泛函分析[1]习题参考答案

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[1]泛函分析 第2版 江泽坚 孙善利 高等教育出版社
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文档说明：

1、这份答案的前身是我自己整理的**PPT**版本答案，之所以有这份答案是因为学习**LaTeX**语言，为了实践一下。

2、答案仅供参考，所有答案均来自于我课外的两本习题参考书[1]，以及老师、助教的讲评。

3、会存在打印错误，由于上面**2**的存在，所以思维错误理论上来说应该不存在，如果你发现某道题目不太对，请先想一想，谢谢~当然说不定就是我错了~

4、再次强调，答案仅供参考，如果你因为平时偷懒参考了很多答案，不好意思，期末估计不会太好~

5、第一次学习**LaTeX**，排版还有很多问题，逐步完善。

6、关于题目，答案包括了前三章的大部分习题，第四章的**60%**的样子。如果你发现你需要的答案这里没有，那么有三种情况：

$$\begin{cases}\text{太难了，我不会做}\\ \text{太简单了，以至于不需要解答}\\ \text{过程太长了，我不想输入}\end{cases}$$

7、关于证明过程，有些证明过程如果你看不懂，不要在意，略去的都是计算以及化简过程，没有任何技术含量。

8、泛函的考试可以用勤能补拙来形容，只要把书上的定理证明都背会，习题的解答背下来，考试**80+**没有问题. . . (其实我觉得是理科生学习，文科生考试，仅仅对于考试而言)

9、如果你有任何问题，可以告诉我**mcelibate@163.com**

PS:泛函学的这么苦逼，**Banach**，**Hilbert** 他们知道么？

[1]泛函分析习题集 V.K.Krishnan 著 步尚全、方宜 译 清华大学出版社
泛函分析疑难分析与解题方法 孙清华 孙昊 华中科技大学出版社

目录

第一章　距离线性空间

习题1

习题2

习题3

Proof. 设$\xi_j^{(n)} \to \xi_j$,即

$$\forall \varepsilon > 0, \exists m \in N, s.t \ \sum_{j=m}^{\infty} \frac{1}{2^j} < \frac{\varepsilon}{2}$$

对$j = 1, ..., m-1$时，有

$$\sum_{j=1}^{m-1} \frac{1}{2^j} \frac{\left|\xi_j^{(n)} - \xi_j\right|}{1 + \left|\xi_j^{(n)} - \xi_j\right|} < \sum_{j=1}^{m-1} \frac{1}{2^j} \frac{\varepsilon/2}{1 + \varepsilon/2} < \frac{\varepsilon}{2}$$

$\therefore n > N$时,有

$$d(x_n, x) = (\sum_{j=1}^{m-1} + \sum_{j=m}^{\infty}) \frac{1}{2^j} \frac{\left|\xi_j^{(n)} - \xi_j\right|}{1 + \left|\xi_j^{(n)} - \xi_j\right|} < \varepsilon$$

即$d(x_n, x) \to 0$ □

习题4

Proof. 在 c 中，令

$$A = \{x = (\xi_i) | \xi_i \text{为有理数,存在自然数} N \text{使得} \forall i \geq N \text{有} \xi_i = \xi_N\}$$

则 A 为可数集,且在 c 中稠密。 □

习题5

Proof.

$$\begin{aligned} |d(x_m, y_m) - d(x_n, y_n)| &= |d(x_m, y_m) - d(x_m, y_n) + d(x_m, y_n) - d(x_n, y_n)| \\ &\leq |d(x_m, y_m) - d(x_m, y_n)| + |d(x_m, y_n) - d(x_n, y_n)| \\ &\leq |d(y_m, y_n)| + |d(x_m, x_n)| \end{aligned}$$

□

习题6

Proof. $\{x_n\}$为$Cauchy$列$\Rightarrow \forall \varepsilon > 0, \exists N > 0, s.t \quad \forall n, m > N, d(x_m, x_n) < \varepsilon$,
令$r = \max\limits_{1 \le i \le N} \{d(x_{N+1}, x_i), \varepsilon\}, \{x_n\} \subset B(x_{N+1}, r) \Rightarrow$ 有界 □

习题7

习题8

习题9

Proof. 当$1 \le p < \infty$时
对于任何$Cauchy$列,$\{x_n\} \subset l^p : x_n = (\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_i^{(n)}, \ldots), n = 1, 2, \ldots,$
有$\forall \varepsilon > 0$,存在$N > 0, m, n > N$时

$$d(x_m, x_n) = \sum_{i=1}^{\infty} \left| \xi_i^{(m)} - \xi_i^{(n)} \right|^p < \varepsilon^p$$

从而对每个$i = 1, 2, \ldots, \left\{ \xi_i^{(n)} \right\}$ 是R中的$Cauchy$列,由 R 的完备性知,存在$\xi_i \in R$,使得$\xi_i^{(n)} \rightarrow \xi_i, n \rightarrow \infty$.同时对于任何自然数$s, \sum\limits_{i=1}^{s} \left| \xi_i^{(n)} - \xi_i^{(m)} \right|^p < \varepsilon^p$,令$m \rightarrow \infty$,得$\sum\limits_{i=1}^{s} \left| \xi_i^{(n)} - \xi_i \right|^p \le \varepsilon^p$,从而$\sum\limits_{i=1}^{\infty} \left| \xi_i^{(n)} - \xi_i \right|^p \le \varepsilon^p$.令$x = (\xi_1, \xi_2, \ldots, \xi_i, \ldots)$,则由

$$(\sum_{i=1}^{\infty} |\xi_i|^p)^{\frac{1}{p}} \le (\sum_{i=1}^{\infty} \left| \xi_i^{(n)} \right|^p)^{\frac{1}{p}} + (\sum_{i=1}^{\infty} \left| \xi_i^{(n)} - \xi_i \right|^p)^{\frac{1}{p}}$$

可知$x \in l^p$,由$d(x_n, x) \le \varepsilon$,可知$x_n \rightarrow x$.从而$l^p (1 \le p < \infty)$完备.
当$p = \infty$时
取$\{x_n\}$是l^∞中的$Cauchy$列,$x_n = (\xi_1^{(n)}, \xi_2^{(n)}, \xi_3^{(n)}, \ldots), \therefore \forall \varepsilon > 0, \exists N > 0$,当$m, n > N$时,有$d(x_m, x_n) = \sup\limits_i |\xi_i^{(m)} - \xi_i^{(n)}| < \varepsilon$对每个固定的$i$,当$m, n > N$时,有$|\xi_i^{(m)} - \xi_i^{(n)}| < \varepsilon (1) \therefore$ 数列$(\xi_i^{(1)}, \xi_i^{(2)}, \xi_i^{(3)}, \xi_i^{(4)}, \ldots)$,是$C$中的$Cauchy$列,即当$m \rightarrow \infty$时,$\xi_i^{(m)} \rightarrow \xi_i \in C$由此得$x = (\xi_1, \xi_2, \xi_3, \xi_4, \ldots)$由(1)中,令$n \rightarrow \infty$,则当$m > N$时,有$|\xi_i^{(m)} - \xi_i| \le \varepsilon$.,又因为$x_m = \{\xi_i^{(m)}\} \in l^\infty$故存在实数$k_m$,对所有的$i$,满足$|\xi_i^{(m)}| \le k_m$,从而对每个$i$有$|\xi_i| \le |\xi_i - \xi_i^{(m)}| + |\xi_i^{(m)}| \le \varepsilon + k_m$,即$\{\xi_i\}$是有界数列,$x = \{\xi_i\} \in l^\infty$,又$|\xi_i^{(m)} - \xi_i| \le \varepsilon$,有$d(x_m, x) = \sup\limits_i |\xi_i^{(m)} - \xi_i| \le \varepsilon$,故当$m \rightarrow \infty$时,$x_n \rightarrow x$,从而$l^\infty$完备. □

