# Expansions for ground state in the transverse field Edwards-Anderson model

C. Itoi<sup>1</sup>, K. Horie<sup>1</sup>, H. Shimajiri<sup>1</sup> and Y. Sakamoto<sup>2</sup>, <sup>1</sup>Department of Physics, GS & CST, Nihon University, Kandasurugadai, Chiyoda, Tokyo 101-8308, Japan <sup>2</sup> Laboratory of Physics, CST, Nihon University, Narashinodai, Funabashi-city, Chiba 274-8501, Japan

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#### Abstract

It is proven that the ground state in the transverse field Edwards-Anderson model is unique for weak transverse fields or weak exchange interactions in a d-dimensional finite cubic lattice. Recently proven uniqueness property of the ground state in the Edwards-Anderson model and the Kirkwood-Thomas expansion method developed by Datta-Kennedy enable us to show that this property of ground state in the Edwards-Anderson model is preserved against a perturbation by sufficiently weak site-dependent transverse fields. This convergent expansion method shows also that the uniqueness of the ground state in the free spin model under site-dependent transverse fields is preserved against a perturbation by sufficiently weak bond-dependent exchange interactions.

### 1 Introduction

The transverse Ising model with random exchange interactions is a simple interesting quantum spin model. Many researchers of information, mathematics and physics have been studied this model extensively, since D-Wave Systems actually devised and produced a quantum annealer based on this model [7], which is theoretically proposed by Finnila-Gomez-Sebenik-Stenson-Dol [5] and Kadowaki-Nishimori [8]. On the other hand, the Edwards-Anderson (EA) model is a well-known Ising model with short range random exchange interactions [3]. Quite recently, it is proven that the ground state in the Edwards-Anderson (EA) model is unique in a finite cubic lattice for almost all continuous random exchange interactions under a boundary condition which breaks the global  $\mathbb{Z}_2$  symmetry [6]. Uniqueness of the ground state in the EA model at zero temperature agrees with the claims of Fisher- Huse and Newman-Stein [4, 10].

In the present paper, it is proven that the uniqueness of ground state is preserved also in the transverse field Edwards-Anderson model with weak site-depending transverse fields or weak bond-depending exchange interactions. The obtained uniqueness theorem for the ground state without transverse field gives a convergent perturbative expansion for sufficiently weak transverse fields. A convergent perturbative expansion proposed by Kirkwood and Thomas [9] and developed by Datta and Kennedy [1, 2] enables us to obtain a rigorous result for the unique ground state under weak transverse fields. Also for sufficiently weak exchange interactions in this model, another convergent perturbative expansion shows the uniqueness theorem for the ground state. The expansion method given by Kirkwood and Thomas can be applied only to a restricted class of systems whose Hamiltonians satisfy the Perron-Frobenius condition. Datta and Kennedy have improved the Kirkwood-Thomas method by removing this condition. Finally, we remark that the Perron-Frobenius theorem is applicable to the transverse field EA model.

# 2 Definitions and main theorem

Consider d-dimensional hyper cubic lattice  $\Lambda_L = \mathbb{Z}^d \cap (-L/2, L/2]^d$  with an even integer L > 0. Note that the lattice  $\Lambda_L$  containts  $L^d$  sites. Define a set of nearest neighbor bonds by

$$B_{\Lambda} = \{\{i, j\} | i, j \in \Lambda_L, |i - j| = 1\}.$$

Note  $|B_{\Lambda}| = |\Lambda_L|d$ . A bond spin  $\sigma_b$  denotes

$$\sigma_b^a = \sigma_i^a \sigma_i^a$$

for a bond  $b = \{i, j\} \in B_{\Lambda}$  and a = x, y, z. Let  $\mathbf{J} := (J_b)_{b \in B_{\Lambda}}$  and  $\mathbf{h} := (h_i)_{i \in \Lambda'_L}$  be sequences of arbitrary real numbers, where  $\Lambda'_L := \Lambda_L \setminus (\Lambda_L \setminus \Lambda_{L-2})$ . Although these numbers can be random variables, here we do not assume it. The Hamiltonian of this model is given by

$$H_{\Lambda}(\sigma, \boldsymbol{h}, \boldsymbol{J}) = -\sum_{b \in B_{\Lambda}} J_b \sigma_b^z - \sum_{i \in \Lambda_L'} h_i \sigma_i^x, \tag{1}$$

with a set of sequences (h, J). This Hamiltonian is invariant under a discrete SU(2) transformation  $U := \exp[i\pi \sum_{i \in \Lambda_L} \sigma_i^x/2]$  acting on each spin  $\sigma_i^z \mapsto U^{\dagger} \sigma_i^z U = -\sigma_i^z$ . This corresponds to  $\mathbb{Z}_2$  symmetry in the Edwards-Anderson model. Define a state  $|\sigma\rangle$  with a spin configuration  $\sigma \in \Sigma_{\Lambda} := \{1, -1\}^{\Lambda_L}$  by

$$\sigma_i^z |\sigma\rangle = \sigma_i |\sigma\rangle.$$

For any  $\beta > 0$ , the partition function as a function of  $(\beta, h, J)$  is defined by

$$Z_{\Lambda}(\beta, \boldsymbol{h}, \boldsymbol{J}) = \operatorname{Tr} e^{-\beta H_{\Lambda}(\sigma, \boldsymbol{h}, \boldsymbol{J})} = \sum_{\sigma \in \{1, -1\}^{\Lambda_L}} \langle \sigma | e^{-\beta H_{\Lambda}(\sigma, \boldsymbol{h}, \boldsymbol{J})} | \sigma \rangle, \tag{2}$$

The average of an arbitrary function  $f: \Sigma_{\Lambda} \to \mathbb{R}$  of the spin configuration in the Gibbs state is given by

$$\langle f(\sigma) \rangle = \frac{1}{Z_{\Lambda}(\beta, \boldsymbol{h}, \boldsymbol{J})} \text{Tr} f(\sigma) e^{-\beta H_{\Lambda}(\sigma, \boldsymbol{h}, \boldsymbol{J})}.$$

Note that the expectation  $\langle \sigma_i \rangle$  of spin at each site *i* vanishes in the  $\mathbb{Z}_2$  symmetric Gibbs state. To remove the trivial two-fold degeneracy due to the global  $\mathbb{Z}_2$  symmetry, assume + boundary condition, such that

$$\sigma_i = 1, \tag{3}$$

for  $i \in \Lambda_L \setminus \Lambda_{L-2}$ .  $\Sigma_{\Lambda}^+ \subset \{1, -1\}^{\Lambda_L}$  denotes the set of spin configurations satisfying the boundary condition (3). We provide two theorems in convergent perturbative expansions around h = 0 and J = 0.

