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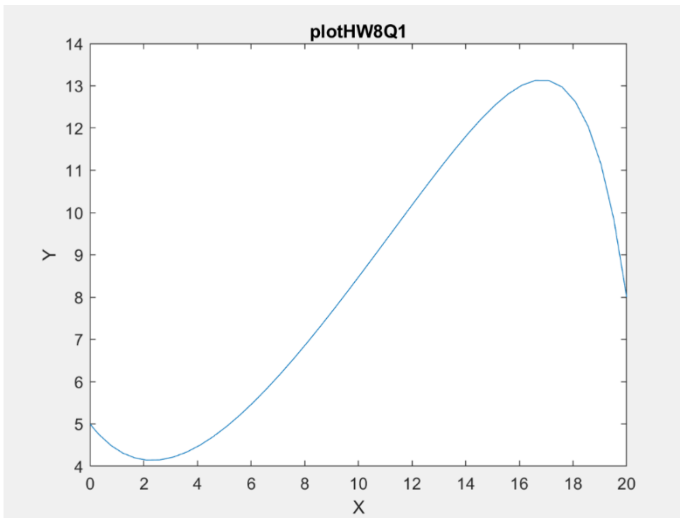
PKL4FR

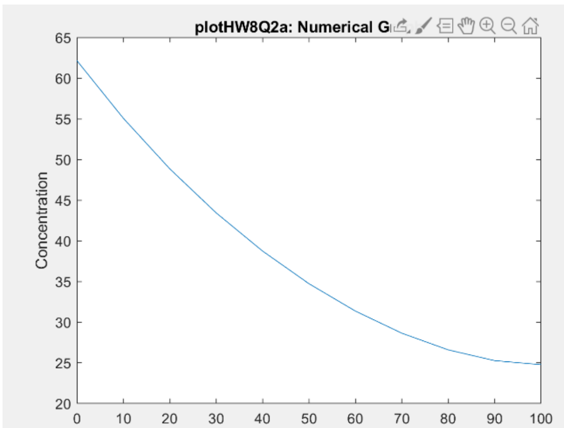
CHE2216

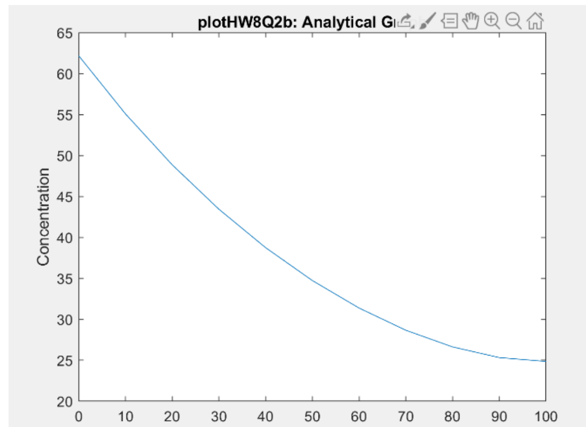
HW8_textsolutions

1)



2)


plotHW8Q2a: Numerical G
Concentration
65
60
55
50
45
40
35
30
25
20
0
10
20
30
40
50
60
70
80
90
100


plotHW8Q2b: Analytical G
Concentration
65
60
55
50
45
40
35
30
25
20
0
10
20
30
40
50
60
70
80
90
100

3)

(3) $f(x) = 2x \qquad L = [-4, 4]$ ($-L$, L)

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{2L}\int_{-4}^{4} 2x$$

$$= \frac{1}{4}\int_{-4}^{4} x$$

$$= \frac{1}{4}\left[\frac{1}{2}x^2\right]_{-4}^{4}$$

$$= \frac{1}{8}[16 - 16]$$

$$\boxed{A_0 = 0}$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{4}\int_{-4}^{4} (2x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{2}\int_{-4}^{4} x\sin\left(\frac{n\pi x}{L}\right)dx$$