习题10

Proof. $\Rightarrow$由A完全有界知,$\forall \varepsilon > 0, \exists A$的有限$\varepsilon-$网,$N = \{x_1, ...x_n\}$.
令$M = span\{N\}, \forall x \in A, d(x, M) \le \|x - x_k\| < \varepsilon$.
$\Leftarrow \forall x \in A, \forall \varepsilon > 0, d(x, M) < \varepsilon / A$有界,则$\|x\| \le C, \forall x \in A$.令$N = \{x \in M, \|x\| \le C + 1\}$为$M$中有界闭集,$\because$ 有限维空间中有界闭集紧,$\therefore \exists N$的有限$\varepsilon-$网$\{x_1, ...x_n\}$,下证$\{x_1, ...x_n\}$, 为A 的$2\varepsilon-$网.$\forall x \in A, d(x, M) < \varepsilon, \exists y \in M, s.t \quad \|x - y\| \le \varepsilon \therefore \|y\| \le \|x\| + \varepsilon \le C + \varepsilon (\varepsilon = 1), y \in N \therefore \exists x_k \in s.t \quad \|y - x_k\| \le \varepsilon \therefore \|x - x_k\| \le \|x - y\| + \|y - x_k\| < 2\varepsilon$ □

习题11

习题12

习题13

习题14

习题15

Proof. $\Rightarrow$若S是列紧集,则S是完全有界集,(i)显然成立.
设$\{x_n=(\xi_i^{(n)})|1\leq n\leq m\}$是$S$的有限$\frac{\varepsilon}{2}-$网,则存在自然数$N$,使$\sum\limits_{i=N+1}^{\infty}|\xi_i^{(n)}|^p$
$<(\frac{\varepsilon}{2})^p$,对$\forall n=1,2,\ldots,m$.对$\forall x=(\xi_i)\in S$ 存在$n_0:1\leq n_0\leq m$,使得$d(x_{n_0},x)<\frac{\varepsilon}{2}$.那么$(\sum\limits_{i=N+1}^{\infty}|\xi_i|^p)^{\frac{1}{p}}\leq(\sum\limits_{i=N+1}^{\infty}|\xi_i-\xi_i^{(n_0)}|^p)^{\frac{1}{p}}+(\sum\limits_{i=N+1}^{\infty}|\xi_i^{(n_0)}|^p)^{\frac{1}{p}}$
$<d(x_{n_0},x)+\frac{\varepsilon}{2}<\varepsilon$.从而(ii)成立.
$\Leftarrow$假设S集合满足这两个条件,设$\{x_n=(\xi_i^{(n)})|n=1,2,\ldots\}$是$S$中任意一个点列,由$(i)$对任意的$i=1,2,\ldots$,数集$\{\xi_i^{(n)}\}$有界,存在自然数列的子列$\{n_k^1\}$,使$\{\xi_1^{(n_k^1)}\}$ 收敛于ξ_1.又存在$\{n_k^1\}$的子列$\{n_k^2\}$使$\{\xi_2^{(n_k^2)}\}$收敛于ξ_2, 等等如此下去.令$x_{n_j^j}=(\xi_i^{(n_j^j)})$,利用$(ii)$容易证明$\{x_{n_j^j}\}$是基本列.令$x=(\xi_i)$, 利用$(i)$可以证明$x\in l^p$,并且$\{x_{n_j^j}\}$收敛于$x$,即可在$\{x_n=(\xi_i^{(n)})|n=1,2,\ldots\}$ 中选出收敛子列,$\therefore$集合S是列紧集. □

习题16

Proof. $\Rightarrow$若存在自然数i,对任意的$M>0$,存在$x=(\xi_i)\in S$使得$|\xi_i|>M$.这样就可以做一个S 中的序列$x_n=(\xi_i{}^{(n)})$使得$|\xi_i{}^{(n)}|>n$.若$\{x_n\}$有子列$\{x_{n_k}\}$收敛.设其极限为$y=(\eta_i)$.则因$\frac{1}{2^i}\frac{|\xi_i{}^{(n_k)}-\eta_i|}{1+|\xi_i{}^{(n_k)}-\eta_i|}\to\frac{1}{2^i}\neq0$, $\therefore d(x_{n_k},y)$不收敛于0, 得到矛盾. $\therefore\{x_n\}$没有收敛子列，S 不是列紧集.
$\Leftarrow$设对任意自然数n,存在$M_n>0$,使得对任意的$x=(\xi_n)\in S$,有$|\xi_n|\leq M_n$.设$\{x_n\}$是S中任一序列存在$\{x_n\}$的子列$\{x_{n,1}=(\xi_i^{(n,1)})\}$,使$\xi_1^{(n,1)}\to\eta_1$. 下一步,存在$\{x_{n,1}\}$ 的子列$\{x_{n,2}=(\xi_i^{(n,2)})\}$,使得$\xi_2^{(n,2)}\to\eta_2$. 依次做下去,然后考虑$\{x_n\}$的子列$\{x_{n,n}\}$,则它的第$i$个坐标收敛于$\eta_i$.令$y=\{\eta_i\}$,显然$\{x_{n,n}\}$ 收敛于$y\in s$, $\therefore S$是列紧集. □

习题17

习题18

习题19

Proof. 只要X不完备即可,取$X=(0,1], d(x,y)=|x-y|, (0,0.5)\subset X$但是不列紧.$\{x_n=\frac{1}{n+2}|n=1,2,...\}\subset(0,0.5)$无收敛子列. □

习题20

Proof. 设$\{x_n\}$的任意子列收敛于x_0.如果$\{x_n\}$不收敛于x_0.则必存在$\varepsilon_0>0$,使对任意的自然数N都存在$n>N$, 使$d(x_n,x_0)\geq\varepsilon_0$,于是可选取$\{x_n\}$的一个子列$\{x_{n_k}\}$, 使$d(x_{n_k},x_0)\geq\varepsilon_0$,显然$\{x_{n_k}\}$不收敛于$x_0$,矛盾. □

习题21

Proof. $\Rightarrow$设F为Y中的闭集,任取$\{x_n\}\subset f^{-1}(F), x_n\to x, x\in X$,则$f(x_n)\in F$,由$T$的连续性可知,$f(x_n)\to f(x)$,从而$f(x)\in F$即$x\in f^{-1}(F)$.
$\Leftarrow$设$x\in X$,任取$\{x_n\}\subset X, x_n\to x$.假设$f(x_n)\to f(x)$不成立,则存在$\varepsilon_0>0$和子列$\{x_{n_k}\}\subset X$, 使得$d(f(x_{n_k}),f(x))\geq\varepsilon_0$.令$F=\{y|d(y,f(x))\geq\varepsilon_0\}$,则 $f(x_{n_k})\subset F$ 并且F为Y中的闭集，从而$f^{-1}(F)$是X中的闭集.由$\{x_{n_k}\}\subset X$可得,$x\in f^{-1}(F)$,即$f(x)\in F$,由此可得$0=d(f(x),f(x_{n_k}))\geq\varepsilon_0>0$,这一矛盾说明,$f(x_n)\to f(x)$,即$T$为连续映射. □

习题22

Proof. $2n$个$, e_1,...,e_n, ie_1,...,ie_n$ □

习题23

Proof. $\Rightarrow X$为$Banach$空间,假设$\sum\limits_{k=1}^{\infty}x_k$绝对收敛$\Rightarrow\sum\limits_{k=1}^{\infty}\|x_k\|$收敛$\Rightarrow\exists N, s.t$

$$\sum_{k=N+1}^{\infty}\|x_k\|<\varepsilon$$

设$S_n=\sum\limits_{k=1}^{n}x_k, n>m$时,

$$\|S_n-S_m\|=\left\|\sum_{k=m+1}^{n}x_k\right\|\leq\sum_{k=m+1}^{n}\|x_k\|\leq\sum_{k=m+1}^{\infty}\|x_k\|<\varepsilon$$

则S_n为$Cauchy$序列,由X完备知收敛性成立.
$\Leftarrow$设$\{x_n\}$为X中$Cauchy$序列,$\exists$子序列$\{x_{n_k}\}s.t\ \ \|x_{n_k+1}-x_{n_k}\|<\frac{1}{2^k}$
则

$$\sum_{k=1}^{\infty}\|x_{n_k+1}-x_{n_k}\|\leq\sum_{k=1}^{\infty}\frac{1}{2^k}$$

$\therefore\sum\|x_{n_k+1}-x_{n_k}\|$收敛,
由条件$x_{n_1}+\sum\limits_{k=1}^{\infty}(x_{n_k+1}-x_{n_k})$收敛,即$\{x_{n_k+1}\}$ 收敛,由习题20知$\{x_n\}$收敛,
$\therefore X$为$Banach$空间. □

