Programming Design

Digital Systems

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Thank you

- Most of the materials in this set of slides are adopted from the teaching materials of Professor Yuh-Jzer Joung's (莊裕澤).
  - Who failed the instructor in the course "Introduction to Computer Science".

https://www.stpi.narl.org.tw/public/leader.htm

Road map

- **Number systems**
- Complements
- Miscellaneous things

Number systems

- Decimal numbers: $7397 = 7 \times 10^3 + 3 \times 10^2 + 9 \times 10^1 + 7 \times 10^0$.
- In general,

$$a_3 a_2 a_1 a_0 . a_{-1} a_{-2}$$
$$= a_3 \times 10^3 + a_2 \times 10^2 + a_1 \times 10^1 + a_0 \times 10^0 + a_{-1} \times 10^{-1} + a_{-2} \times 10^{-2}.$$

  - a_i: **coefficient**.
  - 10: **base** or **radix**.
  - a_3: most significant bit (**msb**).
  - a_{-2}: least significant bit (**lsb**).

Base-r system

- In general, number X in a **base-*r* system** is represented as

$$X = (a_n a_{n-1} \cdots \cdots a_1 a_0 . a_{-1} a_{-2} \cdots \cdots a_{-m})_r .$$

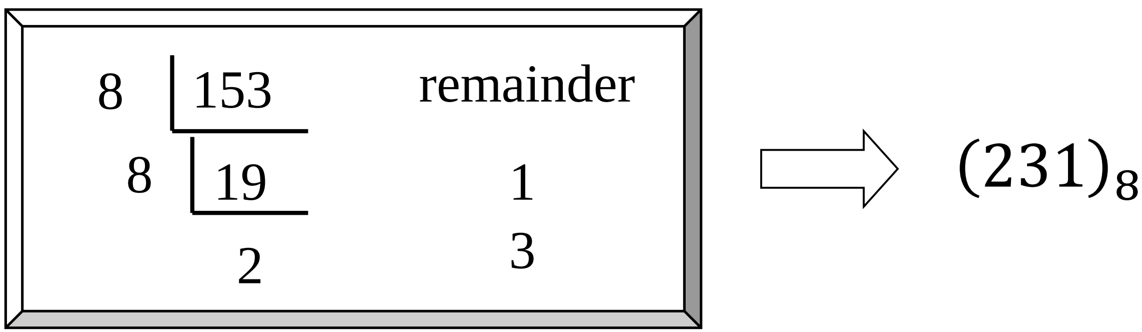
- Its value in the base-10 system is

$$X = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 + a_{-1} r^{-1} + a_{-2} r^{-2} + \cdots + a_{-m} r^{-m}.$$

  - In a **binary** system, $r = 2, a_i \in \{0,1\}$.
  - In an **octal** system, $r = 8, a_i \in \{0,1, \dots, 7\}$.
  - In a **decimal** system, $r = 10, a_i \in \{0,1, \dots, 9\}$.
  - In a **hexadecimal** system, $r = 16, a_i \in \{0,1, \dots 9, A, B, C, D, E, F\}$.

Base conversion

- Base-r to base-10 conversion: Straightforward !
- Base-r to base-s conversion: By **repeated division for integers** and **repeated multiplication for fractions**.
- **Example.** Converting $(153.513)_{10}$ to an octal number.
  Integer part: $(153)_{10} = (231)_8$.



Base conversion

- fractional part: 0.513
  - $0.513 \times 8 = \mathbf{4}.104.$
  - $0.104 \times 8 = \mathbf{0}.832.$
  - $0.832 \times 8 = \mathbf{6}.656.$
  - $0.656 \times 8 = \mathbf{5}.24.$
  - $\cdots$
- $(0.513)_{10} \approx (0.\mathbf{4065})_8$
- All together:

$$(153.513)_{10} \approx (231.4065)_8.$$

General base conversion: integer part

- $X = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 + a_{-1} r^{-1} + a_{-2} r^{-2} + \cdots + a_{-m} r^{-m}$.
- Consider the integer part: $XI = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0$.
- By dividing XI by r, we obtain

$$\frac{XI}{r} = a_n r^{n-1} + a_{n-1} r^{n-2} + a_{n-2} r^{n-3} + \cdots + a_2 r + a_1,$$

  - The **remainder** is $\boldsymbol{a_0}$.
- By dividing XI/r by r, we obtain

$$\frac{XI}{r^2} = a_n r^{n-2} + a_{n-1} r^{n-3} + \cdots + a_3 r + a_2,$$

  - The **remainder** is $\boldsymbol{a_1}$.

