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Optimal placement in a limit order book: an analytical approach

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Abstract This paper proposes and studies an optimal placement problem in a limit order book. Under a correlated random walk model with mean-reversion for the best ask/bid price, optimal placement strategies for both static and dynamic cases are derived. In the static case, the optimal strategy involves only the market order, the best bid, and the second best bid; the optimal strategy for the dynamic case is shown to be of a threshold type depending on the remaining trading time, the market momentum, and the price mean-reversion factor. Critical to the analysis is a generalized reflection principle for correlated random walks, which enables a significant dimension reduction.

Keywords Market making · Optimal placement · Correlated random walk · Markov decision problem · Reflection principle

JEL Classification C020 · C61 · C65

1 Introduction

Automatic and electronic order-driven trading platforms have largely replaced the traditional floor-based trading for virtually all financial markets. In an electronic order-driven market, orders arrive at the exchange and wait in the *Limit Order Book* to be executed. In most

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exchanges, order flow is heavy with thousands of orders in seconds and tens of thousands of price changes in a day for a liquid stock. Meanwhile, the time for the execution of a market order has dropped below one millisecond. This new era of trading is commonly referred to as *high-frequency trading* or *algorithmic trading*. In US, high-frequency trading firms represent 2 % of the approximately 20,000 firms operating today, but account for 73 % of all equity orders volume.

1.1 Limit order book (LOB)

In an order-driven market, there are two types of buy/sell orders for market participants to post: market orders and limit orders. A *limit order* is an order to trade a certain amount of security (stocks, futures, etc.) at a specified price. The lowest price for which there is an outstanding limit sell order is called the *best ask price* and the highest limit buy price is called the *best bid price*. Limit orders are collected and posted in the LOB, which contains the quantities and the prices at different levels for all limit buy and sell orders. A *market order* is an order to buy/sell a certain amount of the equity at the best available price in the LOB. It is then matched with the best available price and a trade occurs immediately and the LOB is updated accordingly. A limit order stays in the LOB until it is executed against a market order or until it is canceled; cancellation is allowed at any time before getting executed. In essence, the closer a limit order is to the best bid/ask, the faster it may be executed. Most exchanges are based on the first-in-first-out (FIFO) policy for orders on the same price level, although some derivatives on some exchanges have the pro-rate microstructure, i.e., an incoming market order is dispatched on all active limit orders at the best price, with each limit order contributing to execution in proportion to its volume. In this paper, unless otherwise specified, the focus is on the FIFO market, with extensions to the pro-rate microstructure whenever appropriate (Table 1).

1.2 The optimal placement problem

Optimal placement studies how to place an order in an LOB for optimizing an objective such as minimizing the cost. Given a number of shares to buy or sell, traders must decide between using market orders, limit orders, or both, decide on the number of orders to place at different price levels, and decide on the optimal sequence of order placement in a give time frame

Table 1 A market sell order with size of 1200, a limit ask order with size of 400 at 9.08, and a cancellation of 23 shares of limit ask order at 9.10, in sequence

	Price	Size
Ask	9.12	1,525
	9.11	3,624
	9.10	4,123
	9.09	1,235
	9.08	3,287
Bid	9.07	4,895
	9.06	3,645
	9.05	2,004
	9.04	3,230
	9.03	7,246

⇒

	Price	Size
Ask	9.12	1,525
	9.11	3,624
	9.10	4,123
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Bid	9.07	3,695
	9.06	3,645
	9.05	2,004
	9.04	3,230
	9.03	7,246

⇒

	Price	Size
Ask	9.12	1,525
	9.11	3,624
	9.10	4,123
	9.09	1,235
	9.08	3,687
Bid	9.07	3,695
	9.06	3,645
	9.05	2,004
	9.04	3,230
	9.03	7,246

⇒

	Price	Size
Ask	9.12	1,525
	9.11	3,624
	9.10	4,100
	9.09	1,235
	9.08	3,687
Bid	9.07	3,695
	9.06	3,645
	9.05	2,004
	9.04	3,230
	9.03	7,246

with multi-trades. Specifically, when using limit orders, traders do not need to pay the spread and most of the time even get a *rebate*.[1] This rebate, however, comes with an execution risk as there is no guarantee of execution for limit orders. On the other hand, when using market orders, one has to pay both the spread between the limit and the market orders and the fee in exchange for a guaranteed immediate execution. Essentially, traders have to balance between paying the spread and fees when placing market orders versus execution/inventory risks when placing limit orders.

Technically, the optimal placement problem can be stated as follows. Consider a setting where N shares are to be bought by time $T > 0$ (T $\approx 1/5$ min). One may split the N shares into $(N_{0,t}, N_{1,t}, \ldots)$, where $N_{0,t} \geq 0$ is the number of shares placed as market order at time $t = 0, 1, \ldots, T$, $N_{1,t}$ is the number of shares placed at the best bid at time t, $N_{2,t}$ is the number of shares placed at the second best bid at time t, and so on. If the limit orders are not executed by time T, then one has to buy the non-executed orders at the market price at time T. When one share of limit order is executed, the market gives a rebate $r > 0$ and when a share of market order is submitted, a fee $f > 0$ is incurred. Although no intermediate selling is allowed at any time, one nevertheless can cancel any non-executed order and replace it with a new order at a later time. Now, given N and T, the goal is to find the optimal strategy $(N_{0,t}, N_{1,t}, \ldots, N_{k,t})_{t=0,1,\ldots,T}$ to minimize the total expected cost.

1.3 Relation to the optimal execution problem

Optimal placement problem is closely related to the optimal execution problem that have been well studied in the mathematical finance literature. In some sense, the two problems correspond to two phases of algorithmic tradings. The latter studies how to slice big orders into smaller ones on a daily/weekly basis in order to minimize the price impact or to maximize some expected utility function. The former, on the other hand, deals with the smaller orders on a smaller (10–100 s) time scale and mostly for different type of (i.e., HFT) traders. (See Kirilenko et al. [37] for further discussions on distinction between these two problems.)[2] While it is it is important to formulate and analyze the combined problems by focusing on a few key features, see for instance Guéant et al. [27] and Bayraktar and Ludkovski [10], it is also necessary to explore these two problems separately.

1.4 Relationship to the market-making problem

Optimal placement problem is also closely related to the well-known market-making problem, one of the central problems in algorithmic trading that studie how to simultaneously placing limit and market orders to buy and sell. The goal of the market maker is to maximize the profit by playing with the spread between the bid and ask prices, while controlling the inventory risk and the execution risk. Essentially, the market-making problem is the optimal placement problem with the added possibility of intermediate selling, hence more difficult to analyze with explicit optimal strategies. (See, for example, Ho and Stoll [32], Avellaneda and Stoikov [8], Bayraktar and Ludkovski [10], Cartea and Jaimungal [15], Cartea et al. [16],

[1] This rebate structure varies from exchange to exchange and leads to different optimization problems. In some exchanges successful executions of limit orders get a discount (i.e., a fixed percentage of the execution price) whereas in other places the discount may be a fixed amount.

[2] Literature on the optimal execution problem is big and growing rapidly fast, see for instance, Bertsimas and Lo [12], Almgren and Chriss [5,6], Almgren [4], Almgren and Lorenz [7], Schied and Schöneborn [48], Weiss [51], Alfonsi et al. [2,3], Predoiu et al. [44], Schied et al. [49], Gatheral and Schied [24], Forsyth et al. [23], Bouchard et al. [13], Obizhaeva and Wang [43], and most recently Becherer et al. [11], Horst et al. [40], Cheridito and Sepin [19], Guo and Zervos [29], Huitema [35], Kratz [38], and Moallemi and Yuan [41].

Veraarta [50], Guilbaud and Pham [28], Guéant et al. [27], and Horst et al. [40]). Given the complexity of the market-making problem, one natural question is: can we derive more structural results on the simpler optimal placement problem? If so, can we obtain any useful insights from the results? This is the motivation for our work.

1.5 Our contributions

Most literature on market making problems start with a continuous time Brownian-motion-based model. Recently, Abergel and Jedidi [1] showed that when the volume of the order book is modeled by a continuous Markov-chain with independent order flow process, the mid-price has a diffusion limit. Moreover, Horst and Paulsen [34] and Horst and Kreher [33] derived diffusion and fluid limit in a very general mathematical setting for the whole limit order books including both volume and price. We start by proposing a discrete time model for the bid/ask price: a correlated random walk model. This model is closely connected to the Brownian motion, yet with an explicit feature of mean-reversion. (See Remark 2 for technical details). We also assume that the execution probability for a limit order within any trading period between t and $t+1$ is a constant q. This differs also from works in the market making problem where such a probability is usually assumed to be dependent on the distance from the bid price to the best ask price. Nevertheless, in addition to its apparent simplicity and intuitive appeal, our proposed model is strong enough to capture several key LOB characteristics such as the mean-reversion nature of algorithmic trading, the depth of the LOB, and the LOB imbalance. (See Remarks 1 and 3).

Under this model, the optimal strategy for the static case is proved to involve placing orders only at three levels: the second best bid, the best bid, and the market order, by Theorem 6. This result significantly reduces the complexity and dimensionality of the optimal placement problem. The optimal placement strategy for the dynamic multi-step case is shown to be a threshold type, with two thresholds explicitly given. Moreover, as time goes by, the optimal strategy shifts from the more aggressive types to the more conservative ones. (See Theorem 11). Mathematically, our results are not completely surprising given the Markovian structure and the mean-reversion nature of the underlying price model. Nevertheless, it provides some useful and intuitive insight: (i) the optimal placement strategy is sensitive to the model parameters for the LOB. The strategy becomes more conservative as the remaining trading time decreases and when the price is more likely to go down; (ii) as the transaction cost or the monetary benefit of using the limit order decreases, the optimal trading strategy shifts towards the market order; and (iii) when the mean reversion is less likely, the optimal trading strategy becomes more pessimistic and involves only the market order and best bid order, by Proposition 7.

Technically speaking, the analysis with the correlated random walk model is surprisingly difficult. The static case is unexpectedly the hardest. The main difficulty is to establish the partial reflection principle for the correlated random walk and the monotonicity property for its running maximal process. This part of analysis is critical for the dimension reduction: it allows us to concentrate on the top three levels of limit order books to search for the optimal strategy instead of comparing the expected cost at all levels, which would be computationally and statistically infeasible. The threshold-type optimal trading strategy for the dynamic case is obtained by the Markov decision theory and by exploiting carefully the specific model structure with some detailed analysis.

