1 Algorithms for Lecture 11

## Algorithm 1.1

LATTICE ALGORITHM FOR PREDICTION			Number of operations/			Number of operations/		
		ERRORS		*	equa /	$\frac{\text{tion}}{+}$	Step	
1.	Initia	dize at $(N=0)$			/			
		$a=0,\ldots,n_{max}-1$						
		(0) = 0						
	- 70	<u>( )</u>						
	$\sigma_0(0)$	$=\sigma_0^b(0)=0$						
2.	Recu	rsions in time					$\mathcal{O}(n_{max}^2)$	
	For I	$N = 1, 2, \dots, N_{max}$						
	2.1	Initialize recursions in order						
		$e_0^f(N) = \varepsilon_0^f(N) = y(N)$						
		$e_0^b(N) = \varepsilon_0^b(N) = y(N-1)$						
		$\sigma_0(N) = \sigma_0(N-1) + \varepsilon_0^{f^2}(N)$	(1)	1	0	1		
		$\sigma_0^b(N) = \sigma_0(N-1)$	(2)					
	2.2	Recursions in order:					$\mathcal{O}(n_{max})$	
		For $n = 0, \dots, n_{max} - 1$						
		$\alpha_n(N) = \alpha_n(N-1) + e_n^f(N)\varepsilon_n^b(N)$	(3)	1	0	1		
		$K_{n+1}^f(N) = -\alpha_n(N)/\sigma_n^b(N)  (\star)$	(4)	0	1	0		
		$K_{n+1}^b(N) = -\alpha_n(N)/\sigma_n(N)  (\star)$	(5)	0	1	0		
		$\sigma_{n+1}(N) = \sigma_n(N)(1 - K_{n+1}^f(N)K_{n+1}^b(N))$	(6)	2	0	1		
		$\sigma_{n+1}^b(N) = \sigma_n^b(N-1)(1 - K_{n+1}^f(N-1)K_{n+1}^b(N-1))$	(7)	1	0	0		
		$e_{n+1}^f(N) = e_n^f(N) + K_{n+1}^f(N-1)e_n^b(N)$	(8)	1	0	1		
		$e_{n+1}^b(N) = e_n^b(N-1) + K_{n+1}^b(N-2)e_n^f(N-1)$	(9)	1	0	1		
		$\varepsilon_{n+1}^f(N) = \varepsilon_n^f(N) + K_{n+1}^f(N)\varepsilon_n^b(N)$	(10)	1	0	1		
		$\varepsilon_{n+1}^b(N) = \varepsilon_n^b(N-1) + K_{n+1}^b(N-1)\varepsilon_n^f(N-1)$	(11)	1	0	1		
	$(\star)$ :	If $\sigma_n^b(N) = 0$ assign $K_{n+1}^f(N) = 0$						
		If $\sigma_n(N) = 0$ assign $K_{n+1}^b(N) = 0$						
	Total: $8n_{max}$ multiplications; $2n_{max}$ divisions; $6n_{max}$ additions							

## Extension of Algorithm 1.1

	N	umb	er		
OPTIONAL SECTION FOR ORDER		of operations/			
RECURSION OF PREDICTORS		equation			
	*	/	+		
2.3 If the predictors are desired at time $N$					
Do:					
2.3.1 Initialize order recursions:					
$\theta_1(N) = -\alpha_0(N)/\sigma_0^b(N)$		1			
$\theta_1^b(N) = -\alpha_0(N-1)/\sigma_0(N-1)$		1			
$K_1^*(N) = -\varepsilon_0^b(N)/\sigma_0^b(N)$		1			
2.3.2 Order recursions:					
For $n = 1, \dots, n_{max} - 1$					
$\theta_{n+1}(N) = \begin{vmatrix} \theta_n(N) \\ 0 \end{vmatrix} - \frac{\alpha_n(N)}{\sigma_n^b(N)} \begin{vmatrix} \theta_n^b(N) \\ 1 \end{vmatrix}$	n	1	n		
