# Least Squares Lattice Estimation

## Overview

- Recursive in order computation of predictions
- Recursive in time and order computation of predictors
- Initialization

Reference : Chapter 15 from *S. Haykin- Adaptive Filtering Theory - Prentice Hall, 1996.*

# 1 Recursive in time and order Least Squares Estimation

## Notations and terminolgy

For the time series $\{y(t)\}$, we define the *prediction* of order $n$ over $k$ steps, $\hat{y}_n(t+k|t)$:

$$\hat{y}_n(t+k|t) = -a_1 y(t) - a_2 y(t-1) - \ldots - a_n y(t-n+1). \tag{1}$$

Denote the observation vector

$$\varphi_n(t) = \begin{bmatrix} -y(t-1) & -y(t-2) & \ldots & -y(t-n) \end{bmatrix}^{\mathrm{T}} \tag{2}$$

the forward prediction

$$\begin{aligned} \hat{y}_n(t) &\triangleq \hat{y}_n(t|t-1) = -a_1 y(t-1) - a_2 y(t-2) - \ldots - a_n y(t-n) = \\ &= \theta_n^{\mathrm{T}} \varphi_n(t) \end{aligned} \tag{3}$$

and the forward predictor of order $n$,

$$\theta_n = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}^{\mathrm{T}}. \tag{4}$$

The forward prediction error is

$$\begin{aligned} \varepsilon_n^f(t) &= y(t) - \hat{y}_n(t) = y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) = \\ &= y(t) - \theta_n^{\mathrm{T}} \varphi_n(t). \end{aligned} \tag{5}$$

The backward prediction of order $n$ over $k = -n$ steps

$$\begin{aligned} \hat{y}_n^b(t-n) &\triangleq \hat{y}_n(t-n|t, \ldots, t-n+1) = -b_1 y(t) - b_2 y(t-1) - \ldots - b_n y(t-n+1) = \\ &= \theta_n^{b\mathrm{T}} \varphi_n(t+1) \end{aligned} \tag{6}$$

represents the extrapolation in the past of the data segment $\varphi_n(t)$. The backward predictor is

$$\theta_n^b = \left[ \begin{array}{cccc} b_1 & b_2 & \dots & b_n \end{array} \right]^{\mathrm{T}}$$

and the backward prediction error is

$$\begin{aligned} \varepsilon_n^b(t) &= y(t-n-1) - \hat{y}_n^b(t-n-1) = y(t-n-1) + b_1 y(t-1) + \ldots + b_n y(t-n) = \\ &= y(t-n-1) - \theta_n^{b\mathrm{T}} \varphi_n(t) \end{aligned} \tag{7}$$

**Problem 1.**

Given the time series $\{y(t)\}$, find $\{\hat{y}_n(N)\}_{n=1}^{n_{max}}$ obtained with the optimal forward predictors minimizing

$$V_n(\theta) = \sum_{t=n}^{N} (y(t) - \hat{y}_n(t))^2 = \sum_{t=n}^{N} (y(t) - \varphi_n^{\mathrm{T}}(t)\theta)^2 \tag{8}$$

and backward predictions $\{\hat{y}_n^b(N-n-1)\}_{n=1}^{n_{max}}$ obtained with the optimal backward predictors minimizing

$$V_n^b(\theta^b) = \sum_{t=n}^{N} (y(t-n-1) - \hat{y}_n^b(t-n-1))^2 = \sum_{t=n}^{N} (y(t-n-1) - \varphi_n^{\mathrm{T}}(t)\theta^b)^2, \tag{9}$$

by using recursive in time algorithms.

*Comments*

1. This problem does not require the stationarity of $y(t)$ (as it was required for Yule-Walker equations and Levinson algorithm).
2. Even if backward prediction is less interesting in practice, it is needed to get efficient updates.

**Lemma 1.1** *Knowing the sequence* $\{y(t)\}_{t=-1}^{N}$ *we denote:*

$$
\begin{aligned}
R_n(N) &= \sum_{t=n}^{N} \varphi_n(t)\varphi_n^{\mathrm{T}}(t), && (10)\\
r_n(N) &= \sum_{t=n}^{N} \varphi_n(t)y(t), && (11)\\
r_n^b(N) &= \sum_{t=n}^{N} \varphi_n(t)y(t-n-1), && (12)\\
\rho_0(N) &= \sum_{t=n}^{N} y^2(t) && (13)
\end{aligned}
$$

*and assume the matrice* $R_n(N)$*nonsingular.*

*i) The forward predictor minimizing (8) is*

$$\theta_n(N) = R_n^{-1}(N)r_n(N) \tag{14}$$

*and the minimum value of criterion (8) is*

$$\sigma_n(N) = \rho_0(N) - r_n^{\mathrm{T}}(N)\theta_n(N) \triangleq V_n(\theta_n(N)). \tag{15}$$

*ii) The backward predictor, minimizing (9) is*

$$\theta_n^b(N) = R_n^{-1}(N)r_n^b(N) \tag{16}$$

*and the minimum value of criterion (9) is*

$$\sigma_n^b(N) = \rho_0(N-n-1) - r_n^{b\mathrm{T}}(N)\theta_n^b(N) \triangleq V_n^b(\theta_n^b(N)). \tag{17}$$

To prove the lemma

$$
\begin{aligned}
V_n(\theta) &= \sum_{t=n}^{N}(y(t)-\varphi_n^{\mathrm{T}}(t)\theta)^2 = \sum_{t=n}^{N} y^2(t) - 2\sum_{t=n}^{N} y(t)\varphi_n^{\mathrm{T}}(t)\theta + \theta^{\mathrm{T}}\sum_{t=n}^{N}\varphi_n(t)\varphi_n^{\mathrm{T}}(t)\theta = \\
&= \rho_0(N) - 2r_n^{\mathrm{T}}(N)\theta + \theta^{\mathrm{T}}R_n(N)\theta = \\
&= (\theta - R_n^{-1}(N)r_n(N))^{\mathrm{T}}R_n(N)(\theta - R_n^{-1}(N)r_n(N)) + \\
&\quad +\rho_0(N) - r_n^{\mathrm{T}}(N)R_n^{-1}r_n(N).
\end{aligned}
\tag{18}
$$

and similarly

$$
\begin{aligned}
V_n^b(\theta^b) &= \sum_{t=n}^{N}(y(t-n-1)-\varphi_n^{\mathrm{T}}(t)\theta^b)^2 = \\
&= \sum_{t=n}^{N} y^2(t-n-1) - 2\sum_{t=n}^{N} y(t-n-1)\varphi_n^{\mathrm{T}}(t)\theta^b + \theta^{b\mathrm{T}}\sum_{t=n}^{N}\varphi_n(t)\varphi_n^{\mathrm{T}}(t)\theta^b = \\
&= (\theta^b - R_n^{-1}(N)r_n^b(N))^{\mathrm{T}}R_n(N)(\theta^b - R_n^{-1}(N)r_n^b(N)) + \\
&\quad +\rho_0(N-n-1) - r_n^{b\mathrm{T}}(N)R_n^{-1}r_n^b(N).
\end{aligned}
\tag{19}
$$

The matrix $R_n(N)$ is positive semidefinite,

$$
\alpha^{\mathrm{T}}R_n(N)\alpha = \alpha^{\mathrm{T}}\sum_{t=n}^{N}\varphi_n(t)\varphi_n^{\mathrm{T}}(t)\alpha = \sum_{t=n}^{N}(\alpha^{\mathrm{T}}\varphi_n(t))^2 \geq 0,
\tag{20}
$$

and therefore the lemma is proved.