The dimension reduction result may be useful beyond the optimal placement problem analyzed here with focus on one exchange with a particular fee structure. Indeed, Cont and Kukanov [20] considered an optimal splitting problem across multiple exchanges in a one-period model, where they assumed that one can only choose the market order and the best limit order at each trading venue. Our results provide direct analytical support for such an ad-hoc assumption, and are also consistent with the empirical work by de Larrard [22] which showed that most of the trading activities are concentrated at the top two levels of the LOB.

Mathematically, we hope the technique developed here in analyzing the correlated random walk may be of independent mathematical interest. In particular, the "mapping" technique exploited in establishing the partial reflection principle differs from the standard scaling technique for the reflection principle for random walk or Brownian motion. The reflection principle, one of the most well-known results in probability theory, is believed to be initially introduced by W. Feller as a combinatorial trick for counting and comparing sample paths; however, it relies heavily on the symmetry of sample paths, shown also in Bayraktar and Nadtochiy [9] for the Lévy process.

1.6 Outline of the paper

Section 2 starts with the model and presents some preliminary analysis. Section 3 provides the main results concerning the optimal placement strategy; the analysis consists of two parts, for both the static and the dynamic cases.

For ease of exposition, all major proofs are given in Appendix.

2 The model and the preliminary analysis

2.1 The model

2.1.1 The correlated random walk

To analyze the optimal placement problem, we will first propose a correlated random walk model for the bid/ask price dynamics. In this model, we will assume that

1. The spread between the best bid price and the best ask price is always 1 tick;
2. The best ask price increases or decreases 1 tick at each time step $t = 0, 1, \ldots, T$.

Moreover, let A_t be the best ask price at time t, expressed as the ticks. Then we will assume that

$$A_t = \sum_{i=1}^{t} X_i, \quad A_0 = 0, \tag{2.1}$$

where X_t $(1 \leq t \leq T)$ is a Markov chain on $\{\pm 1\}$ with $\mathbb{P}(X_1 = 1) = \bar{p} = 1 - \mathbb{P}(X_1 = -1)$ for some $\bar{p} \in [0, 1]$, and

$$\mathbb{P}(X_{i+1} = 1 \mid X_i = 1) = \mathbb{P}(X_{i+1} = -1 \mid X_i = -1) = p < \frac{1}{2}, \quad \text{for } i = 1, \ldots, T-1. \tag{2.2}$$

Such a model is also called a correlated random walk model.

Remark 1 One may view the initial probability $\bar{p}$ in the price dynamics as an indicator of the market momentum or the imbalance of the LOB. The particular choice of $p < \frac{1}{2}$ makes the price "mean revert", a phenomenon often observed in high-frequency trading. For instance, in Cont and de Larrard [21], it was shown that $p < \frac{1}{2}$ is equivalent to the negative autocorrelation of limit order book price at the first lag, and they confirmed this mean-reversion with most of US equity data; independently, Chen and Hall [18] did some statistical analysis for high frequency data with mean-reversion.

Remark 2 One can verify that this correlated random walk has a diffusion limit when appropriately rescaled. Indeed, define $S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} X_i$ for $t \geq 0$ and note that $\{X_i\}_{i \geq 1}$ is a strictly stationary sequence and ϕ-mixing, see e.g., [47] and [14]. Now by stationarity, one can compute via induction that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] = \sigma^2 := \mathbb{E}[X_1^2] + 2\sum_{i=1}^{\infty}\mathbb{E}[X_1 X_{i+1}] = \frac{p}{1-p}.$$

Therefore, the invariance principle holds, see e.g., [36]. That is, $S_n(\cdot)$ converges to $\sigma\mathbf{B}$ in distribution with the Skorokhod topology, where $\mathbf{B}$ denotes a standard Brownian motion and $\sigma^2 = \frac{p}{1-p}$ in this case.

2.1.2 The execution probability of limit orders

Besides the price dynamics, another key element in a LOB is the probability of a limit order being executed by time T. This probability will depend on the order position, the depth of the limit order book, the frequency of price changes, among others. Ideally, the complete characterization of such a probability would require modeling the entire limit order book dynamics in addition to the price dynamics, as shown in Horst and Paulsen [34]. See also Guo et al. [31]. In this paper, for analytical tractability, we assume that

- if $A_t \leq -k$ for some $t \leq T$, then a limit order at price $-k$ will be executed with probability 1;
- if $A_t > -k+1$ for all $t \leq T$, then a limit order at price $-k$ will be executed with probability 0;
- If $A_t = -k+1$ and $A_{t+1} = -k+2$, then a limit order at price $-k$ has a chance q to be executed between t and $t+1$.

From q, one can easily compute the probability that a best bid is executed within a certain time, say T. This probability is an important indicator for the depth of the LOB. (See also Remark 3).

We will see that the initial state of the market (i.e., X_t being -1 or 1 at time t), the mean-reversion property, and the execution probability q, play important roles in decisions of order placement.

2.2 Preliminary analysis

To analyze the optimal placement problem under this model, it is critical to characterize A_t and Y_t $(0 \leq t \leq T)$, where

$$Y_t = \min_{0\leq s\leq t} A_s. \tag{2.3}$$

In the subsequent analysis, when there is little risk of confusion, we call it ω instead of $\{X_t(\omega)\}_{1\leq t\leq T}$ for a particular sample path. Consistent with this convention, $A_t(\omega)$ is the (best) ask price on a sample path ω at time t, $X_t(\omega)$ is the price change at the tth step of a particular sample path ω, and $Y_T(\omega)$ is the lowest level that a particular sample path ω has ever hit by time T.

2.2.1 Probability distribution of A_t and Y_t

Clearly from Eq. (2.2), $\{(A_i, A_{i-1})\}_{i\geq 1}$ is a two-dimensional Markov chain. Compared to the simple random walk, the key to analyzing A_t is to differentiate the sequences with different number of direction changes when they have the same number of "upward" edges (i.e., with the same value for A_T). We call it a *direction change* when a sequence of 1s is followed by a -1 or when a sequence of -1s is followed by a 1. For instance, both $1, 1, 1, -1, -1, 1$ and $1, 1, -1, 1, -1, 1$ have four 1s (upward edges) and two -1s (downward edges), yet the former one has two direction changes while the latter has four direction changes.

It is easy to see that if the numbers of 1s and -1s and the *position* of the direction changes are given, then the sequence is determined as long as the first edge is also given. For instance, if $T = 5$, $A_T = 1$, $X_1 = 1$, and the number of direction changes is 2, then we need to know the positions of the direction changes in order to identify the sequence from the two possible choices: $1, 1, -1, -1, 1$ or $1, -1, -1, 1, 1$.

After some calculations, it is clear that

$$\mathbb{P}(A_T = k \text{ with } i \text{ direction changes}) = \begin{cases} p^{T-i-1}(1-p)^i L^i_{T,k}, & \frac{T+k}{2} \in \mathbb{N} \text{ and } |k| \leq T, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$L^i_{T,k} = (1-\bar{p})\binom{\frac{T+k}{2}-1}{\lfloor\frac{i+1}{2}\rfloor - 1}\binom{\frac{T-k}{2}-1}{\lfloor\frac{i+2}{2}\rfloor - 1} + \bar{p}\binom{\frac{T+k}{2}-1}{\lfloor\frac{i+2}{2}\rfloor - 1}\binom{\frac{T-k}{2}-1}{\lfloor\frac{i+1}{2}\rfloor - 1}. \tag{2.4}$$

From which, we have

Proposition 1

$$\mathbb{P}(A_T = k) = \begin{cases} \sum_{i=1}^{T-|k|} p^{T-i-1}(1-p)^i L^i_{T,k}, & \frac{T+k}{2} \in \mathbb{N} \textit{ and } |k| \leq T, \\ 0, & \textit{otherwise}. \end{cases}$$

Using this, we obtain the following.

Theorem 2 (Partial Reflection Principle) *If* $\frac{T-k}{2} \in \mathbb{N}$ *and* $k > 0$*, then* $\mathbb{P}(Y_T = -k) = \mathbb{P}(A_T = -k)$.

In the limiting case of $p = 1/2$ when a correlated random walk becomes a simple random walk, this theorem is consistent with the classical reflection principle (see, for instance, Redner and Sidney [45, pp. 97–100]). For the more general case, however, we are not aware of any prior results like ours despite rich literature on the correlated random walk. Unlike the simple random walk or related work for reflection principles, direct rescaling or reflection of sample path does not seem to work, as seen from the proof in the Appendix. In fact, the proof requires some careful design of mappings to enable efficient counting and comparison of different sample paths. (For earlier works on correlated random walk model, see Goldstein [26], Mohan [42], Gillis [25], and Renshaw and Henderson [46]).

Next, we show that the distribution of Y_t satisfies a monotone property, which seems intuitively clear although its proof is not that obvious. Note the "non-intuitive" part comes from $\mathbb{P}(Y_T = -k)$, which is different from $\mathbb{P}(Y_T \leq -k)$.

Proposition 3 $\mathbb{P}(Y_T = -k)$ *is a decreasing function of k for* $k = 1, \ldots, T$.

The idea for the proof of this proposition yields a critical lemma for the subsequent analysis of the optimal placement problem.

Lemma 4 *For* $1 \leq k \leq T-2$*,*

$$\begin{aligned}&\mathbb{P}(Y_T = -k)(k + \mathbb{E}[A_T | Y_T \\ &\quad = -k]) - \mathbb{P}(Y_T = -k-1)(k + 2 + \mathbb{E}[A_T | Y_T = -k-1]) \geq 0.\end{aligned} \tag{2.5}$$

3 Optimal strategy for the optimal placement problem

With the above model setup, we can proceed to solve the optimal placement problem. We will focus on the optimal placement problem *without price impact of a large trade*. With this constraint, it is without loss of generality to assume $N = 1$.

3.1 The static case

We will first consider the problem in a static case. That is, a trading strategy where the investor needs to decide where to place her buy order, only at time $t = 0$. If the order is placed as a limit order and the limit bid order is not executed at some time $t < T$, then she has to finish the task by buying with a market order at time $t = T$.