General base conversion: integer part

- By dividing XI/r^2 by r, we obtain

$$\frac{XI}{r^3} = a_n r^{n-3} + a_{n-1} r^{n-4} + \cdots + a_4 r + a_3,$$

  – The **remainder** is $\boldsymbol{a_2}$.
- Continually in this fashion, we eventually obtain the coefficients

$$a_n \, a_{n-1} a_{n-2} \cdots a_1 a_0.$$

General base conversion: fraction part

- Consider the faction part:

$$XF = a_{-1}r^{-1} + a_{-2}r^{-2} + \cdots + a_{-m}r^{-m}.$$

- By multiplying XF by r, we obtain

$$XF \cdot r = \underbrace{\boldsymbol{a_{-1}}}_{\substack{\text{integer in between} \\ 0 \cdots r-1}} + \underbrace{a_{-2}r^{-1} + a_{-3}r^{-1} + \cdots + a_{-m+1}r^{-m+2} + a_{-m}r^{-m+1}}_{<1}.$$

Because the maximum value of this part is

$$(r-1)r^{-1} + (r-1)r^{-2} + \cdots + (r-1)r^{-m+2} + (r-1)r^{-m+1}$$

$$\leq (r-1)\left(r^{-1} + r^{-2} + r^{-3} + \cdots + r^{-m+1}\right)$$

$$< (r-1)\left(\frac{1}{r-1}\right) = 1.$$

General base conversion: fraction part

- By continually multiplying the fraction part of $XF \times r$, we obtain

$$XF_1 = \underbrace{a_{-2}}_{\substack{\text{integer in between} \\ 0 \cdots r-1}} + \underbrace{a_{-3}r^{-1} + \cdots + a_{-m+1}r^{-m+3} + a_{-m}r^{-m+2}}_{<1}.$$

  So we obtain the **second digit a_2**.

- Similarly, by continually multiplying the fraction part of XF_1, we can obtain the third digit a_{-3}, then a_{-4}, then a_{-5}, and so on.

Base 2^i to base 2^j

- Conversion between **base 2^i and base 2^j** can be done more quickly.
- **Example.** Convert $(10111010011)_2$ to octal and hexadecimal:

$$10111010011 \Rightarrow \underbrace{10}_{2}\ \underbrace{111}_{7}\ \underbrace{010}_{2}\ \underbrace{011}_{3} \Rightarrow (2\ 7\ 2\ 3)_8$$

$$10111010011 \Rightarrow \underbrace{101}_{5}\ \underbrace{1101}_{D}\ \underbrace{0011}_{3} \Rightarrow (5\ D\ 3)_{16}$$

Base 2^i to base 2^j

- **Example.** Convert $(12A7F)_{16}$ to binary and octal.

$$(12A7F)_{16} \Rightarrow (0001\ 0010\ 1010\ 0111\ 1111)_2$$

$$(00\ \underbrace{010}_{2}\ \underbrace{010}_{2}\ \underbrace{101}_{5}\ \underbrace{001}_{1}\ \underbrace{111}_{7}\ \underbrace{111}_{7})_2 \Rightarrow (2\ \ 2\ 5\ 1\ 7\ 7)_8$$

Road map

- Number systems
- **Complements**
- Miscellaneous things

Addition/subtraction for binary numbers

- In our **mind**, **subtraction** appears to take a different approach from **addition**.
- The difference will complicate the design of a logical circuit.

Addition

$$\begin{array}{r} 9 \\ +\ 13 \\ \hline 22 \end{array} \qquad \begin{array}{r} 1001 \\ +\ 1101 \\ \hline 10110 \end{array}$$

Subtraction

$$\begin{array}{r} 9 \\ -\ 13 \\ \hline -\ 4 \end{array} \qquad \begin{array}{r} 1001 \\ -\ 1101 \\ \hline ? \end{array}$$

- The problem can be solved if we can represent “negative” numbers so that subtraction becomes **addition to a negative number**.
- How may we represent negative numbers with just 0s and 1s?

