Institute of Technology of Cambodia
Department of Industrial and Mechanical Engineering

TD3 of Lecture 3: Laplace Transform
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Questions and Answers

1. Find the Laplace transform of $f(t) = u(t)$
   Answer: $F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty u(t)e^{-st}dt$
   Unit step function:
$$u(t) = \begin{cases} 1; t > 0 \\ 0; t < 0 \end{cases}$$
$$F(s) = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

2. Find the Laplace transform of $f(t) = tu(t)$
   Answer: $F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty tu(t)e^{-st}dt$
   Unit step function:
$$u(t) = \begin{cases} 1; t > 0 \\ 0; t < 0 \end{cases}$$
$$F(s) = \int_0^\infty te^{-st}dt$$
   Let $u = t \rightarrow du = dt \text{ and } dv = e^{-st}dt \rightarrow v = -\frac{1}{s}e^{-st}$
$$F(s) = -\frac{t}{s}e^{-st}\Big|_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}dt = \frac{1}{s^2}$$

3. Find the Laplace transform of $f(t) = e^{-at}u(t)$
   Answer: $F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-at}u(t)e^{-st}dt$
   Unit step function:
$$u(t) = \begin{cases} 1; t > 0 \\ 0; t < 0 \end{cases}$$
$$F(s) = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt = -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^\infty = \frac{1}{s+a}$$

4. Find the Laplace transform of $f(t) = Ae^{-at}u(t)$
   Answer: $F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty Ae^{-at}u(t)e^{-st}dt$
   Unit step function:

$$u(t) = \begin{cases} 1; t > 0 \\ 0; t < 0 \end{cases}$$

$$F(s) = \int_0^\infty Ae^{-at}e^{-st}dt = A\int_0^\infty e^{-(s+a)t}dt = -\frac{A}{s+a}e^{-(s+a)t}\Big|_0^\infty = \frac{A}{s+a}$$

5. Find the Laplace transform of $f(t) = \sin(\omega t)u(t)$

   Answer: $F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty \sin(\omega t)u(t)e^{-st}dt$

   Unit step function:

$$u(t) = \begin{cases} 1; t > 0 \\ 0; t < 0 \end{cases}$$

$$F(s) = \int_0^\infty \sin(\omega t)e^{-st}dt$$

   Let $u = \sin(\omega t) \rightarrow du = \omega\cos(\omega t)dt\ and\ dv = e^{-st}dt \rightarrow v = -\frac{1}{s}e^{-st}$

$$F(s) = (\sin(\omega t))(-\frac{1}{s}e^{-st})\Big|_0^\infty + \frac{\omega}{s}\int_0^\infty e^{-st}\cos(\omega t)dt$$

$$F(s) = 0 + \frac{\omega}{s}\int_0^\infty e^{-st}\cos(\omega t)\,dt$$

   Let $u = \cos(\omega t) \rightarrow du = -\omega\sin(\omega t)dt\ and\ dv = e^{-st}dt \rightarrow v = -\frac{1}{s}e^{-st}$

$$F(s) = \frac{\omega}{s}(\cos(\omega t))(-\frac{1}{s}e^{-st})\Big|_0^\infty - \frac{\omega^2}{s^2}\int_0^\infty e^{-st}\sin(\omega t)\,dt$$

$$F(s) = \frac{\omega}{s^2} - \frac{\omega^2}{s^2}F(s) \rightarrow F(s) = \frac{\omega}{s^2 + \omega^2}$$

6. Find the Laplace transform of $f(t) = \cos(\omega t)u(t)$

   Answer: $F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty \cos(\omega t)u(t)e^{-st}dt$