3.1.1 Comparing expected costs at each level of LOB.

Evidently, solving the optimal placement problem in the static case amounts to comparing the expected costs of placing one order at each level of the LOB, which depends on all the parameters $f, r, T, p, q, \bar{p}$. (Here recall that r is the rebate for using the limit order and f is the fee for using a market order). For simplicity, however, we will highlight only variables $k, q, T, \bar{p}$ to show the dependence of the cost on those variables. In particular, the expected cost of a limit order placed at the price k ticks lower than the initial best ask price, given q, given the total number of price changes T, and given the probability of the first price change being upward $\bar{p}$, is denoted by $C(k, q, T, \bar{p})$. Here $C(0, q, T, \bar{p})$ is the expect cost of a market order. Since all limit orders placed below $-T-1$ will not be executed until T and will have to be filled by market orders at time T, their expected costs are the same and will be denoted by $C(T+1, q, T, \bar{p})$.

Therefore, it suffices to consider the following minimization problem:

$$\min_{0 \leq k \leq T+1} C(k, q, T, \bar{p}) \left(= \min_{0 \leq k \leq \infty} C(k, q, T, \bar{p}) \right).$$

Remark 3 Note that when $q = 0$, for any given sample path ω there is no chance for a limit order placed at $Y_T(\omega) - 1$ to be executed; when $q = 1$, a limit order placed at $Y_T(\omega) - 1$ is guaranteed for execution. For a general $0 < q < 1$, it suffices to count $n(\omega)$, the number of times $A_t(\omega) = Y_T(\omega)$ before T. That is,

$$n(\omega) = |\{t \leq T-1 : A_t(\omega) = Y_T(\omega)\}|. \tag{3.1}$$

Now, for a given sample path ω, $Q(\omega)$ the probability of a limit order placed at $Y_T(\omega) - 1$ being executed along ω is given by $Q(\omega) = 1 - (1-q)^{n(\omega)}$. Evidently, $Q(\omega)$ increases as $n(\omega)$ increases, meaning that the longer a limit order stays at the best bid queue, the higher its chance of being executed. $Q(\omega)$ is an increasing function of q as well.

In fact, one can show that as the chance of execution increases for a limit order, its expected cost would decrease. That is,

Proposition 5 *$C(k, q, T, \bar{p})$ is a decreasing function of q. That is, if $0 \leq q_1 < q_2 \leq 1$, then*

$$C(k, q_1, T, \bar{p}) > C(k, q_2, T, \bar{p}).$$

Proposition 5 and Lemma 4 lead to the following two results concerning a partial order of the expected costs at different bid levels.

Theorem 6 *Given r, f, T, p, and $\bar{p}$,*

$$C(2, q, T, \bar{p}) < C(3, q, T, \bar{p}) < \cdots < C(T+1, q, T, \bar{p}),$$

for general $q \neq 0$.

This result is crucial: it reduces significantly the complexity of the optimal placement problem. Instead of comparing the expected cost at each single level of LOB, an optimal placement strategy will in general involve comparing the expected costs at only the top three levels: $C(0, q, T, \bar{p})$ for the market order, $C(1, q, T, \bar{p})$ for the best bid, and $C(2, q, T, \bar{p})$ for the second best bid.

In the extreme case when $p = 0$ or when $\bar{p} \geq 1 - p$, the comparison can be further reduced according to the following.

Proposition 7 *Given r, f, T, p, and $\bar{p}$.*

(i) If $q = 0$,

$$C(1, 0, T, \bar{p}) < C(2, 0, T, \bar{p}) < \cdots < C(T+1, 0, T, \bar{p}).$$

That is, the optimal placement strategy involves comparing the best bid order and the market order.

(ii) If $\bar{p} \geq 1 - p$, then $C(1, q, T, \bar{p}) < C(2, q, T, \bar{p})$. That is, the optimal placement strategy involves only the market order and the best bid order.

Finally, we will show that comparison among $C(0, q, T, \bar{p})$, $C(1, q, T, \bar{p})$, and $C(2, q, T, \bar{p})$ is computationally straightforward if focusing on the parameter $\bar{p}$.

Proposition 8 *For fixed values of r, f, p, q, both $C(1, q, T, \bar{p})$ and $C(2, q, T, \bar{p})$ are increasing linear functions of $\bar{p}$. Moreover, the optimal placement strategy is a threshold type when focusing only on the parameter $\bar{p}$. Specifically, the decision to use market order or limit order will depend on at most two of the three intersections $\bar{p}_1^*$, $\bar{p}_2^*$, $\bar{p}_3^*$, with*

$$\begin{cases} \bar{p}_1^* = \dfrac{r + f + 1}{(2 + r + C(2, q, T-1, p))(1-q)}, \\ \bar{p}_2^* = \dfrac{1 + f - C(1, q, T-1, 1-p)}{2 + C(3, q, T-1, p) - C(1, q, T-1, 1-p)}, \\ \bar{p}_3^* = \dfrac{r + C(1, q, T-1, 1-p)}{(2+r+C(2, q, T-1, p))(1-q) - (2 + C(3, q, T-1, p) - C(1, q, T-1, 1-p))}. \end{cases}$$

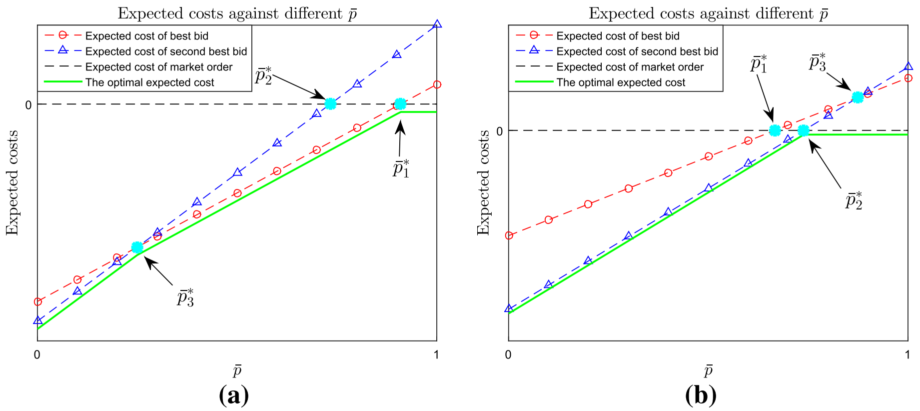


Fig. 1 Expected costs against different $\bar{p}$. **a** Two thresholds, **b** one threshold

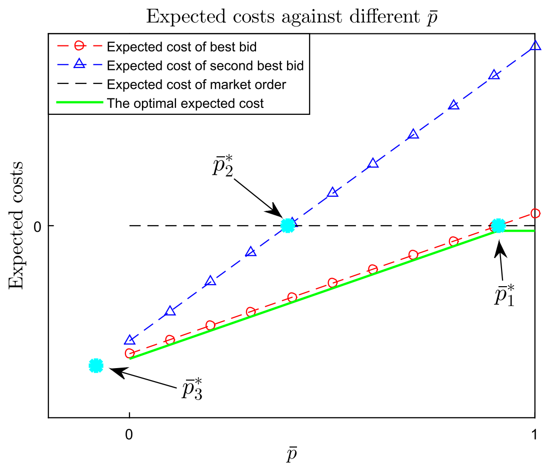


Fig. 2 Expected costs against different $\bar{p}$

Here $\bar{p}_1^*$ is the intersection of $C(1, q, T, \bar{p})$ and $C(0, q, T, \bar{p})$, $\bar{p}_2^*$ is the intersection of $C(2, q, T, \bar{p})$ and $C(0, q, T, \bar{p})$, and $\bar{p}_3^*$ is the intersection of $C(1, q, T, \bar{p})$ and $C(2, q, T, \bar{p})$.

As Figs. 1 and 2 illustrate, when there are two thresholds as in Picture (a), the optimal strategy is to use the second best bid when $\bar{p} \le \bar{p}_3^*$, switch to the best bid order when $\bar{p}_3^* \le \bar{p} \le \bar{p}_1^*$, and switch to the market order when $\bar{p} \ge \bar{p}_1^*$; when there is only one threshold as in Picture (b), the best strategy is to use the second best bid when $\bar{p} \le \bar{p}_2^*$, and then switch to the market order when $\bar{p} \ge \bar{p}_2^*$. Figure 2 gives yet another scenario for the one-threshold type optimal strategy.

In case all of the intersection points are outside [0, 1], then it means we will use only one type of order for all $\bar{p} \in [0, 1]$.

It is worth pointing out that in the expression of $\bar{p}_1^*$, the fourth parameter of $C(2, q, T-1, p)$ is p instead of $\bar{p}$. This is because $C(2, q, T-1, p)$ denotes the expected cost after the price moves up at the first step, therefore the new probability of the "initial" price moving up becomes p. Similar explanation holds for $\bar{p}_2^*$ and $\bar{p}_3^*$.

3.2 The dynamic case

Next, we consider the optimal placement problem in a dynamic case where trades are allowed at any discrete step t for $t \in [0, T]$. That is, we assume that an investor needs to get one share of stock by time T, and she is allowed to place an order at any level at any time t and subsequently modify it by either canceling or changing of limit order to market order between 0 and T. At time T, the previously unexecuted limit order will automatically be replaced by a market order.[3]

Compared to the static case where one has to choose among all possible price levels, in the dynamics setting only two price levels will be sufficient at any given time t: the best bid or the market order, or no order at all. This is because at each time period the price movement is at most one tick, placing an order at the level below the best bid is equivalent to placing no order at all, as this order will not be executed by the next time period.

Given the Markov structure of (A_t, X_t) the optimal placement problem is a Markov decision problem where the expected cost for taking each action at each step can be solved recursively. We will show that one can in fact derive explicitly the optimal solution by further exploring the Markov structure of (A_t, X_t) and the homogeneity of the value function.