*Observations*

1. The matrix $R_n(N)$ is not Toeplitz, but has a structure close to a Toeplitz matrix. If we assume the "Prewindow" case

$$
y(t) = 0 \quad \text{for} \quad t \leq 0,
\tag{21}
$$

we may replace the lower limit $t = n$ in the summations to $t = 0$, and efficient solutions can be found for the time recursions.

The observation vector has a recursive in order structure:

$$\varphi_{n+1}(t) = \begin{bmatrix} -y(t-1) \\ -y(t-2) \\ \cdot \\ \cdot \\ \cdot \\ -y(t-n-1) \end{bmatrix} = \begin{bmatrix} \varphi_n(t) \\ -y(t-n-1) \end{bmatrix} = \tag{22}$$

$$= \begin{bmatrix} -y(t-1) \\ \varphi_n(t-1) \end{bmatrix}. \tag{23}$$

Replacing in (10) the equations (22) and (23), we get the following block partitions:

$$\begin{aligned} R_{n+1}(N) &= \begin{bmatrix} \sum_{t=n}^{N} \varphi_n(t)\varphi_n^{\mathrm{T}}(t) & -\sum_{t=n}^{N} \varphi_n(t)y(t-n-1) \\ -\sum_{t=n}^{N} \varphi_n^{\mathrm{T}}(t)y(t-n-1) & \sum_{t=n}^{N} y^2(t-n-1) \end{bmatrix} = \\ &= \begin{bmatrix} R_n(N) & -r_n^b(N) \\ -r_n^{b\mathrm{T}}(N) & \rho_0(N-n-1) \end{bmatrix} = \\ &= \begin{bmatrix} \sum_{t=n}^{N} y^2(t-1) & -\sum_{t=n}^{N} \varphi_n(t-1)y(t-1) \\ -\sum_{t=n}^{N} \varphi_n(t-1)y(t-1) & \sum_{t=n}^{N} \varphi_n(t-1)\varphi_n^{\mathrm{T}}(t-1) \end{bmatrix} = \end{aligned} \tag{24}$$

$$= \begin{bmatrix} \rho_0(N-1) & -r_n^{\mathrm{T}}(N-1) \\ -r_n(N-1) & R_n(N-1) \end{bmatrix}. \tag{25}$$

Using in (11) the equation (22), we get

$$r_{n+1}(N) = \begin{bmatrix} \sum_{t=n}^{N} \varphi_n(t)y(t) \\ -\sum_{t=n}^{N} y(t-n-1)y(t) \end{bmatrix} = \begin{bmatrix} r_n(N) \\ -\rho_{n+1}(N) \end{bmatrix}, \tag{26}$$

where we denoted

$$\rho_n(N) = \sum_{t=n}^{N} y(t)y(t-n). \tag{27}$$

Using (23) in (12),

$$r_{n+1}^b(N) = \begin{bmatrix} -\sum_{t=n}^{N} y(t-1)y(t-n-2) \\ -\sum_{t=n}^{N} \varphi_n(t-1)y(t-n-2) \end{bmatrix} = \begin{bmatrix} -\rho_{n+1}(N-1) \\ r_n^b(N-1) \end{bmatrix}. \tag{28}$$

Using (25), the forward predictor and the "variance" are shown to be the solutiuons of the augmented system

$$R_{n+1}(N) \begin{bmatrix} 1 \\ \theta_n(N-1) \end{bmatrix} = \begin{bmatrix} \rho_0(N-1) & -r_n^{\mathrm{T}}(N-1) \\ -r_n(N-1) & R_n(N-1) \end{bmatrix} \begin{bmatrix} 1 \\ \theta_n(N-1) \end{bmatrix} = \begin{bmatrix} \sigma_n(N-1) \\ 0_n \end{bmatrix}, \tag{29}$$

and the augmented system for the backward predictor rezults using partition (24) and relations (16), (17)

$$R_{n+1}(N) \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} = \begin{bmatrix} R_n(N) & -r_n^b(N) \\ -r_n^{b\mathrm{T}}(N) & \rho_0(N-n-1) \end{bmatrix} \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} = \begin{bmatrix} 0_n \\ \sigma_n^b(N) \end{bmatrix}. \tag{30}$$

**Lemma 1.2** *The solutions of order $n$ and $n+1$ of the system (29), (30) are connected by*

$$\theta_{n+1}(N) = \begin{bmatrix} \theta_n(N) \\ 0 \end{bmatrix} - \frac{\alpha_n(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix}, \tag{31}$$

$$\theta_{n+1}^b(N) = \begin{bmatrix} 0 \\ \theta_n^b(N-1) \end{bmatrix} - \frac{\alpha_n(N-1)}{\sigma_n(N-1)} \begin{bmatrix} 1 \\ \theta_n(N-1) \end{bmatrix}, \tag{32}$$

$$\sigma_{n+1}(N) = \sigma_n(N) - \frac{\alpha_n^2(N)}{\sigma_n^b(N)}, \tag{33}$$

$$\sigma_{n+1}^b(N) = \sigma_n^b(N-1) - \frac{\alpha_n^2(N-1)}{\sigma_n(N-1)}, \tag{34}$$

*where*

$$\alpha_n(N) = \rho_{n+1}(N) - r_n^{\mathrm{T}}(N)\theta_n^b(N). \tag{35}$$

*Proof.* By using (25),(26) and (30)

$$\begin{aligned} R_{n+2}(N) \left[\begin{array}{c} 0 \\ \hline \theta_n^b(N-1) \\ 1 \end{array}\right] &= \left[\begin{array}{c|c} \rho_0(N-1) & -r_n^{\mathrm{T}}(N-1) \quad \rho_{n+1}(N-1) \\ \hline -r_n(N-1) & \\ \rho_{n+1}(N-1) & R_{n+1}(N-1) \end{array}\right] \left[\begin{array}{c} 0 \\ \hline \theta_n^b(N-1) \\ 1 \end{array}\right] \\ &= \begin{bmatrix} \rho_{n+1}(N-1) - r_n^{\mathrm{T}}(N-1)\theta_n^b(N-1) \\ 0_n \\ \sigma_n^b(N-1) \end{bmatrix} = \begin{bmatrix} \alpha_n(N-1) \\ 0_n \\ \sigma_n^b(N-1) \end{bmatrix} \end{aligned} \tag{36}$$

and from (24),(28) and (29) it results

$$R_{n+2}(N) \begin{bmatrix} 1 \\ \theta_n(N-1) \\ 0 \end{bmatrix} = \left[\begin{array}{c|c} & \rho_{n+1}(N-1) \\ R_{n+1}(N) & -r_n^b(N-1) \\ \hline \rho_{n+1}(N-1) \quad -r_n^{b\mathrm{T}}(N-1) & \rho_0(N-n-2) \end{array}\right] \left[\begin{array}{c} 1 \\ \theta_n(N-1) \\ \hline 0 \end{array}\right]$$

$$
= \begin{bmatrix} \sigma_n(N-1) \\ 0_n \\ \rho_{n+1}(N-1) - r_n^{b\mathrm{T}}(N-1)\theta_n(N-1) \end{bmatrix} = \begin{bmatrix} \sigma_n(N-1) \\ 0_n \\ \alpha_n(N-1) \end{bmatrix} \tag{37}
$$

