Weicheng Wang MATH3310 HW1

Proof:

1. a) From the question it is clear that we are working inside same set X.

Based on the definition of complement of a set, ~~as~~ and the definition of $\emptyset$, for $\emptyset \subseteq X$, $\emptyset^c = X \setminus \emptyset = X$

$\therefore \emptyset^c = X$.

By the same idea, for $X \subseteq X$, $X^c = X \setminus X = \emptyset$

$\therefore X^c = \emptyset$

based on the definition of a set, which indicates that a set is a subset of itself.

$\square$

b) Proof:

assume $x \in B^c$, then according to definition of set complement $x \notin B$. $\because A \subset B$, and $x \notin B$ (definition of subset)

$\therefore x \notin A$

$\therefore x \in A^c$ (definition of set complement)

$\therefore$ If $x \in B^c$, it also belongs to A^c, based on $A \subset B$.

$\therefore B^c \subset A^c$

$\therefore A \subset B \longrightarrow B^c \subset A^c$.

c) Proof:

① Goal: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

let $x \in A \cup (B \cap C)$

$\Longrightarrow x \in A$ or $x \in (B \cap C)$

$\Longrightarrow$ as $x \in (B \cap C)$, $x \in B$ and $x \in C$

$\Longrightarrow$ both ways ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)

$\Longrightarrow x \in (A \cup B) \cap (A \cup C)$

② Goal: $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

let $y \in (A \cup B) \cap (A \cup C)$

$\Longrightarrow y \in (A \cup B)$ and $y \in (A \cup C)$

$\Longrightarrow y \in A$ or $y \in B$, and, $y \in A$ or $y \in C$

$\Longrightarrow y \in A$ or, $y \in B$ and $y \in C$

$\Longrightarrow y \in A \cup (B \cap C)$

$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2. Proof: $x \in (\bigcup_{i\in I} A_i)^C \iff x \notin \bigcup_{i\in I} A_i$

$$\iff \forall i \in I,\ x \notin A_i$$
$$\iff x \notin \bigcap_{i\in I} A_i$$
$$\iff x \in \bigcap_{i\in I} A_i^C$$

$$\therefore \bigcup_{i\in I} A_i^C = \bigcap_{i\in I} A_i^C$$

3. According to the problem, the sequence has the relationship:
$A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_5 \ldots$

Lemma: If $A_{i+1} \subset A_i$ and $A_{i+2} \subset A_{i+1}$, then $A_{i+2} \subset A_i$ for all $i \in \mathbb{N}$.

Proof: assume $x \in A_{i+2}$, since $A_{i+2} \subset A_{i+1}$, then $x \in A_{i+1}$. Since $A_{i+1} \subset A_i$, then $x \in A_i$. (by definition of subset) As we assumed an arbitrary $x \in A_{i+2}$ and proved $x \in A_i$, this implication is true for all $x \in A_{i+2}$, $i \in \mathbb{N}$.

Proof:

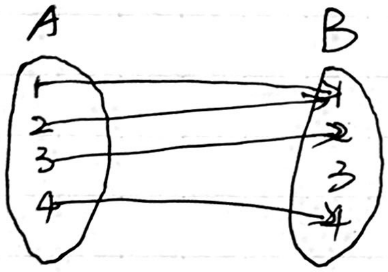
By repetitively applying the lemma and the fact that $A_{i+1} \subset A_i$ and the definition of a set we could derive that for all $i \in \mathbb{N}$, ~~$A_i \in$~~ $A_i \subset A_1$. i.e. every set in the sequence is a subset of A_1. ~~Therefore~~ $\bigcap_{i\in \mathbb{N}} A_i$ is ~~the intersection of A_1 and~~ only a subset of A_1 according to the definition of intersection. ~~As A_1 is finite and nonempty, its subset must also be finite~~ ~~As~~ Therefore, $\bigcap_{i\in \mathbb{N}} A_i$ is finite.

