tau)}\right)\times T$ (which are scaled up to be "visible") as well as the MSE (square of bias plus the variance) for both parameters (scaled up by a factor of T as well). Also provide estimation bias and MSE for both b and the variance of ε_t (for $\tau = i, ii$ and iii) using the time domain maximum likelihood estimation method using your simulation study. As usual, submit your codes and results. Provide some insights/discussions on the performance of different estimators.

Hint 1: if $z \sim t_{(v)}$ then $var(z) = \frac{v}{v-2}$ for $v > 2$, and if $z \sim unif\,(0,1)$ then $var(z) = \frac{1}{12}$.

Hint 2: for maximization problem, you may want to use the first order conditions to concentrated the variance out.

Question 16 *Consider* $y_t = (1-L)^{-d}\varepsilon_t$ *with* $\varepsilon_t \sim iid\mathcal{N}(0,1)$ *for* $t = 1, ..., M, M+1, ..., M+T$ *and* $\varepsilon_t = 0$ *for* $t < 1$. *Generate* $\{y_t\}$ *with* $T = 1000$ *and* $M = 2000$ *for* $d \in \{-0.45, -0.25, 0, 0.25, 0.45\}$ *then discard the first* M *observations. Further, estimate and plot the autocorrelation function of* y_t *for different values of* d *for the first 25 lags and compare them with the theoretical values of the autocorrelation function. Provide also plots for* $\{y_t\}$.

Assignment 10 (due on 25.05.2020 at 13:45)

Question 17 *In this question you are to document the coverage probability (at 95% level) of the local Whittle estimator and the PR in a Monte Carlo study. Generate* y_t *as in the previous question [*$T = 10^3$ *and* $M = 2 \times 10^3$*] with* $d = 0.25$ *but replace* ε_t *with* x_t *where* $(1-0.5L)\,x_t = (1+0.3L)\varepsilon_t$ *for* $S = 1000$ *replications. For the estimation consider* $m_\alpha = \lfloor T^\alpha \rfloor$ *with* $\alpha = 0.2, 0.3, ...0.8$*; here you should then provide the coverage probability corresponding to all values for* m_α*. Further, document the effect of bandwidth choice on the bias and variance of each estimator.*

Note: The coverage probability in this exercise is the share of estimated values that lie in the confidence band which is predicted by the theory for each estimator.

Appendix

Solution to question 4

Let $N_T = \frac{1}{T}\sum_{t=2}^{T}\left(x_t - \overline{x}_T\right)\left(x_{t-1} - \overline{x}_T\right)$ and $D_T = \frac{1}{T}\sum_{t=1}^{T}\left(x_t - \overline{x}_T\right)^2$ such that

$$\widehat{\rho}_x\left(1\right) = \frac{N_T}{D_T}.$$

First note that

$$\begin{aligned}
\overline{x}_T &= \frac{1}{T}\sum_{t=1}^{T} x_t \\
&= \frac{1}{T}\sum_{t=1}^{\lfloor \tau T\rfloor -1} x_t + \frac{1}{T}\sum_{t=\lfloor \tau T\rfloor}^{T} x_t \\
&= \frac{1}{T}\sum_{t=1}^{\lfloor \tau T\rfloor -1} \left(\delta_1 + \varepsilon_t\right) + \frac{1}{T}\sum_{t=\lfloor \tau T\rfloor}^{T} \left(\delta_2 + \varepsilon_t\right) \\
&= \frac{\lfloor \tau T\rfloor - 1}{T}\delta_1 + \frac{T - \lfloor \tau T\rfloor + 1}{T}\delta_2 + \overline{\varepsilon}_T \\
&= \mu_{\tau,T} + \overline{\varepsilon}_T \\
&\xrightarrow{p} \tau\delta_1 + \left(1-\tau\right)\delta_2, \text{ since by LLN1 } \overline{\varepsilon}_T \xrightarrow{p} 0.
\end{aligned}$$

Now consider the numerator:

$$\begin{aligned}
N_T &= \frac{1}{T}\sum_{t=1}^{\lfloor \tau T\rfloor -1}\left(\delta_1 + \varepsilon_t - \mu_{\tau,T} - \overline{\varepsilon}_T\right)\left(\delta_1 + \varepsilon_{t-1} - \mu_{\tau,T} - \overline{\varepsilon}_T\right) \\
&+\frac{1}{T}\left(\delta_2 + \varepsilon_t - \mu_{\tau,T} - \overline{\varepsilon}_T\right)\left(\delta_1 + \varepsilon_{t-1} - \mu_{\tau,T} - \overline{\varepsilon}_T\right) \\
&+\frac{1}{T}\sum_{t=\lfloor \tau T\rfloor +1}^{T}\left(\delta_2 + \varepsilon_t - \mu_{\tau,T} - \overline{\varepsilon}_T\right)\left(\delta_2 + \varepsilon_{t-1} - \mu_{\tau,T} - \overline{\varepsilon}_T\right).
\end{aligned}$$