To start, note that all the transactions are made in a short time, we will ignore the discount factor without much loss of generality. At at each time t, one can take actions from the set $A = \{\text{Act}^N, \text{Act}^L, \text{Act}^M\}$. Let $V_t((A_t, X_t), \alpha^t)$ be the expected cost for purchasing one share of stock by time T when taking policy α^t at time t. By symmetry, $V_t((A_t, X_t), \alpha^t) = V_t((0, X_t), \alpha^t) + A_t$. Therefore, we simply use $V_t(X_t, \alpha^t)$ for $V_t((0, X_t), \alpha^t)$. We call α^{t^*} an optimal policy if for any policy α^t,

$$V_t(X_t, \alpha^{t^*}) \leq V_t(X_t, \alpha^t). \tag{3.2}$$

Note that $(X_t)_{1\leq t\leq T}$ is a Markov chain and $V_t(X_t, \alpha^t)$ only depends on X_t and α^t, therefore the optimal policy α^{t^*} at time t only depends on X_t, i.e., α^{t^*} at time t could be degenerated into $(\alpha_t^*, \alpha_{(t+1)}^*, \ldots, \alpha_T^*)$, where $\alpha_s^*, t \leq s \leq T$ is the optimal action taken at time s. Mathematically, it could be defined as

$$\begin{cases} V_t^*(x_s) = \min\limits_{a\in A}\{V_t(x_t, (a, \alpha_{t+1}^*, \ldots, \alpha_T^*))\}, \\ \alpha_t^*(x_t) = \arg\min\limits_{a\in A}\{V_t(x_t, (a, \alpha_{t+1}^*, \ldots, \alpha_T^*))\}, \end{cases} \tag{3.3}$$

$$\text{with} \qquad \begin{cases} \alpha_T^*(x_T) = \text{Act}^M, \\ V_T^*(x_T) = f, \end{cases} \tag{3.4}$$

as at time T only market orders are allowed. Then, the Dynamic Programming Principle leads to the following backward recursion for the value function: for $1 \leq t < T$,

$$\begin{cases} V_t(1, \text{Act}^L) = p(1-q)(V_{t+1}^*(1) + 1) + (1 - p + pq)(-1 - r), \\ V_t(1, \text{Act}^N) = pV_{t+1}^*(1) + (1-p)V_{t+1}^*(-1) - 1 + 2p, \\ V_t(-1, \text{Act}^L) = (1 - p - q + pq)(V_{t+1}^*(1) + 1) + (p + q - pq)(-1 - r), \\ V_t(-1, \text{Act}^N) = pV_{t+1}^*(1) + (1-p)V_{t+1}^*(-1) + 1 - 2p, \\ V_t(1, \text{Act}^M) = V_t(-1, \text{Act}^M) = f. \end{cases} \tag{3.5}$$

[3] Clearly, in practice, traders may change their strategies without any price change; also the clock for the price movement in general differs from the usual time clock. Adding these features would be worthy future research topics.

It is clear that at time $t-1$, $\alpha^{t-1} = (\alpha^{t^*}, \alpha^M)$ gives $V_{t-1}(x_{t-1}, \alpha^{t-1}) = V_t^*(x_{t-1})$. That is, $V_{t-1}^*(1) \le V_t^*(1)$ and $V_{t-1}^*(-1) \le V_t^*(-1)$. Therefore we have the monotonicity for $V_t^*(1)$ and $V_t^*(-1)$.

Proposition 9 *Both $V_t^*(-1)$ and $V_t^*(1)$ are non-decreasing functions of t.*

The following Proposition is crucial for deriving explicitly analytic expressions for the optimal policy.

Proposition 10 *For any t, $1 \le t \le T-1$, the following inequalities hold:*

$$\begin{cases} f \ge V_t^*(1) > -r - 2 + \frac{p}{(1-p)(p+q-pq)}, \\ V_t(1, \mathrm{Act}^L) < V_t(1, \mathrm{Act}^M), \\ V_t(-1, \mathrm{Act}^L) < V_t(-1, \mathrm{Act}^N). \end{cases}$$

Now with this proposition, at time t, it is clear that if $X_t = 1$, then one should either wait or use the best bid order; if $X_t = -1$, then one should use either the market order or the best bid. Taking into account the Markov properties of (A_t, X_t) and (X_t), we can derive explicitly the optimal strategy and the corresponding value function. The optimal strategy is shown to be a threshold type.

Theorem 11 *(Optimal policy for $t \ge 1$) There exist two integers t_1^*, t_2^* with $t_1^* < t_2^*$ such that*

(i) if $t < t_1^*$, $\alpha_t^*(-1) = \mathrm{Act}^L$, $\alpha_t^*(1) = \mathrm{Act}^N$;
(ii) if $t_1^* \le t < t_2^*$, $\alpha_t^*(-1) = \alpha_t^*(1) = \mathrm{Act}^L$;
(iii) if $t_2^* \le t$, $\alpha_t^*(-1) = \mathrm{Act}^M$, $\alpha_t^*(1) = \mathrm{Act}^L$.

Here,

$$t_1^* = T - \left\lceil \frac{1}{\log(p-pq)} \cdot \log \frac{q(1-2p)}{((1-p+pq)(f+2+r)-1)(1-q)(p^2q+1+p^2-2p)} \right\rceil - 1, \tag{3.6}$$

and

$$t_2^* = T - \left\lceil \frac{1}{\log(p-pq)} \cdot \log \frac{(f+r+1)(1-p+pq) - (1-p)(1-q)}{(1-p)(1-q)((f+2+r)(1-p+pq)-1)} \right\rceil. \tag{3.7}$$

Moreover, one has explicit expressions for $V_t^*(1)$ and $V_t^*(-1)$, $t \le T$, as follows:

$$V_t^*(-1) = \begin{cases} f, & \text{for } t \ge t_2^*, \\ (p+(1-p)q)(-1-r) + (1-p)(1-q)(1+V_{t+1}^*(1)), & \text{for } t < t_2^*. \end{cases} \tag{3.8}$$

$$V_t^*(1) = \begin{cases} (p-pq)^{(T-t)}\left(f+2+r-\frac{1}{1-p+pq}\right) - 2 - r + \frac{1}{1-p+pq} & \text{for } t_1^* \le t \le T, \\ A \cdot \left(\frac{a_1+\sqrt{a_1^2+4a_2}}{2}\right)^{t_1^*-t+1} + B \cdot \left(\frac{a_1-\sqrt{a_1^2+4a_2}}{2}\right)^{t_1^*-t+1} + a_4 & \text{for } t < t_1^*, \end{cases} \tag{3.9}$$

where $a_1 = p$, $a_2 = (1-p)(1-p-q+pq)$, $a_3 = 2p-1-(1+r)(1-p)(p+q-pq)+(1-p)(1-p-q+pq)$, $a_4 = \frac{a_3}{1-a_1-a_2}$, *and coefficients* A *and* B *are given by the following linear equation system:*

$$A + B = V^*_{t_1^*+1}(1) - a_4,$$
$$A\frac{a_1+\sqrt{a_1^2+4a_2}}{2} + B\frac{a_1-\sqrt{a_1^2+4a_2}}{2} = V^*_{t_1^*}(1) - a_4. \tag{3.10}$$

Note that the optimal placement policy is sensitive to the both the remaining trading time and the market momentum because of the price mean-reversion. The policy becomes more conservative when t is closer to T: if $X_t = 1$, then the price is more likely to go down in the next time period, thus one will use best bid or the market order depending on $T-t$ and can not "afford" placing no orders; if $X_t = -1$, then the price is more likely to go up in the next time period, thus the strategy will be more aggressive: either place no order or use the best bid order depending on $T-t$. Note that both t_1^* and t_2^* could be negative in these expressions, leading to degenerate optimal placement policies. For instance, if $t_1^* < 0$, the optimal placement policy is to wait till time T to use the market order; and if $t_2^* < 0$, the optimal placement policy is to use the best bid until time T to switch to the market order if the limit order is unexecuted by then (Figs. 3, 4).

In the extreme case of $t = 0$, the expected costs at time 0 as well as the optimal placement strategy can be derived in terms of $\bar{p}$. Denote cost^L, cost^N, cost^M as the expected costs for purchasing one share if taking Act^L, Act^N, Act^M at $t = 1$, respectively. Simple calculations show that cost^L and cost^N are linear functions with respect to $\bar{p}$ with positive first order coefficients. Since cost^M is a constant, the optimal placement strategy will be of a threshold type as derived in the static one-period case, if focusing only on the parameter $\bar{p}$. That is,

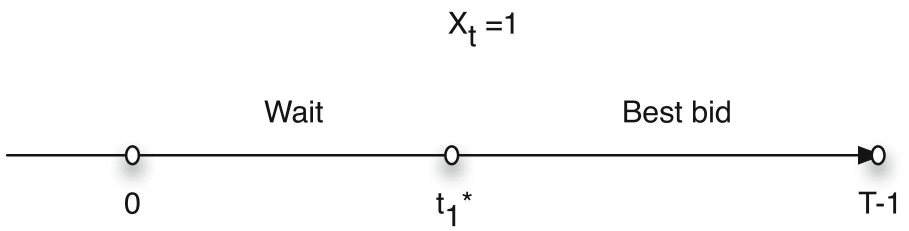


Fig. 3 Illustration of the optimal trading strategy for $1 < t \le T-1$ when $X_t = 1$

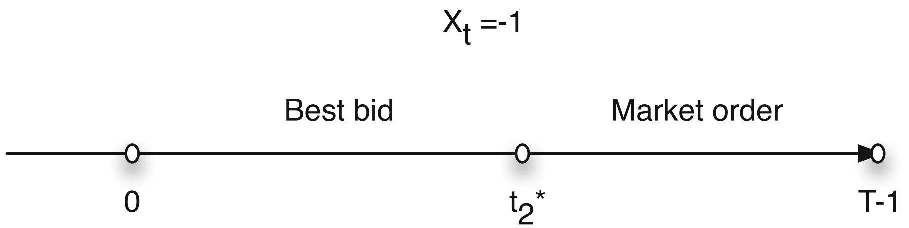


Fig. 4 Illustration of the optimal trading strategy for $1 < t \le T-1$ when $X_t = -1$

Corollary 12 *The optimal strategy at time* 0 *is a threshold type, based on comparison among the following three expressions.*

$$\begin{cases} \mathrm{cost}^L = (1-\bar{p}+\bar{p}q)(-1-r)+(\bar{p}-\bar{p}q)(1+V_1^*(1)) = \bar{p}(1-q)(2+r+V_1^*(1))-1-r, \\ \mathrm{cost}^N = (1-\bar{p})(-1+V_1^*(-1))+\bar{p}(1+V_1^*(1)) = \bar{p}(V_1^*(1)+2-V_1^*(-1))-1+V_1^*(-1), \\ \mathrm{cost}^M = f. \end{cases} \tag{3.11}$$