**Theorem 1.** The ground state of the transverse field EA model defined by the Hamiltonian (1) is unique for sufficiently small  $h := \sup_{i \in \Lambda'_r} |h_i|$ .

**Theorem 2.** The ground state of the transverse field EA model defined by the Hamiltonian (1) is unique for sufficiently small  $J := \sup_{b \in B_{\Lambda}} |J_b|$ .

## 3 Proof of Theorem 1

#### 3.1 Correlation functions at zero temperature

Define the Duhamel function (A, B) between linear operators A and B by

$$(A,B) := \int_0^1 dt \langle e^{\beta t H} A e^{-\beta t H} B \rangle.$$

**Lemma 1.** Consider the transverse field EA model. Let  $f(\sigma)$  be an arbitrary uniformly bounded function of spin operators  $(\sigma_i^a)_{i\in\Lambda_L}$ , such that  $||f(\sigma)|| \leq C$ . For any bond  $b\in B_\Lambda$  and for almost all  $J_b$ , the infinite volume limit and the zero temperature limit of the connected correlation function vanishes

$$\lim_{\beta \to \infty} [(\sigma_b^z, f(\sigma)) - \langle \sigma_b^z \rangle \langle f(\sigma) \rangle] = 0.$$
 (4)

Another connected correlation function vanishes

$$\lim_{\beta \to \infty} [(\sigma_i^x, f(\sigma)) - \langle \sigma_i^x \rangle \langle f(\sigma) \rangle] = 0, \tag{5}$$

for almost all  $h_i$  also for any site  $i \in \Lambda'_L$ .

**Proof.** The derivative of one point function is represented in terms of the following Duhamel function

$$\frac{1}{\beta} \frac{\partial}{\partial J_b} \langle f(\sigma) \rangle = (\sigma_b^z, f(\sigma)) - \langle \sigma_b^z \rangle \langle f(\sigma) \rangle. \tag{6}$$

The integration over an arbitrary interval  $(J_1, J_2)$  is

$$\frac{1}{\beta} [\langle f(\sigma) \rangle_{J_2} - \langle f(\sigma) \rangle_{J_1}] = \int_{J_1}^{J_2} dJ_b [(\sigma_b^z, f(\sigma)) - \langle \sigma_b^z \rangle \langle f(\sigma) \rangle].$$

Uniform bounds  $||f(\sigma)|| \leq C$  in the left hand side,  $-2C \leq |(\sigma_b^z, f(\sigma)) - \langle \sigma_b^z \rangle \langle f(\sigma) \rangle| \leq 2C$  on the integrand in the right hand side, and the dominated convergence theorem imply the following commutativity between the limiting procedure and the integration

$$0 = \lim_{\beta \to \infty} \int_{J_1}^{J_2} dJ_b[(\sigma_b^z, f(\sigma)) - \langle \sigma_b^z \rangle \langle f(\sigma) \rangle]$$
 (7)

$$= \int_{J_1}^{J_2} dJ_b \lim_{\beta \to \infty} [(\sigma_b^z, f(\sigma)) - \langle \sigma_b^z \rangle \langle f(\sigma) \rangle]. \tag{8}$$

Since the integration interval  $(J_1, J_2)$  is arbitrary, the following limit vanishes

$$\lim_{\beta \to \infty} [(\sigma_b^z, f(\sigma)) - \langle \sigma_b^z \rangle \langle f(\sigma) \rangle] = 0, \tag{9}$$

for any  $b \in B_{\Lambda}$  for almost all  $J_b \in \mathbb{R}$ . Another identity (5) can be proven as in the same way.  $\square$ 

## 3.2 Unperturbed system

The following lemma guarantees the uniqueness of ground state in the case for h = 0, where the model defined by the Hamiltonian (1) becomes the classical EA model. This has been obtained in [6].

**Lemma 2.** Consider the transverse field Edwards-Anderson (EA) model at  $\mathbf{h} = \mathbf{0}$  in d-dimensional hyper cubic lattice  $\Lambda_L$  under the boundary condition (3). Let  $f(\sigma^z)$  be a real valued function of a spin operators. For almost all  $\mathbf{J}$ , there exists a unique spin configuration  $s^+ \in \Sigma_{\Lambda}^+$ , such that the following limit is given by

$$\lim_{\beta \to \infty} \langle f(\sigma) \rangle = f(s^+). \tag{10}$$

**Proof.** Consider the model at h = 0. In this case, the model becomes the classical Edwards-Anderson model, then the Duhamel function becomes normal correlation function

$$(\sigma_b^z, f(\sigma)) = \langle \sigma_b^z f(\sigma) \rangle$$

Lemma 1 indicates the following consistent property of the bond spin configuration at zero temperature. Eq.(4) in Lemma 1 for an arbitrary bond  $b \in B_{\Lambda}$  and  $f(\sigma) = \sigma_b$  implies

$$\lim_{\beta \to \infty} (1 - \langle \sigma_b^z \rangle^2) = 0. \tag{11}$$

The above identity can be represented in terms of a probability  $p_b := \lim_{\beta \to \infty} \langle \delta_{\sigma_b^z, 1} \rangle$ 

$$0 = \lim_{\beta \to \infty} \left[ 1 - \langle (2\delta_{\sigma_b^z, 1} - 1) \rangle^2 \right] = 1 - (2p_b - 1)^2$$
  
=  $4p_b(1 - p_b)$ . (12)