习题24

Proof. 设X是n维的,$e_1,\ldots,e_n$是X的一个基,取$M>0$,使对任意$x=\sum\limits_{i=1}^{n}\alpha_ie_i$ 都有

$$\sum_{i=1}^{n}|\alpha_i|\leq M\|x\|\,(Lemma7.1),$$

则

$$\begin{aligned} ||Tx|| &= ||\sum_{i=1}^{n}\alpha_i Te_i|| \\ &\le \sum_{i=1}^{n}|\alpha_i|||Te_i|| \\ &\le \left(\sum_{i=1}^{n}|\alpha_i|\right)\max_{1\le i\le n}\|Te_i\| \\ &\le M\max_{1\le i\le n}\|Te_i\|\,\|x\| \end{aligned}$$

$\therefore T$有界. □

习题25

Proof. 11级考题，高代数分知识可解答。 □

习题26

Proof. $\|T\| \le \sup\limits_{\|x\|_1=1}\|Tx\|_2 \le \sup\limits_{\|x\|_1\le 1}\|Tx\|_2 = \sup\limits_{\|x\|_1\le 1}\|T\|\,\|x\|_1 \le \|T\|$ □

习题27

Proof. 重要结论，经常使用，第四章尤见。 □

习题28

Proof. 设T压缩映射,则有$0<\alpha<1$,使得$d(Tx,Ty)\le\alpha d(x,y)$,从而,

$$d(T^n x, T^n y) \le \alpha^n d(x,y), 0<\alpha^n<1$$

$\therefore T^n$为压缩映射.
设$T: R^2\to R^2 (x_1,x_2)\mapsto(x_2,0)$不是压缩映射,但是$T^2: R^2\to R^2(x_1,x_2)\mapsto(0,0)$为压缩映射. □

习题29

Proof. 令$f(x)=d(x,Tx),\forall x\in F$,则由

$$|f(x)-f(y)|\le d(x,y)+d(Tx,Ty)\le 2d(x,y)$$

可知f是F上的连续线性泛函,$\because F$是有界闭集,$\therefore f$在F可达到最小值m.
下证$m=0$.
设$f(x')=m$,即$d(x',Tx')=m$,若$m\neq 0$,则

$$f(Tx')=(Tx',T(Tx'))<d(x',Tx')=m$$

此与$f(x)$的最小值m 矛盾. $\therefore m=0$,即$x'=Tx'$.
若另有$x^*\in F, x^*\neq x'$亦满足$x^*=Tx^*$.则

$$d(x',x^*)=d(Tx',Tx^*)<d(x',x^*).$$

此矛盾说明满足$Tx'=x'$的x'是唯一的. □

习题*30*

Proof. $L^2[0,1]$中的范数定义为

$$\|x\|=(\int_0^1|x(t)|^2dt)^{\frac{1}{2}},\forall x(t)\in L^2[0,1]$$

且

$$\int_0^1\left|\int_0^1K(t,s)x(s)ds\right|^2dt\leq\int_0^1(\int_0^1|K(t,s)|^2ds\int_0^1|x(s)|^2ds)dt<\infty$$

则

$$\int_0^1K(t,s)x(s)ds\in L^2[0,1]$$

因此定义$T:L^2[0,1]\to L^2[0,1]$

$$(Tx)(t)=f(t)+\lambda\int_0^1K(t,s)x(s)ds,\forall x(t)\in L^2[0,1].$$

下证T压缩,$\forall x,y\in L^2[0,1]$,

$$\begin{aligned}\|Tx-Ty\|^2&=\lambda^2\int_0^1\{\int_0^1K(t,s)[x(s)-y(s)]ds\}^2dt\\&\leq\lambda^2\int_0^1\int_0^1K^2(t,s)dsdt\int_0^1[x(s)-y(s)]^2ds\\&\leq\lambda^2M\|x-y\|^2\end{aligned}$$

$\lambda^2M<1,|\lambda|<\frac{1}{\sqrt{M}}\ \therefore T$压缩,由...知... □

第二章　Hilbert空间

习题1

Proof. $(iii), \Leftarrow$取$\alpha = -\frac{(x,y)}{\|y\|^2}$, 有

$$\begin{aligned} 0 &\leq (x+\alpha y, x+\alpha y) - \|x\|^2 \\ &= \overline{\alpha}(x,y) + \alpha(y,x) + |\alpha|^2\|y\|^2 \\ &= -|(x,y)|^2\|y\|^{-2} \leq 0 \end{aligned}$$

□

习题2

Proof. $\sum\limits_{n=1}^{\infty}|(x,e_n)(y,e_n)| \leq [\sum\limits_{n=1}^{\infty}|(x,e_n)|^2]^{\frac{1}{2}}[\sum\limits_{n=1}^{\infty}|(y,e_n)|^2]^{\frac{1}{2}} \leq \|x\|\,\|y\|$ □

习题3

Proof. 令$x_n = \sum\limits_{k=1}^{n}\alpha_k e_k, y_n = \sum\limits_{j=1}^{n}\beta_j e_j$,

$$(x_n, y_n) = (\sum_{k=1}^{n}\alpha_k e_k, \sum_{j=1}^{n}\beta_j e_j),$$

让$n \to \infty$,绝对收敛由范数得出. □

习题4

Proof. $\|x\| < \infty, \|y\| < \infty, x = \sum\limits_{n=1}^{\infty}(x,e_n)e_n, y = \sum\limits_{m=1}^{\infty}(y,e_m)e_m$
$\therefore (x,y) = \sum((x,e_n)e_n, (y,e_m)e_m) =$待证结论. □

习题5
习题6

Proof. $p \neq 2, x = (1,1,0,...), y = (-1,1,0,...)$
$\|x\| = \|y\| = 2^{\frac{1}{p}}, \|x+y\| = \|x-y\| = 2$不满足平行四边形法则. □

习题7

Proof. 直接验证，可参照高代知识。 □

习题8

Proof. 反证。 □

习题9

Proof. (i)设$x,y\in M^{\perp}$,则对任意的$z\in M,(x,z)=0,(y,z)=0$, 所以

$$(x+y,z)=0,(\alpha x,z)=\alpha(x,z)=0,$$

$\therefore M^{\perp}$是H的线性流形.
现在设$x_n\in M^{\perp},x_n\to x,z\in M$,则

$$(x,z)=\lim_{n\to\infty}(x_n,z)=0,$$

$\therefore x\in M^{\perp},M^{\perp}$是子空间.
$(ii)M\subset\overline{M}\ \therefore M^{\perp}\supset(\overline{M})^{\perp}$.
对$x\in M^{\perp},y\in M$,取$\{y_n\}\subset M, s.t\ \ y_n\to y$, 则

$$(x,y)=\lim_{n\to\infty}(x,y_n)=0,(x,y_n)=0$$

$\Rightarrow x\in(\overline{M})^{\perp}\Rightarrow M^{\perp}\subset(\overline{M})^{\perp}$
$(iii)\forall x\in {M_1}^{\perp}$和一切$y\in M_1$,有$(x,y)=0$.由$M\subset M_1$,知$\forall y\in M$, 有$(x,y)=0$, $\therefore x\in M^{\perp}$,即$x\in {M_1}^{\perp}\Rightarrow x\in M^{\perp}\Rightarrow {M_1}^{\perp}\subset M^{\perp}$. □

习题10

Proof. 设$\{f_n\}$为H^*中$Cauchy$序列,$\forall\varepsilon>0,\exists N,\forall m,n>N$有

$$\|f_n-f_m\|\le\varepsilon\Rightarrow\sup_{\|x\|\le1}|(f_n-f_m)(x)|<\varepsilon(*)$$

$\forall y\in H$,

$$\left|f_n(\frac{y}{\|y\|})-f_m(\frac{y}{\|y\|})\right|<\varepsilon\Rightarrow|f_n(y)-f_m(y)|<\varepsilon\|y\|$$

则$\{f_n(y)\}$收敛.
设$f_n(y)\to f(y)$令$(*)$中$m\to 0$,
则$\sup\limits_{\|x\|\le1}|f_n(y)-f(y)|<\varepsilon$,$f_n(x)\rightrightarrows f(x)$ 则$f(x)$连续,$f\in H^*,H^*$ 完备. □