The decision to use market order, the best bid order, or to wait, will depend on at most two of the three intersections $\bar{p}_1^{**}$, $\bar{p}_2^{**}$, $\bar{p}_3^{**}$, *where*

$$\begin{cases} \bar{p}_1^{**} = \dfrac{f+r+1}{(1-q)(2+r+V_1^*(1))}, \\ \bar{p}_2^{**} = \dfrac{f+1-V_1^*(-1)}{V_1^*(1)+2-V_1^*(-1)}, \\ \bar{p}_3^{**} = \dfrac{V_1^*(-1)+r}{V_1^*(-1)+(1-q)r-2q-qV_1^*(1)}, \end{cases}$$

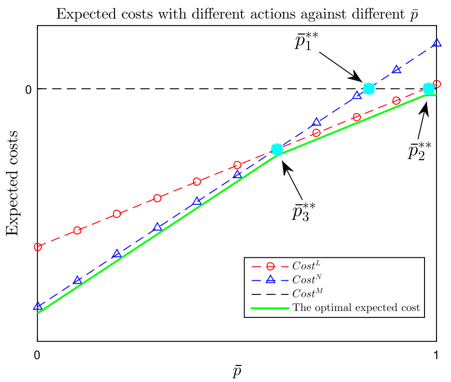
with $\bar{p}_1^{**}$ *the intersection of* cost^L *and* cost^M, $\bar{p}_2^{**}$ *the intersection of* cost^N *and* cost^M, *and* $\bar{p}_3^{**}$ *the intersection of* cost^N *and* cost^L.

As implied by Proposition 10, $\mathrm{cost}^L < \mathrm{cost}^M$ when $\bar{p} = p$ and $\mathrm{cost}^L < \mathrm{cost}^N$ when $\bar{p} = 1-p$. Therefore $\bar{p}_1^{**}$ and $\bar{p}_3^{**}$ mush lie within $(0, 1)$. However, $\bar{p}_2^{**}$ may be outside $[0, 1]$ (Figs. 5, 6).

Note that if $T = 1$, then both the single-step model and multi-step model yield the same optimal trading strategy: use the best bid if $\bar{p} < \frac{r+f+1}{(r+f+2)(1-q)}$ and use the market order otherwise.

4 Conclusions

Clearly our analytical solutions of the optimal placement strategies are obtained with the comprise for analytical tractability. There are several immediate generalizations, including



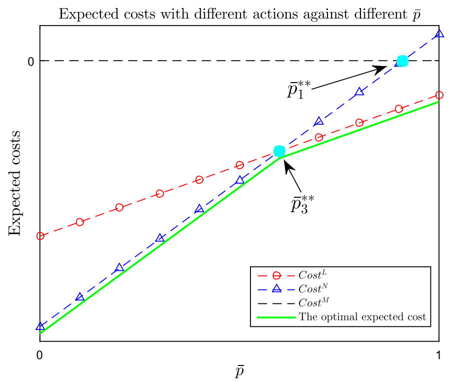


Fig. 5 Expected costs against different $\bar{p}$. **a** $0 \leq \bar{p}_2^{**} \leq 1$, **b** $\bar{p}_2^{**} > 1$

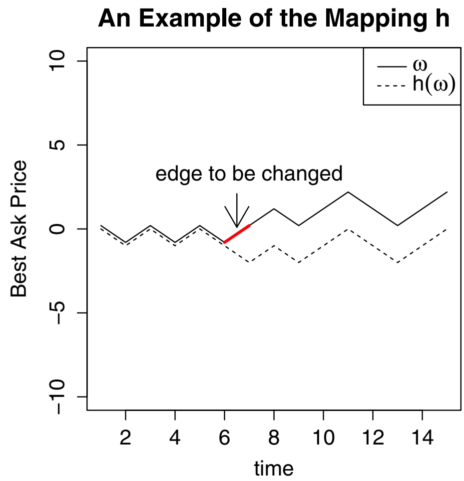


Fig. 6 Illustration of the mapping h

more realistic models for the probability of order execution in a dynamic order-driven market, the possibility of combining price dynamics and order book dynamics, and order placement using multiple trading platforms. Interested readers are referred to Lehalle and Laruelle [39], Cartea et al. [17], Guo et al. [30], and the references therein for more detailed discussions on relevant issues.

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Appendix: Proofs

Proof of Theorem 2

Proof Clearly

$$\begin{aligned}\mathbb{P}(Y_T \leq -k) &= \mathbb{P}(Y_T \leq -k, A_T \leq -k) + \mathbb{P}(Y_T \leq -k, A_T > -k)\\ &= \mathbb{P}(A_T \leq -k) + \mathbb{P}(Y_T \leq -k, A_T > -k),\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(Y_T = -k) &= \mathbb{P}(Y_T \leq -k) - \mathbb{P}(Y_T \leq -k-1)\\ &= \mathbb{P}(A_T = -k) + \mathbb{P}(Y_T \leq -k, A_T \geq -k+1)\\ &\quad - \mathbb{P}(Y_T \leq -k-1, A_T \geq -k).\end{aligned}$$

Since $\frac{T-k}{2} \in \mathbb{N}$, $\{A_T \geq -k+1\} = \{A_T \geq -k+2\}$, it suffices to show that

$$\mathbb{P}(Y_T \leq -k, A_T \geq -k+2) = \mathbb{P}(Y_T \leq -k-1, A_T \geq -k).$$

Consider $\forall \omega \in \{Y_T \leq -k, A_T \geq -k+2\}$ and let $\tau(\omega) = \sup\{t : A_t(\omega) = Y_T(\omega)\}$. We have $\tau(\omega) \leq T-2$ from $A_T(\omega) \geq -k+2$ and $Y_T(\omega) \leq -k$. Now, define a mapping

$h : \{Y_T \le -k, A_T \ge -k+2\} \to \{Y_T \le -k-1, A_T \ge -k\}$ as follows

$$X_t(h(\omega)) = \begin{cases} X_t(\omega), & t \ne \tau(\omega)+1, \\ -1, & t = \tau(\omega)+1. \end{cases}$$

We will show that h is well-defined and is in fact a bijection.

First, we show that h is well-defined, i.e., $h(\omega) \in \{Y_T \le -k-1, A_T \ge -k\}$. By the definition of $\tau(\omega)$, together with $A_T(\omega) - Y_T(\omega) \ge 2$, we see that $X_{\tau(\omega)}(\omega) = -1$ and $X_{\tau(\omega)+1}(\omega) = X_{\tau(\omega)+2}(\omega) = 1$. Since $h(\omega)$ only changes the movement at the step $\tau(\omega)+1$ from 1 to -1, with $A_{\tau(\omega)+1}(h(\omega)) = -Y_T(\omega) - 1$ and $A_T(h(\omega)) = A_T(\omega) - 2 \ge -k$, we have $h(\omega) \in \{Y_T \le -k-1, A_T \ge -k\}$. Moreover, it is easy to see that h does not alter the number of direction changes from ω to $h(\omega)$. Also note that $\tau(\omega) \ge 1$ for all $k > 0$, that means the first price movement will not be changed by h. Then we have $X_1(\omega) = X_1(h(\omega))$ and

$$\mathbb{P}(\omega) = \mathbb{P}(h(\omega)). \tag{4.1}$$

Second, we show that h is an injection. Suppose $\omega_1, \omega_2 \in \{Y_T \le -k, A_T \ge -k+2\}$ with $h(\omega_1) = h(\omega_2)$. If $\tau(\omega_1) = \tau(\omega_2)$, then obviously $\omega_1 = \omega_2$. If $\tau(\omega_1) \ne \tau(\omega_2)$, wlog., we assume that $\tau(\omega_1) < \tau(\omega_2)$. Then for any $t \le \tau(\omega_1)$ and $t > \tau(\omega_2)$, we have $X_t(\omega_1) = X_t(\omega_2)$. For $\tau(\omega_1)+1$, we have $X_{\tau(\omega_1)+1}(\omega_1) = -1$ and $X_{\tau(\omega_1)+1}(\omega_2) = 1$. Thus $A_{\tau(\omega_1)+1}(\omega_2) = A_{\tau(\omega_1)}(\omega_1) - 1 = Y_T(\omega_1) - 1$. Therefore,

$$Y_T(\omega_2) \le A_{\tau(\omega_1)+1}(\omega_2) \le Y_T(\omega_1) - 1.$$

Note that $A_{\tau(\omega_2)+1}(\omega_1) = A_{\tau(\omega_2)+1}(\omega_2) = Y_T(\omega_2) + 1$, we have $A_{\tau(\omega_2)+1}(\omega_1) \le Y_T(\omega_1)$, contradictory to the definition of $\tau(\omega_1)$. Thus $\tau(\omega_1) = \tau(\omega_2)$ and $\omega_1 = \omega_2$, i.e., h is an injection.

Next, we show that h is a surjection, i.e., for any $\tilde{\omega} \in \{Y_T \le -k-1, A_T \ge -k\}$, there exists $\omega \in \{Y_T \le -k, A_T \ge -k+2\}$ such that $h(\omega) = \tilde{\omega}$. Let $\tilde{\tau}(\tilde{\omega}) = \inf\{t : A_t(\omega) = Y_T(\omega)\}$, and define ω by

$$X_t(\omega) = \begin{cases} X_t(\tilde{\omega}), & t \ne \tilde{\tau}(\tilde{\omega}), \\ 1, & t = \tilde{\tau}(\tilde{\omega}). \end{cases}$$

By the definition of $\tilde{\tau}(\tilde{\omega})$, $A_t(\omega) \ge Y_T(\omega) + 1$ for any $t < \tilde{\tau}(\tilde{\omega})$, and $A_t(\omega) \ge A_t(\tilde{\omega}) + 2 \ge Y_T(\omega) + 2$ for any $t \ge \tilde{\tau}(\tilde{\omega})$, Thus $\tilde{\tau}(\tilde{\omega}) - 1$ is the last time that $A_t(\omega)$ reaches its lowest position. Then by the definition of h, $h(\omega) = \tilde{\omega}$. Hence h is a surjection (Figs. 7, 8).