Since either  $p_b = 1$  or  $p_b = 0$  is valid, either a ferromagnetic  $\sigma_b = 1$  or an antiferromagnetic  $\sigma_b = -1$  bond spin configuration appears almost surely on any bond  $b \in B_{\Lambda}$  for almost all J at zero temperature. Consider a plaquette (i, j, k, l) with an arbitrary  $i \in \Lambda_L$  and j = i + e, k = i + e', l = i + e + e' for unit vectors with |e| = |e'| = 1. Lemma for  $b = \{i, j\}, \{i, k\}$  and  $f(\sigma) = \sigma_i^z \sigma_l^z, \sigma_k^z \sigma_l^z$  implies

$$\lim_{\beta \to \infty} \left[ \langle \sigma_i^z \sigma_j^z \sigma_i^z \sigma_l^z \rangle - \langle \sigma_i^z \sigma_j^z \rangle \langle \sigma_j^z \sigma_l^z \rangle \right] = 0, \tag{13}$$

$$\lim_{\beta \to \infty} \left[ \langle \sigma_i^z \sigma_k^z \sigma_k^z \sigma_l^z \rangle - \langle \sigma_i^z \sigma_k^z \rangle \langle \sigma_k^z \sigma_l^z \rangle \right] = 0. \tag{14}$$

These and  $(\sigma_i^z)^2 = (\sigma_k^z)^2 = 1$  give the consistent property of the bond spin configuration

$$\lim_{\beta \to \infty} \langle \sigma_i^z \sigma_j^z \rangle \langle \sigma_j^z \sigma_l^z \rangle \langle \sigma_l^z \sigma_k^z \rangle \langle \sigma_k^z \sigma_i^z \rangle = 1.$$

For any site  $i \in \Lambda_L$  and for  $b = \{i, j\} \in B_{\Lambda}$ , Lemma 1 and  $f(\sigma) = \sigma_i^z$  imply

$$\lim_{\beta \to \infty} \langle \sigma_j^z \rangle = \lim_{\beta \to \infty} \langle \sigma_i^z \sigma_j^z \rangle \langle \sigma_i^z \rangle$$

for almost all J. For any sites  $i, j \in \Lambda_L$  and i, j are connected by bonds in  $B_{\Lambda}$ . Then, the + boundary condition  $\sigma_i = 1$  given by (3) and a bond spin configuration  $(\sigma_i \sigma_j)_{\{i,j\} \in B_{\Lambda}}$  fix a spin configuration  $s^+ \in \Sigma_{\Lambda}^+$  uniquely at zero temperature for any L. This spin configuration  $s^+$  gives

$$\lim_{\beta \to \infty} \langle \sigma_i^z \rangle = s_i^+,$$

for  $i \in \Lambda_L$  and also gives

$$\lim_{\beta \to \infty} \langle f(\sigma^z) \rangle = f(s^+),$$

for a real valued function  $f(\sigma^z)$ . This completes the proof.  $\square$ 

#### 3.3 Expansion method

Datta and Kennedy develop the Kirkwood-Thomas expansion method and study transverse field Ising model and XXZ model. Here we employ their method to the EA model in a weak transverse field. To obtain ground state in the transverse EA model, consider the following unitary transformed Hamiltonian

$$\tilde{H}_{\Lambda}(\sigma, \boldsymbol{h}, \boldsymbol{J}) := U H_{\Lambda} U^{\dagger} = -\sum_{b \in B_{\Lambda}} J_b \sigma_b^x - \sum_{i \in \Lambda_I'} h_i \sigma_i^z, \tag{15}$$

where  $U\sigma_i^x U^{\dagger} = \sigma_i^z$  and  $U\sigma_i^z U^{\dagger} = -\sigma_i^x$ . Define

$$\sigma_X := \prod_{i \in X} \sigma_i.$$

Note the following identity for  $X,Y\subset \Lambda_L'$ 

$$2^{-|\Lambda'_L|} \sum_{\sigma \in \Sigma_A^+} \sigma_X \sigma_Y = I(X = Y) =: \delta_{X,Y}, \tag{16}$$

where an indicator I(e) for an arbitrary event e is defined by I(true) = 1 and I(false) = 0. Let  $\psi(\sigma)$  be a function  $\psi : \{-1,1\}^{\Lambda'_L} \to \mathbb{R}$  of spin configuration, and express ground state of the Hamiltonian

$$|GS\rangle = \sum_{\sigma \in \Sigma_{\Lambda}} \sigma_D \psi(\sigma) |\sigma\rangle,$$

where  $D := \{i \in \Lambda'_L | s_i^+ = -1\}$ . Note that  $\psi(\sigma) = 1$  for  $h_i = 0$  is given by the ground state spin configuration  $s^+ \in \Sigma_{\Lambda}$  defined by Lemma 2. The eigenvalue equation defined by

$$\tilde{H}_{\Lambda}(\sigma, \boldsymbol{h}, \boldsymbol{J})|GS\rangle = E_0|GS\rangle$$

is written in

$$-(\sum_{b\in B_{\Lambda}}J_{b}\sigma_{b}^{x}+\sum_{i\in \Lambda_{L}^{\prime}}h_{i}\sigma_{i}^{z})|GS\rangle=E_{0}|GS\rangle.$$

If  $\sigma_i^x |\sigma\rangle = |\tau\rangle$ ,  $\tau_i = -\sigma_i$  and  $\tau_j = \sigma_j$  for  $j \neq i$ .

$$\sigma_b^x |\sigma\rangle = \sigma_i^x \sigma_i^x |\sigma\rangle = |\sigma^{(i,j)}\rangle,$$

where  $\sigma^{(i,j)}$  denotes a spin configuration replaced by  $(\sigma_i, \sigma_j) \to (-\sigma_i, -\sigma_j)$ . This eigenvalue equation can be represented in terms of  $\psi(\sigma)$ .

$$\sum_{b \in B_{\Lambda}} J_b \sigma_D^{(b)} \psi(\sigma^{(b)}) + \sum_{i \in \Lambda_L'} h_i \sigma_i \sigma_D \psi(\sigma) = -E_0 \sigma_D \psi(\sigma). \tag{17}$$

Therefore

$$\sum_{b \in B_{\Lambda}} J_b \frac{\sigma_D^{(b)} \psi(\sigma^{(b)})}{\sigma_D \psi(\sigma)} + \sum_{i \in \Lambda_{\sigma}'} h_i \sigma_i = -E_0.$$
(18)