Combining (36) andi (37), we get

$$
R_{n+2}(N) \left\{ \begin{bmatrix} 1 \\ \theta_n(N-1) \\ 0 \end{bmatrix} - \frac{\alpha_n(N-1)}{\sigma_n^b(N-1)} \begin{bmatrix} 0 \\ \theta_n^b(N-1) \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} \sigma_n(N-1) - \frac{\alpha_n^2(N-1)}{\sigma_n^b(N-1)} \\ 0_n \\ 0 \end{bmatrix}, \tag{38}
$$

which has the same form as (29) written for $(n+1) \leftarrow n$

$$
R_{n+2}(N) \begin{bmatrix} 1 \\ \theta_{n+1}(N-1) \end{bmatrix} = \begin{bmatrix} \sigma_{n+1}(N-1) \\ 0_{n+1} \end{bmatrix}. \tag{39}
$$

Due to the nonsingularity of $R_{n+2}(N)$, the solution of (39) is unique, and by identifying (38) and (39), we get (31) and (33).

The extended system (30)can be written for $(n+1) \leftarrow n$

$$
R_{n+2}(N) \begin{bmatrix} \theta_{n+1}^b(N) \\ 1 \end{bmatrix} = \begin{bmatrix} 0_{n+1} \\ \sigma_{n+1}^b(N) \end{bmatrix}. \tag{40}
$$

similar to the equation resulting from combining (36) and (37)

$$
R_{n+2}(N) \left\{ \begin{bmatrix} 0 \\ \theta_n^b(N-1) \\ 1 \end{bmatrix} - \frac{\alpha_n(N-1)}{\sigma_n(N-1)} \begin{bmatrix} 1 \\ \theta_n(N-1) \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0_n \\ \sigma_n^b(N-1) - \frac{\alpha_n^2(N-1)}{\sigma_n(N-1)} \end{bmatrix}. \tag{41}
$$

Since the solution of (40) is unique, we get by identifying (40) and (41) the equations (32) and(34).

□

Remark that in (32) and (34) the current time changes from $N-1$ in $N$, therefore we do not have an recursive-in-order algorithm yet.

Now define the apriori forward prediction error

$$e_n^f(N) = y(N) - \varphi_n^{\mathrm{T}}(N)\theta_n(N-1), \tag{42}$$

and the aposteriori forward prediction error

$$\varepsilon_n^f(N) = y(N) - \varphi_n^{\mathrm{T}}(N)\theta_n(N). \tag{43}$$

the apriori backward prediction error

$$e_n^b(N) = y(N-n-1) - \varphi_n^{\mathrm{T}}(N)\theta_n^b(N-1) \tag{44}$$

and the aposteriori backward prediction error

$$\varepsilon_n^b(N) = y(N-n-1) - \varphi_n^{\mathrm{T}}(N)\theta_n^b(N). \tag{45}$$

**Lema 1.1** *The prediction errors (42)–(45) are connected by the following recursive-in-order formula:*

$$\begin{aligned}
e_{n+1}^f(N) &= e_n^f(N) + K_{n+1}^f(N-1)e_n^b(N), &(46)\\
e_{n+1}^b(N) &= e_n^b(N-1) + K_{n+1}^b(N-2)e_n^f(N-1), &(47)\\
\varepsilon_{n+1}^f(N) &= \varepsilon_n^f(N) + K_{n+1}^f(N)\varepsilon_n^b(N), &(48)\\
\varepsilon_{n+1}^b(N) &= e_n^b(N-1) + K_{n+1}^b(N-1)\varepsilon_n^f(N-1), &(49)
\end{aligned}$$

*where the forward reflection coefficient is*

$$K^f_{n+1}(N) = -\frac{\alpha_n(N)}{\sigma^b_n(N)} \tag{50}$$

*and the backward reflection coefficient is*

$$K^b_{n+1}(N) = -\frac{\alpha_n(N)}{\sigma_n(N)}. \tag{51}$$

*Proof.* Using (31) and (22) in definition (42)

$$\begin{aligned} e^f_{n+1}(N) &= y(N) - \varphi^{\mathrm{T}}_{n+1}(N)\theta_{n+1}(N-1) = \\ &= y(N) - \left[\begin{array}{cc} \varphi^{\mathrm{T}}_n(N) & -y(N-n-1) \end{array}\right] \left\{ \left[\begin{array}{c} \theta_n(N-1) \\ 0 \end{array}\right] + K^f_{n+1}(N-1) \left[\begin{array}{c} \theta^b_n(N-1) \\ 1 \end{array}\right] \right\} \\ &= e^f_n(N) + K^f_{n+1}(N-1) e^b_n(N). \end{aligned}$$

Using (31) and (22) in definition (43), we get

$$\begin{aligned} \varepsilon^f_{n+1}(N) &= y(N) - \varphi^{\mathrm{T}}_{n+1}(N)\theta_{n+1}(N) = \\ &= y(N) - \left[\begin{array}{cc} \varphi^{\mathrm{T}}_n(N) & -y(N-n-1) \end{array}\right] \left\{ \left[\begin{array}{c} \theta_n(N) \\ 0 \end{array}\right] + K^f_{n+1}(N) \left[\begin{array}{c} \theta^b_n(N) \\ 1 \end{array}\right] \right\} = \\ &= \varepsilon^f_n(N) + K^f_{n+1}(N) \varepsilon^b_n(N). \end{aligned}$$

Similarly, we get from (32) and (23):

$$e^b_{n+1}(N) = y(N-n-2) - \varphi^{\mathrm{T}}_{n+1}(N)\theta^b_{n+1}(N-1) =$$

$$
\begin{aligned}
&= y(N-n-2) - \left[\begin{array}{cc} -y(N-1) & \varphi_n^{\mathrm{T}}(N-1) \end{array}\right] \left\{ \left[\begin{array}{c} 0 \\ \theta_n^b(N-2) \end{array}\right] + \right. \\
&+ \left. K_{n+1}^b(N-2) \left[\begin{array}{c} 1 \\ \theta_n(N-2) \end{array}\right] \right\} = e_n^b(N-1) + K_{n+1}^b(N-2) e_n^f(N-1),
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{n+1}^b(N) &= y(N-n-2) - \varphi_{n+1}^{\mathrm{T}}(N)\theta_{n+1}^b(N) = \\
&= y(N-n-2) - \left[\begin{array}{cc} -y(N-1) & \varphi_n^{\mathrm{T}}(N-1) \end{array}\right] \left\{ \left[\begin{array}{c} 0 \\ \theta_n^b(N-1) \end{array}\right] + \right. \\
&+ \left. K_{n+1}^b(N-1) \left[\begin{array}{c} 1 \\ \theta_n(N-1) \end{array}\right] \right\} = \varepsilon_n^b(N-1) + K_{n+1}^b(N-1) \varepsilon_n^f(N-1).
\end{aligned}
$$

□ In order to get a complete systeme of equations for prediction error propagation, it is necessary to propagate $\alpha_n(N)$, given in next lemma.