Signed magnitude

0	0000 0000
1	0000 0001
2	0000 0010
…	
127	0111 1111

all positive numbers begin with 0

−127	1111 1111
…	
−2	1000 0010
−1	1000 0001
−0	1000 0000

all negative numbers begin with 1

Subtraction: signed-magnitude numbers

$$\begin{array}{rr} + & 9 \\ + & 13 \\ \hline + & 22 \end{array} \Rightarrow \begin{array}{rr} & 00001001 \\ + & 00001101 \\ \hline & 00010110 \end{array}$$

$$\begin{array}{rr} - & 9 \\ + & 13 \\ \hline + & 4 \end{array} \Rightarrow \begin{array}{rr} & 10001001 \\ + & 00001101 \\ \hline & \end{array} \Rightarrow \begin{array}{rr} & 00001101 \\ - & 00001001 \\ \hline & 00000100 \end{array}$$

$$\begin{array}{rr} + & 9 \\ - & 13 \\ \hline - & 4 \end{array} \Rightarrow \begin{array}{rr} & 00001001 \\ + & 10001101 \\ \hline & \end{array} \Rightarrow \begin{array}{rr} & 10001101 \\ - & 00001001 \\ \hline & 10000100 \end{array}$$

Subtraction: signed-magnitude numbers

$$\begin{array}{r} -\ \ 9 \\ -\ 13 \\ \hline -\ 22 \end{array} \quad \Rightarrow \quad \begin{array}{r} 10001101 \\ +\ 10001001 \\ \hline 10010110 \end{array}$$

Summary：
to perform arithmetic operations on signed magnitude numbers, we need to compare the signs and the magnitudes of the two numbers, and then perform either addition or subtraction, much like what we were taught to do in primary school. So **this does not simplify the problem**.

Complement: for simplifying subtraction

- Two types of complements for each base-r system：
  - **$(r-1)$'s complement**.
  - **r's complement**.
- The $(r-1)$'s complement of an n-digit number $X = \boldsymbol{(r^n - 1) - X}$.
- **Example.** In a decimal system, the 9's complement of 546700 is

$$(10^6 - 1) - 546700 = 999999 - 546700 = 453299.$$

- **Example.** The 9's complement of 012398 is $999999 - 012398 = 987601$.
- **Example.** The 1's complement of 01011000 is

$$(2^8 - 1) - 01011000 = 11111111 - 01011000 = 10100111.$$

- **Example.** The 1's complement of 0101101 is 1010010.

8-bit 1's complement numbers

0	0000 0000
1	0000 0001
2	0000 0010
…	…
127	0111 1111
−127	1000 0000
…	…
−2	1111 1101
−1	1111 1110
−0	1111 1111

all positive numbers begin with 0

all negative numbers begin with 1

r's complement

- The *r*'s complement of an *n*-digital number X is $\begin{cases} r^n - X & \text{if } X \neq 0 \\ 0 & \text{if } X = 0 \end{cases}$.
- **Example.**
  - The 10's complement of 012398 is 987602.
  - The 10's complement of 2467000 is 7533000.
  - The 2's complement of 1101100 is 0010100.
  - The 2's complement of 0110111 is 1001001.
- To compute the complement of a number having radix point, first, **remove the radix point**, compute the complement of the new number, and **restore the radix point**.
  - The 1's complement of 01101.101 is 10010.010
  - The 2's complement of 01101.101 is 10010.011

8-bit 2's complement numbers

0	0000 0000
1	0000 0001
2	0000 0010
...	...
127	0111 1111
−128 ?	1000 0000
−127	1000 0001
...	...
−2	1111 1110
−1	1111 1111
−0	0000 0000

all positive numbers begin with 0

all negative numbers begin with 1

Complement of the complement

- The complement of the complement of X is X.
- $(r-1)$'s complement of $(2^n-1)-X = (2^n-1)-\left((2^n-1)-X\right) = X$.
- r 's complement of $2^n - X = 2^n - (2^n - X) = X$ if $X \neq 0$.
  - Recall that we define 2's complement of 0 as 0.

Representation of signed numbers

- Three possible representations of –9 with 8 bits：
  - signed magnitude: **1**0001001.
  - signed-1's-complement of +9 (00001001): 11110110.
  - signed-2's-complement +9 (00001001): 11110111.
- Values that **can be represented** in *n* bits：
  - signed magnitude: $(-2^{n-1}+1) \sim (2^{n-1}-1)$.
  - signed-1's-complement: $(-2^{n-1}+1) \sim (2^{n-1}-1)$.
  - signed-2's-complement: $(-2^{n-1}) \sim (2^{n-1}-1)$.