To obtain the Kirkwood-Thomas equation for the ground state, represent the function  $\psi(\sigma)$  in terms of a complex valued function g(X) of an arbitrary subset  $X \subset \Lambda'_L$ ,

$$\psi(\sigma) = \exp\left[-\frac{1}{2} \sum_{X \subset \Lambda_L'} g(X)\sigma_X\right]. \tag{19}$$

Note the following relations

$$\psi(\sigma^{(b)}) = \exp\left[-\frac{1}{2} \sum_{X \subset \Lambda_L'} g(X)\sigma_X + \sum_{X \subset \Lambda_L'} I(b \in \partial X)g(X)\sigma_X\right]. \tag{20}$$

$$\frac{\psi(\sigma^{(b)})}{\psi(\sigma)} = \exp\left[\sum_{X \subset \Lambda_T'} I(b \in \partial X) g(X) \sigma_X\right]$$

where a set  $\partial X$  is defined by

$$\partial X := \{\{i, j\} \in B_{\Lambda} | i \in X, j \notin X \text{ or } j \in X, i \notin X\}.$$

Note also,

$$\frac{\sigma_D^{(b)}}{\sigma_D} = s_b^+.$$

These and the eigenvalue equation (35) give

$$\sum_{b \in B_{\Lambda}} J_b s_b^+ \exp\left[\sum_{X \subset \Lambda_L'} I(b \in \partial X) g(X) \sigma_X\right] + \sum_{i \in \Lambda_L'} h_i \sigma_i = -E_0.$$
 (21)

We expand the exponential function in power series. The first order term in the exponential function is given by

$$\sum_{b \in B_{\Lambda}} J_b s_b^+ \sum_{X \subset \Lambda_L'} I(b \in \partial X) g(X) \sigma_X = \sum_{X \subset \Lambda_L'} \sum_{b \in \partial X} J_b s_b^+ g(X) \sigma_X, \tag{22}$$

then we have

$$\sum_{b \in B_{\Lambda}} J_b s_b^+ + E_0 + \sum_{X \subset \Lambda_L'} \sum_{b \in \partial X} J_b s_b^+ g(X) \sigma_X$$

$$+ \sum_{b \in B_{\Lambda}} J_b s_b^+ \exp^{(2)} \left[ \sum_{X \subset \Lambda_L'} I(b \in \partial X) g(X) \sigma_X \right] + \sum_{i \in \Lambda_L'} h_i \sigma_i = 0, \tag{23}$$

where

$$\exp^{(2)} x := e^x - 1 - x = \sum_{k=2}^{\infty} \frac{x^k}{k!}.$$

The orthonormalization property (16) gives

$$E_0 = -\sum_{b \in B_{\Lambda}} J_b s_b^+ - \sum_{c \in B_{\Lambda}} J_c s_c^+ \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_1, \dots, X_k \subset \Lambda_L'} \delta_{X_1 \triangle \dots \triangle X_k, \phi} \prod_{l=1}^k g(X_l) I(c \in \partial X_l), \tag{24}$$

and

$$g(X) = \frac{-1}{\sum_{b \in \partial X} J_b s_b^+} \left[ \sum_{c \in B_\Lambda} J_c s_c^+ \sum_{k=2}^\infty \frac{1}{k!} \sum_{X_1, \dots, X_k \subset \Lambda_L'} \delta_{X_1 \triangle \dots \triangle X_k, X} \prod_{l=1}^k g(X_l) I(c \in \partial X_l) \right] + \sum_{i \in \Lambda_L'} h_i \delta_{X, \{i\}} ,$$

$$=: F(g)(X)$$

$$(25)$$

where  $X \triangle Y := (X \cup Y) \setminus (X \cap Y)$  for arbitrary sets X, Y, and we have used  $\sigma_X \sigma_Y = \sigma_{X \triangle Y}$ . The first term in the ground state energy is identical to that of the ground state spin configuration  $s^+$  for  $h_i = 0$ , and the excited energy of a spin configuration  $\sigma$  for  $h_i = 0$  is  $2 \sum_{b \in \partial X} J_b s_b^+$ , where  $X := \{i \in \Lambda'_L | \sigma_i \neq s_i^+\}$ . Lemma 2 guarantees the positivity  $\sum_{b \in \partial X} J_b s_b^+ > 0$ . To prove uniqueness of the ground state in the transverse field EA model with a given J for sufficiently small  $h_i$ , define a norm for the function g(X) by

$$||g|| := \sup_{c \in B_{\Lambda}} \sum_{X \subset \Lambda_{t}'} I(c \in \partial X) \sum_{b \in B_{\Lambda}} J_{b} s_{b}^{+} |g(X)| I(b \in \partial X) (hM)^{-w(X)}, \tag{26}$$

where w(X) is a number of elements of the smallest connected set which contains X and  $h := \sup_{i \in \Lambda'_L} |h_i|$ . We say that a set X is connected, if for any  $i, j \in X$  there exists a sequence  $i_1, i_2, \dots, i_n \in X$ , such that  $i_1 = i$ ,  $i_n = j$  and  $\{i_k, i_{k+1}\} \in B_{\Lambda}$  for  $k = 1, \dots, n-1$ . Then, the following lemma can be proven.

**Lemma 3.** There exist a constant M>0 and define  $\delta=\frac{4}{M}$ , such that if  $hM\leq 1$ 

$$||F(g) - F(g')|| \le ||g - g'||/2$$
, for  $||g||, ||g'|| \le \delta$ , (27)

and  $||F(g)|| \le \delta$  for  $||g|| \le \delta$ .