   Unit step function:

$$u(t) = \begin{cases} 1; t > 0 \\ 0; t < 0 \end{cases}$$

$$F(s) = \int_0^\infty \cos(\omega t)e^{-st}dt$$

   Let $u = \cos(\omega t) \rightarrow du = -\omega\sin(\omega t)dt\ and\ dv = e^{-st}dt \rightarrow v = -\frac{1}{s}e^{-st}$

$$F(s) = (\cos(\omega t))(-\frac{1}{s}e^{-st})\Big|_0^\infty - \frac{\omega}{s}\int_0^\infty e^{-st}\sin(\omega t)dt$$

$$F(s) = \frac{1}{s} - \frac{\omega}{s}\int_0^\infty e^{-st}\sin(\omega t)\,dt$$

Let $u = \sin(\omega t) \rightarrow du = \omega\cos(\omega t)dt \; and \; dv = e^{-st}dt \rightarrow v = -\frac{1}{s}e^{-st}$

$$F(s) = \frac{1}{s} - \frac{\omega}{s}(\sin(\omega t))(-\frac{1}{s}e^{-st})\Big|_0^{\infty} - \frac{\omega^2}{s^2}\int_0^{\infty} e^{-st}\cos(\omega t)\,dt$$

$$F(s) = \frac{1}{s} - \frac{\omega^2}{s^2}F(s) \rightarrow F(s) = \frac{s}{s^2 + \omega^2}$$

7. Given the differential equation, solve for y(t) if all initial conditions are zero. Use the Laplace transform.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dx} + 32y = 32u(t)$$

Answer: solve for y(t)

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)} = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)}$$

* Calculate K_1

>> We have:

$$\frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)}$$

>> Multiply s to both sides, we get:

$$\frac{32}{(s+4)(s+8)} = K_1 + \frac{sK_2}{(s+4)} + \frac{sK_3}{(s+8)}$$

>> Let s = 0:

$$\frac{32}{(0+4)(0+8)} = K_1 + \frac{0 \times K_2}{(0+4)} + \frac{0 \times K_3}{(0+8)}$$

$$\frac{32}{4 \times 8} = K_1 ==> K_1 = 1$$

* Calculate K_2

>> We have:

$$\frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)}$$

>> Multiply (s + 4) to both sides, we get:

$$\frac{32}{s(s+8)} = \frac{(s+4)K_1}{s} + K_2 + \frac{(s+4)K_3}{(s+8)}$$

>> Let s = -4:

$$\frac{32}{-4(-4+8)} = \frac{(-4+4)K_1}{-4} + K_2 + \frac{(-4+4)K_3}{(s+8)}$$

$$\frac{32}{-4 \times 4} = K_2 ==> K_2 = -2$$

* Calculate K_3
>> We have:

$$\frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)}$$

>> Multiply (s + 8) to both sides, we get:

$$\frac{32}{s(s+4)} = \frac{(s+8)K_1}{s} + \frac{(s+8)K_2}{(s+4)} + K_3$$

>> Let s = -8:

$$\frac{32}{-8(-8+4)} = \frac{(-8+8)K_1}{-8} + \frac{(-8+8)K_2}{(-8+4)} + K_3$$

$$\frac{32}{-8 \times (-4)} = K_3 ==> K_3 = 1$$

We get:

$$Y(s) = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)} = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)}$$

After apply Inverse Laplace:

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t)$$

8. Given the Y(s), solve for y(t)

$$Y(s) = \frac{2}{(s+1)(s+2)^2}$$

Answer: solve for y(t)

$$Y(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

* Calculate K_1
>> We have:

$$\frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

>> Multiply (s + 1) to both sides, we get:

$$\frac{2}{(s+2)^2} = K_1 + \frac{K_2(s+1)}{(s+2)^2} + \frac{K_3(s+1)}{(s+2)}$$

>> Let s = -1 ==> $K_1 = 2$

* Calculate K_2
>> We have:

$$\frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

>> Multiply $(s+2)^2$ to both sides, we get:

$$\frac{2}{(s+1)} = \frac{K_1(s+2)^2}{(s+1)} + \frac{K_2(s+2)^2}{(s+2)^2} + \frac{K_3(s+2)^2}{(s+2)}$$

>> Let s = -2 ==> $K_1 = -2$