*Integration by parts

$u = x \qquad v = \frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

$du = 1 \qquad dv = \sin\left(\frac{n\pi x}{L}\right)$

$$x\,\frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-4}^{4} - \int_{-4}^{4}\frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$$

$$\left[\frac{-L4}{n\pi}\cos(n\pi) - \frac{L4}{n\pi}\cos(-n\pi)\right]$$

$$\frac{-L^2}{n\pi}\left(\cos(n\pi) + 1\right)$$

$$= \frac{+L}{n\pi}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{L}{n\pi}\left[\frac{+L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L}$$

$$= \frac{L^2}{n^2\pi^2}\left[\cancel{0}^{\,0} + (0)\right]$$

$$\boxed{A_n = \frac{-L^2}{n\pi}\left[\cos(n\pi) + 1\right]}$$

$$\therefore f(x) = \sum A_n \sin\left(\frac{n\pi x}{L}\right)$$

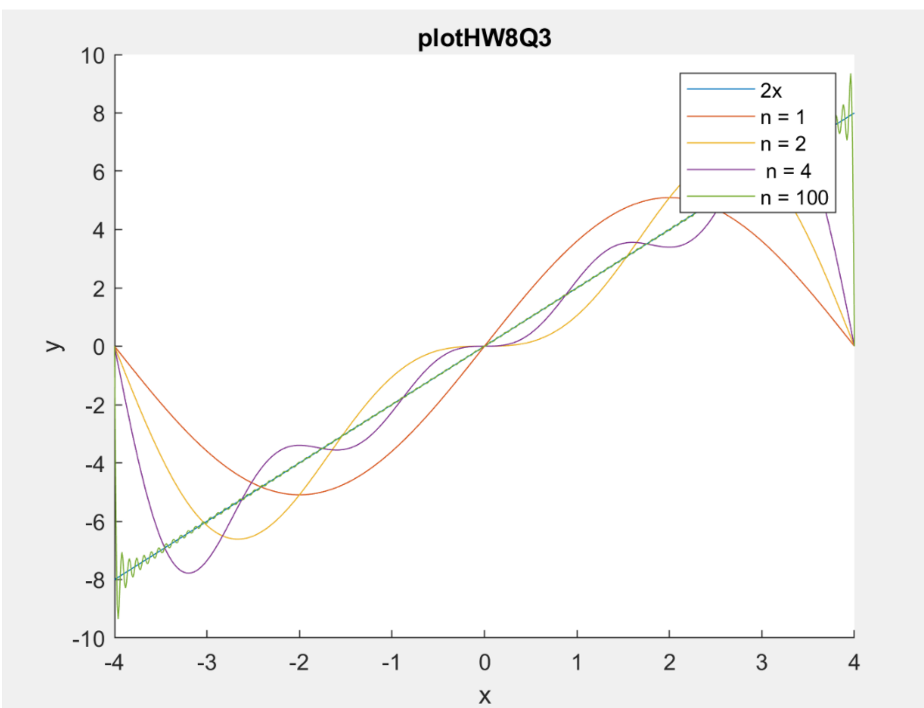
$$\boxed{f(x) = \sum_{n=1}^{\infty}\frac{-L^2}{n\pi}\left[\cos(n\pi) + 1\right]\sin\left(\frac{n\pi x}{L}\right)}$$

Numerical Data

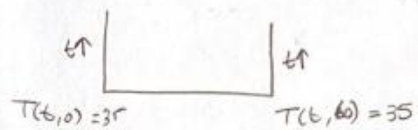
	1
1	62.1767
2	55.0818
3	48.8634
4	43.4390
5	38.7437
6	34.7300
7	31.3708
8	28.6615
9	26.6258
10	25.3223
11	24.8552

Analytical Data

	1
1	62.1490
2	55.0648
3	48.8536
4	43.4334
5	38.7391
6	34.7234
7	31.3585
8	28.6396
9	26.5894
10	25.2653
11	24.7700



4) Copper: 1059Sec. Aluminum: 1368Sec.



IC = $t=0$

$T(x) = 485 - 15|x-30|$

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial \theta(t)}{\partial t} = -D\theta\lambda^2 \qquad \frac{\partial^2 X}{\partial x^2} + \lambda^2 X = 0$$

$$T(t,x) = X(x)\,\theta(t) = e^{(-D\lambda^2 t)}(A\sin(\lambda x) + B\cos(\lambda x))$$