习题11

Proof. 对$\|y\|\le1,|(x,y)|\le\|x\|\,\|y\|\le\|x\|\,.\therefore\sup\limits_{\|y\|\le1}|(x,y)|\le\|x\|$.
令$z=\frac{x}{\|x\|}$

$$\sup_{\|y\|\le1}|(x,y)|\ge|(x,z)|=\left|(x,\frac{x}{\|x\|})\right|=\|x\|$$

□

习题12

Proof. 取$\|x\| < 1, \|y\| < 1$.又$|\phi(x,y)| = |(Ax,y)| \le \|Ax\|\,\|y\| \le \|A\|\,\|x\|\,\|y\| \le \|A\| \therefore$ $\|\phi\| \le \|A\|$.令$y = Ax$,则$\phi(x,Ax) = (Ax,Ax)$,$\|Ax\|^2 = |\phi(x,Ax)| \le \|\phi\|\,\|x\|\,\|Ax\| \Rightarrow$ $\|Ax\| \le \|\phi\|\,\|x\| \Rightarrow \|A\| \le \|\phi\|$ □

习题*13*

Proof. 如果$\{f_n\}$不完备,则$\exists x \in H, x \neq 0$,使$(x,f_n) = 0(n = 1,2,\ldots)$.于是
$0 < \|x\|^2 = \sum\limits_{i=1}^{\infty} |(x,e_n)|^2 = \sum\limits_{i=1}^{\infty} |(x,e_n) - (x,f_n)|^2 = \sum\limits_{i=1}^{\infty} |(x,e_n - f_n)|^2 \le \sum\limits_{i=1}^{\infty} \|x\|^2 \|e_n - f_n\|^2 <$ $\|x\|^2$. 此矛盾说明$\{f_n\}$是完备的. □

习题*14*

Proof. $(Tx,y) = (x,T^*y) \therefore$取$x = T^*y$,则$|(TT^*y,y)| = |(T^*y,T^*y)|$,$\therefore \|T^*y\|^2 \le ||(TT^*y,y)|| \le$ $\|T^*y\|\,\|T\|\,\|y\|$. □

习题*15*

Proof. $((T{+}T^*)(x{+}y),(x{+}y)) = 0 \Rightarrow ((T{+}T^*)x,x){+}((T{+}T^*)y,y){+}2\mathrm{Re}((T{+}T^*)x,y) =$ $0 \Rightarrow \mathrm{Re}((T{+}T^*)x,y) = 0, \mathrm{Im}((T{+}T^*)x,y) = \mathrm{Re}((T{+}T^*)x,iy) = -i\mathrm{Re}((T{+}T^*)x,y) = 0$ $\therefore ((T + T^*)x,y) = 0 \Leftrightarrow T + T^* = 0$ □

习题*16*

Proof. 设$x_k = (0,...,\xi_k,0,..), Tx_k = (a_{1k}\xi_k, a_{2k}\xi_k,...)$
$(Tx_n,x_k) = a_{kn}\xi_n\xi_k = (x_n,T^*x_k) = \overline{a^*_{nk}}\xi_n\xi_k \Rightarrow \overline{a}_{kn} = a^*_{nk}$ □

第三章　Banach空间上的有界线性算子

习题1

Proof. $\|Tx\| = \sup\limits_i |\eta_i| = \sup\limits_i \left|\sum\limits_{j=1}^{\infty} a_{ij}\xi_j\right| \le \sup\limits_i \sum\limits_{j=1}^{\infty} |a_{ij}| \|x\| \therefore \|T\| \le \sup\limits_i \sum\limits_{j=1}^{\infty} |a_{ij}|$. 又$\forall \varepsilon > 0, \exists k > 0, s.t$

$$\sum_{j=1}^{\infty} |a_{kj}| > \sup_i \sum_{j=1}^{\infty} |a_{ij}| - \varepsilon.$$

取$x = \{\xi_i\}$,其中$\xi_i = \mathrm{sgn}(a_{ij})$,于是$\|x\| = 1$, 且

$$\|Tx\| = \sup_i |\eta_i| = \sup_i \left|\sum_{j=1}^{\infty} a_{ij}\xi_j\right| \ge \left|\sum_{j=1}^{\infty} a_{kj}\xi_j\right| = \sum_{j=1}^{\infty} |a_{kj}|.$$

$\because \|Tx\| \le \|T\| \|x\| \therefore$

$$\|T\| \ge \|Tx\| \ge \sum_{j=1}^{\infty} |a_{kj}| \ge \sup_i \sum_{j=1}^{\infty} |a_{ij}| - \varepsilon.$$

令$\varepsilon \to 0$,则$\|T\| \ge \sup\limits_i \sum\limits_{j=1}^{\infty} |a_{ij}|$ □

习题2

Proof. 记$\alpha = \sup\limits_{n\ge 1} |\alpha_n|$,则$\forall x = \{\xi_i\} \in l^1$,

$$\|Tx\| = \|\{\alpha_i\xi_i\}\| = \sum_{i=1}^{\infty} |\alpha_i\xi_i| \le \alpha \sum_{i=1}^{\infty} |\xi_i| = \alpha \|x\|,$$

$\therefore \|T\| \le \alpha$.
当$\alpha = 0$时,显然有$\|T\| = \alpha$.
当$\alpha > 0$ 时,对任意给定的$0 < \varepsilon < \alpha$, 必有某个α_{i_0},满足$|\alpha_{i_0}| > \alpha - \varepsilon$, 取

$$x = \{\delta_{ii_0}\}(\delta_{ii_0} = \begin{cases} 0 & i \ne i_0 \\ 1 & i = i_0 \end{cases})$$

则$\|x\| = 1$,且$|\alpha_{i_0}| = \|Tx\| \le \|T\| \|x\| = \|T\|$,故$\|T\| \ge \alpha - \varepsilon$,因为$\varepsilon > 0$ 是任意的,故$\|T\| \ge \alpha$. □

习题3

Proof. 设$\beta = \inf\limits_{n\geq 1} |\alpha_n|$,若$T$有有界逆$T^{-1}$,则$\forall x = (\xi_i) \in l^1$, $\left\|T^{-1}x\right\| \leq \left\|T^{-1}\right\| \cdot \|x\|$,设$e_i$为单位向量,则$Te_i = (0,\ldots,\overset{i}{\alpha_i},0,\ldots) = \alpha_i e_i$,故$\left\|T^{-1}(\alpha_i e_i)\right\| = \left\|T^{-1}Te_i\right\| = \|e_i\| = 1$.但$\left\|T^{-1}(\alpha_i e_i)\right\| \leq \left\|T^{-1}\right\| \cdot |\alpha_i| \cdot \|e_i\| = \left\|T^{-1}\right\| \cdot |\alpha_i|$.故$|\alpha_i| \geq \left\|T^{-1}\right\|^{-1}$, $\therefore \inf\limits_{n\geq 1} |\alpha_n| \geq \left\|T^{-1}\right\|^{-1} > 0$. 反之,设$\beta > 0$,则$\sup\limits_{n\geq 1} |\frac{1}{\alpha_n}| \leq \frac{1}{\beta}$,令$Sx = \left(\frac{1}{\alpha_i}\xi_i\right) \forall x = (\xi_i) \in l^1$,则$S$是有界线性算子,显然$TS = ST = I$ 故T有有界逆S. □

习题4

Proof. 反复运用$Minkowski$与$Holder$不等式。 □

习题5

Proof. $\|ABx\| \leq \|A\| \|B\| \|x\| \therefore \|AB\| \leq \|A\| \|B\|, \left\|B^{-1}A^{-1}\right\| \leq \left\|B^{-1}\right\| \left\|A^{-1}\right\|$
又$(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB)$. □

习题6

Proof. 设$x_n \in N, x_n \to x$,由T有界知连续知$Tx = \lim\limits_{n\to\infty} Tx_n = 0,, \therefore x' \in N$, 即$N$是闭集,$N$显然$X$是的子空间.
X为$C[a,b]$中连续可微函数全体,定义$T: X \to C[a,b], (Tx)(t) = \frac{d}{dx}x(t)$.
则$N(T) = \{x_n | x_n(t) \equiv C, C$为常数$\}$,显然$N(T)$闭,但是$T$无界. □

习题7

Proof. 假设$x \neq y$,则$x - y \neq 0$,由$Hahn - Banach$定理知,$\exists f \in x', \|f\| = 1, f(x-y) = \|(x-y)\| \neq 0$.由$f(x-y) = f(x) - f(y) = 0$,导出矛盾,$\therefore$ ... □

习题8

Proof. $|f(x)| \leq \|f\| \|x\| \leq \|x\| \Rightarrow \sup |f(x)| \leq \|x\|$,对$x \neq 0$,由$Hahn - Banach$定理知$\exists f_0 \in X', \|f_0\| = 1, f_0(x) = \|x\|, \sup |f(x)| \geq |f_0(x)| = \|x\|$. □

