Coupling exponents of three-state Potts model on two dimensional square lattice

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The key idea behind studying critical behavior of physical systems with phase transition is determining their critical exponents respective to the coupling parameters, both numerically and theoretically. In a numerical aspect, methods like Finite Size Scaling and Monte Carlo Renormalization Group is utilized to characterize those universal parameters while we can also apply Real-Space Renormalization Group and Conformal Field Theory to directly calculate the exponents. Here we explicitly derive a critical exponent for the three-state Potts Model in two dimensional square lattice, and compare the numerical and theoretical results with the known value. We concluded that both Finite Size Scaling and Real-Space Renormalization Group shows an accurate results within a reasonable margin.

Keywords: Renormalization Group, 3-state Potts model, Critical Exponents, Universality Class

I. 3-STATE 2D POTTS MODEL

As a generalization of the Ising model, the Potts model named after Renfrey Potts, is a physical system of interacting spins on a crystalline lattice which shows phase transition with ferromagnetic phase. It can be generalized into several other important models in statistical physics, with examples being, but not limited to, the XY model, the Heisenberg Model, and the N-vector model.

The Potts model consists of discrete spins that are placed on a lattice. In this study, we will only consider two-dimensional square Euclidean lattice with continuous boundary condition.

The classical Hamiltonian for this model can be written by a combination of coupling parameters and spins.

$$\mathcal{H} = -\mathcal{K}\sum_{\langle i,j\rangle}\delta(s_i, s_j) - h\sum_i \delta(s_i, s_{ghost}) \tag{1}$$

While s_i is a spin respective to the location i on lattice with value from 0 to $q-1$ for q-state Potts model. $\mathcal{K}$ and h represents the coupling parameters in association with the spin and external field, respectively. In this paper, we will assume s_{ghost}; a spin interacting with an external field, to be 0 without losing generality.

As a result, the partition function and bulk free energy density for such system is given as

$$\mathcal{Z} = \sum_{\{s_i\}}\exp\left[\mathcal{K}\sum_{\langle i,j\rangle}\delta(s_i, s_j) + h\sum_i \delta(s_i, 0)\right] \tag{2}$$

$$f_{bulk} = f := -\lim_{V\to\infty}\left[\frac{\ln \mathcal{Z}}{V}\right] \tag{3}$$

It is known that the q-state Potts model on 2D square lattice has a critical fixed point on the following basis.

$$(\mathcal{K}_c, h_c) = (1/T_c, h_c) = (\ln[1+\sqrt{q}], 0) \tag{4}$$

Eq. (4) can be easily proven based on a duality relation between two coupling parameter $\mathcal{K}$ and $\mathcal{K}^*$ on 2D square lattice: $\mathcal{K} = \mathcal{K}^* = \mathcal{K}_c$ is satisfied on a critical fixed point

$$e^{-\mathcal{K}^*} = \frac{e^{\mathcal{K}} - 1}{e^{\mathcal{K}} + q - 1} \tag{5}$$

Note that the standard ferromagnetic Potts model in 2d shows continuous phase transition for $1 \le q \le 4$ and first-order phase transition for $q > 4$. In this research, we will only consider cases where q is 3.

Following the definition of Hamiltonian in Eq. (1), we determine the order parameter m to be

$$m = \frac{q\rho - 1}{q-1}, \text{ where } \rho = \frac{1}{V}\sum_i \delta(s_i, 0) \tag{6}$$

and correlation function $G(i,j)$ and correlation length $\xi(T)$ of the system to be

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$$G(i,j) = \left\langle \frac{q\delta(s_i,s_j)-1}{q-1} \right\rangle - \left\langle \frac{q\delta(s_i,0)-1}{q-1} \right\rangle \left\langle \frac{q\delta(s_j,0)-1}{q-1} \right\rangle \propto \frac{1}{|i-j|^{d-2+\eta}} \exp\left[-\frac{|i-j|}{\xi(T)}\right] \tag{7}$$

It is straight forward to prove that correlation function is 0 for both completely ordered and dis-ordered phases.

II. CRITICAL EXPONENTS AND SCALING RELATIONS

The basic approach to quantitatively analyzing the system in statistical physics is to follow the variation of physical quantities as we increase the length scale. Such approach is known as coarse graining and rescaling, which allows us to systematically take into account fluctuations near the critical point by tracing out short-range fluctuations and shift our attention to longer-length behavior of the system.

The basis of coarse graining and rescaling analysis is the rule that the partition function is invariant.

$$\mathcal{Z}(\mathcal{H}) = \sum_{s'}\sum_{s} e^{-\mathcal{H}} = \sum_{s'} e^{-\mathcal{H}'} = \mathcal{Z}'\left(\mathcal{H}'\right) \tag{8}$$

$$\text{where } e^{-\mathcal{H}'} = \sum_{s} e^{-\mathcal{H}} \leftrightarrow \mathcal{H}' = -\ln\left[\sum_{s} e^{-\mathcal{H}}\right] \tag{9}$$

Combining Eqs. (3) and (8), the bulk free energy density of the system changes as we rescale the system with scale factor $b > 1$.

$$f(\mathcal{K}) = b^{-d} f(\mathcal{K}') \tag{10}$$

Eq. (10) is known as the fundamental relation in thermodynamics and statistical physics. For $\mathcal{K}$ consisting of nearest-neighbor coupling and external magnetic field, we arrive at

$$f(t,h) = b^{-d} f(b^{y_t}t, b^{y_h}h), \text{ where } t = \frac{T-T_c}{T_c} \tag{11}$$

Where y_t and y_h are the thermal and magnetic coupling exponents, respectively. It is worth noting that Eq. (11) leads to several important scaling relations in critical phenomena.