Now let's focus on the first term:

$$\begin{aligned}
N_{1,T} &= \frac{1}{T}\sum_{t=1}^{\lfloor \tau T\rfloor -1}\left(\delta_1 - \mu_{\tau,T} + \varepsilon_t - \overline{\varepsilon}_T\right)\left(\delta_1 - \mu_{\tau,T} + \varepsilon_{t-1} - \overline{\varepsilon}_T\right) \\
&= \frac{1}{T}\sum_{t=1}^{\lfloor \tau T\rfloor -1}\left(\delta_1 - \mu_{\tau,T}\right)^2 \\
&\quad + \left(\delta_1 - \mu_{\tau,T}\right)\frac{1}{T}\sum_{t=1}^{\lfloor \tau T\rfloor -1}\left(\varepsilon_{t-1} - \overline{\varepsilon}_T\right) + \left(\delta_1 - \mu_{\tau,T}\right)\frac{1}{T}\sum_{t=1}^{\lfloor \tau T\rfloor -1}\left(\varepsilon_t - \overline{\varepsilon}_T\right) \\
&\quad +\frac{1}{T}\sum_{t=1}^{\lfloor \tau T\rfloor -1}\left(\varepsilon_t - \overline{\varepsilon}_T\right)\left(\varepsilon_{t-1} - \overline{\varepsilon}_T\right)
\end{aligned}$$

$$\begin{aligned} &\xrightarrow{p} \tau\left(\delta_1 - \tau\delta_1 - (1-\tau)\,\delta_2\right)^2, \text{ [using } LLN1 \text{ for all the stochastic terms]} \\ = &\ \tau\,(1-\tau)^2\,(\delta_1-\delta_2)^2. \end{aligned}$$

For the second term we have

$$\begin{aligned} N_{2,T} &= \frac{1}{T}\left(\delta_2+\varepsilon_t-\mu_{\tau,T}-\bar{\varepsilon}_T\right)\left(\delta_1+\varepsilon_{t-1}-\mu_{\tau,T}-\bar{\varepsilon}_T\right) \\ &= \frac{1}{T}\left(\delta_2-\mu_{\tau,T}\right)\left(\delta_1-\mu_{\tau,T}\right) \\ &\quad+\frac{1}{T}\left(\delta_2-\mu_{\tau,T}\right)\left(\varepsilon_{t-1}-\bar{\varepsilon}_T\right) \\ &\quad+\frac{1}{T}\left(\varepsilon_t-\bar{\varepsilon}_T\right)\left(\delta_1-\mu_{\tau,T}\right) \\ &\quad+\frac{1}{T}\left(\varepsilon_t-\bar{\varepsilon}_T\right)\left(\varepsilon_{t-1}-\bar{\varepsilon}_T\right), \end{aligned}$$

where the first term converges to zero, for the second terms we have that $\bar{\varepsilon}_T \xrightarrow{p} 0$ by LLN1 and that $\frac{1}{T}\varepsilon_{t-1} \xrightarrow{p} 0$ since $\Pr\left\{\left|\frac{1}{T}\varepsilon_{t-1}\right| > a\right\} \leq \frac{E\left|\frac{1}{T}\varepsilon_{t-1}\right|}{a} < \frac{\sigma}{aT} \to 0$, therefore $\frac{1}{T}\left(\delta_2-\mu_{\tau,T}\right)\left(\varepsilon_{t-1}-\bar{\varepsilon}_T\right) \xrightarrow{p} 0$, similarly the third term is $o_p(1)$ and for the fourth term we have that

$$\begin{aligned} \frac{1}{T}\left(\varepsilon_t-\bar{\varepsilon}_T\right)\left(\varepsilon_{t-1}-\bar{\varepsilon}_T\right) &= \frac{1}{T}\varepsilon_t\varepsilon_{t-1}+\frac{1}{T}\varepsilon_t\bar{\varepsilon}_T-\frac{1}{T}\bar{\varepsilon}_T\varepsilon_{t-1}+\frac{1}{T}\left(\bar{\varepsilon}_T\right)^2 \\ &= o_p(1), \end{aligned}$$

using LLN1 and arguments similar to those used for the third term.

Finally for the last part of N_T we have

$$\begin{aligned} N_{3,T} &= \frac{1}{T}\sum_{t=\lfloor\tau T\rfloor+1}^{T}\left(\delta_2+\varepsilon_t-\mu_{\tau,T}-\bar{\varepsilon}_T\right)\left(\delta_2+\varepsilon_{t-1}-\mu_{\tau,T}-\bar{\varepsilon}_T\right) \\ &\xrightarrow{p} (1-\tau)\,\tau^2\,(\delta_2-\delta_1)^2, \end{aligned}$$

which follows using the same arguments as used for $N_{1,T}$.

Therefore

$$N_T \xrightarrow{p} \tau\,(1-\tau)\,(\delta_1-\delta_2)^2 \tag{4}$$

Now let's turn to the denominator.

$$\begin{aligned} D_T &= \frac{1}{T}\sum_{t=1}^{T}\left(x_t-\bar{x}_T\right)^2 \\ &= \frac{1}{T}\sum_{t=1}^{\lfloor\tau T\rfloor-1}\left(\delta_1+\varepsilon_t-\mu_{\tau,T}-\bar{\varepsilon}_T\right)^2+\frac{1}{T}\sum_{t=\lfloor\tau T\rfloor}^{T}\left(\delta_2+\varepsilon_t-\mu_{\tau,T}-\bar{\varepsilon}_T\right)^2 \\ &= \frac{1}{T}\sum_{t=1}^{\lfloor\tau T\rfloor-1}\left(\delta_1-\mu_{\tau,T}\right)^2+\frac{1}{T}\sum_{t=1}^{\lfloor\tau T\rfloor-1}\left(\varepsilon_t-\bar{\varepsilon}_T\right)^2+\frac{2}{T}\sum_{t=1}^{\lfloor\tau T\rfloor-1}\left(\delta_1-\mu_{\tau,T}\right)\left(\varepsilon_t-\bar{\varepsilon}_T\right) \end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{T}\sum_{t=\lfloor\tau T\rfloor}^{T}\left(\delta_2-\mu_{\tau,T}\right)^2+\frac{1}{T}\sum_{t=\lfloor\tau T\rfloor}^{T}\left(\varepsilon_t-\overline{\varepsilon}_T\right)^2+\frac{2}{T}\sum_{t=\lfloor\tau T\rfloor}^{T}\left(\delta_2-\mu_{\tau,T}\right)\left(\varepsilon_t-\overline{\varepsilon}_T\right) \\
= \; & \frac{\lfloor\tau T\rfloor-1}{T}\left(\delta_1-\mu_{\tau,T}\right)^2+\frac{T-\lfloor\tau T\rfloor+1}{T}\left(\delta_2-\mu_{\tau,T}\right)^2+\frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_t-\overline{\varepsilon}_T\right)^2 \\
& +\frac{2\left(\delta_1-\mu_{\tau,T}\right)}{T}\sum_{t=1}^{\lfloor\tau T\rfloor-1}\left(\varepsilon_t-\overline{\varepsilon}_T\right)+\frac{2\left(\delta_2-\mu_{\tau,T}\right)}{T}\sum_{t=\lfloor\tau T\rfloor}^{T}\left(\varepsilon_t-\overline{\varepsilon}_T\right) \\
= \; & \tau\left(1-\tau\right)\left(\delta_1-\delta_2\right)^2+\sigma^2+o_p\left(1\right). \qquad (5)
\end{aligned}
$$