习题9

Proof. $\forall \varepsilon > 0, \forall x, x_0 \in X, P(x) - P(x_0) \leq P(x - x_0), P(x_0) - P(x) \leq P(x_0 - x), \because$在0连续,$\therefore \lim\limits_{|x-x_0|\to 0} |P(x) - P(x_0)| \leq |P(x - x_0)| \leq \varepsilon$ □

习题10

习题11

习题12

习题13

Proof. 设$y \in B(0,1) \triangleq I$,由$Riesz$表现定理知,对于$f_y \in H^*$,有$f_y(x) = (Ax, y) = (x, Ay), \forall x \in H, |f_y(x)| = |(x, Ay)| = |(Ax, y)| \leq \|Ax\| \|y\| \Rightarrow \sup\limits_{\|y\|\leq 1} |f_y(x)| \leq \sup\limits_{\|y\|\leq 1} \|Ax\| \|y\| \leq \|Ax\|$.由一致有界定理知$\sup\limits_{y\in I} \|f_y\| < \infty, \|A\| = \sup\limits_{y\in I} \|Ay\| = \|f_y\| < \infty$. □

习题14

Proof. 设$x_n \to x_0, Ax_n \to y_0$.则$\forall y \in H$,有$(y_0, y) = \lim\limits_{x\to\infty}(Ax_n, y) = \lim\limits_{x\to\infty}(x_n, By) = (x_0, By) = (Ax_0, y)$. $\therefore y_0 = Ax_0$. 故A为闭算子,由闭图形定理,得证. □

习题15

Proof. 令$y_n \to y_0, T^{-1}y_n \to x_0, x_n \to x_0, Tx_n \to y_n$,两边极限

$$\lim_{n\to\infty} Tx_n = T\lim_{n\to\infty} x_n = Tx_0 = \lim_{n\to\infty} y_n = y_0,$$

故$Tx_0 = y_0$,即$T^{-1}y_0 = x_0 \therefore T^{-1}$闭算子. □

习题16

Proof. T是闭算子,$x_n \to x_0, Tx_n \to y_0 \Rightarrow x_0 \in D$且$Tx_0 = y_0$.当$(x_n, Tx_n) \to (x_0, y_0)$时,$x_0 \in D$且$Tx_0 = y_0 \therefore (x_0, y_0) \in G$,即$G$是闭的. □

习题17

Proof. 定义$T_n \in L(X', l^1), T_n f = ((f(x_1)), ..., (f(x_n)), 0, ...)$,
则

$$\begin{aligned}\sup_{n\in N}\|T_n f\| = \sum_1^n |f(x_i)| < \infty \quad &\Rightarrow \\ \sup_{n\in N}\|T_n\| < \infty \quad &\Rightarrow \\ |f(x_n)| \le \sup_{n\in N}\|T_n f\| \le \sup_{n\in N}\|T_n\|\,\|f\| &\end{aligned}$$

令$\mu = \sup\limits_{n\in N}\|T_n\|$. □

习题18

Proof. 设E是无穷维赋范线性空间,如果E'是有限维的,设E'是n维的,则E'中只能有n个线性无关的元,现在因为E是无穷维的,故E中有$n+1$ 个线性无关元,设$x_1, \ldots, x_{n+1}$是E中无关的元,$x_1, \ldots, x_{n+1}$ 张成的$n+1$ 维子空间E_0,设$f_1, \ldots, f_{n+1}$是E_0上的线性泛函满足

$$f_i(x_j) = \begin{cases} 0 & j \ne i \\ 1 & j = i \end{cases}$$

则$f_i(1 \le i \le n+1)$是E_0上的连续线性泛函,把它们保范延拓成E上的连续线性泛函,仍记为$f_1, \ldots, f_{n+1}$, 设$\sum\limits_{i=1}^{n+1}\alpha_i f_i = 0$,则$\sum\limits_{i=1}^{n+1}\alpha_i f_i(x_i) = 0$,但$\sum\limits_{i=1}^{n+1}\alpha_i f_i(x_i) = \alpha_i$, 故$\alpha_i = 0 (i = 1, \ldots, n+1)$.这说明$f_1, \ldots, f_{n+1}$线性无关,此与$E'$是$n$维的矛盾.
有限维课本$P115$. □

习题19

Proof. 定义典型映射$\tau : X \to X''$以及$\tau' : X' \to X'''$
$\Rightarrow$设${x_0}''' \in X'''$,定义$x' : X \to C$,${x_0}'(x) = {x_0}'''(\tau(x))$,则${x_0}' = {x_0}''' \circ \tau \in X'$.$\because X$自反,$\therefore$取$x \in X\ s.t\ \ \tau(x) = x''.\tau'({x_0}')(x'') = x''({x_0}') = \tau(x)({x_0}') = {x_0}'(x) = {x_0}'''(\tau(x)) = {x_0}'''(x'')$即$\forall x'' \in X'', \tau'({x_0}')(x'') = {x_0}'''(x''), \tau'({x_0}') = {x_0}''', \tau'$满射$X'$自反.
$\Leftarrow$反证:X不自反,$\tau(X) \neq X''$,由$Hahn-Banach$定理,$\exists {x_0}'''(\neq 0) \in X''', s.t$
${x_0}'''(\tau(x)) = 0, \forall x \in X$,$\because$ 若有${x_0}' \in X' s.t\ \ \ \tau'({x_0}') = {x_0}'''$,则${x_0}'(x) = \tau(x)({x_0}') = {x_0}'''(\tau(x)) = 0 \therefore {x_0}' = 0$ 从而${x_0}''' = \tau'({x_0}') = 0$, 导出矛盾,$\therefore X$自反. □

习题20

Proof. $T' : Y' \to X'$
(1)取$y' \in Y', s.t\ T'y' = 0, (T'y')x = y'(Tx) = 0, \forall x \in X$,$\because T$同构,$Tx = y, \therefore y'(Tx) = 0 \Rightarrow y' = 0$
(2)取$x' \in X', (T'y')x = y'(Tx) = x'(x)$.令$y'T = x'$,则$y' = (T^*)^{-1}x$,又$\because T$保范,则$||Tx|| = ||x||, \therefore ||T'y'|| = \sup\limits_{||x||=1} ||y'(x)|| = ||y'(x)||$ □

习题21

习题22

Proof. *Lemma*6.3
$\because \overline{R(A)}$为子空间,$\therefore {}^{\circ}\left(\left(\overline{R(A)}\right)^{\circ}\right) = {}^{\circ}N(A') \Leftrightarrow \overline{R(A)} = {}^{\circ}N(A')$
*Lemma*6.4
$\forall x' \in \overline{R(A')}, \exists y' \in Y'\ s.t\ \ A'y' = x'.\forall x \in N(A)\ Ax = 0$ 又$N(A) \subset X$
$\therefore x'(x) = A'y'(x) = y'(Ax)0. \therefore x' \in N(A)^{\circ} \Rightarrow \overline{R(A')} \subset N(A)^{\circ}$ □

习题23

习题24

Proof. 设$[a,b] = [0,\pi]$,在$L^p[0,\pi]$中令$f_n(x) = (\mathrm{sgn} \sin nx)\, |\sin nx|_p^{2}$, 则$\|f\| = \{\int_0^{\pi} \sin^2(nx)dx\}^{1/p} = (\frac{\pi}{2})^{1/p}$. 易证对一切$t \in [0,\pi]$,有$\int_0^t f_n(x)dx \to 0$. 故知$f_n \xrightarrow{w} 0$,但$f_n$不强收敛于0. □

习题25

Proof. 对固定的$t_0 \in [a,b]$,在$C[a,b]$上定义有界线性泛函f_{t_0},有$f_{t_0}(x) = x(t_0)$.由$x_n \xrightarrow{w} x$则$f_{t_0}(x_n) \to f_{t_0}(x)$,即$x_n(t_0) \to x(t_0) \therefore \{x_n\}$在$[a,b]$ 上逐点收敛. □

习题26

Proof. 任取$h \in Y'$,定义f为$f(x) = h(Tx), x \in X$,则f线性是显然的.$\because h \in Y'$,且T有界,有$|f(x)| = |h(Tx)| \le \|h\|\,\|Tx\| \le \|h\|\,\|T\|\,\|x\| \therefore f \in X'$.又$\because x_n \xrightarrow{w} x \Rightarrow f(x_n) \to f(x)$即$h(Tx_n) \to h(Tx)$, 而$h$是任意的,从而$Tx_n \xrightarrow{w} Tx_0$. □