The degree of singularity of physical quantities near the critical point is described by the critical exponents defined below, note that each exponents represents the asymptotic behavior of each physical quantities as the system reaches criticality.

$$\lim_{t\to 0} C_v \propto |t|^{-\alpha}, \text{ where } C_v := -T\left.\frac{\partial^2 f(t,h)}{\partial t^2}\right|_{h=0} \text{ is a Specific Heat} \tag{12}$$

$$\lim_{t\to 0-} M(h=0) \propto (-t)^{\beta}, \text{ where } M := -\frac{\partial f(t,h)}{\partial h} \text{ is a Order Parameter} \tag{13}$$

$$\lim_{t\to 0} X_T \propto |t|^{-\gamma}, \text{ where } X_T := -\left.\frac{\partial^2 f(t,h)}{\partial h^2}\right|_{h=0} \text{ is a Susceptibility} \tag{14}$$

$$\lim_{h\to 0} M(t=0) \propto |h|^{1/\delta} \tag{15}$$

$$\lim_{t\to 0} \xi(t) \propto |t|^{-\nu} \tag{16}$$

$$\lim_{t\to 0} G(i,j) \propto \frac{1}{|i-j|^{d-2+\eta}}, \text{ where } d \text{ and } \eta \text{ is a dimension and an anomalous dimension, respectively} \tag{17}$$

Combining Eq. (11) with Eqs. (12) to (15), we can determine the critical exponents α, β, γ, and δ as follows:

$$-\alpha y_t + 2y_t - d = 0 \to \alpha = 2 - d/y_t \tag{18}$$

$$\beta y_t + y_h - d = 0 \to \beta = (d - y_h)/y_t \tag{19}$$

$$-\gamma y_t + 2y_h - d = 0 \to \gamma = (2y_h - d)/y_t \tag{20}$$

$$y_h/\delta + y_h - d = 0 \to \delta = y_h/(d - y_h) \tag{21}$$

One can also derive the relations for ν and η by applying fluctuation-dissipation theorem and a trivial scaling relation for correlation length $\xi(t) = b\xi(tb^{y_t})$, please refer to Appendix A for further details regarding the relationship between the correlation function and susceptibility.

$$-\nu y_t + (2y_h - d)/(2-\eta) = 0 \to \nu = 1/y_t \tag{22}$$

$$(2y_h - d)/(2-\eta) = 1 \to \eta = d - 2y_h + 2 \tag{23}$$

III. NUMERICAL METHOD 1: FINITE SIZE SCALING

A. Monte Carlo Simulation and Heat-Bath Method

Monte Carlo simulations are realized as the numerical implementation of stochastic dynamics represented by the master equation, which describes how the set of probabilities evolves with time.

For spin configuration $a = \{s_1, s_2, ...s_N\}$, let us denote the state of a system using a probability that the system has a configuration a at time t, $P(a,t)$. For the ising model, the total number of possible configuration a is 2^N while for q-state Potts model, it is q^N. Suppose that the configuration changes from a to b with the transition probability $w(a \to b)\Delta t$ in a small time interval Δt. The net change of the probability for the system in configuration a should satisfy the following master equation.

$$P(a, t + \Delta t) - P(a, t) = -\sum_{b \neq a} w(a \to b)P(a, t)\Delta t + \sum_{b \neq a} w(b \to a)P(b, t)\Delta t \tag{24}$$

It is critical to choose appropriate transition probabiltiles in Monte Carlo simulations so that the equilibrium distribution is of the Gibbs-Boltzmann form $P(a, t) = e^{-H(a)}/\mathcal{Z} := P_{eq}(a)$. Suppose that an equilibrium has been achieved in the master equation Eq. (24) with $P(a, t) = P_{eq}(a)$. Then, the left-hand side vanishes and consequently

$$\sum_{b \neq a} w(a \to b)P_{eq}(a) = \sum_{b \neq a} w(b \to a)P_{eq}(b) \tag{25}$$

A sufficient condition for the above relation to hold is to equate both sides term by term,

$$w(a \to b)P_{eq}(a) = w(b \to a)P_{eq}(b) \tag{26}$$

$$\frac{w(a \to b)}{w(b \to a)} = e^{-\beta[H(b) - H(a)]} \tag{27}$$

This relation is called the detailed balance condition that the transition probability should satisfy. A common choide of the transition probability that satisfied such condition is a Heat-Bath method described below.

$$w(a \to b) = \frac{e^{-\beta H(b)}}{e^{-\beta H(a)} + e^{-\beta H(b)}} \tag{28}$$

Note that in Monte Carlo simulations one regards the calculation of the expectation value of a physical quantity O as an average over the configurations generated by the stochastic dynamics.

B. Finite Size Scaling

To extract coupling exponents from numerical data, we would have to run simulations where critical phenomena take place in macroscopic system. We can, however, carry out numerical computations only for finite-size systems. Therefore, it is not possible for our physical quantities to show authentic singular behavior since critical phenomena only happens at the thermodynamic limit, i.e. either when system size it infinite or the temperature is zero.