Putting (4) and (5) together we obtain

$$
\widehat{\rho}_x\left(1\right)\xrightarrow{p}\frac{\tau\left(1-\tau\right)\left(\delta_1-\delta_2\right)^2}{\tau\left(1-\tau\right)\left(\delta_1-\delta_2\right)^2+\sigma^2}.
$$

Solution to Question 7

We shall first write out the OLS estimator which we want to analyze:

$$
\widehat{a}_1=\frac{\sum_{t=2}^{T}\left(x_t-\overline{x}_T\right)\left(x_{t-1}-\overline{x}_T\right)}{\sum_{t=2}^{T}\left(x_{t-1}-\overline{x}_T\right)^2}, \text{ where } \overline{x}_T=\frac{1}{T}\sum_{t=1}^{T}x_t.
$$

[Here I repeat/review the OLS algebra] Using the model, $x_t=\mu+a_1x_{t-1}+\varepsilon_t$ we have

$$
\begin{aligned}
\widehat{a}_1 &= \frac{\sum_{t=2}^{T}\left(\mu+a_1x_{t-1}+\varepsilon_t-\overline{x}_T\right)\left(x_{t-1}-\overline{x}_T\right)}{\sum_{t=2}^{T}\left(x_{t-1}-\overline{x}_T\right)^2} \\
&= \frac{\sum_{t=2}^{T}\left(\mu+a_1\left(x_{t-1}-\overline{x}_T\right)+\varepsilon_t-\overline{x}_T+a_1\overline{x}_T\right)\left(x_{t-1}-\overline{x}_T\right)}{\sum_{t=2}^{T}\left(x_{t-1}-\overline{x}_T\right)^2} \\
&= a_1+\frac{\sum_{t=2}^{T}\left(\mu+\varepsilon_t-\overline{x}_T+a_1\overline{x}_T\right)\left(x_{t-1}-\overline{x}_T\right)}{\sum_{t=2}^{T}\left(x_{t-1}-\overline{x}_T\right)^2},
\end{aligned}
$$

and keeping in mind the "$\sqrt{T}$-consistency" of OLS we may write

$$
\sqrt{T}\left(\widehat{a}_1-a_1\right)=\frac{\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\left(\mu+\varepsilon_t-\overline{x}_T+a_1\overline{x}_T\right)\left(x_{t-1}-\overline{x}_T\right)}{\frac{1}{T}\sum_{t=2}^{T}\left(x_{t-1}-\overline{x}_T\right)^2}.
$$

The denominator is easy to analyze: the conditions of LLN1 are satisfied hence we have

$$
\frac{1}{T}\sum_{t=2}^{T}\left(x_{t-1}-\overline{x}_T\right)^2\xrightarrow{p}\gamma_x\left(0\right)=\frac{\sigma^2}{1-a_1^2};
$$

the limit is deterministic, hence the denominator can be set aside. In what follows we shall analyze the behavior of numerator.

At this point, you should ask yourself "is there any vanishing term in the numerator?". The answer to this question is crucial in simplifying your asymptotic analysis. Indeed we should be able to sort out terms that vanish so that we do not spend any "energy" analyzing them. This shall make us efficient in doing asymptotic analysis. Generally we should be scrutinizing our expressions to find terms that converge to zero [vanish]. One such term is $\mu - \overline{x}_T + a_1\overline{x}_T \xrightarrow{p} 0$ (by LLN1: the conditions are satisfied for x_t such that $\overline{x}_T \xrightarrow{p} Ex_t = \frac{\mu}{1-a_1}$) that is we have $\mu - \overline{x}_T + a_1\overline{x}_T = o_p(1)$. Keeping this in mind we may write

$$\sqrt{T}(\widehat{a}_1 - a_1) = \frac{\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t(x_{t-1} - \overline{x}_T)}{\frac{1}{T}\sum_{t=2}^{T}(x_{t-1} - \overline{x}_T)^2} + \frac{(\mu - \overline{x}_T + a_1\overline{x}_T)\frac{1}{\sqrt{T}}\sum_{t=2}^{T}(x_{t-1} - \overline{x}_T)}{\frac{1}{T}\sum_{t=2}^{T}(x_{t-1} - \overline{x}_T)^2}. \quad (6)$$