$$T(t,x) = e^{-D\lambda^2 t}[A\sin(\lambda x) + B\cos(\lambda x)]$$

$$U(t,x) = T(t,x) - 35 \quad \swarrow \text{Transformation}$$

$U(t,0) = 0 \longrightarrow 0 = A\sin(\lambda\cdot 0) + B\cos(\lambda\cdot 0)$

$U(t,60) = 0$

$\boxed{B = 0}$ $\quad U(x,t) = e^{-D\lambda^2 t} A\sin(\lambda x)$

$U(0,x) = 485 - 15|x-30| - 35$

$U(x,60) = e^{-D\lambda t} A\sin(60\lambda) = 0$

$\hookrightarrow A\sin(60\lambda) = 0$

$60\lambda = n\pi$

$\boxed{\lambda_n = \frac{n\pi}{L}}$

$$\boxed{U(x,t) = e^{-D\lambda^2 t}\sum A_n \sin(\lambda_n x)}$$

$$450 - 15|x-30| = \sum e^{-D\lambda^2(0)} A_n \sin(\lambda_n x)$$

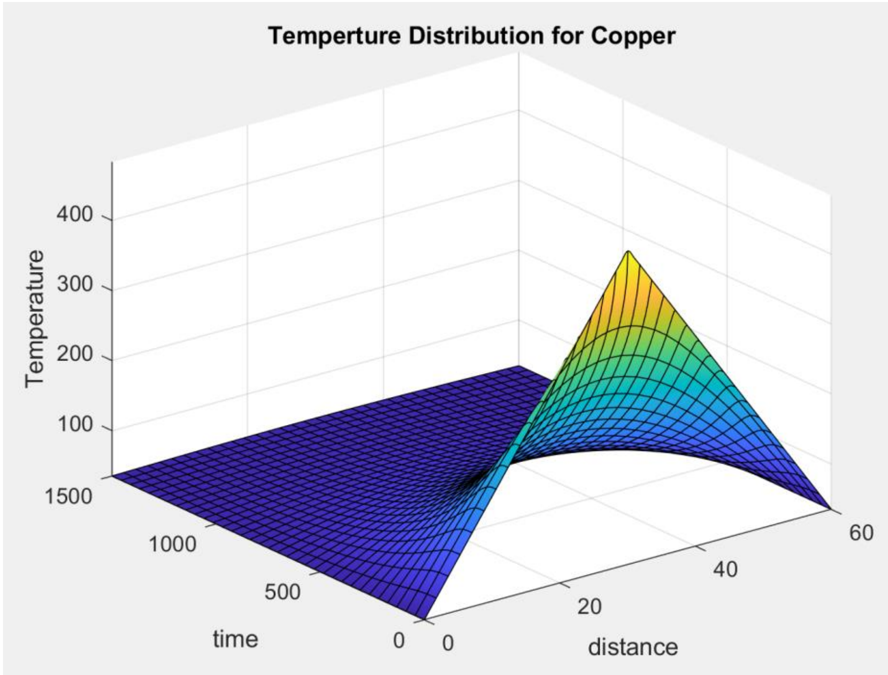
$$450 - 15|x-30| = \sum A_n \sin(\lambda_n x)$$

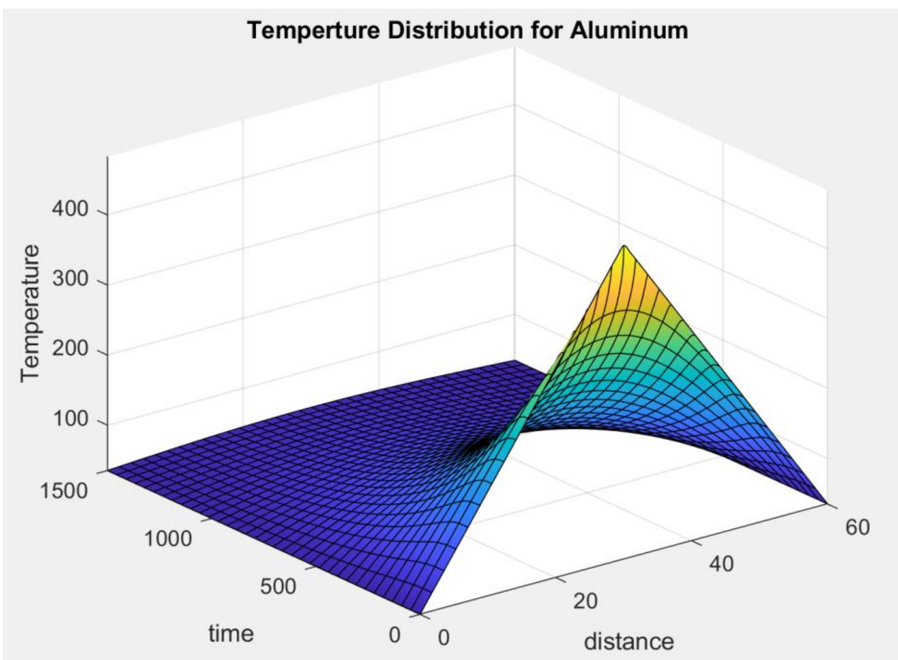
$$A_n = \frac{2}{L}\int_0^L (450 - 15|x-30|)\sin(\lambda_n x)$$