Subtraction: 1's-complement numbers

- Assume that $X, Y \geq 0$, and they have n bits (including their signs).
- **Case $X + Y$**: normal binary addition.

$$\begin{array}{r} +\ \ 9 \\ +\ 13 \\ \hline +\ 22 \end{array} \Rightarrow \begin{array}{r} 00001001 \\ +\ 00001101 \\ \hline 00010110 \end{array}$$

Subtraction: 1's-complement numbers

- **Case $-X + Y$:**

$$(1\text{'s complement of } X) + Y$$
$$= [(2^n - 1) - X] + Y$$
$$= (2^n - 1) - (X - Y).$$

Sub-case: $X - Y \geq 0$: there will be no carry.

Sub-case: $X - Y < 0$:

$$(2^n - 1) - (X - Y) = \underset{\text{carry bit}}{2^n} + \underbrace{(Y - X)}_{\geq 0} - 1$$

So there will be **a carry**. To obtain $(Y - X)$, we discard the carry and **add 1** to the result.

Subtraction: 1's-complement numbers

$$\begin{array}{r} -\ \ 9 \\ +\,13 \\ \hline +\ \ 4 \end{array} \Rightarrow \begin{array}{r} 11110110 \\ +\ 00001101 \\ \hline 100000011 \\ +\qquad\quad 1 \\ \hline 00000100 \end{array}$$

end-around carry

The 1's complement of 9 (00001001) is 11110110

Subtraction: 1's-complement numbers

- **Case $(-X) + (-Y)$:**

$$
\begin{aligned}
&(1\text{'s complement of } X) + (1\text{'s complement of } Y) \\
&\quad = [(2^n - 1) - X] + [(2^n - 1) - Y] \\
&\quad = (2^n - 1) + \underbrace{(2^n - 1) - (X + Y)}_{1\text{'s complement of } (X+Y)}
\end{aligned}
$$

carry bit

extra -1

So there will be **a carry**. If we discard the carry, then the result is the 1's complement of $(X + Y)$ minus 1. To obtain the correct result, we need to **add 1** to the result of $[-1 + (2^n - 1) - (X + Y)]$.

Subtraction: 1's-complement numbers

$$\begin{array}{r} -\ \ 9 \\ -13 \\ \hline -22 \end{array} \Rightarrow \begin{array}{r} 11110110 \\ +\ 11110010 \\ \hline 111101000 \\ +\qquad\quad 1 \\ \hline 11101001 \end{array}$$

1's complement of 9 (00001001) is 11110110

1's complement of 13 (00001101) is 11110010

1's complement of 22 (00010110)

Subtraction: 2's-complement numbers

- Case $X + Y$:

$$\begin{array}{r} +\ \ 9 \\ +\ 13 \\ \hline +\ 22 \end{array} \quad \Rightarrow \quad \begin{array}{r} 00001001 \\ +\ 00001101 \\ \hline 00010110 \end{array}$$

- Case $X - Y$ where $X < Y$:

$$\begin{array}{r} 9 \\ -\ 13 \\ \hline -\ 4 \end{array} \quad \Rightarrow \quad \begin{array}{r} 00001001 \\ -\ 00001101 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} 00001001 \\ +\ 11110011 \\ \hline 11111100 \end{array}$$

2's complement of 13

2's complement of 4

Subtraction: 2's-complement numbers

- Case $-X + Y$ where $X > Y$:

$$\begin{array}{r} -\ 9 \\ +\ 13 \\ \hline 4 \end{array} \Rightarrow \begin{array}{r} 11110111 \\ +\ 00001101 \\ \hline 100000100 \end{array}$$

2's complement of 9 (00001001)

discard this carryout

- In general: $-X + Y = (2^n - X) + Y = 2^n - (X - Y)$.
  - Case $X > Y$: there will be no carry out, and the result $2^n - (X - Y)$ is the 2's complement of $(X - Y)$.
  - Case $X \leq Y$: there will be a carry out, and the result $2^n + (Y - X)$ after discarding the carry out is $(Y - X)$.

Subtraction: 2's-complement numbers

- Case $-X - Y$:

$$\begin{array}{r} -\ \ 9 \\ -\ 13 \\ \hline -\ 22 \end{array} \Rightarrow \begin{array}{r} 11110111 \\ +\ 11110011 \\ \hline 111101010 \end{array}$$

2's complement of 22 (00010110)

discard this carryout

- In general, $(-X) + (-Y) \Rightarrow (2^n - X) + (2^n - Y) = 2^n + [2^n - (X + Y)]$.
  - There will be a carry out, and the result after discarding the carry out is the 2's complement of $(X + Y)$.