**Proof.** For lighter notations, define  $\triangle_k := X_1 \triangle \cdots \triangle X_k$ . The norm ||F(g) - F(g')|| is represented in

$$||F(g) - F(g')||$$

$$= \Big| \sup_{c \in B_{\Lambda}} \sum_{X \subset \Lambda'_{L}} I(c \in \partial X) \sum_{b \in B_{\Lambda}} J_{b} s_{b}^{+} \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_{1}, \dots, X_{k} \subset \Lambda'_{L}} \delta_{\Delta_{k}, X}$$

$$\times [\prod_{l=1}^{k} g(X_{l}) - \prod_{l=1}^{k} g'(X_{l})] [\prod_{l=1}^{k} I(b \in \partial X_{l})] (hM)^{-w(X)} \Big|$$

$$\leq \sup_{c \in B_{\Lambda}} \sum_{b \in B_{\Lambda}} J_{b} s_{b}^{+} \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_{1}, \dots, X_{k} \subset \Lambda'_{L}} I(c \in \partial \Delta_{k})$$

$$\times \Big| \prod_{l=1}^{k} g(X_{l}) - \prod_{l=1}^{k} g'(X_{l}) \Big| [\prod_{l=1}^{k} I(b \in \partial X_{l})] (hM)^{-w(\Delta_{k})}$$

$$\leq \sup_{c \in B_{\Lambda}} \sum_{b \in B_{\Lambda}} J_{b} s_{b}^{+} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{X_{1}, \dots, X_{k} \subset \Lambda'_{L}} I(c \in \partial X_{1})$$

$$\times \Big| \prod_{l=1}^{k} g(X_{l}) - \prod_{l=1}^{k} g'(X_{l}) \Big| [\prod_{l=1}^{k} I(b \in \partial X_{l})] (hM)^{-w(\Delta_{k})},$$

where  $I(c \in \partial \triangle_k) \leq \sum_{l=1}^k I(c \in \partial X_l)$  and permutation symmetry in the summation over  $X_1, \dots, X_k$  have been used. Inequalities  $w(\triangle_k) \leq \sum_{l=1}^k w(X_l)$ , and

$$\left| \prod_{l=1}^{k} g(X_l) - \prod_{l=1}^{k} g'(X_l) \right| \le \sum_{l=1}^{k} \prod_{j=1}^{l-1} |g(X_j)| |g(X_l) - g'(X_l)| \prod_{j=l+1}^{k} |g'(X_j)|$$

enable us to evaluate the norm as follows:

$$||F(g) - F(g')||$$

$$\leq \sup_{c \in B_{\Lambda}} \sum_{b \in B_{\Lambda}} J_{b} s_{b}^{+} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{X_{1}, \dots, X_{k} \subset \Lambda_{L}'} I(c \in \partial X_{1})$$

$$\times \left[ \sum_{l=1}^{k} \prod_{j=1}^{l-1} |g(X_{j})| |g(X_{l}) - g'(X_{l})| \prod_{j=l+1}^{k} |g'(X_{j})| \right] \left[ \prod_{l=1}^{k} I(b \in \partial X_{l}) \right] (hM)^{-w(\triangle_{k})}$$

$$\leq \sup_{c \in B_{\Lambda}} \sum_{b \in B_{\Lambda}} J_{b} s_{b}^{+} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{X_{1}, \dots, X_{k} \subset \Lambda_{L}'} I(c \in \partial X_{1})$$

$$\times \left[ \sum_{l=1}^{k} \prod_{j=1}^{l-1} |g(X_{j})| |g(X_{l}) - g'(X_{l})| \prod_{j=l+1}^{k} |g'(X_{j})| \right] \left[ \prod_{l=1}^{k} I(b \in \partial X_{l}) \right] (hM)^{-\sum_{j=1}^{k} w(X_{j})}$$

$$\begin{split} &= \sup_{c \in B_{\Lambda}} \sum_{b \in B_{\Lambda}} J_b s_b^+ \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{X_1, \cdots, X_k \subset \Lambda'_L} I(c \in \partial X_1) \\ &\times \left[ \sum_{l=1}^k \prod_{j=1}^{l-1} |g(X_j)| I(b \in \partial X_j) (hM)^{-w(X_j)} |g(X_l) - g'(X_l)| I(b \in \partial X_l) (hM)^{-w(X_l)} \\ &\times \prod_{j=l+1}^k |g'(X_j)| I(b \in \partial X_j) (hM)^{-w(X_j)} \right] \\ &= \sup_{c \in B_{\Lambda}} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{l=1}^k \sum_{X_1 \subset \Lambda'_L} I(c \in \partial X_1) \sum_{b \in B_{\Lambda}} J_b s_b^+ |g(X_1)| I(b \in \partial X_1) (hM)^{-w(X_1)} \\ &\times \prod_{j=2}^{l-1} \sum_{X_j \subset \Lambda'_L} |g(X_j)| I(b \in \partial X_j) (hM)^{-w(X_j)} \\ &\times \sum_{X_l \subset \Lambda'_L} |g(X_l) - g'(X_l)| I(b \in \partial X_l) (hM)^{-w(X_l)} \\ &\times \prod_{j=l+1}^k \sum_{X_j \subset \Lambda'_L} |g'(X_j)| I(b \in \partial X_j) (hM)^{-w(X_j)} \\ &\leq \|g - g'\| \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{l=1}^k \prod_{j=1}^{l-1} \|g\|/\Delta \prod_{j=l+1}^k \|g'\|/\Delta \\ &= \|g - g'\| \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{l=1}^k (\|g\|/\Delta)^{l-1} (\|g'\|/\Delta)^{k-l} \\ &\leq \|g - g'\| \sum_{k=2}^{\infty} \frac{k(\delta/\Delta)^{k-1}}{(k-1)!} = K \|g - g'\|, \end{split}$$

where the energy gap  $\Delta > 0$  above the unique ground state in the unperturbed model is defined by

$$\Delta := \inf_{g} \left[ \sup_{c \in B_{\Lambda}} \sum_{X \subset \Lambda'_{L}} I(c \in \partial X) |g(X)| (hM)^{-w(X)} \right]^{-1}$$

$$\times \sup_{c \in B_{\Lambda}} \sum_{X \subset \Lambda'_{L}} I(c \in \partial X) \sum_{b \in B_{\Lambda}} J_{b} s_{b}^{+} |g(X)| I(b \in \partial X) (hM)^{-w(X)},$$
(30)

and  $K := e^{\delta/\Delta}(1 + \delta/\Delta) - 1$ . The condition  $K = \frac{1}{2}$  fixes  $\delta/\Delta$ , and then the following inequality for any g

$$\sup_{c \in B_{\Lambda}} \sum_{X \subset \Lambda'_L} I(c \in \partial X) |g(X)| (hM)^{-w(X)} \le \frac{\|g\|}{\Delta} \le \frac{\delta}{\Delta},$$