Since the parameter should be carefuly tuned for the system to reach the critical point, we should include the inverse of system size L^{-1} in the argument of free energy density.

$$f(t, h, L^{-1}) = b^{-d} f(b^{y_t} t, b^{y_h} h, bL^{-1}) \tag{29}$$

If we combine Eq. (29) with the definition described in Eq. (12), (13), and (14), we obtain

$$C_v(t, 0, L^{-1}) = b^{2y_t - d} \partial_t^2 f(b^{y_t} t, 0, bL^{-1}) \tag{30}$$

$$m(t, h, L^{-1}) = b^{y_h - d} \partial_h f(b^{y_t} t, b^{y_h} h, bL^{-1}) \tag{31}$$

$$X_T(t, 0, L^{-1}) = b^{2y_h - d} \partial_h^2 f(b^{y_t} t, 0, bL^{-1}) \tag{32}$$

If we insert $b = L$ and rewrite the equation,

$$L^{d-2y_t} C_v(t, L^{-1}) = \Phi_1(L^{y_t} t) \tag{33}$$

$$L^{d-y_h} m(t, L^{-1}) = \Phi_2(L^{y_t} t) \tag{34}$$

$$L^{d-y_h} m(h, L^{-1}) = \Phi_3(L^{y_h} h) \tag{35}$$

$$L^{d-2y_h} X_T(t, L^{-1}) = \Phi_4(L^{y_t} t) \tag{36}$$

Therefore, we can estimate the exact value of coupling exponents by plotting physical quantities on graph with appropriate x and y axis in respective to Eqs. (33) to (36).

The result of the numerical computation is displayed from Fig. 1 to 8, each containing a corresponding physical quantities introduced in Section II. In general, the critical exponents that forms the universality class of 3-state 2D Potts Model coincides with the numerical data generated by Monte Carlo simulations, refer to Table I in Appendix C for the full list of critical exponents.

IV. THEORETICAL METHOD 1: REAL-SPACE RENORMALIZATION GROUP

In this section, we calculate the coupling exponents of 3-state Potts model in 2 dimension by utilizing Block-Spin Transformation technique, a well-known analytic realization in Real-space Renormalization Group. To avoid further complication, we will be applying this technique to the Potts model on triangular lattice since the topological aspect of the lattice, unlike the dimension, has no affect on the value of coupling exponents.

As sdisplayed in Fig. 9, we first group sites on the triangular lattice into triples on triangles and then coarse grain the system b choosing a representative single spin value for each of the triples forming a new triangular lattice with scale factor $b = \sqrt{3}$.

For newly denoted block spin S grouped from σ_1, σ_2, and σ_3, the value are assigned in accordance to the majority rule. For σ spin configuration $\sigma_I = \{\sigma_1\sigma_2\sigma_3\}$ in block lattice I, the block spin S_I that forms the coarsed-grain system is transformed as

$$\{000, 001, 010, 100, 002, 020, 200, (012)\} \to S_I = 0 \quad (37)$$
$$\{111, 110, 101, 011, 112, 121, 211, (012)\} \to S_I = 1 \quad (38)$$
$$\{222, 220, 202, 022, 221, 212, 122, (012)\} \to S_I = 2 \quad (39)$$

where for configuration $\{012\}$, the block spin is assigned randomly with equal probability.

The block-spin S forms a new Hamiltonian H' scaled from the original Hamiltonian H.

$$H = -K \sum_{\langle i,j \rangle} \delta(\sigma_i, \sigma_j) - h \sum_i \delta(\sigma_i, 0) \quad (40)$$

$$H' = -K' \sum_{\langle I,J \rangle} \delta(S_I, S_J) - h' \sum_I \delta(S_I, 0) \quad (41)$$

The coupling exponent y_t and y_h is given as a derivative betweem original and scaled coupling parameter K, h and $K'(K,h),\ h'(K,h)$.

$$b^{y_t} = \left.\frac{\partial K'}{\partial K}\right|_{K_c, h_c} \text{ and } b^{y_h} = \left.\frac{\partial h'}{\partial h}\right|_{K_c, h_c} \quad (42)$$

Therefore, our goal is to calculate the scaled coupling parameter K' and h' from the following relation which is a combination of Eq. (9) and (41).

$$\sum_{\{S\}} e^{K' \sum_{\langle I,J \rangle} \delta(S_I, S_J) + h' \sum_I \delta(S_I, 0)} = \sum_{\{\sigma\}} e^{-H} \quad (43)$$

$$H' = -\ln\left[\sum_{\{\sigma^S\}} e^{K \sum_{\langle i,j \rangle} \delta(\sigma_i, \sigma_j) + h \sum_i \delta(\sigma_i, 0)}\right] \quad (44)$$

where the term $\{\sigma^S\}$ represents the set of spin configuration σ when S is given.

The RHS of Eq. (44) can be elaborated as follows to obtain a useful version of the renormalization group equation that describes the flow.

$$\sum_{\{\sigma^S\}} e^{K \sum_{\langle i,j \rangle} \delta(\sigma_i, \sigma_j) + h \sum_i \delta(\sigma_i, 0)} = \sum_{\{\sigma^S\}} e^{K \sum_{\langle i,j \in I \rangle} \delta(\sigma_i, \sigma_j) + h \sum_i \delta(\sigma_i, 0) + K \sum_{\langle i \in I, j \in J \rangle} \delta(\sigma_i, \sigma_j)} \quad (45)$$

$$= \sum_{\{\sigma^S\}} e^{K \sum_{\langle i,j \in I \rangle} \delta(\sigma_i, \sigma_j) + h \sum_i \delta(\sigma_i, 0)} \times \frac{\sum_{\{\sigma^S\}} e^{K \sum_{\langle i,j \in I \rangle} \delta(\sigma_i, \sigma_j) + h \sum_i \delta(\sigma_i, 0)} \times e^{K \sum_{\langle i \in I, j \in J \rangle} \delta(\sigma_i, \sigma_j)}}{\sum_{\{\sigma^S\}} e^{K \sum_{\langle i,j \in I \rangle} \delta(\sigma_i, \sigma_j) + h \sum_i \delta(\sigma_i, 0)}} \quad (46)$$