$$A_n = \frac{2}{L}\left[\int_0^{60} 450\sin(\lambda_n x) + 15\int_0^{30}(x-30)\sin(\lambda x)\,dx - 15\int_{30}^{60}(x-30)\sin(\lambda x)\,dx\right]$$

$$\boxed{A_n = \frac{2}{L}\left(\frac{30\sin(30x) - 15\sin(30y)}{x^2}\right) \qquad \lambda_n = \frac{n\pi}{L}}$$

$$\therefore\ T(x,t) = \sum_{n=}^{\infty} e^{-D\lambda^2 n t} A_n \sin(\lambda_n x) + 35$$


Temperture Distribution for Copper
Temperature
400
300
200
100
1500
1000
500
0
time
0
20
40
60
distance


Temperture Distribution for Aluminum
Temperature
400
300
200
100
1500
1000
500
0
time
0
20
40
60
distance

5)

(5) ~~$U(x,y) = [(A\sin(\lambda x) + B\cos(\lambda x))][(C\sinh(\lambda y) + D\cosh(\lambda y))]$~~ $* \eta = 10 - y$

$U(x,\eta) = T(x,\eta) - 400$

$U(x,y) = T(x,y) - (axy + bx + cy + d)$

$\frac{du}{dy} = \frac{dT}{dy} - (ax+c) = 0$

$= 0 - (ax+c) = 0$

$= -ax = c$

$\boxed{a = 0 \quad c = 0}$

du

$U(x,y) = T(x,y) - 400$

$\frac{du}{dy}(x,10) = 0$

$U(x,0) = -100$

$U(0,y) = 0$

$U(20,y) = 0$

$U(x,y) = T(x,y) - (bx + d)$

$U(0,y) = T(0,y) - (d) = 0$

$400 - d = 0$

$\boxed{d = 400}$

$U(x,y) = T(x,y) - (bx + 400)$

$U(20,y) = T(20,y) - b(20) - 400$

$400 - 20b - 400 = 0$

$\boxed{b = 0}$

$U(x,y) = T(x,y) - 400$

$* \eta = 10 - y$

$U(0,\eta) = 0 \qquad U(20,\eta) = 0$

$U(x,10) = -100 \qquad \frac{du}{d\eta}(x,0) = 0$

$y = 0 \qquad y = 10$

$U(x,\eta) = [A\sin(\lambda x) + B\cos(\lambda x)][C\sinh(\lambda\eta) + D\cosh(\lambda\eta)]$

(1) $U(0,\eta) = 0 = B(1)$

$\boxed{B = 0}$

(2) $u(20,\eta) = 0 = A\sin(20\lambda)$

$20\lambda = (n - 1/2)\pi$

$\lambda_n = \frac{(n - 1/2)\pi}{20}$

$M(x) = \sum A_n \sin(20\lambda_n)$

(3) $\frac{du}{d\eta}(x,0) = \lambda C\cosh(\lambda\eta) + \lambda D\sinh(\lambda\eta)$

$0 = \lambda C$

$\boxed{C = 0}$

$N(\eta) = D\cosh(\lambda\eta)$

$$U(x,\eta) = \Sigma E_n \sin\left(\frac{n\pi}{20} x\right)\cosh\left(\frac{n\pi}{20}\eta\right)$$

$$U(x,10) = -100 = \Sigma E_n \sin\left(\frac{n\pi}{20}x\right)\cosh\left(\frac{n\pi}{20}\right)$$

$$-100 = \Sigma E_n \sin\left(\frac{n\pi}{20}x\right)\cosh\left(\frac{n\pi}{20}\eta\right)$$