习题27

Proof. 反证:若$x_0 \notin M$,则由题设知,$d = \rho(x_0, M) > 0$.于是依$Hahn - Banach$定理必可取到$f \in X', s.t \ \ f(x_0) = d, f(x) = 0, x \in M$.从而$f(x_n) = 0$.但是$x_n \xrightarrow{w} x_0$,$\therefore f(x_0) = \lim\limits_{n\to\infty} f(x_n) = 0$导出矛盾,$\therefore$必有$x_0 \in M$. □

习题28

Proof. 在X中取非零元a,$\exists f \in X' \ s.t \ \ f(a) = \|a\|$且$\|f\| = 1$.设$\{y_n\}$为$Y$ 中$Cauchy$列.定义$F_n : X \to Y$为$F_n(x) = f(x)y_n, x \in X$,$\because f$是线性的,$\therefore F_n$ 是线性的且$\|F_n(x)\| = |f(x)| \|y_n\| \leq \|f\| \|y_n\| \|x\|, x \in X$. 因此$F_n \in L(X, Y)$. 进而对所有$x \in X$,以及$n, m \geq 1$, 有$\|F_n(x) - F_m(x)\| = \|f(x)(y_n - y_m)\| = |f(x)| \|y_n - y_m\| \leq \|f\| \|x\| \|y_n - y_m\|$ $\therefore$ $\|F_n - F_m\| \leq \|f\| \|y_n - y_m\| = \|y_n - y_m\|$这就证明了$\{F_n\}$是$L(X, Y)$中的$Cauchy$列.$\therefore$在$L(X, Y)$的范数下它收敛到某个$F$.$\therefore \|F_n(a) - F(a)\| \leq \|F_n - F\| \|a\| \to 0$.即$\|a\| y_n = f(a)y_n = F_n(a) \to F(a)$这就证明了$\{y_n\}$收敛到$Y$中的$(1/\|a\|)F(a)$. □

习题29

习题30

Proof. $\because T \in L(X, Y), \therefore N(T) = \{x \in X | T(x) = 0\}$为$X$的子空间.$\therefore$商空间$(X/N(T), || \cdot ||_0)$是$Banach$空间,其中$|| \cdot ||_0$为商范数.$\because T$满射.$\therefore$诱导了线性同构$T_0 : X/N(T) \to Y, [x] \longmapsto y \triangleq T(x)$.要说明的是$T_0$是双射，且为有界线性算子.由逆算子定理知，${T_0}^{-1}$也是有界线性算子,$\therefore$ $\exists C > 0, s.t \ \ \forall y \in Y, ||{T_0}^{-1}(y)||_0 \leq C||y||$.由商范数定义:$||{T_0}^{-1}(y)|| = \inf\limits_{x \in {T_0}^{-1}(y)} ||x||$,可知，$\exists x \in {T_0}^{-1}$,满足$||x|| \leq 2||{T_0}^{-1}(y)||_0$, 综合上面不等式得到$||x|| \leq 2C||y||$,令$M = 2C$,得证. □

第四章　有界线性算子谱论

习题1

习题2

Proof. $R(\lambda, T_1) - R(\lambda, T_2) = (\lambda I - T_1)^{-1} - (\lambda I - T_2)^{-1}$
$= (\lambda I - T_1)^{-1} \left[(\lambda I - T_2) - (\lambda I - T_1)\right] (\lambda I - T_2)^{-1}$
$= (\lambda I - T_1)^{-1}(T_1 - T_2)(\lambda I - T_2)^{-1}$.
因为T_1与T_2可交换,故T_1与$R(\lambda, T_2)$,T_2 与$R(\lambda, T_1)$,$R(\lambda, T_1)$与$R(\lambda, T_1)$ 皆可交换,$\therefore$ $R(\lambda, T_1) - R(\lambda, T_2) = (T_1 - T_2)(\lambda I - T_1)^{-1}(\lambda I - T_2)^{-1}$
$= (T_1 - T_2)R(\lambda, T_1)R(\lambda, T_2)$ □

习题3

Proof. (1)即证$\mu \in \rho(A) \Leftrightarrow \lambda \in \rho(T)$
$\mu \in \rho(A) \Rightarrow R(\mu, A) = \frac{1}{\mu - A} = \frac{1}{(\alpha-\lambda)^{-1} - (\alpha - T)^{-1}} = R(\lambda, T)(\alpha - \lambda)(\alpha - T)$
$\lambda \in \rho(T) \Rightarrow R(\lambda, T) = \frac{1}{(\alpha-\lambda)(\alpha-T)} R(\mu, A)$
(2) $\mu + R(\lambda, T) = \frac{1}{\alpha-\lambda} + \frac{1}{\lambda - T} = \frac{1}{\alpha-\lambda}\frac{\alpha-T}{\lambda-T}$
$\mu^2 R(\mu, A) = \frac{1}{(\alpha-\lambda)^2}\frac{1}{(\mu-A)} = \frac{1}{\alpha-\lambda}\frac{\alpha-T}{\lambda-T}$
本题过程省略较多，只要验证即可。 □

习题4

Proof. $\because$ $(\lambda_0 I - T_n) = [I - \frac{T_n - T}{\lambda_0 I - T}](\lambda_0 I - T), \|T_n - T\| \to 0$,则对$0 < \varepsilon < 1, \exists N \in N, s.t \forall n > N$时,$\left\|(T_n - T)(\lambda_0 I - T)^{-1}\right\| \le \varepsilon < 1$

$$(\lambda_0 I - T_n)^{-1} = (\lambda_0 I - T)^{-1}\sum_{m=0}^{\infty}\left[(T_n - T)(\lambda_0 I - T)^{-1}\right]^m, (\lambda_0 I - T_n)^{-1} \in L(X)$$

即$\forall n > N$时,$\lambda_0 \in \rho(T_n)$ 且

$$\begin{aligned}\left\|(\lambda_0 I - T_n)^{-1} - (\lambda_0 I - T)^{-1}\right\| &\le \left\|(\lambda_0 I - T)^{-1}\right\| \sum_{m=1}^{\infty}(\|T_n - T\| \left\|(\lambda_0 I - T)^{-1}\right\|)^m \\ &\le \left\|(\lambda_0 I - T)^{-1}\right\| \frac{\|T_n - T\| \, \|\lambda_0 I - T\|^{-1}}{1 - \|T_n - T\| \left\|(\lambda_0 I - T)^{-1}\right\|},\end{aligned}$$

$\therefore \lim\limits_{n\to\infty}(\lambda_0 I - T_n)^{-1} = (\lambda_0 I - T)^{-1}$ □

习题5

Proof. 设$\lambda_1,\ldots,\lambda_n$为$\lambda$的$n$个$n$次方根,$x\in X$为$A^n$的一个对应于$\lambda$的特征向量,则$(A^n-\lambda I)x=0$,即

$$(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)x=0$$

若$(A-\lambda_1 I)x=0$,则λ_1 为A特征值,否则必有某个$i,s.t$

$$(A-\lambda_i I)(A-\lambda_{i-1} I)\cdots(A-\lambda_1 I)x\neq 0$$

$$(A-\lambda_{i+1} I)(A-\lambda_i I)\cdots(A-\lambda_1 I)x=0.$$

则λ_{i+1}为A特征值. □

习题*6*

Proof. $(1)\|Tx\|=\max\limits_n|\alpha_n\xi_n|\leq\max\limits_n|\alpha_n|\max\limits_n|\xi_n|\leq M\|x\|$

(2)取$e_n=(0,...,0,1,0,...),Te_n=\alpha_n e_n\Rightarrow\alpha_n$为一个特征值

(3)由$P146TH1.2$知$\sigma(T)$闭,$\therefore\overline{\{\alpha_n\}}=F\subset\sigma(T)$.若$\lambda\notin F$,则$d(\lambda,F)>0$,定义$R_\lambda:x=(\xi_1,...,\xi_n,...),R_\lambda(x)=(\frac{1}{\lambda-\alpha_1}\xi_1,...,\frac{1}{\lambda-\alpha_n}\xi_n,...),\|R_\lambda x\|\leq\frac{1}{d(\lambda,F)}\|x\|$,
且$R_\lambda(\lambda I-T)=(\lambda I-T)R_\lambda=I$,,则$\lambda I-T$双射,$\lambda\in\rho(T),\lambda\notin\sigma(T)\therefore F\supset\sigma(T)$.