**Lemma 1.3**

$$
\alpha_n(N) = \alpha_n(N-1) + e_n^f(N)\varepsilon_n^b(N). \tag{52}
$$

*Proof.* The definition (35) can be rewritten

$$
\alpha_n(N) = \rho_{n+1}(N) - r_n^{\mathrm{T}}(N) R_n^{-1}(N) r_n^b(N). \tag{53}
$$

We may use the factorization

$$
\left[\begin{array}{ccc} 1 & -y(N-n-1) & \varphi_n^{\mathrm{T}}(N) \\ -y(N) & \rho_{n+1}(N) & -r_n^{\mathrm{T}}(N) \\ \varphi_n(N) & -r_n^b(N) & R_n(N) \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ -y(N) & 1 & 0 \\ \varphi_n(N) & 0 & I \end{array}\right] \cdot
$$

$$\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho_{n+1}(N-1) & -r_n^{\mathrm{T}}(N-1) \\ 0 & -r_n^b(N-1) & R_n(N-1) \end{bmatrix}\begin{bmatrix} 1 & -y(N-n-1) & \varphi_n^{\mathrm{T}}(N) \\ 0 & 1 & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{54}$$

Multiplying at the left with the inverse of the first matrix in the right hand side we get

$$\begin{bmatrix} 1 & 0 & 0 \\ y(N) & 1 & 0 \\ \varphi_n(N) & 0 & I \end{bmatrix}\begin{bmatrix} 1 & -y(N-n-1) & \varphi_n^{\mathrm{T}}(N) \\ -y(N) & \rho_{n+1}(N) & -r_n^{\mathrm{T}}(N) \\ \varphi_n(N) & -r_n^b(N) & R_n(N) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho_{n+1}(N-1) & -r_n^{\mathrm{T}}(N-1) \\ 0 & -r_n^b(N-1) & R_n(N-1) \end{bmatrix}\begin{bmatrix} 1 & -y(N-n-1) & \varphi_n^{\mathrm{T}}(N) \\ 0 & 1 & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{55}$$

Multiplying (55) at right with $\begin{bmatrix} 0 & 1 & \theta_n^{b\mathrm{T}}(N) \end{bmatrix}^{\mathrm{T}}$ and at left with $\begin{bmatrix} 0 & 1 & \theta_n^{\mathrm{T}}(N-1) \end{bmatrix}$ it results

$$\begin{bmatrix} e_n^f(N) & 1 & \theta_n^{\mathrm{T}}(N-1) \end{bmatrix}\begin{bmatrix} -\varepsilon_n^b(N) \\ \alpha_n(N) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha_n(N-1) & 0 \end{bmatrix}\begin{bmatrix} -\varepsilon_n^b(N) \\ 1 \\ \theta_n^b(N) \end{bmatrix}, \tag{56}$$

and therefore

$$\alpha_n(N) = \alpha_n(N-1) + e_n^f(N)\varepsilon_n^b(N).$$

Grouping (33),(34),(46)–(51) and (52) we get the Algorithm 1.1.

The algorithm contains two lattice structures, one for apriori errors, the other for aposteriori errors. The lattice is particular in the sense that the reflection coefficients are different, $K_{n+1}^f \neq K_{n+1}^b$.

The equations (33) and (34) for the error "variances" form a third lattice structure:

$$\begin{aligned} \sigma_{n+1}(N) &= \sigma_n(N) - K_{n+1}^{f2}(N)\sigma_n^b(N), && (57)\\ \sigma_{n+1}^b(N) &= \sigma_n^b(N-1) - K_{n+1}^{b2}(N-1)\sigma_n(N-1), && (58) \end{aligned}$$

but that will require 4 multiplications and 2 subtractions, compared to 3 multiplications and 1 subtraction in (6) and (7) of Algorithm 1.1.

Initializing the recursions at order 0 ensures for order $n = 1$ the *exact* value of all variables, e.g.

$$\sigma_0(N) = \rho_0(N); \quad \sigma_0^b(N) = \rho_0(N-1); \quad \alpha_0(N) = \rho_1(N),$$

will give:

$$\begin{aligned} K_1^f(N) &= -\frac{\rho_1(N)}{\rho_0(N-1)} = \frac{r_1(N)}{R_1(N)} = \theta_1(N), && (59)\\ K_1^b(N) &= -\frac{\rho_1(N)}{\rho_0(N)} = \frac{r_1^b(N+1)}{R_1(N+1)} = \theta_1^b(N+1), && (60)\\ \sigma_1^b(N) &= \rho_0(N-2) - \frac{\rho_1^2(N-1)}{\rho_0(N-1)} = \rho_0(N-2) - r_1^b(N)\theta_1^b(N), && (61)\\ e_1^f(N) &= y(N) + \theta_1^f(N-1)y(N-1). && (62) \end{aligned}$$

*Observations*

1. From (29),(30) we observe that when $\sigma_n(N)$ or $\sigma_n^b(N)$ are zero, the matrix $R_{n+1}$ is singular. The algorithm 1.1 treats the situations $\sigma_n = 0$ or $\sigma_n^b = 0$, in an exact LS sense.

## 2 Lattice algorithm for predictors

In this section we discuss other solutions of Problem 1 and also we present algorithms for the following problem.

Problem 2 Given the time series $\{y(t)\}$, find the forward predictor $\{\theta_n(N)\}_{n=1}^{n_{max}}$ minimizing (8) and backward predictor $\{\theta_n^b(N)\}_{n=1}^{n_{max}}$ minimizing (9), using block recursive algorithms.

Finding efficiently all optimal predictors of orders between 1 and $n_{max}$ can be realized through the recursions (31)–(32) in $\mathcal{O}(n_{max}^2)$ operations; when iterating these relations we need the reflection coefficients, of cost $\mathcal{O}(n_{max})$ operations. Therefore this method requires $\mathcal{O}(n_{max}^2)$ operations for each processed sample $y(N)$.

In some applications the AR models are required periodically, once for each frame. We call these applications block recursive in time.

The combination: lattice algorithm + relations (31),(32) is not a block recursive method, since $\theta_{n+1}^b(N)$ is computed in (32) as a function of $\theta_n^b(N-1)$, which is available only if the predictors are computed at each time moment.

The principle is to propagate in time the usefull information (e.g. reflection coefficients in $\mathcal{O}(n_{max})$ operations) and at seldom time moments one runs a recursive in order procedure to get the predictors in $\mathcal{O}(n_{max}^2)$ operations.

## 2.1 Recursive in time equations

We define the following auxilliary variables

- $K_n(N)$, the apriori Kalman gain vector, which is a solution of

$$R_n(N-1)K_n(N) = \varphi_n(N); \tag{63}$$

- $K_n^*(N)$, the aposteriori Kalman gain vector, which is a solution of

$$R_n(N)K_n^*(N) = \varphi_n(N); \tag{64}$$

- $\gamma_n^*(N)$, the aposteriori gain (a scalar), defined by

$$\gamma_n^*(N) = \varphi_n^{\mathrm{T}}(N)K_n^*N); \tag{65}$$

- $\gamma_n(N)$, the apriori gain (a scalar), defined by

$$\gamma_n(N) = \varphi_n^{\mathrm{T}}(N)K_n(N); \tag{66}$$

- $\gamma_n^0(N)$, the ratio aposteriori/apriori, defined as

$$\gamma_n^0(N) = 1 - \gamma_n^*(N). \tag{67}$$

*Observation.* We will discuss latter about $K_n(N)$ as about a predictor, relevant to the following prediction problem.