Now the second term will vanish only if $\frac{1}{\sqrt{T}}\sum_{t=2}^{T}(x_{t-1} - \overline{x}_T)$ is bounded in probability, i.e. it should be $O_p(1)$. We check this next:

$$\begin{aligned}
\frac{1}{\sqrt{T}}\sum_{t=2}^{T}(x_{t-1} - \overline{x}_T) &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T-1}(x_t - \overline{x}_T) \\
&= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}(x_t - \overline{x}_T) - \frac{1}{\sqrt{T}}(x_T - \overline{x}_T) \\
&= -\frac{1}{\sqrt{T}}(x_T - \overline{x}_T),
\end{aligned}$$

which is $o_p(1)$:

$$\begin{aligned}
\Pr\left(\left|\frac{1}{\sqrt{T}}x_T\right| > a\right) &\le \frac{E|x_T|}{\sqrt{T}a} \to 0, \text{ for } a > 0, \\
\frac{1}{\sqrt{T}}\overline{x}_T &= \frac{1}{\sqrt{T}}\mu_x + \frac{1}{\sqrt{T}}(\overline{x}_T - \mu_x) = o_p(1), \text{ since } \overline{x}_T \xrightarrow{p} \mu_x \text{ (by LLN1).}
\end{aligned}$$

Therefore we have for equation (6) that

$$\sqrt{T}(\widehat{a}_1 - a_1) = \frac{\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t(x_{t-1} - \overline{x}_T)}{\frac{1}{T}\sum_{t=2}^{T}(x_{t-1} - \overline{x}_T)^2} + o_p(1).$$

Now we should focus on $\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t(x_{t-1} - \overline{x}_T)$. The summand $y_t = \varepsilon_t(x_{t-1} - \overline{x}_T)$ is not a linear process hence CLT2 can't be used here. y_t is also not an mds since the expectation of y_t conditional on its own past is not zero. Therefore, we can not use CLT1 directly here. Here you could ask yourself "what is the reason that the conditional expectation of y_t given its own past is not zero?". Indeed, y_t is a function of $\{x_1, x_2, ..., x_T\}$ for any value of t since it is a function of $\overline{x}_T$, implying that y_t must be serially correlated (hence not an mds). The problem then seems to lie at dependence on $\overline{x}_T$. So we may try getting rid of it by writing

$$\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t(x_{t-1} - \overline{x}_T) = \frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t(x_{t-1} - \mu_x + \mu_x - \overline{x}_T)$$

$$= \frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t\left(x_{t-1}-\mu_x\right)+\left(\mu_x-\overline{x}_T\right)\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t.$$

Now as $\mu_x - \overline{x}_T = o_p(1)$ and by a classical CLT $\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t = O_p(1)$ we have

$$\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t\left(x_{t-1}-\overline{x}_T\right)=\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t\left(x_{t-1}-\mu_x\right)+o_p(1),$$

i.e.

$$\sqrt{T}\left(\widehat{a}_1-a_1\right)=\frac{\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t\left(x_{t-1}-\mu_x\right)}{\frac{1}{T}\sum_{t=2}^{T}\left(x_{t-1}-\overline{x}_T\right)^2}+o_p(1).\ \text{[compare with eq. (6)]}$$

Now redefine $y_t = \varepsilon_t\left(x_{t-1}-\mu_x\right)$ and check that y_t is an mds [if so we may hope to be able to use CLT1]. Let $I_t = \sigma\left(y_s; s \le t\right)$.

$$\begin{aligned}
E\left|y_t\right| &= E\left|\varepsilon_t\right|E\left|x_{t-1}-\mu_x\right|<\infty\\
E\left(y_t|I_{t-1}\right) &= E\left(\varepsilon_t\left(x_{t-1}-\mu_x\right)|I_{t-1}\right)\\
&= \left(x_{t-1}-\mu_x\right)E\left(\varepsilon_t|I_{t-1}\right)\\
&= 0,
\end{aligned}$$

i.e. y_t is mds. We then check the other four conditions of CLT1:

(2) $Ey_t^4 = E\varepsilon_t^4 E\left(x_{t-1}-\mu_x\right)^4<\infty$

(3) $Ey_t^2 = E\varepsilon_t^2 E\left(x_{t-1}-\mu_x\right)^2=\sigma^2\frac{\sigma^2}{1-a_1^2}>0$

(4) $\frac{1}{T}\sum_{t=1}^{T}Ey_t^2=\sigma^4\left(1-a_1^2\right)^{-1}$

(5) $\frac{1}{T}\sum_{t=1}^{T}y_t^2\xrightarrow{p}\sigma^4\left(1-a_1^2\right)^{-1}$, since

$$\frac{1}{T}\sum_{t=1}^{T}y_t^2=\frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_t^2-\sigma^2\right)\left(x_{t-1}-\mu_x\right)^2+\frac{\sigma^2}{T}\sum_{t=1}^{T}\left(x_{t-1}-\mu_x\right)^2$$

with the conditions of LLN2 being satisfied for the first summand such that

$$\frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_t^2-\sigma^2\right)\left(x_{t-1}-\mu_x\right)^2\xrightarrow{p}0,$$

and by LLN1 we have $\frac{1}{T}\sum_{t=1}^{T}\left(x_{t-1}-\mu_x\right)^2\xrightarrow{p}\sigma^2\left(1-a_1^2\right)^{-1}$.

Therefore all the conditions of CLT1 are satisfied and we have

$$\begin{aligned}
\sqrt{T}\left(\widehat{a}_1-a_1\right) &= \frac{\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t\left(x_{t-1}-\mu_x\right)}{\frac{1}{T}\sum_{t=2}^{T}\left(x_{t-1}-\overline{x}_T\right)^2}+o_p(1)\\
&\xrightarrow{d}\sigma^{-2}\left(1-a_1^2\right)\mathcal{N}\left(0,\sigma^4\left(1-a_1^2\right)^{-1}\right)\\
&\sim \mathcal{N}\left(0,1-a_1^2\right).
\end{aligned}$$

Solution to Question 8

This question highlights a principle that generally pops up when looking at "a" central limit theorem for a summation. Indeed a CLT requires an appropriate normalization factor. This factor is proportional - for most cases - to the square root of the variance of the summation. Recall here that asymptotic analysis of OLS estimators from your previous regression analysis courses all focus on using a normalization factor proportional to $\sqrt{\#Obs.}$.