$$= \sum_{\{\sigma^S\}} e^{-H_0} \times \frac{\sum_{\{\sigma^S\}} e^{-H_0} \times e^{-V}}{\sum_{\{\sigma^S\}} e^{-H_0}} = \prod_I Z_{\text{block}} \times \left\langle e^{-V} \right\rangle_0 = \prod_I Z_{\text{block}} \times \left\langle e^{K \sum_{\langle i \in I, j \in J \rangle} \delta(\sigma_i, \sigma_j)} \right\rangle_0 \quad (47)$$

$$K' \sum_{\langle I,J \rangle} \delta(S_I, S_J) + h' \sum_I \delta(S_I, 0) = \sum_I \ln\left[Z_{\text{block}}\right] + \ln\left[\left\langle e^{K \sum_{\langle i \in I, j \in J \rangle} \delta(\sigma_i, \sigma_j)} \right\rangle_0\right] \quad (48)$$

$$\approx \sum_I \ln\left[Z_{\text{block}}\right] + \left\langle K \sum_{\langle i \in I, j \in J \rangle} \delta(\sigma_i, \sigma_j) \right\rangle_0 + \cdots \approx \sum_I \ln\left[Z_{\text{block}}\right] + K \sum_{\langle I,J \rangle} \left\langle \delta(\sigma_{I1}, \sigma_{J2}) + \delta(\sigma_{I3}, \sigma_{J2}) \right\rangle_0 \quad (49)$$

$$\approx \sum_I \ln\left[Z_{\text{block}}\right] + 2K \sum_{\langle I,J \rangle} \left\langle \delta(\sigma_{I1}, \sigma_{J2}) \right\rangle_0 \quad (50)$$

$$\approx \sum_I \ln\left[Z_{\text{block}}\right] + 2K \sum_{\langle I,J \rangle} \Big[\left\langle \delta(\sigma_{I1}, 0) \right\rangle_0 \left\langle \delta(\sigma_{J2}, 0) \right\rangle_0 + \left\langle \delta(\sigma_{I1}, 1) \right\rangle_0 \left\langle \delta(\sigma_{J2}, 1) \right\rangle_0 + \left\langle \delta(\sigma_{I1}, 2) \right\rangle_0 \left\langle \delta(\sigma_{J2}, 2) \right\rangle_0 \Big] \quad (51)$$

While $\langle\cdots\rangle_0$ is the expectation value with respect to the weight of internal block Hamiltonian H_0. Fig 10 shows the inter-block relation between block indices $I = 1$ and $J = 2$, note that in triangular lattice, there are 6 nearest-neighbor indices after block-spin transformation.

If we denote $Z_{\text{block}} = \exp[C_1 + D\delta(S_I, 0)]$ and $\langle\delta(\sigma_{I1}, \sigma_{J2})\rangle_0 = A\delta(S_I, S_J) + B\delta(S_I, 0) + B\delta(S_J, 0) + C_2$, the renormalization group equation is written as

$$K' = 2KA(K, h) \tag{52}$$

$$h' = D(K, h) + 6 \times 2KB(K, h) \tag{53}$$

Explicit calculations through majority rule and rescaling in triangular lattice reveals the following result.

$$A(K, h) = \left(\frac{e^{3K} + 2e^{K+h} + e^{K}}{e^{3K} + 3e^{K+h} + 3e^{K} + 2e^{h}}\right)^2 \tag{54}$$

$$B(K, h) = \frac{1}{2} \times \left[\frac{\left(e^{3K+3h} + 4e^{K+2h} + 2/3e^{h}\right)^2 + 2\left(e^{K+2h} + 2/3e^{h}\right)^2}{\left(e^{3K+3h} + 6e^{K+2h} + 2e^{h}\right)^2} - \frac{\left(e^{3K} + 2e^{K+h} + 2e^{K} + 2/3e^{h}\right)^2 + \left(e^{K+h} + 2/3e^{h}\right)^2 + \left(e^{K} + 2/3e^{h}\right)^2}{\left(e^{3K} + 3e^{K+h} + 3e^{K} + 2e^{h}\right)^2}\right] \tag{55}$$

$$D(K, h) = \ln\left[\frac{e^{3K+3h} + 6e^{K+2h} + 2e^{h}}{e^{3K} + 3e^{K+h} + 3e^{K} + 2e^{h}}\right] \tag{56}$$

One can determine the critical fixed point with infinite correlation length by applying $K = K' = K_c$ and $h = h' = h_c$ in Eq. (52) and (53).

$$(K_c, h_c) = (0.911, 0) \tag{57}$$

As a result, the coupling parameter for temperature and external magnetic field is

$$y_t = \ln\left[\left.\frac{\partial K'}{\partial K}\right|_{(0.911,\ 0)}\right] / \ln\sqrt{3} = 1.102 \tag{58}$$

$$y_h = \ln\left[\left.\frac{\partial h'}{\partial h}\right|_{(0.911,\ 0)}\right] / \ln\sqrt{3} = 2(?) \tag{59}$$

which is pretty close to the exact value of $(6/5,\ 28/15)$ for 3-state Potts model in 2 dimension.