Consider the prediction of the time series

$$x(t) = \begin{cases} 0 & t = 0, \ldots, N-1 \\ 1 & t = N \end{cases}$$

using a linear combination of entries in $\varphi_n(t)$

$$\hat{x}(t) = K^{\mathrm{T}} \varphi_n(t).$$

and evaluated by the criterion

$$\sum_{t=0}^{N} (x(t) - \hat{x}(t))^2 = \sum_{t=0}^{N} (x(t) - K^{\mathrm{T}} \varphi_n(t))^2.$$

The minimization of the criterion is realized by

$$K_n(N) = R_n^{-1}(N) \varphi_n(N),$$

which has the "variance" of prediction errors

$$\gamma_n^0(N) = 1 - \varphi_n^{\mathrm{T}}(N) K_n(N).$$

The time recurrence of predictors and error variances is shown in Table 2. The proofs are sketched in the following.

The time recurrence of the matrix $R_n(N)$ can be described by the factorization:

$$\begin{bmatrix} 1 & \varphi_n^{\mathrm{T}}(N+1) \\ \varphi_n(N+1) & R_n(N+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \varphi_n(N+1) & I_n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_n(N) \end{bmatrix} \begin{bmatrix} 1 & \varphi_n^{\mathrm{T}}(N+1) \\ 0 & I_n \end{bmatrix}, \tag{77}$$

$$\theta_n(N) = \theta_n(N-1) + K_n(N)\varepsilon_n^f(N) \tag{68}$$

$$\theta_n^b(N) = \theta_n^b(N-1) + K_n(N)\varepsilon_n^b(N) \tag{69}$$

$$\sigma_n(N) = \sigma_n(N-1) + e_n^f(N)\varepsilon_n^f(N) \tag{70}$$

$$\sigma_n^b(N) = \sigma_n^b(N-1) + e_n^b(N)\varepsilon_n^b(N) \tag{71}$$

$$\alpha_n(N) = \alpha_n(N-1) + e_n^f(N)\varepsilon_n^b(N) \tag{72}$$

$$K_n^*(N) = K_n(N)\gamma_n^0(N) \tag{73}$$

$$\varepsilon_n^f(N) = e_n^f(N)\gamma_n^0(N) \tag{74}$$

$$\varepsilon_n^b(N) = e_n^b(N)\gamma_n^0(N) \tag{75}$$

$$\gamma_n^0(N) = 1 - \gamma_n^*(N) = \frac{1}{1+\gamma_n(N)} \tag{76}$$

Table 1: Basic relations for time recurrence.

By left-multiplication by the the inverse of the first matrix in the right hand side,

$$\begin{bmatrix} 1 & 0 \\ -\varphi_n(N+1) & I_n \end{bmatrix} \begin{bmatrix} 1 & \varphi_n^{\mathrm{T}}(N+1) \\ \varphi_n(N+1) & R_n(N+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R_n(N) \end{bmatrix} \begin{bmatrix} 1 & \varphi_n^{\mathrm{T}}(N+1) \\ 0 & I_n \end{bmatrix}, \tag{78}$$

we obtain the basic relation, from which we obtain all equations in Table 2 by multiplying at the left and right with convenient vectors, as specified in Table 3.

As an example, the first line in Table 3 asks the following operations

$$\begin{bmatrix} 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ y(N) & 1 & 0 \\ -\varphi_n(N) & 0 & I_n \end{bmatrix} \left[\begin{array}{c|c} 1 & -y(N) \quad \varphi_n^{\mathrm{T}}(N) \\ \hline -y(N) & \\ \varphi_n(N) & R_{n+1}(N+1) \end{array}\right] \left[\begin{array}{c} 0 \\ \hline 1 \\ \theta_n(N) \end{array}\right] = \tag{79}$$

$$= \begin{bmatrix} 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho_0(N-1) & -r_n^{\mathrm{T}}(N-1) \\ 0 & -r_n(N-1) & R_n(N-1) \end{bmatrix} \begin{bmatrix} 1 & -y(N) & \varphi_n^{\mathrm{T}}(N) \\ 0 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \theta_n(N) \end{bmatrix}, \tag{80}$$

resulting in

$$\begin{bmatrix} -\varphi_n(N) & 0 & I_n \end{bmatrix} \begin{bmatrix} -\varepsilon_n^f(N) \\ \sigma_n(N) \\ 0_n \end{bmatrix} = \begin{bmatrix} 0 & -r_n(N-1) & R_n(N-1) \end{bmatrix} \begin{bmatrix} -\varepsilon_n^f(N) \\ 1 \\ \theta_N(n) \end{bmatrix}, \tag{81}$$

or

$$\varphi_n(N)\varepsilon_n^f(N) = -r_n(N-1) + R_n(N-1)\theta_n(N), \tag{82}$$

and by using the definitions (14) and (63) we get

$$R_n(N-1)K_n(N)\varepsilon_n^f(N) = -R_n(N-1)\theta_n(N-1) + R_n(N-1)\theta_n(N). \tag{83}$$

| Equation | Partitioning of matrix $R_{n+1}$ | Multiplication at left | Multiplication at right | Resulting equation |
|---|---|---|---|---|
| (78) $(n+1 \leftarrow n)$ | (25),(23) | $\begin{bmatrix} 0 & 0 & I \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & \theta_n^{\mathrm{T}}(N) \end{bmatrix}^{\mathrm{T}}$ | (68) |
| (78) $(n+1 \leftarrow n)$ | (25),(23) | $\begin{bmatrix} 0 & 1 & \theta_n^{\mathrm{T}}(N-1) \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & \theta_n^{\mathrm{T}}(N) \end{bmatrix}^{\mathrm{T}}$ | (70) |
| (78) $(n+1 \leftarrow n)$ | (24),(22) | $\begin{bmatrix} 0 & I & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & \theta_n^{b\mathrm{T}}(N+1) & 1 \end{bmatrix}^{\mathrm{T}}$ | (69) |
| (78) $(n+1 \leftarrow n)$ | (24),(22) | $\begin{bmatrix} 0 & \theta_n^{b\mathrm{T}}(N) & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & \theta_n^{b\mathrm{T}}(N+1) & 1 \end{bmatrix}^{\mathrm{T}}$ | (71) |
| (78) $(n+1 \leftarrow n)$ | (25),(23) | $\begin{bmatrix} 0 & 0 & I \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & -K_n^{*\mathrm{T}}(N) \end{bmatrix}^{\mathrm{T}}$ | (73) |
| (78) $(n+1 \leftarrow n)$ | (25),(23) | $\begin{bmatrix} 0 & 1 & \theta_n^{\mathrm{T}}(N-1) \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & -K_n^{*\mathrm{T}}(N) \end{bmatrix}^{\mathrm{T}}$ | (74) |
| (78) $(n+1 \leftarrow n)$ | (24),(22) | $\begin{bmatrix} 0 & \theta_n^{b\mathrm{T}}(N) & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & -K_n^{*\mathrm{T}}(N+1) & 0 \end{bmatrix}^{\mathrm{T}}$ | (75) |

Table 2: The operations needed in deriving the recursive in time equations.