In this question, the normalization factor is parameterized as $T^{-\alpha}$ and the summation is $\sum_{t=1}^{T} t\varepsilon_t$. Clearly the variance of $\sum_{t=1}^{T} t\varepsilon_t$ is $\sigma^2 \sum_{t=1}^{T} t^2 = \sigma^2 \frac{2T^3+3T^2+T}{6}$. Indeed, to set the value of α, one may rely on intuition: if the limit of S_T is to be a [non-degenerate] random variable, then its asymptotic variance shall be bounded and positive. The variance of S_T can be calculated as

$$ES_T^2 = \frac{\sigma^2}{T^{2\alpha}} \frac{2T^3 + 3T^2 + T}{6}.$$

Therefore if $\alpha = \frac{3}{2}$ then we have $ES_T^2 \to \frac{\sigma^2}{3}$. This choice for α is equivalent to choosing a normalization factor proportional to $\sqrt{T^3}$. This is just one simple intuition.

Let's then proceed with analyzing $\frac{1}{T^{3/2}} \sum_{t=1}^{T} t\varepsilon_t$ which we may write as

$$\frac{1}{T^{3/2}} \sum_{t=1}^{T} t\varepsilon_t = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_t,$$

with $x_t = \frac{t}{T}\varepsilon_t$. Now note that

(1) x_t is an mds since $E|x_t| \leq E|\varepsilon_t| < \infty$ and $E(x_t|\sigma(\varepsilon_{t-1}, \varepsilon_{t-2}, ...)) = 0$,

(2) $Ex_t^4 \leq E\varepsilon_t^4 < \infty$,

(3) $Ex_t^2 = \frac{t^2}{T^2}\sigma^2 > 0$,

(4) $\frac{1}{T}\sum_{t=1}^{T} Ex_t^2 = \frac{1}{T}\sum_{t=1}^{T} \frac{t^2}{T^2}\sigma^2 \to \sigma^2 \int_0^1 s^2 ds = \frac{\sigma^2}{3}$,

(5) $\frac{1}{T}\sum_{t=1}^{T} x_t^2 \xrightarrow{p} \frac{\sigma^2}{3}$: first we have

$$\frac{1}{T}\sum_{t=1}^{T} x_t^2 = \frac{1}{T}\sum_{t=1}^{T} \frac{t^2}{T^2}\left(\varepsilon_t^2 - \sigma^2\right) + \frac{\sigma^2}{T}\sum_{t=1}^{T} \frac{t^2}{T^2},$$

where $\frac{t^2}{T^2}\left(\varepsilon_t^2 - \sigma^2\right)$ is an mds with bounded second moment and hence is $o_p(1)$ by LLN2 implying that

$$\frac{1}{T}\sum_{t=1}^{T} x_t^2 = o_p(1) + \frac{\sigma^2}{T}\sum_{t=1}^{T} \frac{t^2}{T^2} \to \sigma^2 \int_0^1 s^2 ds = \frac{\sigma^2}{3}.$$

Therefore the conditions of CLT1 are satisfied and we have

$$\frac{1}{T^{3/2}}\sum_{t=1}^{T} t\varepsilon_t \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{3}\right).$$

Solution to Question 12

Note that $\sum_{j=0}^{q} b_j e^{ij\lambda} \sum_{k=0}^{q} b_k e^{-ik\lambda} = \sum_{j=0}^{q}\sum_{k=0}^{q} b_j b_k e^{ij\lambda - ik\lambda}$ has q^2 terms which are

b_0^2	$b_0b_1e^{i\lambda}$	$b_0b_2e^{2i\lambda}$	$\cdots$	$b_0b_{q-1}e^{i(q-1)\lambda}$	$b_0b_qe^{iq\lambda}$
$b_1b_0e^{-i\lambda}$	b_1^2	$b_1b_2e^{i\lambda}$	$\cdots$	$b_1b_{q-1}e^{i(q-2)\lambda}$	$b_1b_qe^{i(q-1)\lambda}$
$b_2b_0e^{-2i\lambda}$	$b_2b_1e^{-i\lambda}$	b_2^2	$\cdots$	$b_2b_{q-1}e^{i(q-3)\lambda}$	$b_2b_qe^{i(q-2)\lambda}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$b_{q-1}b_0e^{-(q-1)i\lambda}$	$b_{q-1}b_1e^{-(q-2)i\lambda}$	$b_{q-1}b_2e^{-(q-3)i\lambda}$	$\cdots$	b_{q-1}^2	$b_{q-1}b_qe^{i\lambda}$
$b_qb_0e^{-qi\lambda}$	$b_qb_1e^{-(q-1)i\lambda}$	$b_qb_2e^{-(q-2)i\lambda}$	$\cdots$	$b_qb_{q-1}e^{-i\lambda}$	b_q^2

Now let's sum across the diagonals:

$$\begin{aligned}
\sum_{j=0}^{q} b_j e^{ij\lambda} \sum_{k=0}^{q} b_k e^{-ik\lambda} &= \sum_{j=0}^{q} b_j^2 \\
&\quad + e^{i(1)\lambda}\sum_{j=0}^{q-(1)} b_j b_{j+(1)} + \ldots e^{i(q-1)\lambda}\sum_{j=0}^{q-(q-1)} b_j b_{j+(q-1)} + e^{i(q)\lambda}\sum_{j=0}^{q-(q)} b_j b_{j+(q)} \\
&\quad + e^{-i(1)\lambda}\sum_{j=0}^{q-(1)} b_{j+(1)} b_j + \ldots e^{-i(q-1)\lambda}\sum_{j=0}^{q-(q-1)} b_{j+(q-1)} b_j + e^{-i(q)\lambda}\sum_{j=0}^{q-(q)} b_{j+(q)} b_j \\
&= \sum_{j=0}^{q} b_j^2 + \left(e^{i(1)\lambda} + e^{-i(1)\lambda}\right)\sum_{j=0}^{q-(1)} b_j b_{j+(1)} + \ldots + \left(e^{i(q)\lambda} + e^{-i(q)\lambda}\right)\sum_{j=0}^{q-(q)} b_j b_{j+(q)} \\
&= \frac{\gamma(0)}{\sigma^2} + 2\cos(\lambda) \times \frac{\gamma(1)}{\sigma^2} + \ldots + 2\cos(\lambda q) \times \frac{\gamma(q)}{\sigma^2}.
\end{aligned}$$