Note that since h is a bijection from $\{Y_T \le -k, A_T \ge -k+2\}$ to $\{Y_T \le -k-1, A_T \ge -k\}$, using (4.1), we obtain

$$\mathbb{P}(Y_T \le -k, A_T \ge -k+2) = \mathbb{P}(Y_T \le -k-1, A_T \ge -k).$$

□

Proof of Proposition 3

Proof For $\forall \omega \in \{Y_T = -k-1\}$, let $\tau(\omega) = \inf\{t : A_t(\omega) = Y_T(\omega)\}$. Define a mapping g on $\{Y_T = -k-1\}$ as follows

$$X_t(g(\omega)) = \begin{cases} X_t(\omega), & t \ne \tau(\omega), \\ 1, & t = \tau(\omega). \end{cases}$$

In other words, g changes $X_{\tau(\omega)}(\omega)$ from a "downward" edge to an "upward" edge. By the definition of $\tau(\omega)$, it is clear that $Y_T(g(\omega)) = -k$. Thus $g(\omega) \in \{Y_T = -k\}$. Now we

Fig. 7 Illustration of the mapping g

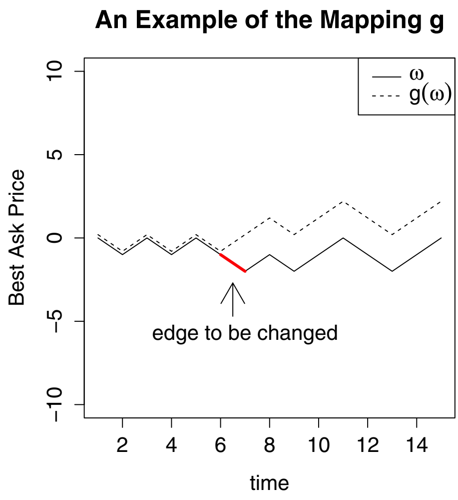
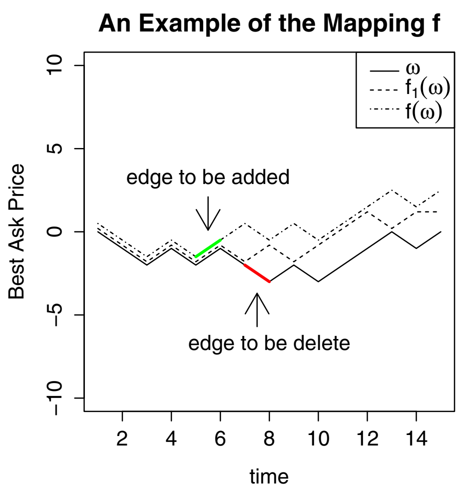


Fig. 8 Illustration of the mapping F



show that g in an injection, i.e., $\forall\omega_1$ and $\omega_2 \in \{Y_T = -k-1\}$, if $g(\omega_1) = g(\omega_2)$, then $\omega_1 = \omega_2$. If $\tau(\omega_1) = \tau(\omega_2)$, then it is easy to see that $X_t(\omega_1) = X_t(\omega_2)$ for $1 \le t \le T$, which is equivalent to $\omega_1 = \omega_2$. If $\tau(\omega_1) \neq \tau(\omega_2)$, wlog., we assume that $\tau(\omega_1) < \tau(\omega_2)$. Then for any $t < \tau(\omega_1)$, $X_t(\omega_1) = X_t(\omega_2)$; $A_t(\omega_1) = A_t(\omega_2) - 2$ for $\tau(\omega_1) \le t < \tau(\omega_2)$; and $X_t(\omega_2) = -1$ and $X_t(\omega_1) = 1$ for $t = \tau(\omega_2)$. Thus $A_{\tau(\omega_2)}(\omega_2) = A_{\tau(\omega_1)}(\omega_2) \ge Y_T(\omega_1)+1 = -k$, contradictory to the definition of ω_2. Hence $\tau(\omega_1) = \tau(\omega_2)$ and $\omega_1 = \omega_2$. Thus g is an injection. Note that since the mapping g does not reduce the number of direction changes, we also have $\mathbb{P}(g(\omega)) \ge \mathbb{P}(\omega)$. Thus $\mathbb{P}\{Y_T = -k-1\} \le \mathbb{P}\{Y_T = -k\}$. □

Proof of Lemma 4

Define the same mapping g from $\{Y_T = -k-1\}$ to $\{Y_T = -k\}$ as in the proof of Proposition 3. Note that g is an injection and $\mathbb{P}(\omega) \le \mathbb{P}(g(\omega))$ for any $\omega \in \{Y_T = -k-1\}$, because $g(\omega)$ has at least as many direction changes as ω has. Moreover, because g changes one "downward" edge to one "upward" edge, we have $A_T(g(\omega)) = A_T(\omega) + 2$. Thus

$$\begin{aligned}
&\mathbb{P}(Y_T=-k-1)(k+2+\mathbb{E}[A_T \mid Y_T=-k-1])\\
&\quad=\sum_{\omega\in\{Y_T=-k-1\}} \mathbb{P}(\omega)(k+2+A_T(\omega))\\
&\quad\le \sum_{g(\omega)\in g(\{Y_T=-k-1\})} \mathbb{P}(g(\omega))(k+A_T(g(\omega)))\\
&\quad\le \sum_{g(\omega)\in\{Y_T=-k\}} \mathbb{P}(g(\omega))(k+A_T(g(\omega))).
\end{aligned}$$

The second to the last inequality holds because g is an injection and $k+2+A_T(\omega)\ge 0$, and the last inequality follows from $A_T(g(\omega))+k\ge 0$ for $\forall g(\omega)\in\{Y_T=-k\}$.

Proof of Proposition 5

This proposition is clear from the following equation 4.2, combined with the observation that $A_T(\omega)\ge Y_T(\omega)=-k+1$ for $\omega\in\{Y_T=-k+1\}$ hence $-k-r-f-A_T(\omega)<0$.

$$C(k,q,T,\bar p)=\sum_{\omega\in\{Y_T=-k+1\}} \mathbb{P}(\omega)Q(\omega)(-k-r-A_T(\omega)-f)+C(k,0,T,\bar p), \quad (4.2)$$

for any $1\le k\le T+1$, with $C(0,q,T,\bar p)=f$.

To see this, note

$$\begin{aligned}
C(k,q,T,\bar p)&=\sum_{\omega\in\{Y_T=-k+1\}} \mathbb{P}(\omega)\big[Q(\omega)(-k-r)+(1-Q(\omega))(A_T(\omega)+f)\big]\\
&\quad+\mathbb{P}(Y_T\le -k)(-k-r)+\mathbb{P}(Y_T>-k+1)(\mathbb{E}[A_T\mid Y_T>-k+1]+f)\\
&=\sum_{\omega\in\{Y_T=-k+1\}} \mathbb{P}(\omega)Q(\omega)(-k-r-A_T(\omega)-f)\\
&\quad+\sum_{\omega\in\{Y_T=-k+1\}} \mathbb{P}(\omega)(A_T(\omega)+f)\\
&\quad+\mathbb{P}(Y_T\le -k)(-k-r)+\mathbb{P}(Y_T>-k+1)(\mathbb{E}[A_T\mid Y_T>-k+1]+f)\\
&=\sum_{\omega\in\{Y_T=-k+1\}} \mathbb{P}(\omega)Q(\omega)(-k-r-A_T(\omega)-f)\\
&\quad+\mathbb{P}(Y_T\le -k)(-k-r)+\mathbb{P}(Y_T\ge -k+1)(\mathbb{E}[A_T\mid Y_T\ge -k+1]+f)\\
&=\sum_{\omega\in\{Y_T=-k+1\}} \mathbb{P}(\omega)Q(\omega)(-k-r-A_T(\omega)-f)+C(k,0,T,\bar p).
\end{aligned}$$

Proof of Theorem 6

Proof For ease of exposition, define the first term on the right hand side of (4.2) as

$$D_k^q=\sum_{\omega\in\{Y_T=-k+1\}} \mathbb{P}(\omega)Q(\omega)(-k-r-A_T(\omega)-f).$$

Since $C(k,0,T,\bar p)$ is increasing in k for $1\le k\le T+1$, it suffices to show that D_k^q is also increasing in k for $2\le k\le T+1$. The key idea here is to define a mapping $F:\{Y_T=-k\}\to\{Y_T=-k+1\}$ such that F is an injection and keeps the number of direction changes and the number of hitting the lowest level. Then we can argue that $D_k^q<D_{k+1}^q$.

To construct F, we first define $N(\omega) = \sum_{t=1}^{T} \mathbb{I}_{A_t(\omega)=Y_T(\omega)}$, and $\tau_n(\omega) = \inf\{i : \sum_{t=1}^{i} \mathbb{I}_{A_t(\omega)=Y_T(\omega)} = n\}$ for $n \leq N(\omega)$. Then F will be defined in two steps. First, construct $F_1 : \{Y_T = -k\} \to \{Y_{T-1} = -k + 1\}$ so that

$$X_t(F_1(\omega)) = \begin{cases} X_t(\omega), & t < \tau_1(\omega), \\ X_{t+1}(\omega), & \tau_1(\omega) \leq t < T, \\ 0, & t = T. \end{cases}$$

Second, construct $F_2 : \{Y_{T-1} = -k + 1\} \times \mathbb{N} \to \{Y_T = -k + 1\}$ so that

$$X_t(F_2(\omega, n)) = \begin{cases} X_t(\omega), & t \leq \tau_n(\omega), \\ 1, & t = \tau_n(\omega) + 1, \\ X_{t-1}(\omega), & t \geq \tau_n(\omega) + 2. \end{cases}$$

Now F is defined as

$$F(\omega) = F_2(F_1(\omega), N(\omega)).$$

In other words, F first deletes the "downward" edge when ω first hits $Y_T(\omega)$ and then adds an "upward" edge after the modified ω hits $Y_T(\omega) + 1$ for $N(\omega)$ times.