One solution of the system (83) (if $R_n(N-1)$ is nonsingular, then the solution is unique) is (68).

The second line in Table 3 asks the following operations:

$$\begin{bmatrix} 0 & 1 & \theta_n^{\mathrm{T}}(N-1) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ y(N) & 1 & 0 \\ -\varphi_n(N) & 0 & I \end{bmatrix} \left[\begin{array}{c|c} 1 & -y(N)\varphi_n^{\mathrm{T}}(N) \\ \hline -y(N) & \\ \varphi_n(N) & R_{n+1}(N+1) \end{array}\right] \left[\begin{array}{c} 0 \\ \hline 1 \\ \theta_n(N) \end{array}\right] = \tag{84}$$

$$= \begin{bmatrix} 0 & 1 & \theta_n^{\mathrm{T}}(N-1) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho_0(N-1) & -r_n^{\mathrm{T}}(N-1) \\ 0 & -r_n(N-1) & R_n(N-1) \end{bmatrix} \begin{bmatrix} 1 & -y(N) & \varphi_n^{\mathrm{T}}(N) \\ 0 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \theta_n(N) \end{bmatrix}$$

and we get

$$\begin{bmatrix} e_n^f(N) & 1 & \theta_n^{\mathrm{T}}(N-1) \end{bmatrix} \begin{bmatrix} -\varepsilon_n^f(N) \\ \sigma_n(N) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma_n(N-1) & 0 \end{bmatrix} \begin{bmatrix} -\varepsilon_n^f(N) \\ 1 \\ \theta_n(N) \end{bmatrix}, \tag{85}$$

or

$$-e_n^f(N)\varepsilon_n^f(N) + \sigma_n(N) = \sigma_n(N-1),$$

which is equivalent to (70).

The equations (69) and (71) are similar.

In the 5th line of the Table 3 the following operations are indicated:

$$\begin{bmatrix} 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ y(N) & 1 & 0 \\ -\varphi_n(N) & 0 & I_n \end{bmatrix} \begin{bmatrix} 1 & -y(N) & \varphi_n^{\mathrm{T}}(N) \\ -y(N) & \rho_0(N) & -r_n^{\mathrm{T}}(N) \\ \varphi_n(N) & -r_n(N) & R_n(N) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -K_n^*(N) \end{bmatrix} = \tag{86}$$

$$= \begin{bmatrix} 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho_0(N-1) & -r_n^{\mathrm{T}}(N-1) \\ 0 & -r_n(N-1) & R_n(N-1) \end{bmatrix} \begin{bmatrix} 1 & -y(N) & \varphi_n^{\mathrm{T}}(N) \\ 0 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -K_n^*(N) \end{bmatrix}. \tag{87}$$

We note that

$$\begin{aligned} -y(N) + r_n^{\mathrm{T}}(N)K_n^*(N) &= -y(N) + \theta_n^{\mathrm{T}}(N)R_n(N)K_n^*(N) = \\ &= -y(N) + \theta_n^{\mathrm{T}}(N)\varphi_n(N) = -\varepsilon_n^f(N), \end{aligned} \tag{88}$$

to transform (86) to

$$\begin{bmatrix} -\varphi_n(N) & 0 & I_n \end{bmatrix} \begin{bmatrix} 1-\gamma_n^*(N) \\ -\varepsilon_n^f(N) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -r_n(N-1) & R_n(N-1) \end{bmatrix} \begin{bmatrix} 1-\gamma_n^*(N) \\ 0 \\ -K_n^*(N) \end{bmatrix}, \tag{89}$$

or

$$-\varphi_n(N)(1-\gamma_n^*(N)) = -R_n(N-1)K_n^*(N), \tag{90}$$

and using the definition (63), it results

$$R_n(N-1)K_n(N)(1-\gamma_n^*(N)) = R_n(N-1)K_n^*(N), \tag{91}$$

having as solution (73).

The operations indicated in the 6th line of Table 3 are:

$$\begin{bmatrix} 0 & 1 & \theta_n^{\mathrm{T}}(N-1) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ y(N) & 1 & 0 \\ -\varphi_n(N) & 0 & I_n \end{bmatrix} \begin{bmatrix} 1 & -y(N) & \varphi_n^{\mathrm{T}}(N) \\ -y(N) & \rho_0(N) & -r_n^{\mathrm{T}}(N) \\ \varphi_n(N) & -r_n(N) & R_n(N) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -K_n^*(N) \end{bmatrix} \tag{92}$$

$$= \begin{bmatrix} 0 & 1 & \theta_n^{\mathrm{T}}(N-1) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho_0(N-1) & -r_n^{\mathrm{T}}(N-1) \\ 0 & -r_n(N-1) & R_n(N-1) \end{bmatrix} \begin{bmatrix} 1 & -y(N) & \varphi_n^{\mathrm{T}}(N) \\ 0 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -K_n^*(N) \end{bmatrix}$$

and after performing the multiplications we get

$$\begin{bmatrix} e_n^f(N) & 1 & \theta_n^{\mathrm{T}}(N-1) \end{bmatrix} \begin{bmatrix} 1-\gamma_n^*(N) \\ -\varepsilon_n^f(N) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma_n(N-1) & 0 \end{bmatrix} \begin{bmatrix} 1-\gamma_n^*(N) \\ 0 \\ -K_n^*(N) \end{bmatrix}, \tag{93}$$

therefore

$$e_n^f(N)(1-\gamma_n^*(N)) - \varepsilon_n^f(N) = 0,$$

a relation equivalent to (74). The operations in the last line of Table 3 are similar.

The last relation from Table 2, (76), immediately results by using (73):

$$\begin{aligned} \gamma_n^*(N) &= \varphi_n^{\mathrm{T}}(N)K_n^*(N) = \varphi_n^{\mathrm{T}}(N)K_n(N)(1-\gamma_n^*(N)) = \\ &= \gamma_n(N)(1-\gamma_n^*(N)), \end{aligned} \tag{94}$$

therefore

$$\gamma_n(N) = \frac{\gamma_n^*(N)}{1-\gamma_n^*(N)}, \tag{95}$$

$$\gamma_n^*(N) = \frac{\gamma_n(N)}{1+\gamma_n(N)}. \tag{96}$$

$$\theta_{n+1}(N) = \begin{bmatrix} \theta_n(N) \\ 0 \end{bmatrix} - \frac{\alpha_n(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} \tag{97}$$

$$\theta_{n+1}^b(N) = \begin{bmatrix} 0 \\ \theta_n^b(N-1) \end{bmatrix} - \frac{\alpha_n(N-1)}{\sigma_n(N-1)} \begin{bmatrix} 1 \\ \theta_n(N-1) \end{bmatrix} \tag{98}$$

$$\sigma_{n+1}(N) = \sigma_n(N) - \frac{\alpha_n^2(N)}{\sigma_n^b(N)} \tag{99}$$

$$\sigma_{n+1}^b(N) = \sigma_n^b(N-1) - \frac{\alpha_n^2(N-1)}{\sigma_n(N-1)} \tag{100}$$

$$e_{n+1}^f(N) = e_n^f(N) - \frac{\alpha_n(N-1)}{\sigma_n^b(N-1)} e_n^b(N) \tag{101}$$