V. NUMERICAL METHOD 2: MONTE CARLO RENORMALIZATION GROUP

Monte Carlo Renormalization Group (MCRG) is a numerical method used to study the behavior of statistical systems in physics and related fields. It is based on the idea of the Renormalization Group, but instead of analytically solving flow equations, MCRG uses computer simulations to generate a sequence of approximate solutions for the system under consideration.

In MCRG, the system is treated as a set of discrete variables, and the values of these variables are updated using a random process based on the Metropolis algorithm. The updates are performed repeatedly, and the resulting configurations are used to calculate various properties of the system. Over time, the RG transformation is carried out by changing the scale at which the system is studied, leading to the creation of a block-spin representation that summarizes the behavior of the system at different scales.

MCRG is commonly used in computational physics and is considered a powerful tool for studying complex systems, especially those that exhibit phase transitions or critical phenomena.

TBA

VI. THEORETICAL METHOD 2: CONFORMAL FIELD THEORY

Conformal field theory (CFT) is a type of field theory that is characterized by the presence of conformal symmetry, which is a set of transformations that preserve angles and ratios of distances in a two-dimensional space. Conformal symmetries are important in many areas of physics and mathematics, including string theory, condensed matter physics, and statistical mechanics.

The Virasoro algebra is a Lie algebra that plays a central role in conformal field theory. It is a symmetry algebra of the conformal group, which is the group of transformations that preserve conformal symmetry. The Virasoro algebra is generated by a set of infinite-dimensional operators, known as Virasoro generators, which encode the symmetries of the theory and describe the behavior of the theory under conformal transformations.

The Virasoro algebra is used to classify and study the symmetries of conformal field theories, and it is an important tool in the study of critical phenomena, such as phase transitions. In particular, it provides a way to study the behavior of two-dimensional field theories under scale transformations, which are important in understanding the critical behavior of systems near a phase transition.

The representation of Virasoro Algebra describes the basis of conformal field theory in 2 dimension. For co-prime integers $p, q \geq 2$ and $r = 1 \sim q-1, s = 1 \sim p-1$,

$$c_{p,q} = 1 - \frac{6(p-q)^2}{pq} \tag{60}$$

$$h_{r,s} = \frac{(pr-qs)^2 - (p-q)^2}{4pq} \tag{61}$$

where $c_{p,q}$ and $h_{r,s}$ is a central charge and conformal weight of the system, respectively. It is worth noting that the total number of primary fields is $(p-1)(q-1)/2$.

The representation is unitary if and only if $|p-q| = 1$, e.g. $p = m+1$ and $q = m$.

$$c = 1 - \frac{6}{m(m+1)} \text{ for m=3,4,5}\cdots \tag{62}$$

$$h_{r,s} = \frac{\left[(m+1)r - ms\right]^2 - 1}{4m(m+1)} \tag{63}$$

The critical ising model in 2 dimension is represented by $m = 3$ while the conformal symmetry of critical 3-state Potts model in 2 dimension is described by the unitary representation of Virasoro Algebra with $m = 5$. One can calculate the temperature and magnetic coupling exponents from their respective conformal weights associated with the primary fields.

$$y_t = 2 - 2h_{2,1} = 2 - \frac{4}{5} = \frac{6}{5} \tag{64}$$

$$y_h = 2 - 2h_{2,3} = 2 - \frac{2}{15} = \frac{28}{15} \tag{65}$$

TBA

VII. CONCLUSIONS

Conclusion.

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The code that generates data for Finite Size Scaling in Section III in this paper is available at https://github.com/LEE-SungBin/potts.git.

Appendix A: Fluctuation-dissipation theorem

Consider an arbitrary system whose Hamiltonian $H(0)$ is modified because of the presence of an external inhomogeneous field $B(r)$ as

$$H = H_0 - \int drO(r)B(r) \tag{A1}$$

where $O(r)$ is the system variable that linearly couples to the external field. The free energy $F = -\beta^{-1}\ln Z$ in terms of the partition function

$$Z = \mathrm{Tr}\left[\exp\left(-\beta H_0 + \beta \int drO(r)B(r)\right)\right] \tag{A2}$$

The generalized isothermal susceptibility is defined as follows:

$$\chi(r, r') := -\frac{\partial^2 F}{\partial B(r)\partial B(r')} \tag{A3}$$

as the second-order functional derivative of the free energy with the result

$$\chi(r, r') = \beta^{-1}\left[\frac{1}{Z}\frac{\partial^2 Z}{\partial B(r)\partial B(r')} - \frac{1}{Z}\frac{\partial Z}{\partial B(r)}\frac{1}{Z}\frac{\partial Z}{\partial B(r')}\right] \tag{A4}$$

Therefore, the generalized isothermal susceptibility is proportional to the correlation function between two corresponding points.

$$\begin{aligned}\chi(r, r') &= \beta G(r, r') \\ &= \beta\left[\langle O(r)O(r')\rangle - \langle O(r)\rangle\langle O(r')\rangle\right]\end{aligned} \tag{A5}$$

If the system is translation invariant, we can sum over the both hand side to get the fluctuation-dissipation theorem.

$$\chi = \int dr\chi(r) = \beta\int drG(r) \tag{A6}$$

Appendix B: Real-Space Renormalization Group for Ising Model on two dimensional square lattice

Real-space Renormalization Group for Ising Model on two dimensional square lattice is more or less similiar to that for the 3-state Potts model. The block-spin transformation satisfies the following majority principle for spin $\sigma = \pm 1$.