It is easy to see that $Y_T(F(\omega)) = Y_T(\omega) + 1$. And for each t where $A_t(\omega) = Y_T(\omega)$, we have $A_{t-1}(F_1(\omega)) = Y_T(\omega) + 1$. Thus $N(F_1(\omega)) \geq N(\omega)$ and F_2 is well-defined on $(F_1(\omega), N(\omega))$, and

$$Q(\omega) = Q(F(\omega)).$$

Moreover, when we remove the "downward" edge in mapping F_1, since $k \geq 2$ (hence $\tau_1(\omega) \geq 2$), we change neither the first step nor the number of direction changes. When adding the "upward" edge in mapping F_2, the number of direction changes of $F_2(F_1(\omega))$ is the same as the number of direction changes of ω and hence

$$\mathbb{P}(\omega) = \mathbb{P}(F(\omega)).$$

Furthermore, since F replaces a "downward" edge by an "upward" edge for ω, $A_T(f(\omega)) = A_T(\omega) + 2$.

Next, we show that F is an injection. Let $\omega_1, \omega_2 \in \{Y_T = -k\}$ and $F(\omega_1) = F(\omega_2)$. First, since $N(\omega) = N(F(\omega))$, we have $N(\omega_1) = N(\omega_2)$. Note that after operating F_2 at $\tau_{N(\omega)}$, $A_t(F(\omega)) > Y_T(f(\omega))$ for all $t > \tau_{N(\omega)}$. In order to have $F(\omega_1) = F(\omega_2)$, we must have $\tau_{N(\omega_1)}(\omega_1) = \tau_{N(\omega_2)}(\omega_2)$. Therefore by the definition of F_2, $F(\omega_1) = F(\omega_2)$ implies $F_1(\omega_1) = F_1(\omega_2)$. Now if $\tau_1(\omega_1) = \tau_1(\omega_2)$, then by the definition of F_1, we have $X_t(\omega_1) = X_t(\omega_2)$ for all $t \neq \tau_1(\omega_1)$. Since ω_1 and ω_2 first hit the lowest level $-k$ at $\tau_1(\omega_1)$, we have $X_{\tau_1(\omega_1)}(\omega_1) = X_{\tau_1(\omega_1)}(\omega_2) = -1$, implying $\omega_1 = \omega_2$. If $\tau_1(\omega_1) \neq \tau_1(\omega_2)$, wlog., we assume that $\tau_1(\omega_1) < \tau_1(\omega_2)$. Then $X_t(\omega_1) = X_t(\omega_2)$ for all $t < \tau_1(\omega_1)$ and $t > \tau_1(\omega_2)$, and $A_t(\omega_1) = A_t(\omega_2)$ for $t \geq \tau_1(\omega_2)$ and $A_t(\omega_1) = A_t(\omega_2) - 1$ for all $\tau_1(\omega_1) \leq t < \tau_1(\omega_2)$. Now $A_{\tau_1(\omega_1)}(\omega_1) = Y_T(\omega_1)$ suggests that $N(\omega_1) \geq N(\omega_2) + 1$, which is a contradiction to the assumption $F(\omega_1) = F(\omega_2)$. Therefore, F is an injection.

Finally, calculating the difference of the sequence D_k^q yields

$$
\begin{aligned}
D_k^q - D_{k+1}^q &= \sum_{\omega\in\{Y_T=-k\}} \mathbb{P}(\omega)Q(\omega)(k+1+r+A_T(\omega)+f)\\
&\quad - \sum_{\omega\in\{Y_T=-k+1\}} \mathbb{P}(\omega)Q(\omega)(k+r+A_T(\omega)+f)\\
&\le \sum_{\omega\in\{Y_T=-k\}} \mathbb{P}(\omega)Q(\omega)(k+1+r+A_T(\omega)+f)\\
&\quad - \sum_{\omega\in\{Y_T=-k\}} \mathbb{P}(F(\omega))Q(F(\omega))(k+r+A_T(F(\omega))+f)\\
&= \sum_{\omega\in\{Y_T=-k\}} \Big(\mathbb{P}(\omega)Q(\omega)(k+1+r+A_T(\omega)+f)\\
&\quad - \mathbb{P}(\omega)Q(\omega)(k+r+A_T(F(\omega))+f)\Big)\\
&= \sum_{\omega\in\{Y_T=-k\}} \mathbb{P}(\omega)Q(\omega)\Big((k+1+r+A_T(\omega)+f)\\
&\quad - (k+r+A_T(\omega)+2+f)\Big)\\
&= - \sum_{\omega\in\{Y_T=-k\}} \mathbb{P}(\omega)Q(\omega) \le 0.
\end{aligned}
$$

Thus D_k^q is increasing as k increases for $k \ge 2$. □

Proof of Proposition 7

Proof We first show the last inequality in the sequence, i.e., $C(T,0,T,\bar{p}) \le C(T+1,0,T,\bar{p})$. Indeed,

$$
\begin{aligned}
&C(T+1,0,T,\bar{p}) - C(T,0,T,\bar{p})\\
&\quad = \mathbb{E}[A_T]+f - \mathbb{P}(Y_T=-T)(-T-r) - \mathbb{P}(Y_T \ge -T+1)(\mathbb{E}[A_T \mid Y_T \ge -T+1]+f)\\
&\quad = \mathbb{P}(Y_T=-T)(r+f+T+\mathbb{E}[A_T \mid Y_T=-T]) = \mathbb{P}(Y_T=-T)(r+f+T-T)\\
&\quad = (f+r)\mathbb{P}(Y_T=-T) \ge 0.
\end{aligned}
$$

Now let $b_k = C(k+1,0,T,\bar{p}) - C(k,0,T,\bar{p})$ for $k=1,2,\ldots,T-1$, then

$$
\begin{aligned}
b_k &= \mathbb{P}(Y_T \le -k-1)(-k-1-r) + \mathbb{P}(Y_T \ge -k)(\mathbb{E}[A_T \mid Y_T \ge -k]+f)\\
&\quad + \mathbb{P}(Y_T \le -k)(-k-r) + \mathbb{P}(Y_T \ge -k+1)(\mathbb{E}[A_T \mid Y_T \ge -k+1]+f)\\
&= -\mathbb{P}(Y_T \le -k-1) + \mathbb{P}(Y_T=-k)(r+f+k+\mathbb{E}[A_T \mid Y_T=-k]).
\end{aligned}
$$

Since $r, f \ge 0$, it suffices to check this for the extreme case of $r=f=0$, i.e., to show

$$
\tilde{b}_k = -\mathbb{P}(Y_T \le -k-1) + k\mathbb{P}(Y_T=-k) + \mathbb{P}(Y_T=-k)\mathbb{E}[A_T \mid Y_T=-k] > 0.
$$

First, by Proposition 3, we have

$$
\begin{aligned}
\tilde{b}_{T-1} &= -\mathbb{P}(Y_T=-T) + (T-1)\mathbb{P}(Y_T=-T+1)\\
&\quad + \mathbb{P}(Y_T=-T+1)\mathbb{E}[A_T \mid Y_T=-T+1]\\
&= -\mathbb{P}(Y_T=-T) + \mathbb{P}(Y_T=-T+1) > 0;
\end{aligned}
$$

and for $k = 1, 2, \ldots, T-2$, by Lemma 4 we have

$$\begin{aligned}\tilde{b}_k - \tilde{b}_{k+1} &= -\mathbb{P}(Y_T = -k-1) + k\mathbb{P}(Y_T = -k) + \mathbb{E}[A_T \mid Y_T = -k]\mathbb{P}(Y_T = -k)\\ &\quad - (k+1)\mathbb{P}(Y_T = -k-1) - \mathbb{E}[A_T \mid Y_T = -k-1]\mathbb{P}(Y_T = -k-1)\\ &= \mathbb{P}(Y_T = -k)(k + \mathbb{E}[A_T \mid Y_T = -k])\\ &\quad - \mathbb{P}(Y_T = -k-1)(k + 2 + \mathbb{E}[A_T \mid Y_T = -k-1])\\ &\geq 0.\end{aligned}$$

Thus recursively $\tilde{b}_k > 0$ for $k = 1, 2, \cdots, T-1$.

To prove the second part of the proposition, note that the proof of the monotonicity of D_k^q with respect to $k \geq 2$ in Proposition 6 does not work for $k = 1$. This is because the first step of ω may be an "upward" edge and may be changed into a "downward" edge by the mapping F. That will decrease the probability of this sample path. However, in the case of $\bar{p} \geq 1-p$, even if the first step of ω is changed from "downward" to "upward", the probability of the sample path will not decrease and the inequality $\mathbb{P}(f(\omega)) = \mathbb{P}(\omega)\frac{\bar{p}p}{(1-\bar{p})(1-p)} \geq \mathbb{P}(\omega)$ as well as the other properties of F still hold. That is, the same method will show that

$$C(1, q, T, \bar{p}) < C(2, q, T, \bar{p}),$$

for $\bar{p} \geq 1-p$. □

Proof of Proposition 8

Proof Notice that $C(2, q, T-1, p) \geq -2-r$, and $C(1, q, T, \bar{p}) = (1 - \bar{p} + \bar{p}q)(-1 - r) + \bar{p}(1-q)(1 + C(2, q, T-1, p)) = \bar{p}(2 + r + C(2, q, T-1, p))(1-q) - 1 - r$, where $C(2, q, T-1, p)$ is independent of $\bar{p}$ and $1 + C(2, q, -1, p) \geq -1-r$. Thus $C(1, q, T, \bar{p})$ is an increasing linear function of $\bar{p}$. Similarly, $C(2, q, T, \bar{p}) = (1-\bar{p})(-1 + C(1, q, T-1, 1-p)) + \bar{p}(1 + C(3, q, T-1, p)) = \bar{p}(2 + C(3, q, T-1, p) - C(1, q, T-1, 1-p)) - 1 + C(1, q, T-1, 1-p)$. Thus $C(2, q, T, \bar{p})$ is a linear function of $\bar{p}$. Moreover, $C(3, q, T-1, p) - C(1, q, T-1, 1-p) > -2$ since for each sample path, the difference between the limit orders placed at -3 and -1 will not be more than 2. Therefore $C(2, q, T, \bar{p})$ is also increasing in $\bar{p}$. □

Proof of Proposition 10

Proof We will show by mathematical induction.
First, for $t = T-1$, simple algebra shows that all $V_{T-1}(1, \mathrm{Act}^L)$, $V_{T-1}(1, \mathrm{Act}^M)(= f)$, $V_{T-1}(1, \mathrm{Act}^N) > -2-r+\frac{p}{(1-p)(p+q-pq)}$, clearly so is $V^*_{T-1}(1) = \min\{V_{T-1}(1, \mathrm{Act}^L)$, $V_{T-1}(1, \mathrm{Act}^M), V_{T-1}(1, \mathrm{Act}^N)\}$.