Comparison of the three systems

- signed-magnitude：
  - useful in ordinary arithmetic, **awkward in computer arithmetic**
- signed-1's-complement：
  - used in old computers, but is now seldom used.
- signed-2's-complement：
  - **used in most computers.**

Road map

- Number systems
- Complements
- **Miscellaneous things**

Overflow

$$\begin{array}{r} +\ 64 \\ +\ 96 \\ \hline +\ 160 \end{array} \quad \Rightarrow \quad \begin{array}{r} 01000000 \\ +\ 01100000 \\ \hline 10100000 \end{array}$$

negative number, representing the 2's complement of 01100000 (96); that is, $64 + 96 = -96$! Wrong!

$$\begin{array}{r} -\ 64 \\ -\ 96 \\ \hline -\ 160 \end{array} \quad \Rightarrow \quad \begin{array}{r} 11000000 \\ +\ 10100000 \\ \hline 101100000 \end{array}$$

if we discard the carryout, then the result becomes a positive number 0110000 (96); that is, $-64 - 96 = +96$! Wrong!

Overflow

- The above problems (called **overflow**) are due to that using 8 bits, we can represent only $-128 \sim +127$! So the results of $64 + 96$ or $-64 - 96$ cannot be represented in a 8-bit system.
- In general, when adding two *n*-bits (including the sign bit) numbers X and Y, **overflow** occurs when:
  - $X, Y \geq 0$ and $X + Y > 2^{n-1} - 1$.
  - $X, Y < 0$ and $X + Y < -2^{n-1}$.
- Note that overflow cannot occur if one of X and Y is positive and the other is negative.

Overflow

- Overflow can be detected by examining the most significant bit of the result.

+ 64	01000000	(positive)
+ 96	+ 01100000	(positive)
+ 160	10100000	(negative)

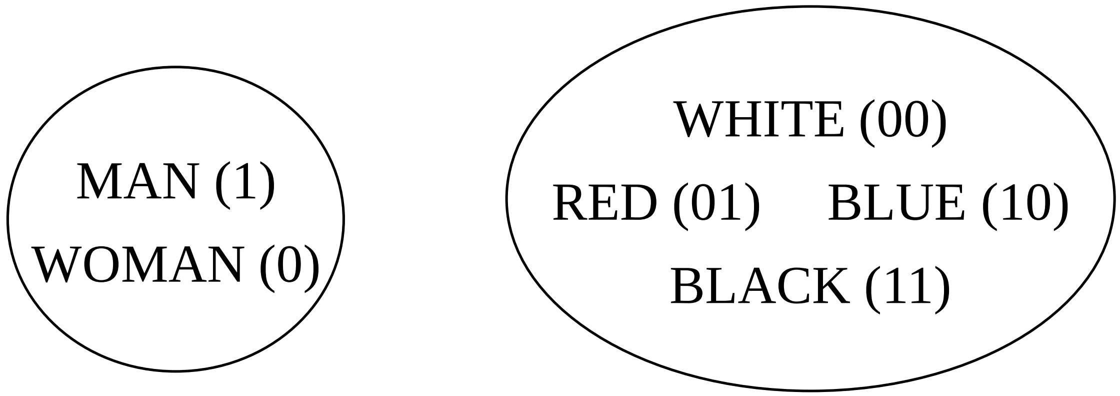
indicate a negative number

− 64	11000000	(negative)
− 96	+ 10100000	(negative)
− 160	101100000	(positive)

indicate a positive number

Binary codes

- Binary codes can be established for any set of discrete elements.



- Using n bits, we can represent at most 2^n distinct elements.
- So, to represent m distinct objects, we need at least $\lceil \log_2 m \rceil$ bits.
  - For example, we need $\lceil \log_2 10 \rceil = 4$ bits to represent $\{0,1,\dots,9\}$.

Alphanumeric code

- **ASCII** (American Standard Code for Information Interchange)
  - Originally using 7 bits to code 128 characters (32 are non-printing)
  - Because most digital systems handle 8-bit (byte) more efficiently, an 8 bit version ASCII has also been developed.

95 printable ASCII characters, numbered 32 to 126.

Error-detecting code

- Binary code that can detect errors during data transmission.
- The most common way to achieve error-detection is by means of a **parity bit**.
- A parity bit is an extra bit included in a binary code to make the total number of 1's transmitted either odd **(odd parity)** or **(even parity)**.

Odd parity	
message	Parity bit
0010	0
0110	1
1110	0
1010	1

Even parity	
message	Parity bit
0010	1
0110	0
1110	1
1010	0

Application of parity bit