$$\{++++, \overline{+++-}, \widehat{++--}, \widehat{+-+-}\} \to S_I = + \quad \text{(B1)}$$
$$\{----, \overline{---+}, \widehat{--++}, \widehat{-+-+}\} \to S_I = - \quad \text{(B2)}$$

The renormalization group equation for ising model is given as

$$K' \sum_{\langle I,J \rangle} S_I S_J + h' \sum_I S_I \approx \sum_I \ln\left[Z_{\text{block}}\right] + 2K \sum_{\langle I,J \rangle} \langle \sigma_{I4} \rangle_0 \langle \sigma_{J1} \rangle_0 \quad \text{(B3)}$$

If we set $Z_{\text{block}} = \exp[C + DS_I]$ and $\langle \sigma_{I4} \rangle_0 = A + BS_I$, The renormalization equation is simplified as follows:

$$K' = 2KB(K,h)^2 \quad \text{(B4)}$$
$$h' = D(K,h) + 4 \times 2KA(K,h)B(K,h) \quad \text{(B5)}$$

After some calculation, it is straight forward to prove

$$A(K,h) = \frac{1}{2}\left[\frac{e^{4K+4h} + 2e^{2h}}{e^{4K+4h} + 4e^{2h} + 2 + e^{-4K}} - \frac{e^{4K-4h} + 2e^{-2h}}{e^{4K-4h} + 4e^{-2h} + 2 + e^{-4K}}\right] \quad \text{(B6)}$$

$$B(K,h) = \frac{1}{2}\left[\frac{e^{4K+4h} + 2e^{2h}}{e^{4K+4h} + 4e^{2h} + 2 + e^{-4K}} + \frac{e^{4K-4h} + 2e^{-2h}}{e^{4K-4h} + 4e^{-2h} + 2 + e^{-4K}}\right] \quad \text{(B7)}$$

$$D(K,h) = \ln\left[\frac{e^{4K+4h} + 4e^{2h} + 2 + e^{-4K}}{e^{4K-4h} + 4e^{-2h} + 2 + e^{-4K}}\right] \quad \text{(B8)}$$

Therefore, we can theoretically derive the critical fixed point and temperature and magnetix exponent of the ising model in 2 dimension.

$$(K_c, h_c) = (0.5, 0) \quad \text{(B9)}$$

Note that the exact critical temperature of ising model in 2 dimensional square lattice is $K_c = \frac{\ln[1+\sqrt{2}]}{2} = 0.441$.

$$y_t = \ln\left[\left.\frac{\partial K'}{\partial K}\right|_{(0.5,0)}\right] / \ln 2 = \quad \text{(B10)}$$

$$y_h = \ln\left[\left.\frac{\partial h'}{\partial h}\right|_{(0.5,0)}\right] / \ln 2 = \quad \text{(B11)}$$

which is pretty close to the exact value of (1, 15/8) for Ising model in 2 dimension.

Appendix C: Universality Class

Universality Class is a set of physical system that shares an identical scale invariant limit under the process of renormalization group flow. While the model may seem to evolve in time differently in most situations, their behavior will converge to a certain classification as they reach the scale limit.

The most prominent examples of the universality class is the critical exponents of the system which describes the asymptotic phenomena of the model as they approach the phase transition point. Table I and II shows the full list of critical exponents and critical coupling exponents for several well-known physical system that exhibit critical phenomena or phase transitions.

TABLE I. A full list of critical exponents of various physical systems that shows phase transition. Each critical exponents describes the asymptotic behavior of the physical quantities as the system approaches phase transition.

Class	d	α	β	γ	δ	ν	η
Ising	2	0	1/8	7/4	15	1	1/4
Ising	3	0.11	0.33	1.24	4.79	0.63	0.04
3-state Potts	2	1/3	1/9	13/9	14	5/6	4/15
4-state Potts	2	2/3	1/12	7/6	15	2/3	1/4
XY	3	-0.02	0.35	1.32	4.78	0.67	0.04
Heisenberg	3	-0.12	0.37	1.40	4.78	0.71	0.04
Mean Field		0	1/2	1	3	1/2	0

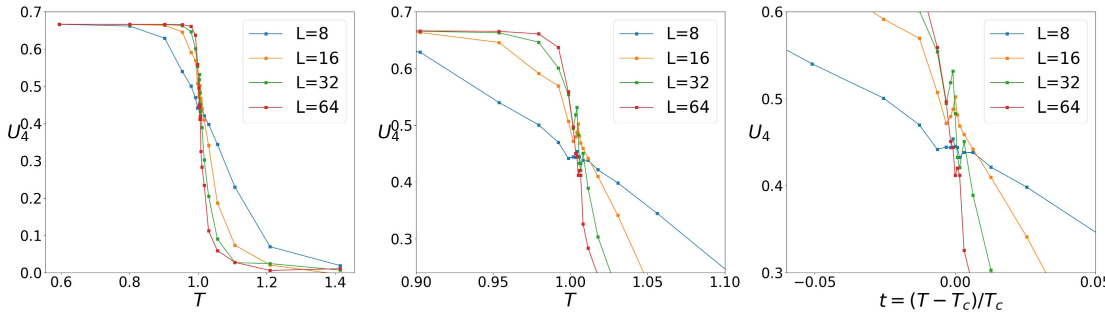

FIG. 1. Binder Cumulant U_4 of the system at Size = 8, 16, 32, and 64. The T_c where the cumulant intersect is 1.005.