Assume $x_2 \in A_{i+1}$, since $A_{i+1} \subset A_i$, $x_2 \subset A_i$. By repetitively applying this implication to the sequence and since A_i is nonempty and finite for $\forall i \in \mathbb{N}$, we could conclude that there's some x that belongs to every set of $A_i, \forall i \in \mathbb{N}$. Therefore, $\bigcap_{i\in \mathbb{N}} A_i$ is never empty.

Thus, we have proved $\bigcap_{i\in \mathbb{N}} A_i$ is nonempty and finite.

4\. a). Let ~~x∈~~ $y \in f(\bigcap_{i\in I} A_i)$. Therefore, $\exists\, a \in \bigcap_{i\in I} A_i$ s.t. $f(a) = y$.
This means $a \in A_i$ for all $i \in I$.
It follows that $f(a) = y \in f(A_i)$ for all $i \in I$.
Therefore $y \in \bigcap_{i\in I} f(A_i)$.
~~Thus, $f(\bigcap_{i\in I} A_i) \subset$~~
Thus, $f(\bigcap_{i\in I} A_i) \subset \bigcap_{i\in I} f(A_i)$. The reverse in false by the example of $f(x) = x^2$. Say $A_1 = \{1\}$ and $A_2 = \{-1\}$. There's no intersection between A_1 & A_2, but $f(A_1) = f(A_2) = 1$. i.e. There's an intersection between $f(A_1)$ & $f(A_2)$.
Therefore, $f(\bigcap_{i\in I} A_i) \subset \bigcap_{i\in I} f(A_i)$

b) Proof by a counterexample:



Say $A_1 = \{1, 3\}$, $A_2 = \{2, 3\}$
Then according to the function depicted by the lines, $f(A_1) = \{1, 2\}$, $f(A_2) = \{1, 2\}$.
It follows that $f(A_1) \cap f(A_2) = \{1, 2\}$.
However, $A_1 \cap A_2 = \{3\}$, and $f(A_1 \cap A_2) = \{2\}$
As $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$, we couldn't say $f(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} f(A_i)$.

c) Proof: let $y \in \bigcap_{i\in I} f(A_i)$, it follows that $y \in f(A_i)$ for all $i \in I$.
Hence, $\exists\, x_i \in A_i$ for all $i \in I$ such that $f(x_i) = y$. ~~As f is injective. Therefore, $x \in \bigcap_{i\in I} A_i$~~ However, as f is injective, there's only one $x_i \in A_i$ for all $i \in I$. i.e. all x_i are the same number, denoted as x.
Hence $x \in A_i$ for all $i \in I$. i.e. $x \in \bigcap_{i\in I} A_i$.
$\therefore y = f(x) \in f(\bigcap_{i\in I} A_i)$ $\therefore \bigcap_{i\in I} f(A_i) \subset f(\bigcap_{i\in I} A_i)$ for injective f.
As proven above in part a) that $f(\bigcap_{i\in I} A_i) \subset \bigcap_{i\in I} f(A_i)$ for the general case of any f, meaning it also applies when f is injective.
Hence, $\because f(\bigcap_{i\in I} A_i) \subset \bigcap_{i\in I} f(A_i)$ and $\bigcap_{i\in I} f(A_i) \subset f(\bigcap_{i\in I} A_i)$ for an injective f.
$\therefore$ If f is injective, $f(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} f(A_i)$.

5. 1). Proof: let an arbitrary $x \in C$. Then $f(x) \in f(C)$; then it follows by definition that $x \in f^{-1}(f(C))$.

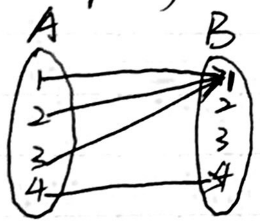
Thus, $C \subset f^{-1}(f(C))$

2) Proof: let an arbitrary $y \in f(f^{-1}(D))$; then $\exists\, x \in f^{-1}(D)$ s.t. $f(x)=y$.
Since $x \in f^{-1}(D)$, by definition, ~~$y=f(x) \in f(f^{-1}(D))$~~ $f(x)=y \in D$
~~Thus,~~

Thus, $f(f^{-1}(D)) \subset D$.