(4)若$\lambda\in F\backslash\{\alpha_n\}_{n=1}^\infty$且$x=(\xi_1,...,\xi_n,...)s.t\quad Tx=\lambda x$.则$\forall n\in N,\lambda\xi_n=\alpha_n\xi_n.\because\lambda\neq\alpha_n\therefore\xi_n=0\Rightarrow x=0$, 则$\lambda$不为特征值,$\therefore\lambda\in\sigma_c(T)$,即$F\backslash\{\alpha_n\}_{n=1}^\infty\subset\sigma_c(T)$. □

习题*7*

习题*8*

Proof. 令$T_nx=(\alpha_1\xi_1,...,\alpha_n\xi_n,0,...)$.其中$x=\{\xi_n\}$,易知$T_n$为有限秩算子,$\forall\varepsilon>0,\exists N,\forall n>N$,有$|\alpha_n|<\varepsilon$且$\|(T_n-T)x\|\leq\varepsilon\|x\|\Rightarrow\|T_n-T\|\to 0$, 即$T$紧. □

习题*9*

Proof. 在X''中找$X_1'',...,X_n'',...$为有界线性算子,则$\forall x'\in X',\left\|X_n''x'\right\|<\infty,\because X_n''x'\in l\ \therefore$有收敛子列,设为$X_{n_j}''x'\to x_0''x'$,令$M=\sup\limits_{n_j}\left\|X_{n_j}''\right\|,\left\|X_0''x'\right\|\leq M\|x'\|\Rightarrow X_0''\in X'',X_{n_j}''x'\to X_0''x'\Rightarrow x'(x_{n_j})\to x'(x_0)$且$Ax_{n_j}\to Ax_0\Rightarrow A$紧. □

习题*10*

习题*11*

习题*12*

Proof. (1)设$Tx=\lambda x,T^nx=\lambda^nx,\therefore\lambda^n\in\sigma_p(T^n)$,由$T^n$紧可知$\sigma_p(T^n)$至多可数,而对于每个$\mu\in\sigma_p(T^n)$至多对应$n$个不同的$\lambda\in\sigma_p(T)\therefore\sigma_p(T)$至多可数
(2)反证:若$\lambda_n\to\lambda\neq 0$,则${\lambda_n}^n\to\lambda^n$,则$\lambda^n$为$\sigma_p(T^n)$的非零聚点,这与$T^n$为紧算子的唯一可能聚点为0矛盾. □

习题13

Proof. 11级考题。$\because T \neq 0, T \neq I \therefore \exists x_1, x_2 \in X, s.t\ Tx_1 \neq 0, Tx_2 \neq Ix_2 = x_2$,设$y_1 = Tx_1, y_2 = Tx_2 - x_2$,则$Ty_1 = T^2x_1 = Tx_1 = y_1, Ty_1 = y_1, y_1 \neq 0, \therefore 1$为$T$特征值.$Ty_2 = T^2x_2 - Tx_2 = Tx_2 - Tx_2 = 0, Ty_2 = 0 \cdot y_2, y_2 \neq 0, \therefore 0$为$T$特征值.设$\lambda \neq 0, 1.Q = \frac{1}{\lambda(1-\lambda)}T - \frac{1}{\lambda}I$,则$Q \in L(X)$,且$TQ = QT = \frac{1}{\lambda(1-\lambda)}T^2 - \frac{1}{\lambda}T = \frac{1}{\lambda(1-\lambda)}T - \frac{1}{\lambda}T = \frac{1}{1-\lambda}T = \lambda Q + I \therefore (T - \lambda I)Q = I = Q(T - \lambda I), \therefore (T - \lambda I)^{-1} = Q$在$L(X)$存在,即$\lambda$不为谱值.$\therefore \{0, 1\} \subset \sigma(T) \subset \{0, 1\}$.

□

习题14

Proof. 由题设知,$P_1^2 = P_1, P_2^2 = P_2, P_1P_2 = P_2P_1$,则$P^2 = (P_1 + P_2 - P_1P_2)^2 = P_1^2 + P_2^2 + P_1^2P_2^2 + 2P_1P_2 - 2P_1^2P_2 - 2P_1P_2^2 = P_1 + P_2 - P_1P_2 = P.\therefore P$ 为射影,又$\forall x, y \in H (Px, (I - P)y) = ((P_1 + P_2 - P_1P_2)x, (I - (P_1 + P_2 - P_1P_2)y) = (P_1x, (I - P_1)(I - P_2)y) + (P_2x, (I - P_2)(I - P_1)y) + (P_1(P_2x), (I - P_1)(I - P_2)y) = 0$,即$P$为正交射影.

$\because PP_1 = P_1^2 + P_1P_2 - P_1^2P_2 = P_1 \therefore P_1 \leq P$

$\because PP_2 = P_2^2 + P_1P_2 - P_1P_2^2 = P_2 \therefore P_2 \leq P$

若$Q \geq P_1, Q \geq P_2, \therefore QP = Q(P_1 + P_2 - P_1P_2) = P_1 + P_2 - P_1P_2 = P$,故$Q \geq P$. □

习题15

Proof. 因为$\|T^*\| = \|T\| \leq 1$,且$(T^*)^* = T$, 故只须证明:若$Tx = x$, 则必有$T^*x = x$. 设$Tx = x, Tx - x = 0, (Tx - x, x) = 0, (x, T^*x - x) = 0$, 即$x \perp (T^*x - x), \|T^*x\|^2 = \|(T^*x - x) + x\|^2 = \|T^*x - x\|^2 + \|x\|^2$,但$\|T^*x\|^2 \leq ||T^*||^2||x||^2 \leq \|x\|^2, \|T^*x - x\|^2 = 0$,即$T^*x = x$.

$\{x|Tx = x\} \subset \{x|T^*x = x\} \subset \{x|T^{**}x = x\} \subset \{x|Tx = x\}$. □

习题16

Proof. 先证$P : H \to L$为线性算子.

$\forall x, y \in H$有$x = x_1 + x_2, y = y_1 + y_2, x_1, y_1 \in L, y_1, y_2 \in L^{\perp}, x + y = (x_1 + y_1) + (x_2 + y_2), x_1 + y_1 \in L, x_2 + y_2 \in L^{\perp}$,从而$P(\alpha x + \beta y) = \alpha Px + \beta Py, \therefore P$为线性算子.

下证$\|P\| = 1$:

$\because \forall x \in H, x = x_1 + x_2$,有$\|Px\|^2 = \|x_1\|^2 \leq \|x_1\|^2 + \|x_2\|^2 = \|x\|^2 \therefore \|P\| \leq 1$.又若取$x_0 \in L$且$\|x_0\| = 1$,则$\|Px_0\| = \|x_0\| = 1$,则$\|P\| \geq \frac{||Px_0||}{||x_0||} = 1$.从而$\|P\| = 1$. □

习题17

Proof. 设$\lambda = m - d, d > 0.\forall x \in H, ((A - \lambda I)x, x) = (Ax, x) - \lambda(x, x) \geq m(x, x) - \lambda(x, x) = d(x, x), |((A - \lambda I)x, x)| \leq \|(A - \lambda I)x\| \|x\|, \therefore \|(A - \lambda I)x\| \geq d \|x\| \Rightarrow A - \lambda I$单射,$R(A - \lambda I)$为闭集,又由前知$\sigma_r(A) = \emptyset, \therefore R(A - \lambda I) = H, \therefore \lambda \in \rho(A), (-\infty, m) \subset \rho(A)$. □

习题18

Proof. 设$x,y\in H$,则$x=\sum\limits_{n=1}^{\infty}(x,e_n)e_n, y=\sum\limits_{n=1}^{\infty}(y,e_n)e_n$,

$\therefore (Tx,y)=(T\sum\limits_{n=1}^{\infty}(x,e_n)e_n,\sum\limits_{m=1}^{\infty}(y,e_m)e_m)=\sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty}(x,e_n)(y,e_m)(Te_n,e_m)$

$=\sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty}(x,e_n)(y,e_m)(e_m,Te_n)=\sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty}(x,e_n)(y,e_m)(e_n,Te_m)$

$=(\sum\limits_{n=1}^{\infty}(x,e_n)e_n,\sum\limits_{m=1}^{\infty}(y,e_m)Te_m)=(x,Ty)$ $\therefore T$是自伴的. □

习题19

习题20

Proof. $m=\inf\limits_{\|x\|=1}(Tx,x), M=\sup\limits_{\|x\|=1}(Tx,x)$.