Therefore

$$f(\lambda) = \frac{\sigma^2}{2\pi}\sum_{j=0}^{q} b_j e^{ij\lambda} \sum_{k=0}^{q} b_k e^{-ik\lambda}$$

$$= \frac{1}{2\pi}\left(\gamma(0) + 2\sum_{h=1}^{q}\gamma(h)\cos(\lambda h)\right).$$

Alternatively, define

$$\widetilde{b}_j = \begin{cases} b_j, \text{ if } 0 \le j \le q \\ \\ 0, \text{ otherwise.} \end{cases}$$

With $\widetilde{b}_j$ we need not worry about the indices becoming negative or bigger than q.

The transfer function reads as

$$\begin{aligned} T(\lambda) &= \sum_{j=0}^{q} b_j e^{-ij\lambda} \times \sum_{k=0}^{q} b_k e^{ik\lambda} \\ &= \sum_{j=0}^{q}\sum_{k=0}^{q} b_j b_k e^{-i\lambda(j-k)} \\ &= \sum_{j=0}^{q}\sum_{k=0}^{q} \widetilde{b}_j \widetilde{b}_k e^{-i\lambda(j-k)}. \end{aligned}$$

Now let $k = j + h$ and rewrite

$$\begin{aligned} T(\lambda) &= \sum_{j=0}^{q}\sum_{h=-q}^{q} \widetilde{b}_j \widetilde{b}_{j+h} e^{i\lambda h} \\ &= \sum_{h=-q}^{q}\sum_{j=0}^{q} \widetilde{b}_j \widetilde{b}_{j+h} e^{i\lambda h} \\ &= \sum_{h=-q}^{q} \frac{\gamma(h)}{\sigma^2} e^{i\lambda h} \\ &= \frac{1}{\sigma^2}\left(\gamma(0) + \sum_{h=1}^{q}\gamma(h)\left(e^{i\lambda h} + e^{-i\lambda h}\right)\right) \\ &= \frac{1}{\sigma^2}\left(\gamma(0) + 2\sum_{h=1}^{q}\gamma(h)\cos(\lambda h)\right). \end{aligned}$$

Therefore

$$\begin{aligned} f(\lambda) &= \frac{\sigma^2}{2\pi} T(\lambda) \\ &= \frac{1}{2\pi}\left(\gamma(0) + 2\sum_{h=1}^{q}\gamma(h)\cos(\lambda h)\right). \end{aligned}$$

Solution to Question 13

With

$$\Psi(\lambda) = \begin{cases} 1, \text{ if } \lambda_l \leq |\lambda| \leq \lambda_h \ (\lambda_l \text{ and } \lambda_h \text{ are in } [0,\pi]), \\ \\ 0, \text{ otherwise.} \end{cases}$$

for $j > 0$ we have

$$\begin{aligned}
c_j &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ij\lambda}\Psi(\lambda)\,d\lambda \\
&= \frac{1}{2\pi}\int_{-\lambda_h}^{-\lambda_l} e^{ij\lambda}d\lambda + \frac{1}{2\pi}\int_{\lambda_l}^{\lambda_h} e^{ij\lambda}d\lambda \\
&= \frac{1}{2\pi ij}\left(e^{-ij\lambda_l} - e^{-ij\lambda_h}\right) + \frac{1}{2\pi ij}\left(e^{ij\lambda_h} - e^{ij\lambda_l}\right) \\
&= \frac{1}{2\pi ij}\left(e^{-ij\lambda_l} - e^{ij\lambda_l}\right) + \frac{1}{2\pi ij}\left(e^{ij\lambda_h} - e^{-ij\lambda_h}\right) \\
&= -\frac{1}{2\pi ij}\left(2i\sin(j\lambda_l)\right) + \frac{1}{2\pi ij}\left(2i\sin(j\lambda_h)\right) \\
&= \frac{\sin(j\lambda_h) - \sin(j\lambda_l)}{\pi j}
\end{aligned}$$

and for $j = 0$ we have

$$\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi(\lambda)\,d\lambda \\
&= \frac{1}{2\pi}\int_{-\lambda_h}^{-\lambda_l} d\lambda + \frac{1}{2\pi}\int_{\lambda_l}^{\lambda_h} d\lambda \\
&= \frac{1}{2\pi}(-\lambda_l + \lambda_h) + \frac{1}{2\pi}(\lambda_h - \lambda_l) \\
&= \frac{\lambda_h - \lambda_l}{\pi}.
\end{aligned}$$

Solution to Question 14

We have

$$\begin{aligned}
f_x(\lambda) &= \frac{\sigma^2}{2(2q+1)^2\pi}\left(\sum_{j=-q}^{q} e^{-ij\lambda}\right)\left(\sum_{j=-q}^{q} e^{ij\lambda}\right) \\
&= \frac{\sigma^2}{2(2q+1)^2\pi}\sum_{j=-q}^{q}\sum_{k=-q}^{q} e^{ik\lambda}e^{-ij\lambda} \\
&= \frac{\sigma^2}{2(2q+1)^2\pi}\left(2q+1+\sum_{h=1}^{2q}(2q+1-h)\left(e^{hi\lambda}+e^{-hi\lambda}\right)\right)
\end{aligned}$$

$$= \frac{\sigma^2}{2\left(2q+1\right)^2\pi}\left(2q+1+2\sum_{h=1}^{2q}\left(2q+1-h\right)\cos\left(\lambda h\right)\right) \tag{7}$$