Moreover,

$$\begin{aligned}V_{T-1}(1, \mathrm{Act}^L) &= (pq + 1 - p)(-1-r) + p(1-q)(1+f)\\ &= -1 - 2pq + 2p - (pq + 1 - p)r\\ &\quad + p(1-q)f < f = V_{T-1}(1, \mathrm{Act}^M),\\ V_{T-1}(-1, \mathrm{Act}^N) &= p(-1+f) + (1-p)(1+f) > (p+q-pq)(-1-r)\\ &\quad + (1-p-q+pq)(1+f) = V_{T-1}(-1, \mathrm{Act}^L).\end{aligned}$$

Hence the desired result holds for $t = T-1$.

Now suppose this lemma holds for some t, $2 \le t \le T-1$. Then it is easy to verify $V_{t-1}(1, Act) > -2 - r + \frac{p}{(1-p)(p+q-pq)}$ for $Act = \mathrm{Act}^M, \mathrm{Act}^L$. Meanwhile,

$$\begin{aligned} V_{t-1}(1, \mathrm{Act}^N) &= (1-p)(-1 + \min\{V_t(-1, \mathrm{Act}^L), V_t(-1, \mathrm{Act}^N), V_t(-1, \mathrm{Act}^M)\}) \\ &\quad + p(1 + V_t^*(1)) \\ &= (1-p)(-1 + \min\{V_t(-1, \mathrm{Act}^L), V_t(-1, \mathrm{Act}^M)\}) + p(1 + V_t^*(1)) \\ &> -2 - r + \frac{p}{(1-p)(p+q-pq)}. \end{aligned}$$

because $V_t(-1, \mathrm{Act}^N) > V_t(-1, \mathrm{Act}^L)$, and $(1-p)(-1 + V_t(-1, Act)) + p(1 + V_t^*(1)) > -2 - r + \frac{p}{(1-p)(p+q-pq)}$ for $Act = \mathrm{Act}^L, \mathrm{Act}^M$.

Therefore, $V_{t-1}^*(1) > -2 - r + \frac{p}{(1-p)(p+q-pq)}$, with $V_{t-1}^*(1) \le f$ from $V_{t-1}(1, \mathrm{Act}^M) = f$.

Finally,

$$\begin{aligned} V_{t-1}(-1, \mathrm{Act}^N) &= p(-1 + V_t^*(-1)) + (1-p)(1 + V_t^*(1)) \\ &= V_{t-1}(-1, \mathrm{Act}^L) + p(r + V_t^*(-1)) + (1-p)q(2 + r + V_t^*(1)) \\ &= V_{t-1}(-1, \mathrm{Act}^L) + p(r + (p + (1-p)q)(-1-r) \\ &\quad + (1-p)(1-q)(1 + V_{t+1}^*(1))) + (1-p)q(2 + r + V_t^*(1)) \\ &> V_{t-1}(-1, \mathrm{Act}^L) + p(-1 + (1-p)(1-q)\frac{p}{(1-p)(p+q-pq)}) \\ &\quad + (1-p)q\frac{p}{(1-p)(p+q-pq)} \\ &= V_{t-1}(-1, \mathrm{Act}^L). \end{aligned}$$

Hence the lemma holds for $t-1$, and for any $1 \ge t \ge T-1$ by induction. □

Proof of Theorem 11

Proof By Proposition 9 and Proposition 10, to show the existence of t_1^*, it suffices to consider

$$\begin{aligned} V_t(1, \mathrm{Act}^L) - V_t(1, \mathrm{Act}^N) &= (pq + 1 - p)(-1 - r) + p(1-q)(1 + V_{t+1}^*(1)) \\ &\quad - (1-p)(-1 + V_{t+1}^*(-1)) + p(1 + V_{t+1}^*(1)) \\ &= -pq(2 + r + V_{t+1}^*(1)) - (1-p)(r + V_{t+1}^*(-1)). \end{aligned}$$

This is a non-increasing function with respect to t. Thus if $V_t(1, \mathrm{Act}^L) \le V_t(1, \mathrm{Act}^N)$ for some t_1^*, then the inequality holds for all $t > t_1^*$. $V_{T-1}(1, \mathrm{Act}^L) - V_{T-1}(1, \mathrm{Act}^N) = -pq(2 + r + f) - (1-p)(r + f) < 0$ implies $t_1^* \le T-1$ if t_1^* exists. Similarly, $V_t(-1, \mathrm{Act}^M) - V_t(-1, \mathrm{Act}^L)$ is also a non-increasing function with respect to t. Therefore if $V_t(-1, \mathrm{Act}^M) \le V_t(-1, \mathrm{Act}^L)$ for some t_2^*, then the inequality holds for all $t > t_2^*$. Notice that when $V_{t+1}(-1, \mathrm{Act}^M) \le V_{t+1}(-1, \mathrm{Act}^L)$, it follows that $V_t(1, \mathrm{Act}^L) - V_t(1, \mathrm{Act}^N) \le 0$ as $V_{t+1}^*(-1) = f \ge 0$. Therefore $t_1^* < t_2^*$ if both of them exist. And the existence will be clear when the expressions of t_1^* and t_2^* are given.

In fact, t_1^* and t_2^* can be computed explicitly as follows. For $t \ge t_1^*$ the sequence $V_t^*(1)$ can be computed recursively from the following equation

$$V_t^*(1) = p(1-q)(V_{t+1}^*(1) + 1) + (1 - p + pq)(-1 - r) \qquad \text{for } t_1^* \le t \le T-1$$

with the boundary condition $V_T^*(1) = f$, yielding

$$\begin{aligned} V_t^*(1) &= (p-pq)^{(T-t)}\left(f + \frac{2p-2pq-1-r(1-p+pq)}{p-1-pq}\right) \\ &\quad + \frac{2p-2pq-1-r(1-p+pq)}{1-p+pq} \\ &= (p-pq)^{(T-t)}\left(f+2+r-\frac{1}{1-p+pq}\right)-2-r \\ &\quad + \frac{1}{1-p+pq} \quad \text{for} \quad t_1^* \le t \le T. \end{aligned}$$

Moreover, $V_{t_1^*}(-1, \mathrm{Act}^L) \le V_{t_1^*}(-1, \mathrm{Act}^M)$ from the showed claim that $V_{t+1}(-1, \mathrm{Act}^M) \le V_{t+1}(-1, \mathrm{Act}^L)$ implies $V_t(1, \mathrm{Act}^L) - V_t(1, \mathrm{Act}^N) \le 0$. Therefore $V_{t_1^*}^*(-1) = V_{t_1^*}(-1, \mathrm{Act}^L)$ and

$$\begin{aligned} &V_{t_1^*-1}(1, \mathrm{Act}^L) - V_{t_1^*-1}(1, \mathrm{Act}^N) \\ &\quad = -pq(2+r+V_{t_1^*}^*(1)) - (1-p)(r+V_{t_1^*}^*(-1)) \\ &\quad = -pq\left(2+r+(p-pq)^{T-t_1^*}\left(f+2+r-\frac{1}{1-p+pq}\right)-2-r+\frac{1}{1-p+pq}\right) \\ &\qquad -(1-p)(r+(p+(1-p)q)(-1-r)+(1-p)(1-q)(1+V_{t_1^*+1}^*(1))) \\ &\quad = \frac{q(1-2p)}{1-p+pq} - \left(f+2+r-\frac{1}{1-p+pq}\right)(p-pq)^{T-t_1^*-1} \\ &\qquad \times (1-q)(p^2q+1+p^2-2p) \\ &\quad \ge 0. \end{aligned}$$

Note that the from the expression of the second last line, it is clear that $V_{t_1^*-1}(1, \mathrm{Act}^L) - V_{t_1^*-1}(1, \mathrm{Act}^N)$ is positive when t_1^* is negative infinity, and negative when $t_1^* = T-1$. Thus the existence of t_1^* is guaranteed. Moreover, we derive the formula for t_1^* as shown in (3.6). Simple calculation from the above expression confirms $t_1^* \le T-1$.

Now, for $t \le t_1^*$ we have

$$\begin{aligned} V_{t-1}^*(1) &= (1-p)(-1+V_t^*(-1)) + p(1+V_t^*(1)) \\ &= pV_t^*(1) + 2p - 1 + (1-p)[(1-p-q+pq)(1+V_{t+1}^*(1)) \\ &\quad + (-1-r)(p+q-pq)] = a_1V_t^*(1) + a_2V_{t+1}^*(1) + a_3 \end{aligned}$$

where $a_1 = p$, $a_2 = (1-p)(1-p-q+pq)$, $a_3 = 2p-1-(1+r)(1-p)(p+q-pq)+(1-p)(1-p-q+pq)$. And the boundary condition for the iteration is

$$\begin{aligned} V_{t_1^*}^*(1) &= (p-pq)^{T-t_1^*}\left(f+2+r-\frac{1}{1-p+pq}\right)-2-r+\frac{1}{1-p+pq}, \\ V_{t_1^*+1}^*(1) &= (p-pq)^{T-t_1^*-1}\left(f+2+r-\frac{1}{1-p+pq}\right)-2-r+\frac{1}{1-p+pq}. \end{aligned}$$

Rewriting the above relation as

$$V_{t-1}^*(1) - a_4 = a_1(V_t^*(1) - a_4) + a_2(V_{t+1}^*(1) - a_4),$$

with $a_4 = \frac{a_3}{1-a_1-a_2}$ leads to the solution for $V_t^*(1)$ and t_2^* (as well as the existence of t_2^*). One can verify from the expression of t_2^* that it is less than or equal to T. Moreover, we can verify that $t_1^* < t_2^*$ from their expression, which is consistent with our previous claim. Finally, from the iterative expression of $V_t^*(-1)$, we can compute it by (3.8).

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