$$e_{n+1}^b(N) = e_n^b(N-1) - \frac{\alpha_n(N-2)}{\sigma_n(N-2)} e_n^f(N-1) \tag{102}$$

$$\varepsilon_{n+1}^f(N) = \varepsilon_n^f(N) - \frac{\alpha_n(N)}{\sigma_n^b(N)} \varepsilon_n^b(N) \tag{103}$$

$$\varepsilon_{n+1}^b(N) = \varepsilon_n^b(N-1) - \frac{\alpha_n(N-1)}{\sigma_n(N-1)} \varepsilon_n^f(N-1) \tag{104}$$

$$K_{n+1}^*(N) = \begin{bmatrix} K_n^*(N) \\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} \tag{105}$$

$$K_{n+1}(N) = \begin{bmatrix} K_n(N) \\ 0 \end{bmatrix} - \frac{e_n^b(N)}{\sigma_n^b(N-1)} \begin{bmatrix} \theta_n^b(N-1) \\ 1 \end{bmatrix} \tag{106}$$

$$K_{n+1}^*(N) = \begin{bmatrix} 0 \\ K_n^*(N-1) \end{bmatrix} - \frac{\varepsilon_n^f(N-1)}{\sigma_n(N-1)} \begin{bmatrix} 1 \\ \theta_n(N-1) \end{bmatrix} \tag{107}$$

$$K_{n+1}(N) = \begin{bmatrix} 0 \\ K_n(N-1) \end{bmatrix} - \frac{e_n^f(N-1)}{\sigma_n(N-2)} \begin{bmatrix} 1 \\ \theta_n(N-2) \end{bmatrix} \tag{108}$$

$$\gamma_{n+1}^*(N) = \gamma_n^*(N) + \frac{\varepsilon_n^{b2}(N)}{\sigma_n^b(N)} \tag{109}$$

$$\gamma_{n+1}(N) = \gamma_n(N) + \frac{e_n^{b2}(N)}{\sigma_n^b(N-1)} \tag{110}$$

$$\gamma_{n+1}^*(N) = \gamma_n^*(N-1) + \frac{\varepsilon_n^{f2}(N-1)}{\sigma_n(N-1)} \tag{111}$$

$$\gamma_{n+1}(N) = \gamma_n(N-1) + \frac{e_n^{f2}(N-1)}{\sigma_n(N-2)} \tag{112}$$

Table 3: Basic relations for order recursion.

## 2.2 Derivation of the recursions in order

The recursive in order equations have been derived in Lemmas 1.2 and 1.1 for prediction errors, their variances and for predictors. We establish now recursive-in-order equations for the Kalman gain vectors and for the gains.

**Lema 2.1** *The following recurrences hold*

$$\begin{aligned}
K^*_{n+1}(N) &= \begin{bmatrix} K^*_n(N) \\ 0 \end{bmatrix} - \frac{\varepsilon^b_n(N)}{\sigma^b_n(N)} \begin{bmatrix} \theta^b_n(N) \\ 1 \end{bmatrix}, & (113) \\
K_{n+1}(N) &= \begin{bmatrix} K_n(N) \\ 0 \end{bmatrix} - \frac{e^b_n(N)}{\sigma^b_n(N-1)} \begin{bmatrix} \theta^b_n(N-1) \\ 1 \end{bmatrix}, & (114) \\
K^*_{n+1}(N) &= \begin{bmatrix} 0 \\ K^*_n(N-1) \end{bmatrix} - \frac{\varepsilon^f_n(N-1)}{\sigma_n(N-1)} \begin{bmatrix} 1 \\ \theta_n(N-1) \end{bmatrix}, & (115) \\
K_{n+1}(N) &= \begin{bmatrix} 0 \\ K_n(N-1) \end{bmatrix} - \frac{e^f_n(N-1)}{\sigma_n(N-2)} \begin{bmatrix} 1 \\ \theta_n(N-2) \end{bmatrix}, & (116) \\
\gamma^*_{n+1}(N) &= \gamma^*_n(N) + \frac{\varepsilon^{b2}_n(N)}{\sigma^b_n(N)}, & (117) \\
\gamma_{n+1}(N) &= \gamma_n(N) + \frac{e^{b2}_n(N)}{\sigma^b_n(N-1)}, & (118) \\
\gamma^*_{n+1}(N) &= \gamma^*_n(N-1) + \frac{\varepsilon^{f2}_n(N-1)}{\sigma_n(N-1)}, & (119) \\
\gamma_{n+1}(N) &= \gamma_n(N-1) + \frac{e^{f2}_n(N-1)}{\sigma_n(N-2)}. & (120)
\end{aligned}$$

*Proof.* The definitions (63)–(66) written for the predictors $K$ and variances $1-\gamma$ of order $n+1$ can be grouped in the following augmented systems:

$$\begin{bmatrix} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N) \end{bmatrix} \begin{bmatrix} 1 \\ -K_{n+1}^{*}(N) \end{bmatrix} = \begin{bmatrix} 1-\gamma_{n+1}^{*}(N) \\ 0_{n+1} \end{bmatrix}, \tag{121}$$

$$\begin{bmatrix} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N) \end{bmatrix} \begin{bmatrix} 1 \\ -K_{n+1}(N) \end{bmatrix} = \begin{bmatrix} 1-\gamma_{n+1}(N) \\ 0_{n+1} \end{bmatrix}. \tag{122}$$

Using the partition (obtained with (22) and (24))

$$\begin{bmatrix} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N) \end{bmatrix} = \begin{bmatrix} 1 & \varphi_{n}^{\mathrm{T}}(N) & -y(N-n-1) \\ \varphi_{n}(N) & R_{n}(N) & -r_{n}^{b}(N) \\ -y(N-n-1) & -r_{n}^{b\mathrm{T}}(N) & \rho_0(N-n-1) \end{bmatrix}, \tag{123}$$

we get the following identities:

$$\begin{bmatrix} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N) \end{bmatrix} \begin{bmatrix} 0 \\ \theta_{n}^{b}(N) \\ 1 \end{bmatrix} = \begin{bmatrix} -\varepsilon_{n}^{b}(N) \\ 0_{n} \\ \sigma_{n}^{b}(N) \end{bmatrix}, \tag{124}$$

$$\begin{bmatrix} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N) \end{bmatrix} \begin{bmatrix} 1 \\ -K_{n}^{*}(N) \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\gamma_{n}^{*}(N) \\ 0_{n} \\ -\varepsilon_{n}^{b}(N) \end{bmatrix}, \tag{125}$$

which can be combined as

$$\begin{bmatrix} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N) \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ -K_{n}^{*}(N) \\ 0 \end{bmatrix} + \frac{\varepsilon_{n}^{b}(N)}{\sigma_{n}^{b}(N)} \begin{bmatrix} 0 \\ \theta_{n}^{b}(N) \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} 1-\gamma_{n}^{*}(N) - \frac{\varepsilon_{n}^{b2}(N)}{\sigma_{n}^{b}(N)} \\ 0_{n+1} \end{bmatrix} \tag{126}$$

and comparing with (121) we get (113) and (117).