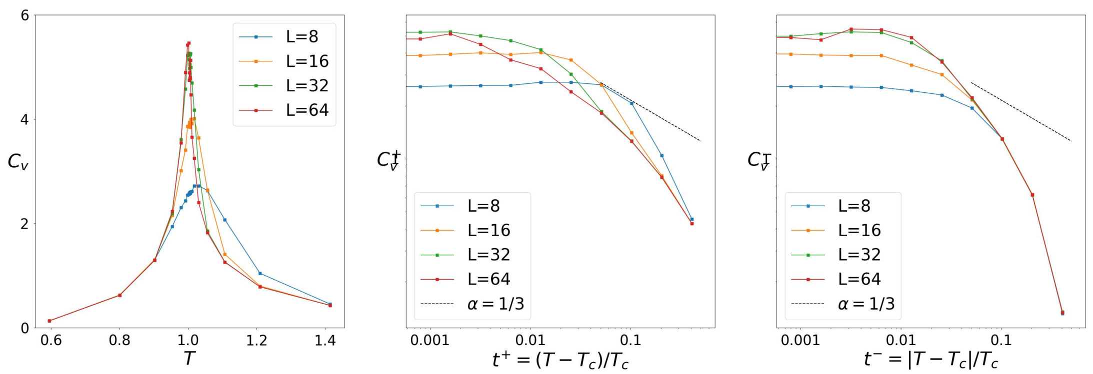

FIG. 2. Specific Heat C_v of the system at Size = 8, 16, 32, and 64. $\alpha = 1/3$

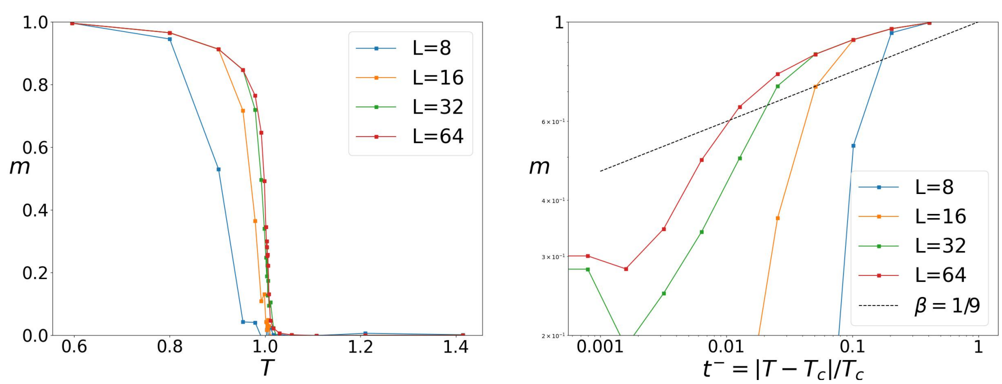

FIG. 3. Order Parameter m of the system with no external field at Size = 8, 16, 32, and 64. $\beta = 1/9$

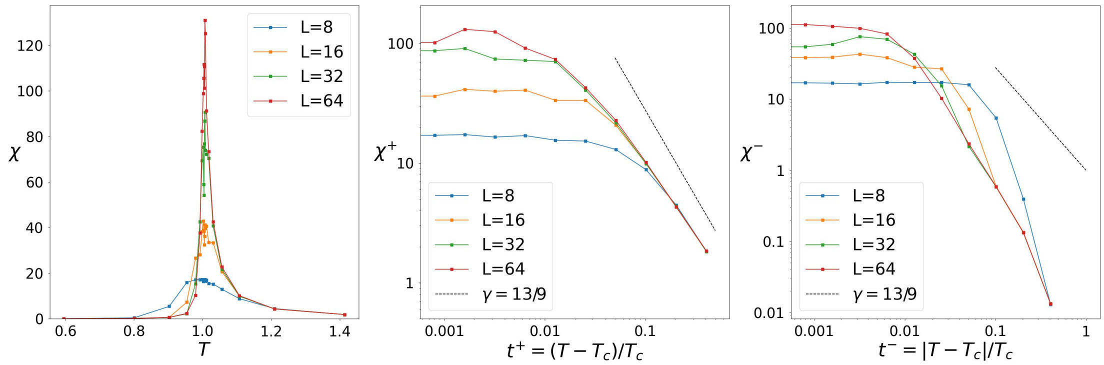

FIG. 4. Susceptibility X_T of the system at Size = 8, 16, 32, and 64. $\gamma = 13/9$

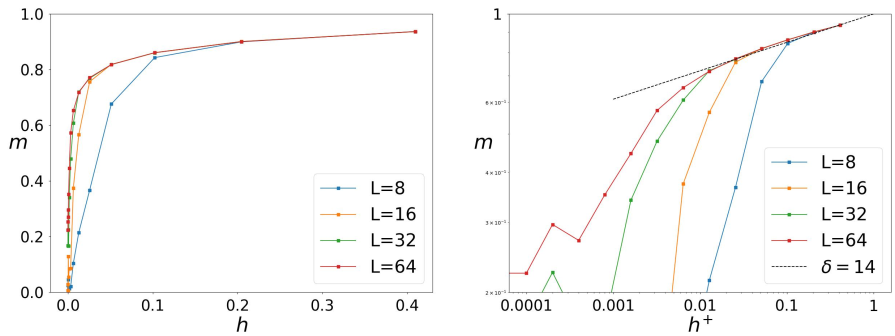

FIG. 5. Order Parameter m of the system at $t = 0$ at Size = 8, 16, 32, and 64. $\delta = 14$