$\Rightarrow\because T$自伴,$\sigma(T)\subset[m,M]$,且$m!M\in\sigma(T)$. 若$T\geq 0$, 知$m\geq 0,\therefore \sigma(T)\subset[m,M]\subset[0,\infty)$,

$\Leftarrow$若$\sigma(T)\subset[0,\infty)$ 则$m\geq 0, m\in\sigma(T)$,得$T\geq 0$. □

习题21

Proof. $\Rightarrow T=0$,显然任取$x\in H,(Tx,x)=0$.

$\Leftarrow$设任取$y\in H$,,有$(Ty,y)=0$.若$y=x+Tx$,则$Ty=Tx+T^2x$.

因而$0=(Ty,y)=(Tx+T^2x,x+Tx)$

$=(Tx,x)+(Tx,Tx)+(T^2x,x)+(T^2x,Tx)$

$=0+(Tx,Tx)+(Tx,Tx)+0\Rightarrow 2(Tx,Tx)=0$.

因此任取$x\in H$,有$Tx=0$,即$T=0$. □

习题22

习题23

习题24

Proof. (1) $T\in L(H)\therefore \|T^*T\|=\|T\|^2(P180)$ $\Pr op4.1$

$\|T^2\|=\sup\limits_{\|x\|=1}\|T^2x\|=\sup\limits_{\|x\|=1}\sqrt{(T^2x,T^2x)}=\sup\limits_{\|x\|=1}\sqrt{(T^*Tx,T^*Tx)}$

$=\sup\limits_{\|x\|=1}\|T^*Tx\|=\|T^*T\|=\|T\|^2$

(2) $r(T)=\lim\limits_{n\to\infty}\|T^n\|^{\frac{1}{n}}=\lim\limits_{n\to\infty}\|T^{2^n}\|^{\frac{1}{2^n}}=\|T\|$ (3)先证T正规$\Leftrightarrow\ \forall x\in H\,\|Tx\|=\|T^*x\|$:(本题只需要用到“$\Rightarrow$”)

$\forall x\in H,(T^*Tx,x)=(Tx,Tx)=\|Tx\|^2,(TT^*x,x)=(T^*x,T^*x)=\|T^*x\|^2\ \therefore \|Tx\|=\|T^*x\|\Leftrightarrow(T^*Tx,x)=(TT^*x,x)$.或者$(Bx,x)=0,B=T^*T-TT^*.\because T$自伴$\Leftrightarrow B=0\Leftrightarrow T^*T=TT^*.T$ 正规$\Rightarrow T-\mu I$正规.

再证$N(T-\mu I)=N(T^*-\overline{\mu}I)$:

$\|(T-\mu I)x\|=\|(T-\mu I)^*x\|=\|(T^*-\overline{\mu}I)x\|$,所有$x\in H,\Rightarrow (T-\mu I)x=0\Leftrightarrow(T^*-\overline{\mu}I)x=0$. 由题意知$Tx=\lambda x, Ty=\mu y$,则$x\in N(T-\lambda I), y\in N(T-\mu I)=N(T^*-\overline{\mu}I)$.

因此$Tx-x=0, T^*y-\overline{\mu}y=0.\therefore\lambda(x,y)=(\lambda x,y)=(Tx,y)=(x,T^*y)=(x,\overline{\mu}y)\mu y)=\mu(x,y)$.由于$\lambda\neq\mu$. 可以得到$\therefore(x,y)=0$,得证. □

习题25

Proof. 记$T_\lambda = T - \lambda I, G(\lambda) = T_\lambda(H), E(\lambda) = \{x \in H | T_\lambda x = 0\}$.则$H = \overline{G(\lambda)} + E(\lambda)$, 特别$H = \overline{G(0)} + E(0)(1), G(0)^\perp = E(0)$,但$E(0) = (E_0 - E_{0-})(H)(2)I - (E_0 - E_{0-}) \perp (E_0 - E_{0-})$,,故$(E_0 - E_{0-})(H)^\perp = (I - (E_0 - E_{0-}))(H)(3)$,由$(1)(2)(3)$得$(I - (E_0 - E_{0-}))(H) = G(0)$,即$T$的值域$G(0)$的闭包是$[(I - E_0) + E_{0-}](H)$. □

习题26

习题27

习题28

习题29

习题30

第五章　课本证明补充

$P11$例6上面的(实为充要条件)

Proof. $\Rightarrow$由$d(x_n,x)=(\sum\limits_{j=1}^{\infty}|\xi_j^{(n)}-\xi_j|^p)^{\frac{1}{p}}\to 0$知$\xi_j^{(n)}\to\xi_j, j=1,2,3\dots$

由$x=(\xi_1,\dots,\xi_j,\dots)\in l^p$可知,$\forall\varepsilon>0,\exists N_1>0,s.t\ \sum\limits_{j=N_1+1}^{\infty}|\xi_j|^p<(\frac{\varepsilon}{2})^p$,并且$n>N_1$时,$\sum\limits_{j=1}^{\infty}|\xi_j^{(n)}-\xi_j|^p<(\frac{\varepsilon}{2})^p$,由此知

$$\sum_{j=N_1+1}^{\infty}|\xi_j^{(n)}|^p\le((\sum_{j=N_1+1}^{\infty}|\xi_j^{(n)}-\xi_j|^p)^{\frac{1}{p}}+(\sum_{j=N_1+1}^{\infty}|\xi_j|^p)^{\frac{1}{p}})^p<\varepsilon^p,n>N_1$$

对于$n=1,2,3,\dots,N_1,\exists N_2>0,\sum\limits_{j=N_2+1}^{\infty}|\xi_j^{(n)}|^p<\varepsilon^p$.取$N=maxN_1,N_2$,则$\sum\limits_{j=N+1}^{\infty}|\xi_j^{(n)}|^p<\varepsilon^p,\forall n.$

$\Leftarrow$由条件知,$\forall\varepsilon>0,\exists K>0,s.t\ \sum\limits_{j=K+1}^{\infty}|\xi_j^{(n)}|^p<(\frac{\varepsilon}{2})^p,\forall n$.并且$\sum\limits_{j=K+1}^{\infty}|\xi_j|^p<(\frac{\varepsilon}{2})^p$.由$\xi_j^{(n)}\to\xi_j$可知,$\exists N>0,s.t\ n>N$时$\sum\limits_{j=1}^{K}|\xi_j^{(n)}-\xi_j|^p<\varepsilon^p$,并且,$d^p(x_n,x)=\sum\limits_{j=1}^{\infty}|\xi_j^{(n)}-\xi_j|^p=\sum\limits_{j=1}^{K}|\xi_j^{(n)}-\xi_j|^p+\sum\limits_{j=K+1}^{\infty}|\xi_j^{(n)}-\xi_j|^p\le\sum\limits_{j=1}^{K}|\xi_j^{(n)}-\xi_j|^p+((\sum\limits_{j=K+1}^{\infty}|\xi_j^{(n)}|^p)^{\frac{1}{p}}+(\sum\limits_{j=K+1}^{\infty}|\xi_j|^p)^{\frac{1}{p}})^p\le 2\varepsilon^p.$ □

$P58$定理3.1上面的易证

Proof. 见习题9. □

$P85$命题2.2的另一种等价形式

设X是赋范线性空间,E是X的子空间,$x_0\in X\backslash E$,则存在X上有界线性泛函f满足

$(1)f(x)=0,x\in E$

$(2)f(x_0)=d,$

$(3)||f||=1,$

这里$d=dist(x_0,E)>0.$

$P86$命题2.3

Proof. $\Rightarrow f(x)=0,\forall x\in M$,又$x_0\in\overline{M}$,由$f$连续,$f(x_0)=\lim\limits_{n\to\infty}f(x_n)=0$.
$\Leftarrow$反证:$x_0\notin\overline{M}$,由$Prop2.2$,$\exists$连续线性泛函$f(x)=0,x\in\overline{M},f(x_0)=1$,矛盾.

□

$P94$命题3.1必要性

Proof. 设$T\neq 0$,则$||T||\neq 0,||Tx||\leq ||T||||x||$.若$||x||\leq\frac{1}{||T||}$,则$||Tx||\leq 1,\therefore\{x\in X|||x||\leq\frac{1}{||T||}\}\subset T^{-1}\{y\in Y|||y||\leq 1\}$. □