An

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$A_n = \frac{2}{20}\int_0^{20}(-100)\sin\left(\frac{n\pi x}{20}\right)$$

$$= \frac{-1}{10}\left[\frac{-20}{n\pi}\cos\left(\frac{n\pi x}{20}\right) - \left(\frac{-20}{n\pi}\cos\frac{n\pi x}{20}\right)\right]_0^{20}$$

$$= \frac{-1}{n\pi}\left[\frac{-20}{n\pi}\cos(n\pi) + \frac{20}{n\pi}\right]$$

$$= \frac{20}{n^2\pi^2}\cos(n\pi) - \frac{20}{n^2\pi^2}$$

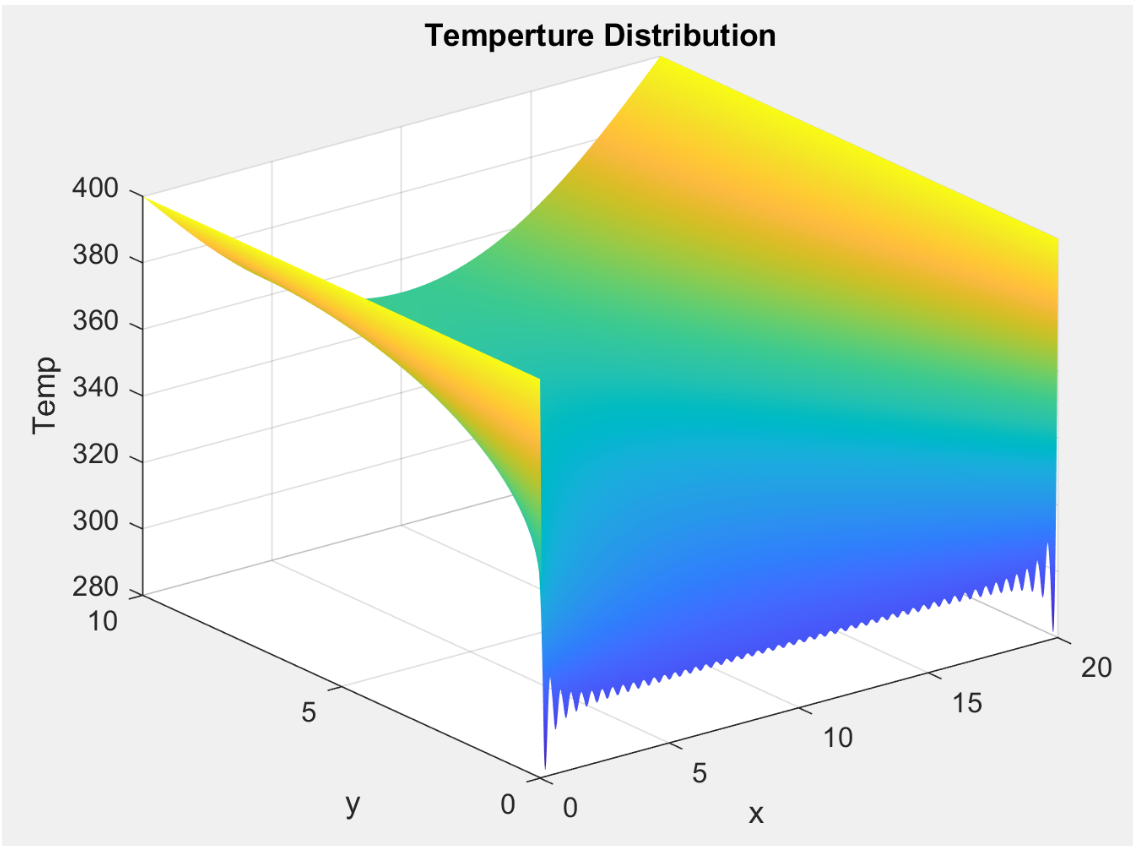
$$= \frac{20\cos(n\pi) - 20}{n^2\pi^2} = E_n \cosh\left(\frac{n\pi}{2}\right)$$

$$\boxed{E = \frac{200(\cos(n\pi) - 1)}{n\pi\cosh\left(\frac{n\pi}{2}\right)}}$$

$$U(x,\eta) = T(x,\eta) - 400$$

$$T(x,y) = U(x,10-y) + 400$$

$$\boxed{T(x,y) = \Sigma E \sin\left(\frac{\pi N x}{20}\right)\cosh\left(\frac{\pi N(10-y)}{20}\right) + 400}$$