requires sufficiently large M > 0. To obtain the bound on ||F(g)||, let us evaluate ||F(0)|| first. Since

$$F(0)(X) = \frac{-\sum_{i \in \Lambda'_L} h_i \delta_{X,\{i\}}}{\sum_{b \in \partial X} J_b s_b^+},$$

the norm is given by

$$||F(0)|| = \sup_{c \in B_{\Lambda}} \sum_{X \subset \Lambda'_{L}} I(c \in \partial X) |\sum_{i \in \Lambda'_{L}} h_{i} \delta_{X,\{i\}} | (hM)^{-w(X)}$$

$$\leq \sup_{c \in B_{\Lambda}} \sum_{X \subset \Lambda'_{L}} I(c \in \partial X) \sum_{i \in \Lambda'_{L}} |h_{i}| \delta_{X,\{i\}} (hM)^{-w(X)}$$

$$\leq \sup_{c \in B_{\Lambda}} \sum_{i \in \Lambda'_{L}} I(c \in \partial \{i\}) h(hM)^{-1} = 2M^{-1} = \frac{\delta}{2}, \tag{31}$$

Hence,

$$||F(g)|| = ||F(g) - F(0)| + F(0)|| \le ||F(g) - F(0)|| + ||F(0)|| \le \frac{||g||}{2} + \frac{\delta}{2} \le \delta.$$

This completes the proof of Lemma 3.  $\square$ 

**Proof of** Theorem 1. Lemma 3 and the contraction mapping theorem enable us to prove that the fixed point equation

$$F(g) = g (32)$$

has unique solution g satisfying  $||g|| \leq \delta$ . This completes the proof of Theorem 1.  $\square$ 

## 4 Proof of Theorem 2

Here, we discuss another convergent expansion around  $J = \mathbf{0}$ . In the case for  $J = \mathbf{0}$ , the model defined by the Hamiltonian (1) becomes the free spin model, and the uniqueness of the ground state is obvious. Assume  $h_i \neq 0$  for any  $i \in \Lambda'_L$ , and define  $D := \{i \in \Lambda'_L | h_i < 0\}$ , such that the following state satisfies

$$\sigma_i^x \sum_{\sigma \in \Sigma_A^+} \sigma_D |\sigma\rangle = \frac{h_i}{|h_i|} \sum_{\sigma \in \Sigma_A^+} \sigma_D |\sigma\rangle. \tag{33}$$

This state gives the unique ground state for J = 0.

Let  $\psi(\sigma)$  be a function  $\psi: \{-1,1\}^{\Lambda'_L} \to \mathbb{R}$  of spin configuration, and express ground state of the Hamiltonian (1) for weak exchange interactions

$$|GS\rangle = \sum_{\sigma \in \Sigma_{\Lambda}^{+}} \sigma_{D} \psi(\sigma) |\sigma\rangle.$$

Note that  $\psi(\sigma) = 1$  for  $J_b = 0$  is given by the ground state. The eigenvalue equation defined by

$$H_{\Lambda}(\sigma, \boldsymbol{h}, \boldsymbol{J})|GS\rangle = E_0|GS\rangle$$

is written in

$$-(\sum_{b\in B_{\Lambda}} J_b \sigma_b^z + \sum_{i\in \Lambda_L'} h_i \sigma_i^x) |GS\rangle = E_0 |GS\rangle.$$

Using  $\sigma_i^x |\sigma\rangle = |\sigma^{(i)}\rangle$  and  $\sigma_b^z |\sigma\rangle = \sigma_b |\sigma\rangle$ , the eigenvalue equation can be represented in terms of  $\psi(\sigma)$ .

$$\sum_{b \in B_{\Lambda}} J_b \sigma_b \sigma_D \psi(\sigma) + \sum_{i \in \Lambda_L'} h_i \sigma_D^{(i)} \psi(\sigma^{(i)}) = -E_0 \sigma_D \psi(\sigma), \tag{34}$$

where  $\sigma^{(i)}$  denotes a spin configuration replaced by  $\sigma_i \to -\sigma_i$ . Therefore

$$\sum_{b \in B_{\Lambda}} J_b \sigma_b + \sum_{i \in \Lambda'_L} h_i \frac{\sigma_D^{(i)} \psi(\sigma^{(i)})}{\sigma_D \psi(\sigma)} = -E_0.$$
(35)

To obtain the Kirkwood-Thomas equation for the ground state, represent the function  $\psi(\sigma)$  in terms of a complex valued function g(X) of an arbitrary subset  $X \subset \Lambda'_L$ ,

$$\psi(\sigma) = \exp\left[-\frac{1}{2} \sum_{X \subset \Lambda_L'} g(X)\sigma_X\right]. \tag{36}$$

Note the following relations

$$\psi(\sigma^{(i)}) = \exp\left[-\frac{1}{2} \sum_{X \subset \Lambda_L'} g(X)\sigma_X + \sum_{X \subset \Lambda_L'} I(i \in X)g(X)\sigma_X\right]. \tag{37}$$

$$\frac{\psi(\sigma^{(i)})}{\psi(\sigma)} = \exp\left[\sum_{X \subset \Lambda_L'} I(i \in X)g(X)\sigma_X\right]$$

Note also,

$$\frac{\sigma_D^{(i)}}{\sigma_D} = \frac{h_i}{|h_i|}.$$

These and the eigenvalue equation (35) give

$$\sum_{b \in B_{\Lambda}} J_b \sigma_b + \sum_{i \in \Lambda'_L} |h_i| \exp\left[\sum_{X \subset \Lambda'_L} I(i \in X) g(X) \sigma_X\right] = -E_0.$$
(38)

We expand the exponential function in power series.

$$\sum_{b \in B_{\Lambda}} J_b \sigma_b + E_0 + \sum_{i \in \Lambda'_L} |h_i| + \sum_{X \subset \Lambda'_L} \sum_{i \in \Lambda'_L} |h_i| I(i \in X) g(X) \sigma_X$$

$$+ \sum_{i \in \Lambda'_L} |h_i| \exp^{(2)} \left[ \sum_{X \subset \Lambda'_L} I(i \in X) g(X) \sigma_X \right] = 0, \tag{39}$$