Using the partition (obtained with relations (22) and (24))

$$\begin{bmatrix} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N-1) \end{bmatrix} = \begin{bmatrix} 1 & \varphi_n^{\mathrm{T}}(N) & -y(N-n-1) \\ \varphi_n(N) & R_n(N-1) & -r_n^b(N-1) \\ -y(N-n-1) & -r_n^{b\mathrm{T}}(N-1) & \rho_0(N-n-2) \end{bmatrix}, \tag{127}$$

we get the identities:

$$\begin{bmatrix} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N-1) \end{bmatrix} \begin{bmatrix} 1 \\ -K_n(N) \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\gamma_n(N) \\ 0_n \\ -e_n^b(N) \end{bmatrix}, \tag{128}$$

$$\begin{bmatrix} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N-1) \end{bmatrix} \begin{bmatrix} 0 \\ \theta_n^b(N-1) \\ 1 \end{bmatrix} = \begin{bmatrix} -e_n^b(N) \\ 0_n \\ \sigma_n^b(N-1) \end{bmatrix}, \tag{129}$$

which can be cobined to get

$$\begin{bmatrix} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N-1) \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ -K_n(N) \\ 0 \end{bmatrix} + \frac{e_n^b(N)}{\sigma_n^b(N-1)} \begin{bmatrix} 0 \\ \theta_n^b(N-1) \\ 1 \end{bmatrix} \right\} =$$

$$= \begin{bmatrix} 1-\gamma_n(N) - \frac{e_n^{b2}(N)}{\sigma_n^b(N-1)} \\ 0_{n+1} \end{bmatrix}. \tag{130}$$

Comparing (130) and (122), we get (114) and (118).

From the partition obtained with (23) and (25)

$$\left[\begin{array}{cc} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N) \end{array}\right] = \left[\begin{array}{ccc} 1 & -y(N-1) & \varphi_n^{\mathrm{T}}(N-1) \\ -y(N-1) & \rho_0(N-1) & -r_n^{\mathrm{T}}(N-1) \\ \varphi_n(N-1) & -r_n(N-1) & R_n(N-1) \end{array}\right], \tag{131}$$

we get the identities:

$$\left[\begin{array}{cc} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N) \end{array}\right] \left[\begin{array}{c} 1 \\ 0 \\ -K_n^*(N-1) \end{array}\right] = \left[\begin{array}{c} 1-\gamma_n^*(N-1) \\ -\varepsilon_n^f(N-1) \\ 0_n \end{array}\right], \tag{132}$$

$$\left[\begin{array}{cc} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N) \end{array}\right] \left[\begin{array}{c} 0 \\ 1 \\ \theta_n(N-1) \end{array}\right] = \left[\begin{array}{c} -\varepsilon_n^f(N-1) \\ \sigma_n(N-1) \\ 0_n \end{array}\right], \tag{133}$$

which can be combined to get

$$\begin{aligned} &\left[\begin{array}{cc} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N) \end{array}\right] \left\{\left[\begin{array}{c} 1 \\ 0 \\ -K_n^*(N-1) \end{array}\right] + \frac{\varepsilon_n^f(N-1)}{\sigma_n(N-1)} \left[\begin{array}{c} 0 \\ 1 \\ \theta_n(N-1) \end{array}\right]\right\} = \\ &= \left[\begin{array}{c} 1-\gamma_n^*(N-1)-\frac{\varepsilon_n^{f2}(N-1)}{\sigma_n(N-1)} \\ 0_{n+1} \end{array}\right] \end{aligned} \tag{134}$$

and comparing with (121) we get (115) and (119).

From the partition similar to (131)

$$\left[\begin{array}{cc} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N-1) \end{array}\right] = \left[\begin{array}{ccc} 1 & -y(N-1) & \varphi_n^{\mathrm{T}}(N-1) \\ -y(N-1) & \rho_0(N-2) & -r_n^{\mathrm{T}}(N-2) \\ \varphi_n(N-1) & -r_n(N-2) & R_n(N-2) \end{array}\right], \tag{135}$$

we get after immediate computations:

$$\begin{bmatrix} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N-1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -K_n(N-1) \end{bmatrix} = \begin{bmatrix} 1-\gamma_n(N-1) \\ -e_n^f(N-1) \\ 0_n \end{bmatrix}, \tag{136}$$

$$\begin{bmatrix} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N-1) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \theta_n(N-2) \end{bmatrix} = \begin{bmatrix} -e_n^f(N-1) \\ \sigma_n(N-2) \\ 0_n \end{bmatrix}, \tag{137}$$

which can be combined to get

$$\begin{bmatrix} 1 & \varphi_{n+1}^{\mathrm{T}}(N) \\ \varphi_{n+1}(N) & R_{n+1}(N-1) \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ 0 \\ -K_n(N-1) \end{bmatrix} + \frac{e_n^f(N-1)}{\sigma_n(N-2)} \begin{bmatrix} 0 \\ 1 \\ \theta_n(N-2) \end{bmatrix} \right\} =$$
$$= \begin{bmatrix} 1-\gamma_n(N-1) - \frac{e_n^{f2}(N-1)}{\sigma_n(N-2)} \\ 0_{n+1} \end{bmatrix} \tag{138}$$

and comparing to (122) we get (116) and (120).

## 2.3 Variants of the lattice prediction error and predictor algorithms

The algorithm 1.1 propagates in time, at a cost of $\mathcal{O}(n_{max})$ operations/sample the lattice parameters, and the prediction errors.

Using only these variables one can obtain the predictors at a certain moment $N$ from (97) and (105) in the Table 3 with order recursive equations. and by using (98) transformed to become a pure in order equation.

From (68),(69), (73)–(75) we get

$$\begin{aligned} \theta_n(N-1) &= \theta_n(N) - K_n^*(N)e_n^f(N), & (139) \\ \theta_n^b(N-1) &= \theta_n^b(N) - K_n^*(N)e_n^b(N), & (140) \end{aligned}$$

which can be replaced in (98) to give

$$\begin{aligned} \theta_{n+1}^b(N) &= \begin{bmatrix} 0 \\ \theta_n^b(N) \end{bmatrix} - \frac{\alpha_n(N-1)}{\sigma(N-1)} \begin{bmatrix} 1 \\ \theta_n(N) \end{bmatrix} - \\ &- (e_n^b(N) - \frac{\alpha_n(N-1)}{\sigma_n(N-1)} e_n^f(N)) \begin{bmatrix} 0 \\ K_n^*(N) \end{bmatrix}. \qquad (141) \end{aligned}$$

The relations (97), (141) and (105) provide recursively in order the predictors $\theta_n(N), \theta_n^b(N)$ and $K_n^*(N)$. Initializing these recursions at the order $n = 1$ results from (59),(60) and (64). The step 2.3, presented in the second algorithm slide needs to be added to Algorithm 1.1 to make it an algorithm for the predictors.

A different version of the Algorithm 1.1 is obtained by using the apriori Kalman gain vector from (106), replacing $\theta_n^b(N-1)$ by

$$\theta_n^b(N-1) = \theta_n^b(N) - K_n(N)\varepsilon_n^b(N),$$

and $\theta_n(N-1)$ by (68).

The most used variant is shown as Algorithm 1.2.

The step 2.2, performed at every sampling time, needs in Algorithm 1.2 $10n_{max}$ multiplications or divisions, and $5n_{max}$ additions, when compared to the $10n_{max}$ multiplications or divisions and $6n_{max}$ additions, which are necessary in Algorithm 1.1.