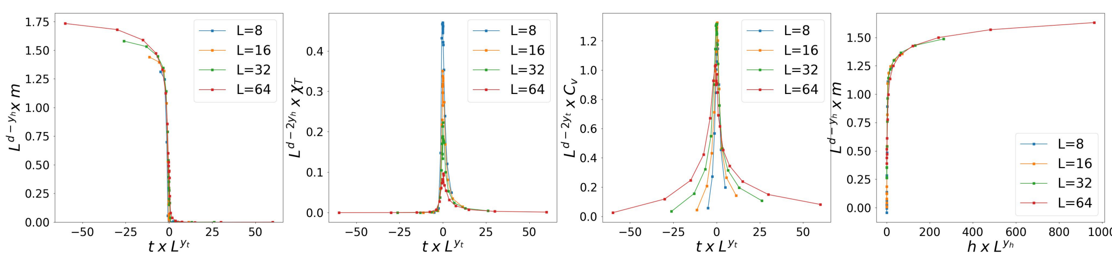

FIG. 6. Finite Size Scaling of the system at Size = 8, 16, 32, and 64. $y_t = 6/5$ and $y_h = 28/15$

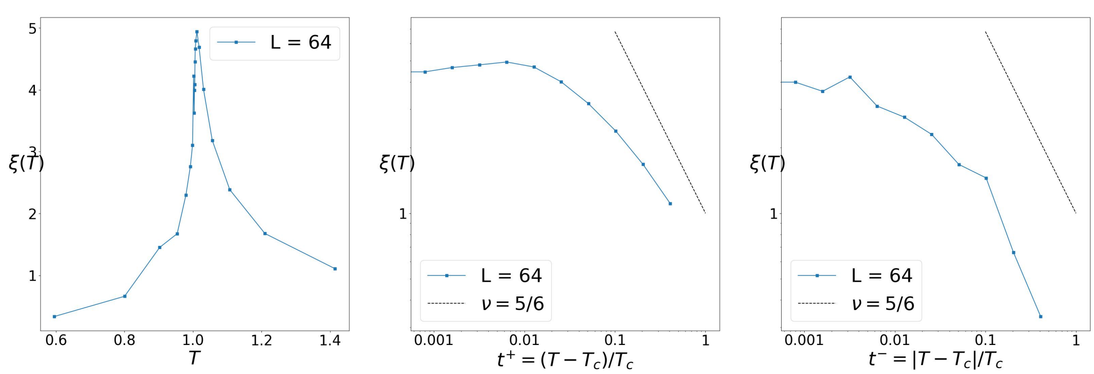

FIG. 7. Correlation Length ξ of the system at Size = 64. $\nu = 5/6$

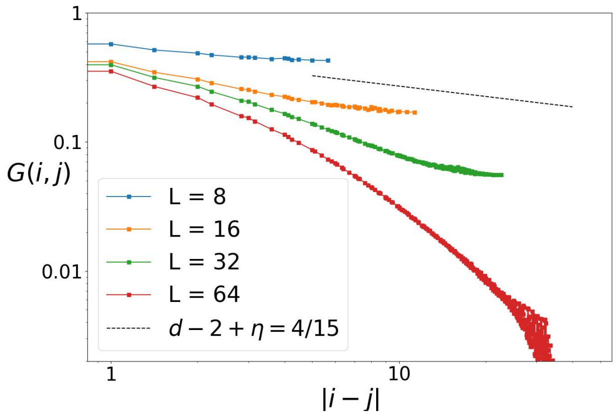

FIG. 8. Correlation Function $G(i, j)$ of the system at critical fixed point at Size = 8, 16, 32, and 64. $\eta = 4/15$ since $d = 2$

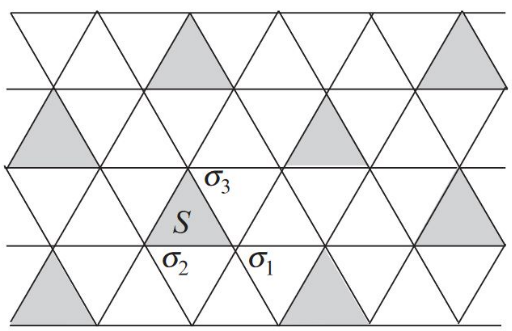

FIG. 9. Three spins on a shaded triangle σ_1, σ_2, and σ_3 are grouped into a scaled block spin S with value determined by the majority rule.

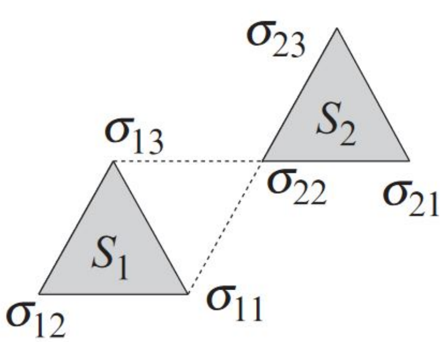

FIG. 10. Inter-block relation between block indices $I = 1$ and $J = 2$. It is worth mentioning that in triangular lattice, there are 6 nearest-neighbor indices after block-spin transformation.

TABLE II. A full list of critical temperature and external field exponents of various physical system that shows phase transition. It is worth noting that every critical exponents which describes the asymptotic behavior of physical system near critical point can be calculated from the coupling parameter via scaling relations.

Class	d	y_t	y_h
Ising	2	1	15/8
Ising	3	1.59	2.48
3-state Potts	2	6/5	28/15
4-state Potts	2	3/2	15/8
XY	3	1.49	2.48
Heisenberg	3	1.41	2.48