The orthonormalization property (16) gives

$$E_0 = -\sum_{i \in \Lambda_T'} |h_i| - \sum_{i \in \Lambda_T'} |h_i| \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_1, \dots, X_k \subset \Lambda_T'} \delta_{X_1 \triangle \dots \triangle X_k, \phi} \prod_{l=1}^k g(X_l) I(i \in X_l), \tag{40}$$

and

$$g(X) = \frac{-1}{\sum_{i \in X} |h_i|} \left[ \sum_{j \in \Lambda'_L} |h_j| \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_1, \dots, X_k \subset \Lambda'_L} \delta_{X_1 \triangle \dots \triangle X_k, X} \prod_{l=1}^k g(X_l) I(j \in X_l) \right] + \sum_{b \in B_{\Lambda}} J_b \delta_{X, \{b\}},$$

$$=: F(g)(X)$$

$$(41)$$

The first term in the ground state energy is identical to that of the ground state eigenvalue configuration for  $J = \mathbf{0}$ , and the excited energy of a spin configuration  $\sigma$  for  $J = \mathbf{0}$  is  $2\sum_{i \in X} |h_i|$ ,

where  $X := \{i \in \Lambda'_L | \sigma_i \neq h_i / |h_i| \}$ . To prove uniqueness of the ground state in the transverse field EA model with a given h for sufficiently small  $J_i$ , define a norm for the function g(X) by

$$||g|| := \sup_{i \in \Lambda'_L} \sum_{X \subset \Lambda'_L} I(i \in X) \sum_{j \in \Lambda'_L} |h_j| |g(X)| I(j \in X) (JM)^{-v(X)}, \tag{42}$$

where v(X) is a number of bonds of the smallest connected set which contains X and  $J := \sup_{b \in B_{\Lambda}} |J_b|$ . Then, the following lemma can be proven.

**Lemma 4.** There exist a constant M>0 and define  $\delta=\frac{4d}{M}$ , such that if  $JM\leq 1$ 

$$||F(g) - F(g')|| \le ||g - g'||/2$$
, for  $||g||, ||g'|| \le \delta$ , (43)

and  $||F(g)|| \le \delta$  for  $||g|| \le \delta$ .

**Proof.** The norm ||F(g) - F(g')|| is represented in

$$||F(g) - F(g')||$$

$$= \left| \sup_{j \in \Lambda'_{L}} \sum_{X \subset \Lambda'_{L}} I(j \in X) \sum_{i \in \Lambda'_{L}} |h_{i}| \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_{1}, \dots, X_{k} \subset \Lambda'_{L}} \delta_{\Delta_{k}, X} \right|$$

$$\times \left[ \prod_{l=1}^{k} g(X_{l}) - \prod_{l=1}^{k} g'(X_{l}) \right] \left[ \prod_{l=1}^{k} I(i \in X_{l}) \right] (JM)^{-v(X)} \left| \right|$$

$$\leq \sup_{j \in \Lambda'_{L}} \sum_{i \in \Lambda'_{L}} |h_{i}| \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{X_{1}, \dots, X_{k} \subset \Lambda'_{L}} I(j \in \Delta_{k})$$

$$\times \left| \prod_{l=1}^{k} g(X_{l}) - \prod_{l=1}^{k} g'(X_{l}) \right| \left[ \prod_{l=1}^{k} I(i \in X_{l}) \right] (JM)^{-v(\Delta_{k})}$$

$$\leq \sup_{j \in \Lambda'_{L}} \sum_{i \in \Lambda'_{L}} |h_{i}| \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{X_{1}, \dots, X_{k} \subset \Lambda'_{L}} I(j \in X_{1})$$

$$\times \left| \prod_{l=1}^{k} g(X_{l}) - \prod_{l=1}^{k} g'(X_{l}) \right| \left[ \prod_{l=1}^{k} I(i \in X_{l}) \right] (JM)^{-v(\Delta_{k})},$$

where  $I(j \in \triangle_k) \leq \sum_{l=1}^k I(j \in X_l)$  and permutation symmetry in the summation over  $X_1, \dots, X_k$  have been used. Inequalities  $v(\triangle_k) \leq \sum_{l=1}^k v(X_l)$ , enables us to evaluate the norm as follows:

$$||F(g) - F(g')||$$

$$\leq \sup_{m \in \Lambda'_{L}} \sum_{i \in \Lambda'_{L}} |h_{i}| \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{X_{1}, \dots, X_{k} \subset \Lambda'_{L}} I(m \in X_{1})$$

$$\times \left[ \sum_{l=1}^{k} \prod_{j=1}^{l-1} |g(X_{j})| |g(X_{l}) - g'(X_{l})| \prod_{j=l+1}^{k} |g'(X_{j})| \right] \left[ \prod_{l=1}^{k} I(i \in X_{l}) \right] (JM)^{-v(\triangle_{k})}$$

$$\leq \sup_{m \in \Lambda'_{L}} \sum_{i \in \Lambda'_{L}} |h_{i}| \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{X_{1}, \dots, X_{k} \subset \Lambda'_{L}} I(m \in X_{1})$$

$$\times \left[ \sum_{l=1}^{k} \prod_{j=1}^{l-1} |g(X_{j})| |g(X_{l}) - g'(X_{l})| \prod_{j=l+1}^{k} |g'(X_{j})| \right] \left[ \prod_{l=1}^{k} I(i \in X_{l}) \right] (JM)^{-\sum_{j=1}^{k} v(X_{j})}$$