3) ① counter-examples for 1): Take $f(x)=x^2$ where $f: \mathbb{R} \to \mathbb{R}$
Take $C=1$, $f(C)=1$
While $f(C)=1$, $f^{-1}(f(C))=\{1,-1\} \neq C$
∴ It doesn't require an equality.

② counter-example for 2): Take the function grapheel below
Take $D=\{1,2\}$
$f^{-1}(D)=\{1,2,3\}$
$f(f^{-1}(D)) = \{1\} \neq \{1,2\}$
∴ It also doesnot require an equality.



6. ① Goal: If $f: A \to B$ is invertible, then f is a bijective.

Proof: assume $g: B \to A$ is the inverse of f.

assume ~~a, b ∈ A s.t.~~ $x_1, x_2 \in A$ s.t. $f(x_1) = f(x_2)$

$\because$ f is invertible, so we can apply g to both sides

$\Rightarrow \; g(f(x_1)) = g(f(x_2))$

$\Rightarrow \; x_1 = x_2$

$\therefore$ f is injective.

assume $y_3 \in B$ and let $x_3 = g(y_3)$

Then $f(x_3) = f(g(y_3)) = y_3$

$\therefore$ f is surjective. $\therefore$ f is bijective.

② Goal: If $f: A \to B$ is bijective, then f is invertible.

Proof: assume $y_4 \in B$ ~~s.t. $\exists x_4$~~

Since f is surjective, $\exists x_4$ s.t. $f(x_4) = y_4$.

Since f is injective, there's only one x_4 existing.

Set a function g s.t. $g(y_4) = x_4$

$\because$ $g(f(x_4)) = x_4$ and $f(g(y_4)) = y_4$ for an arbitrary x_4

$\therefore$ g is the inverse function of f, and f is invertible.

$\therefore$ A function $f: A \to B$ is invertible if and only if it is bijective.

No.

Date

①

7. Construct a bijection between X^n and the set of functions from $\{1, \ldots, n\}$ to X

let $(x_1, \ldots, x_n) \in X^n$ and let $h: i \to x_i$, $i \in \{1, \ldots, n\}$

It follow that $x_k \in X^n$, for $k \in \{1, \ldots, n\}$.

Let $f: [x_k \in X^n, \text{s.t. } k \in \{1, \ldots, n\}] \longrightarrow h(i)$, for all $i \in \{1, \ldots, n\}$

As we could set a bijective ~~relationship~~ relationship by letting $k = i$.

Hence, it follows that $f(x_k) = h(k) = x_k$

It is clearly a bijection.

The inverse of f would be $g(h(k)) = x_k$.

$g: h(k) \to x_k$ for $k \in \{1, \ldots, n\}$.

② If $X = \phi$. X^n would also be ϕ.

Therefore, f would become an empty function or null function. Also, ~~as h(k)~~ as $X^n = \phi$, the codomain of $h(k)$ will become ~~the~~ empty, which leads to a nonexistence. According to the definition of function, we cannot map a nonempty set to an empty set. Therefore, if $X = \phi$, it will lead to a ~~mapping~~ function from an empty set to a non-existing function.

8. Proof: suppose there's a set A which contains all sets.

Assume $N \in A$.

~~let $N=\{$~~

let $N=\{x \in A \mid x \notin x\}$ i.e. elements of N belong to A (definition of subset) such that they don't belong to themselves

There're only 2 ~~scenarios~~ relationships between sets: $\in$ or $\notin$.

① Say $N \in N$, then according to the property of N,
$N \in A$ such that $N \notin N$. Thus, a contradiction exists.

② Say $N \notin N$, then based on the definition of a set's complement
$N \in N^c$. $N^c=\{x \in A \mid x \in x\}$

Therefore, it follows that $N \in A$ s.t. $N \in N$.
Thus, a contradiction exists.

Hence, based on the 2 contradictions, we could infer that our assumption $N \in A$ is wrong, and $N \notin A$.