Temperture Distribution
Temp
400
380
360
340
320
300
280
10
5
0
y
0
5
10
15
20
x

6)

(6) $T=0$ initial
$t=0 \quad T=T_0$.

$$\eta = \frac{x}{h(t)}$$

$$\left.\frac{\partial T}{\partial t}\right|_x = \frac{dT}{d\eta}\left.\frac{\partial \eta}{\partial t}\right|_x = \frac{dT}{d\eta}\left(\frac{-x}{h^2}\frac{dh}{dt}\right)$$

$$\left.\frac{\partial T}{\partial x}\right|_t = \frac{dT}{d\eta}\left.\frac{\partial \eta}{\partial x}\right|_t = \frac{dT}{d\eta}\frac{1}{h}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_x = \frac{d^2T}{d\eta^2}\frac{1}{h^2}$$

$$\frac{dT}{d\eta}\left(\frac{-x}{h^2}\frac{dh}{dt}\right) = D\frac{d^2T}{d\eta^2}\frac{1}{h^2}$$

$$D\frac{d^2T}{d\eta^2}\frac{1}{h^2} + \frac{dT}{d\eta}\frac{x}{h^2}\frac{dh}{dt} = 0$$

$$\frac{dT}{d\eta}\left(\frac{x}{h}\frac{dh}{dt}\frac{h}{D}\right) + \frac{d^2D}{d\eta^2} = 0$$

$$\frac{dT}{d\eta}\left(\eta\frac{dh}{dt}\frac{h}{D}\right) + \frac{d^2T}{d\eta^2} = 0$$

* $\frac{dh}{dt}\frac{h}{D} = \text{constant} = A$

$h\,dh = AD\,dt$

$$\frac{h^2}{2} = ADt + c \xrightarrow[t\to 0]{h(t)\to 0} \frac{h^2}{2} = ADt \rightarrow 0 = 0 + c \quad \boxed{c=0}$$

$$h = \sqrt{2ADt}$$

$$\eta = \frac{x}{\sqrt{2ADt}}$$

* $\frac{dT}{d\eta}(\eta A) + \frac{d^2T}{d\eta} = 0$

Boundaries

$T(t, x=0) = 0 \rightarrow T(\eta=0) = 0$

$T(t, x=\infty) = T_0 \rightarrow T(\eta\to\infty) = T_0$

$T(0,x) = T_0 \rightarrow T(\eta\to\infty) = T_0$) same condition

$$\frac{\partial T}{\partial \eta}(\eta A) + \frac{d^2T}{d\eta^2} = 0$$

(1) $p = \frac{dT}{d\eta}$

(2) $(\eta A)p + \frac{dp}{d\eta} = 0$

* $-p(\eta A) = \frac{dp}{d\eta}$

$$\frac{dp}{p} = -\eta A\,d\eta$$

$$P = Be^{\left(-\frac{A\eta^2}{2}\right)} = \frac{dT}{d\eta} \qquad \underline{B = \text{constant}}$$

$$dT = Be^{\left(-\frac{A\eta^2}{2}\right)}d\eta$$

$$T = \int Be^{\left(\frac{-A\eta^2}{2}\right)}d\eta + c$$

$$0 = \int_0^{\eta=0} Be^{\frac{-A\xi^2}{2}}d\xi + c$$

$$\boxed{C=0}$$

$$T = B\int_0^{\eta} e^{\frac{-A\xi^2}{2}}d\xi$$

$$T_0 = B\int_0^{\infty} e^{\frac{-A\xi^2}{2}}d\xi$$

$$\boxed{B = \frac{T_0}{\int_0^{\infty} e^{-A\xi^2/2}d\xi}}$$

$$T = \frac{T_0\int_0^{\eta} e^{-A\xi^2/2}d\xi}{\underbrace{\int_0^{\infty} e^{-\frac{A\xi^2}{2}}d\xi}}$$

if $A=2$ ↓ becomes

$$\frac{2}{\sqrt{\pi}}$$

$$\longrightarrow \boxed{T = T_0(\mathrm{erf}(\eta))}$$