$$\begin{split} &= \sup_{m \in \Lambda'_L} \sum_{i \in \Lambda'_L} |h_i| \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{X_1, \cdots, X_k \subset \Lambda'_L} I(m \in X_1) \\ &\times \left[ \sum_{l=1}^k \prod_{j=1}^{l-1} |g(X_j)| I(i \in X_j) (JM)^{-v(X_j)} |g(X_l) - g'(X_l)| I(i \in X_l) (JM)^{-v(X_l)} \\ &\times \prod_{j=l+1}^k |g'(X_j)| I(i \in X_j) (JM)^{-v(X_j)} \right] \\ &= \sup_{m \in \Lambda'_L} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{l=1}^k \sum_{X_1 \subset \Lambda'_L} I(m \in X_1) \sum_{i \in \Lambda'_L} |h_i| |g(X_1)| I(i \in X_1) (JM)^{-v(X_1)} \\ &\times \prod_{j=2}^{l-1} \sum_{X_j \subset \Lambda'_L} |g(X_j)| I(i \in X_j) (JM)^{-v(X_j)} \\ &\times \sum_{X_l \subset \Lambda'_L} |g(X_l) - g'(X_l)| I(i \in X_l) (JM)^{-v(X_l)} \\ &\times \prod_{j=l+1}^k \sum_{X_j \subset \Lambda'_L} |g'(X_j)| I(i \in X_j) (JM)^{-w(X_j)} \\ &\leq \|g - g'\| \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{l=1}^k \prod_{j=1}^{l-1} \|g\|/\Delta \prod_{j=l+1}^k \|g'\|/\Delta \\ &= \|g - g'\| \sum_{k=2}^\infty \frac{1}{(k-1)!} \sum_{l=1}^k (\|g\|/\Delta)^{l-1} (\|g'\|/\Delta)^{k-l} \\ &\leq \|g - g'\| \sum_{k=2}^\infty \frac{k(\delta/\Delta)^{k-1}}{(k-1)!} = K \|g - g'\|, \end{split}$$

where the energy gap  $\Delta > 0$  above the unique ground state in the unperturbed model is defined by

$$\Delta := \inf_{g} \left[ \sup_{j \in \Lambda'_{L}} \sum_{X \subset \Lambda'_{L}} I(j \in X) |g(X)| (JM)^{-v(X)} \right]^{-1}$$

$$\times \sup_{j \in \Lambda'_{L}} \sum_{X \subset \Lambda'_{L}} I(j \in X) \sum_{i \in \Lambda'_{L}} |h_{i}| |g(X)| I(i \in X) (JM)^{-v(X)}, \tag{46}$$

and  $K := e^{\delta/\Delta}(1 + \delta/\Delta) - 1$ . The condition  $K = \frac{1}{2}$  fixes  $\delta/\Delta$ , and then the following inequality for any g

$$\sup_{j \in \Lambda'_L} \sum_{X \subset \Lambda'_r} I(j \in X) |g(X)| (JM)^{-v(X)} \le \frac{\|g\|}{\Delta} \le \frac{\delta}{\Delta},$$

requires sufficiently large M > 0. To obtain the bound on ||F(g)||, let us evaluate ||F(0)|| first. Since

$$F(0)(X) = \frac{-\sum_{b \in B_{\Lambda}} J_b \delta_{X,\{b\}}}{\sum_{i \in Y} |h_i|},$$

the norm is given by

$$||F(0)|| = \sup_{j \in \Lambda'_L} \sum_{X \subset \Lambda'_L} I(j \in X) |\sum_{b \in B_{\Lambda}} J_b \delta_{X,\{b\}} | (JM)^{-v(X)}$$

$$\leq \sup_{j \in \Lambda'_L} \sum_{X \subset \Lambda'_L} I(j \in X) \sum_{b \in B_{\Lambda}} |J_b| \delta_{X,\{b\}} (JM)^{-v(X)}$$

$$\leq \sup_{j \in \Lambda'_L} \sum_{b \in B_{\Lambda}} I(j \in b) J(JM)^{-1} = 2dM^{-1} = \frac{\delta}{2},$$
(47)

Hence,

$$||F(g)|| = ||F(g) - F(0)| + F(0)|| \le ||F(g) - F(0)|| + ||F(0)|| \le \frac{||g||}{2} + \frac{\delta}{2} \le \delta.$$

This completes the proof of Lemma 4.  $\square$ 

**Proof of** Theorem 2. Lemma 4 and the contraction mapping theorem enable us to prove that the fixed point equation

$$F(g) = g (48)$$

has unique solution g satisfying  $||g|| \leq \delta$ . This completes the proof of Theorem 2.  $\square$ 

### 5 Remark

For the transverse field EA model, the Perron-Frobenius theorem enables us to prove the uniqueness of the ground state. For an arbitrary sequence  $(h_i)_{i \in \Lambda'_L}$ , define a sequence  $\theta := (\theta_i)_{i \in \Lambda'_L}$  by  $\theta_i = 0$  for  $h_i > 0$  and  $\theta_i = \frac{\pi}{2}$  for  $h_i < 0$ . The following unitary transformation

$$U_{\boldsymbol{\theta}} := \exp\left(i \sum_{i \in \Lambda_L'} \theta_i \sigma_i^z\right)$$

transforms the perturbation Hamiltonian

$$U_{\boldsymbol{\theta}} \sum_{i \in \Lambda'_L} h_i \sigma_i^x U_{\boldsymbol{\theta}}^{\dagger} = \sum_{i \in \Lambda'_L} |h_i| \sigma_i^x.$$

For the Hamiltonian H defined by (1), the transformed Hamiltonian  $H_{\theta} := U_{\theta}HU_{\theta}^{\dagger}$  satisfies (i) non-positivity of all off-diagonal matrix elements  $\langle \sigma | H_{\theta} | \tau \rangle \leq 0$  for  $\sigma \neq \tau$  and (ii) connectivity condition  $\langle \sigma | H_{\theta}^{n} | \tau \rangle \neq 0$  for any  $\sigma \neq \tau$  for some positive integer n. These conditions allow the application of the Perron-Frobenius theorem to the transverse field EA model. This theorem implies that the unique ground state is given by  $|GS\rangle = \sum_{\sigma} \psi(\sigma) | \sigma \rangle$  with positive coefficients  $\psi(\sigma)$ . This result is consistent to Theorem 1 and 2 in the present paper. This Perron-Frobenius argument is valid also for the EA model under an arbitrary sequence of vector-valued fields  $(\vec{h}_i)_{i\in\Lambda_L} := (h_i^x, h_i^y, h_i^z)_{i\in\Lambda_L}$ . Our expansion method is applicable to general perturbation Hamiltonian, which does not satisfy the Perron-Frobenius conditions. For example, the uniqueness of the ground state can be shown in the following EA Hamiltonian perturbed by weak transverse exchange interactions

$$H := -\sum_{b \in B_{\Lambda}} (J_b \sigma_b^z + \epsilon_b \sigma_b^x).$$

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