$FO(\mathbb{Z})$

Section 1.14.1 exercise 1-2.

Textbook Solution 1. Every two numbers has a common divisor in FO(Z)

$$\forall x.\forall y.\exists w.(x \mid w) \wedge (y \mid w) \tag{1}$$

$$\forall x.\forall y.\exists z.x \mid z \wedge y \mid z \wedge \forall w.x \mid w \wedge y \mid w \Rightarrow z \leq w \tag{2}$$

Section 1.14.2 exercise 1.

Textbook Solution 2.

$$[\![\phi \wedge \phi]\!]\sigma = [\![\phi]\!]\sigma$$
$$[\![\phi]\!]\sigma \wedge [\![\phi]\!]\sigma = [\![\phi]\!]\sigma$$

Take ϕto equal True or False. True ∧True = True. False ∧False = False

$$[\![\phi + 0]\!]\sigma = [\![\phi]\!]\sigma$$
$$[\![\phi]\!]\sigma + 0 = [\![\phi]\!]\sigma$$

Take ϕto equal any integer x, add x + 0 = 0 by the meaning of adding nothing to something. In this case x.

Section 1.14.3 exercise 2.

Textbook Solution 3.

$$\forall w \forall x.\forall y \forall z.w > x \wedge y > z \Rightarrow w + y > x + z$$
$$\forall w \forall x.\forall y \forall z.w > x \wedge y > z \Rightarrow w + y > x + z$$

We can say x + z <x + y by left and right monotonicity of addition We can say x + z <x + y <w + y by transitive property of <

Section 1.14.4 exercise 2 ($\star$)

Textbook Solution 4. Use the following properties of addition to prove the following theoroms.

1. 0 + y = y
2. S(x) = S(x + y)

$$\forall x.x + 0 = x \tag{1}$$

By property 1 of addition, any number y + 0 = y

$$\forall x. \forall y. x + S(y) = S(x + y) \tag{2}$$

Take x to be any integer and take S(y) to be any function on y in FO(z), By proprty 2 of addition we can see that any function plus an integer is the function plus the integer

$$\forall x. \forall y. x + y = y + x \tag{3}$$

Problem 1. Using the following definitions of addition and multiplication:

$$0 + y = y \qquad\qquad 0 \times y = 0$$
$$Sx + y = S(x + y) \qquad\qquad Sx \times y = y + x \times y$$

Show the following:

(a) $\forall xyz. x \times z + y \times z = (x + y) \times z$

(b) $\forall xyz. x \times (y \times z) = (x \times y) \times z$

Solution 1. case x = S(a)

$$\begin{aligned}
(IH) a * z + y * z &= (a + y) * z \\
(def*)(z + a * z) + y * z &= (Sa + y) * z \\
(assoc) z + (a * z + y * z) &= (Sa + y) * z \\
(IH) z + ((a + y) * z) &= (Sa = y) * z \\
(def*) S(a + y) * z &= (Sa + y) * z \\
(def+)(Sa + y) * z &= (Sa + y) * z \\
(Sa + y) * z &= (Sa + y) * z
\end{aligned}$$

case y = S(a)

$$\begin{aligned}
(IH) x * z + a * z &= (x + a) * z \\
(def*)(z + a * z) + x * z &= (Sa + x) * z \\
(assoc) z + (a * z + x * z) &= (Sa + x) * z \\
(IH) z + ((a + x) * z) &= (Sa + x) * z \\
(def*) S(a + x) * z &= (Sa + x) * z \\
(def+)(Sa + x) * z &= (Sa + x) * z \\
(Sa + x) * z &= (Sa + x) * z
\end{aligned}$$

Solution 2. case x = S(a)

$$\begin{aligned}
(IH) a * z + y * z &= (a + y) * z \\
(def*)(z + a * z) + y * z &= (Sa + y) * z \\
(assoc) z + (a * z + y * z) &= (Sa + y) * z \\
(IH) z + ((a + y) * z) &= (Sa = y) * z \\
(def*) S(a + y) * z &= (Sa + y) * z \\
(def+)(Sa + y) * z &= (Sa + y) * z \\
(Sa + y) * z &= (Sa + y) * z
\end{aligned}$$

case y = S(a)

$$\begin{aligned}
(IH) x * z + a * z &= (x + a) * z \\
(def*) (z + a * z) + x * z &= (Sa + x) * z \\
(assoc) z + (a * z + x * z) &= (Sa + x) * z \\
(IH) z + ((a + x) * z) &= (Sa + x) * z \\
(def*) S(a + x) * z &= (Sa + x) * z \\
(def+) (Sa + x) * z &= (Sa + x) * z \\
(Sa + x) * z &= (Sa + x) * z
\end{aligned}$$

Solution 3. case x = S(a)

$$\begin{aligned}
(IH) a * (y * z) &= ((a * y) * z \\
prove Sa * (y * z) &= (Sa * y) * z \\
(def*) (y * z) + a(y * z) &= (Sa * y) * z \\
(assoc) z(y + ay) &= (Sa * y) * z \\
(def*) z * (Sa * y) &= (Sa * y) * z \\
(Sa * y) * z &= (Sa * y) * z
\end{aligned}$$

The `while` language

Section 2.10.1 exercises 2-5.

Textbook Problem 1. 2.10.1.1: Write a while command that sets z to max of x and y

Textbook Solution 1. if x - y > 0 then x else if x = y then -1 else y

Textbook Problem 2. 2.10.1.2: Write a while command that sets z to x^y (assume y is non-neg)

Textbook Solution 2. while y ¿ 0; z := x * z; y := y - 1;

Textbook Problem 3. 2.10.1.3: What is the meaning of $x := y$; ; y:= z; z:= x in in state x maps to 0, y maps to 2, z maps to 1

Textbook Solution 3. This means x := 2; y := 1; z := 0

Textbook Problem 4. 2.10.1.4: What is the meaning of if x > 0 then z := y - x else z := y + x in state x maps to 3, y maps to 2, z maps to 1?

Textbook Solution 4. since x maps to 3, it is greater than 0. Therefor z is set to the value of y - x (2-(-3)) = -5

Textbook Problem 5. Write down the meaning of x := y; y := x in some arbitrary state σ

Textbook Solution 5.

$$\sigma \mapsto \sigma[x \mapsto y] \quad (1)$$