Hence we have

$$\begin{aligned}
s\left(\lambda_0, q\right) &= 2\int_0^{\lambda_0} f\left(\lambda\right)d\lambda \\
&= \frac{\sigma^2}{\left(2q+1\right)^2\pi}\left(\left(2q+1\right)\lambda_0+2\sum_{h=1}^{2q}\left(2q+1-h\right)\int_0^{\lambda_0}\cos\left(\lambda h\right)d\lambda\right) \\
&= \frac{\sigma^2}{\left(2q+1\right)^2\pi}\left(\left(2q+1\right)\lambda_0+2\sum_{h=1}^{2q}\left(2q+1-h\right)\frac{\sin\left(\lambda_0 h\right)}{h}d\lambda\right).
\end{aligned}$$

ALTERNATIVELY and as discussed in the class we have

$$\begin{aligned}
\Psi\left(\lambda\right) &= \frac{1}{2q+1}\sum_{j=-q}^{q}\left(e^{-i\lambda}\right)^j \\
&= \frac{1}{2q+1}\sum_{j=0}^{2q}\left(e^{-i\lambda}\right)^{j-q} \\
&= \frac{e^{i\lambda q}}{2q+1}\sum_{j=0}^{2q}\left(e^{-i\lambda}\right)^j \\
&= \frac{e^{i\lambda q}}{2q+1}\frac{1-\left(e^{-i\lambda}\right)^{2q+1}}{1-e^{-i\lambda}} \\
&= \frac{1}{2q+1}\frac{e^{i\lambda q}-e^{-i\lambda(q+1)}}{1-e^{-i\lambda}}\frac{e^{i\lambda/2}}{e^{i\lambda/2}} \\
&= \frac{1}{2q+1}\frac{e^{i\lambda(q+1/2)}-e^{-i\lambda(q+1/2)}}{e^{i\lambda/2}-e^{-i\lambda/2}} \\
&= \frac{1}{2q+1}\frac{\sin\left(\lambda\left(q+1/2\right)\right)}{\sin\left(\lambda/2\right)}
\end{aligned}$$

which implies

$$f_x\left(\lambda; q\right) = \frac{\sigma^2}{2\pi}\left(\frac{1}{2q+1}\frac{\sin\left(\lambda\left(q+1/2\right)\right)}{\sin\left(\lambda/2\right)}\right)^2.$$

The latter expression is not so handy to "integrate" compared to the expression given under equation (7). For a numerical analysis both expressions are of course fine [as they are equivalent].

Question A1 - 27.05.

Let the returns, $\{r_t\}$, for a financial index be generated by $r_t = ah_t + x_t$ with

$$x_t = \sqrt{h_t}\,\varepsilon_t,$$

$$h_t = \alpha_0 + \alpha_1 x_{t-1}^2 + \beta_1 h_{t-1},$$

where $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$, $a \neq 0$ is a constant, x_t is stationary and $\varepsilon_t \sim \text{iid}\mathcal{N}(0,1)$.

- Is r_t a martingale difference series with respect to $\mathcal{I}_{t-1} = \sigma(x_{t-1}, x_{t-2}, ...)$? Justify your answer.

Solution We have

$$\begin{aligned}
\mathrm{E}(r_t|\mathcal{I}_{t-1}) &= \mathrm{E}\left(ah_t + \sqrt{h_t}\,\varepsilon_t|\mathcal{I}_{t-1}\right) \\
&= \mathrm{E}(ah_t\,|\mathcal{I}_{t-1}) + \sqrt{h_t}\mathrm{E}(\varepsilon_t|\mathcal{I}_{t-1}) \\
&= \mathrm{E}(ah_t\,|\mathcal{I}_{t-1}) + \sqrt{h_t}\mathrm{E}(\varepsilon_t) \\
&= ah_t + 0 \neq 0 \text{ hence } r_t \text{ is not mds.}
\end{aligned}$$

Question A2 - 27.05.

Find the kurtosis, κ, of a stationary ARCH(1) process $x_t = \sqrt{h_t}\varepsilon_t$ where $\varepsilon_t \overset{iid}{\sim} \mathcal{N}(0,1)$ and $h_t = \alpha_0 + \alpha_1 x_{t-1}^2$. Discuss wether α_1 must be restricted such that the kurtosis could be defined. [Hint $E\varepsilon_t^4 = 3$ and $\kappa = \frac{Ex_t^4}{E^2 x_t^2}$].

Solution

$$\begin{aligned}
Ex_t^4 &= Eh_t^2\varepsilon_t^4 \\
&= 3Eh_t^2 \\
&= 3E\left(\alpha_0^2 + \alpha_1^2 x_{t-1}^4 + 2\alpha_0\alpha_1 x_{t-1}^2\right)
\end{aligned}$$

hence by stationarity we have $(1 - 3\alpha_1^2)\,Ex_t^4 = 3\alpha_0^2 + 6\alpha_0\alpha_1 Ex_t^2$ [note here that we need $1 - 3\alpha_1^2 > 0$ such that the fourth moment could be defined, i.e. $0 \leq \alpha_1 < 1/\sqrt{3}$]. Further we have

$$\begin{aligned}
Ex_t^2 &= Eh_t \\
&= \alpha_0 + \alpha_1 Ex_{t-1}^2
\end{aligned}$$

and again by stationarity we have $Ex_t^2 = \alpha_0/(1 - \alpha_1)$.

Therefore we have

$$\begin{aligned}
\kappa &= \frac{Ex_t^4}{E^2 x_t^2} \\
&= \frac{3\alpha_0^2 + 6\alpha_0\alpha_1(\alpha_0/(1-\alpha_1))}{(1 - 3\alpha_1^2)}\frac{(1-\alpha_1)^2}{\alpha_0^2} \\
&= 3\frac{1 - \alpha_1^2}{1 - 3\alpha_1^2}.
\end{aligned}$$

Question B - 27.05.

Let x_t be generated by the following model

$$x_t = \beta\mu_t + \varepsilon_t + \frac{1}{2}\varepsilon_{t-1}.$$

where $\mu_t = 0$ for t being even and $\mu_t = 1$ for t being odd. Further $\varepsilon_t \overset{iid}{\sim} (0, \sigma^2)$ and has finite fourth moment. Let $V_T = \sum_{t=2}^T x_t\mu_t / \sum_{t=2}^T \mu_t^2$.

- Show that $V_T \xrightarrow{p} \beta$.
- Find the asymptotic distribution of $\sqrt{T}\,(V_T - \beta)$.

Solution

$$\begin{aligned} \frac{\sum_{t=2}^T x_t\mu_t}{\sum_{t=2}^T \mu_t^2} &= \frac{\sum_{t=2}^T \left(\beta\mu_t + \varepsilon_t + \frac{1}{2}\varepsilon_{t-1}\right)\mu_t}{\sum_{t=2}^T \mu_t^2} \\ &= \beta + \frac{\frac{1}{T}\sum_{t \text{ is odd}} \varepsilon_t + \frac{1}{2}\frac{1}{T}\sum_{t \text{ is odd}} \varepsilon_{t-1}}{\frac{1}{T}\sum_{t \text{ is odd}} 1} \\ &= \beta + \frac{\frac{1}{T}\sum_{t \text{ is odd}} \varepsilon_t + \frac{1}{2}\frac{1}{T}\sum_{t \text{ is even}} \varepsilon_t}{\frac{1}{T}\sum_{t \text{ is odd}} 1} \end{aligned}$$

Hence we have

$$\begin{aligned} \sqrt{T}\,(V_T - \beta) &= \frac{\frac{1}{\sqrt{T}}\sum_{t \text{ is odd}} \varepsilon_t + \frac{1}{2}\frac{1}{\sqrt{T}}\sum_{t \text{ is even}} \varepsilon_t}{\frac{1}{T}\sum_{t \text{ is odd}} 1} \\ &= \frac{\frac{1}{\sqrt{T}}\sum \left(\frac{1}{2}\right)^{1-\mu_t} \varepsilon_t}{\frac{1}{T}\sum_{t \text{ is odd}} 1} \end{aligned}$$

For the denominator we have $\frac{1}{T}\sum_{t \text{ is odd}} 1 \to \frac{1}{2}$. For the numerator we may use CLT1. Define $x_t = \left(\frac{1}{2}\right)^{1-\mu_t} \varepsilon_t$.

a) x_t is mds since $E\left(x_t | \sigma\left(x_{t-1}, x_{t-2}, ...\right)\right) = E\left(x_t\right) = 0$, and $E\left|x_t\right| = \left(\frac{1}{2}\right)^{1-\mu_t} E\left|\varepsilon_t\right| < \infty$

b) $E\left(x_t^4\right) = E\left(\left(\frac{1}{16}\right)^{1-\mu_t} \varepsilon_t^4\right) = \left(\frac{1}{16}\right)^{1-\mu_t} E\left(\varepsilon_t^4\right) < \infty$

c) $E\left(x_t^2\right) = E\left(\left(\frac{1}{4}\right)^{1-\mu_t} \varepsilon_t^2\right) = \left(\frac{1}{4}\right)^{1-\mu_t} \sigma^2 > 0.$

d) $\frac{1}{T}\sum E\left(x_t^2\right) = \frac{1}{T}\sum \left(\frac{1}{4}\right)^{1-\mu_t} \sigma^2 \to \frac{1}{2}\frac{\sigma^2}{4} + \frac{1}{2} \times \sigma^2 = \frac{5}{8}\sigma^2.$

e)

$$\frac{1}{T}\sum x_t^2 = \frac{1}{T}\sum \left(\frac{1}{4}\right)^{1-\mu_t} \varepsilon_t^2$$

$$\begin{aligned}
&= \frac{1}{T}\sum_{t\text{ is odd}}\left(\frac{1}{4}\right)^{1-\mu_t}\varepsilon_t^2+\frac{1}{T}\sum_{t\text{ is even}}\left(\frac{1}{4}\right)^{1-\mu_t}\varepsilon_t^2\\
&= \frac{1}{T}\sum_{t\text{ is odd}}\varepsilon_t^2+\left(\frac{1}{4}\right)\frac{1}{T}\sum_{t\text{ is even}}\varepsilon_t^2\\
&= \frac{1}{2}\frac{1}{T/2}\sum_{t\text{ is odd}}\varepsilon_t^2+\frac{1}{8}\frac{1}{T/2}\sum_{t\text{ is even}}\varepsilon_t^2\xrightarrow{p}\frac{5}{8}\sigma^2
\end{aligned}$$

Hence

$$\begin{aligned}
\sqrt{T}\left(V_T-\beta\right) &= \frac{\frac{1}{\sqrt{T}}\sum_{t\text{ is odd}}\varepsilon_t+\frac{1}{2}\frac{1}{\sqrt{T}}\sum_{t\text{ is even}}\varepsilon_t}{\frac{1}{T}\sum_{t\text{ is odd}}1}\\
&\xrightarrow{d}\left(\frac{1}{2}\right)^{-1}N\left(0,\frac{5}{8}\sigma^2\right)\\
&\sim N\left(0,\frac{5}{2}\sigma^2\right).
\end{aligned}$$