$\theta_{n+1}(N) = \begin{bmatrix} \theta_n(N) \\ 0 \end{bmatrix} - \frac{\alpha_n(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix}$ $\theta_{n+1}^b(N) = \begin{bmatrix} 0 \\ \theta_n^b(N) \end{bmatrix} - \frac{\alpha_n(N-1)}{\sigma_n(N-1)} \begin{bmatrix} 1 \\ \theta_n(N) \end{bmatrix} - \frac{\alpha_n(N)}{\sigma_n(N-1)} \begin{bmatrix} 1 \\ \theta_n(N) \end{bmatrix}$					
$-\left(e_n^b(N) - \frac{\alpha_n(N-1)}{\sigma_n(N-1)}e_n^f(N)\right) \begin{bmatrix} 0 \\ K_n^*(N) \end{bmatrix}$	2n+1	1	2n+1		
$-(e_n^b(N) - \frac{\alpha_n(N-1)}{\sigma_n(N-1)}e_n^f(N)) \begin{bmatrix} 0 \\ K_n^*(N) \end{bmatrix}$ $K_{n+1}^*(N) = \begin{bmatrix} K_n^*(N) \\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix}$	n		n		
Total: $\mathcal{O}(n_{max}^2)$ multiplications; $\mathcal{O}(n_{max})$ divisions; $\mathcal{O}(n_{max}^2)$ additions					

$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	LATTICE ALGORITHM FOR PREDICTION			Num. op.			
$ \left  \begin{array}{c} \text{For } n=0,\dots,n_{max}-1  \alpha_n(0)=0 \\ \sigma_0(0)=\sigma_0^b(0)=0 \end{array} \right  \\ \text{2. Time recursions} \\ \left  \begin{array}{c} \text{For } N=1,2,\dots,N_{max} \\ 2.1  \left  \begin{array}{c} \text{Initialize order recursions} \\ \varepsilon_0^f(N)=y(N);  \varepsilon_0^b(N)=y(N-1) \\ \sigma_0(N)=\sigma_0(N-1)+\varepsilon_0^f(2)(N);  \sigma_0^b(N)=\sigma_0(N-1) \\ \gamma_0^0(N)=1 \end{array} \right  \\ \text{2.2 Recursions in order:} \\ \left  \begin{array}{c} \text{ntru } n=0,\dots,n_{max}-1 \\ \alpha_n(N)=\alpha_n(N-1)+\varepsilon_n^f(N)\varepsilon_n^b(N)/\gamma_n^0(N) & (1) & 1 & 1 & 1 \\ K_{n+1}^f(N)=-\alpha_n(N)/\sigma_n^b(N) & (2) & 0 & 1 & 0 \\ K_{n+1}^b(N)=-\alpha_n(N)/\sigma_n^b(N) & (3) & 0 & 1 & 0 \\ K_{n+1}^b(N)=\sigma_n^b(N)(1-K_{n+1}^f(N)K_{n+1}^b(N)) & (4) & 2 & 0 & 1 \\ \sigma_{n+1}^b(N)=\varepsilon_n^b(N-1)(1-K_{n+1}^f(N-1)K_{n+1}^b(N-1)) & (5) & 1 & 0 & 0 \\ \varepsilon_{n+1}^f(N)=\varepsilon_n^b(N-1)K_{n+1}^f(N)-1K_{n+1}^b(N-1) & (6) & 1 & 0 & 1 \\ \varepsilon_{n+1}^f(N)=\varepsilon_n^b(N)+K_{n+1}^f(N)\varepsilon_n^b(N) & (6) & 1 & 0 & 1 \\ \varepsilon_{n+1}^f(N)=\varphi_n^b(N)-\varepsilon_n^b(N)-1K_{n+1}^b(N-1)\varepsilon_n^f(N-1) & (7) & 1 & 0 & 1 \\ \varepsilon_{n+1}^f(N)=\varphi_n^b(N)-\varepsilon_n^b(N)/\sigma_n^b(N) & \star & (8) & 1 & 0 & 1 \\ \end{array} \right. \\ 2.3  \text{If the predictors are needed at time } N \\ \text{Then:} \\ \\ 2.3.1  \text{Initialize order recursions:} \\ \theta_1(N)=K_1^f(N);  \theta_1^b(N)=K_1^b(N-1) \\ K_1^*(N)=-\varepsilon_0^b(N)/\sigma_0^b(N) & (9) \\ 2.3.2  \text{Recursions in order:} \\ \text{For } n=1,\dots,n_{max}-1 \\ \theta_{n+1}(N)=\begin{bmatrix} \theta_n(N) \\ \theta_n(N) \\ 0 \\ \theta_n^b(N) \end{bmatrix}+K_{n+1}^f(N)\begin{bmatrix} \theta_n^b(N) \\ 1 \\ \theta_n(N) \end{bmatrix} - \\ -\frac{(\varepsilon_n^b(N)+K_{n+1}^b(N-1)\varepsilon_n^f(N))}{\gamma_n^b(N)}\begin{bmatrix} 0 \\ K_n^*(N) \\ 0 \end{bmatrix} -\frac{\varepsilon_n^b(N)}{\gamma_n^b(N)}\begin{bmatrix} 0 \\ K_n^*(N) \\ 1 \end{bmatrix} & (10)  n=0  n \\ \\ \left(\star\right):  \text{If any of } \sigma_n(N),\sigma_n^b(N) \text{ sau } \gamma_n^b(N) \\ \end{array}$		ERRORS AND PREDICTORS	*	/	+		
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	1.	Initialize at time $(N=0)$					
$ \begin{array}{c} \text{2. Time recursions} \\ \text{For } N = 1, 2, \dots, N_{max} \\ \text{2.1} & \text{Initialize order recursions} \\ \varepsilon_0^f(N) = y(N); \; \varepsilon_0^b(N) = y(N-1) \\ \sigma_0(N) = \sigma_0(N-1) + \varepsilon_0^{12}(N); \; \sigma_0^b(N) = \sigma_0(N-1) \\ \gamma_0^0(N) = 1 \\ \text{2.2 Recursions in order:} \\ \text{2.1} & \text{Intru } n = 0, \dots, n_{max} - 1 \\ \alpha_n(N) = \alpha_n(N-1) + \varepsilon_n^f(N)\varepsilon_n^b(N) / \gamma_n^0(N) & (1) & 1 & 1 & 1 \\ K_{n+1}^f(N) = -\alpha_n(N)/\sigma_n^b(N) & (2) & 0 & 1 & 0 \\ K_{n+1}^b(N) = -\alpha_n(N)/\sigma_n^b(N) & (3) & 0 & 1 & 0 \\ \sigma_{n+1}(N) = \sigma_n(N)(1 - K_{n+1}^f(N)K_{n+1}^b(N)) & (4) & 2 & 0 & 1 \\ \sigma_{n+1}^b(N) = \sigma_n^b(N-1)(1 - K_{n+1}^f(N-1)K_{n+1}^b(N-1)) & (5) & 1 & 0 & 0 \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^b(N-1)(1 - K_{n+1}^f(N-1)K_{n+1}^b(N-1)) & (6) & 1 & 0 & 1 \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^b(N-1) + K_{n+1}^b(N) + \varepsilon_n^b(N-1) & (7) & 1 & 0 & 1 \\ \varepsilon_{n+1}^b(N) = \varepsilon_n^b(N) + K_{n+1}^b(N) + \varepsilon_n^b(N-1) & (7) & 1 & 0 & 1 \\ \varepsilon_{n+1}^b(N) = \varepsilon_n^b(N) - \varepsilon_n^b(N)/\sigma_n^b(N) & * & (8) \\ \text{2.3 If the predictors are needed at time } N & \\ \text{Then:} & \\ 2.3.1 \text{ Initialize order recursions:} & \theta_1(N) = K_1^b(N) + K_{n+1}^f(N) - K_1^b(N-1) \\ K_1^*(N) = -\varepsilon_0^b(N)/\sigma_0^b(N) & (9) \\ \text{2.3.2 Recursions in order:} & \\ \text{For } n = 1, \dots, n_{max} - 1 \\ \theta_{n+1}(N) = \begin{bmatrix} \theta_n(N) \\ \theta_n^b(N) \\ 0 \end{bmatrix} + K_{n+1}^f(N-1) \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} & (10) \\ \theta_{n+1}^b(N) = \begin{bmatrix} \theta_n(N) \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^f(N-1) \begin{bmatrix} 1 \\ \theta_n(N) \end{bmatrix} - \\ -\frac{(\varepsilon_n^b(N) + K_{n+1}^b(N) - 1)\varepsilon_n^f(N)}{\gamma_n^b(N)}} \begin{bmatrix} \theta_n^b(N) \\ 0 \end{bmatrix} & (11) \\ K_{n+1}^*(N) = \begin{bmatrix} K_n^*(N) + K_{n+1}^b(N) \\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} \end{bmatrix} \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} & (12) \\ n = 1 & n \end{bmatrix} \\ \text{($\star$): If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N) \end{cases}$		For $n = 0, \dots, n_{max} - 1$ $\alpha_n(0) = 0$					
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		$\sigma_0(0) = \sigma_0^b(0) = 0$					
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	2.	Time recursions					
$\begin{cases} \varepsilon_0^f(N) = y(N); \ \varepsilon_0^b(N) = y(N-1) \\ \sigma_0(N) = \sigma_0(N-1) + \varepsilon_0^{f2}(N); \ \sigma_0^b(N) = \sigma_0(N-1) \\ \gamma_0^o(N) = 1 \end{cases} \\ 2.2  \text{Recursions in order:} \\                                   $		For $N = 1, 2, \dots, N_{max}$					
$ \begin{array}{c} \sigma_0(N) = \sigma_0(N-1) + \varepsilon_0^{f2}(N); \ \ \sigma_0^b(N) = \sigma_0(N-1) \\ \gamma_0^0(N) = 1 \\ 2.2  \text{Recursions in order:} \\ \hline \\ \text{ntru } n = 0, \dots, n_{max} - 1 \\ \alpha_n(N) = \alpha_n(N-1) + \varepsilon_n^f(N) \varepsilon_n^b(N) / \gamma_n^0(N) & (1) \\ K_{n+1}^f(N) = -\alpha_n(N) / \sigma_n^b(N) & (2) \\ K_{n+1}^b(N) = -\alpha_n(N) / \sigma_n(N) & (3) \\ \sigma_{n+1}(N) = \sigma_n(N) / (1 - K_{n+1}^f(N) K_{n+1}^b(N)) & (4) \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^b(N-1) (1 - K_{n+1}^f(N) K_{n+1}^b(N)) & (6) \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^b(N-1) (1 - K_{n+1}^f(N-1) K_{n+1}^b(N-1)) & (5) \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^b(N-1) + K_{n+1}^b(N-1) \varepsilon_n^f(N-1) & (7) \\ \gamma_{n+1}^f(N) = \varepsilon_n^b(N) - \varepsilon_n^b(N) / \sigma_n^b(N) & (8) \\ \end{array} \\ 2.3  \text{If the predictors are needed at time $N$} \\ \text{Then:} \\ \hline \\ 2.3.1  \text{Initialize order recursions:} \\ \theta_1(N) = K_1^f(N);  \theta_1^b(N) = K_1^b(N-1) \\ K_1^*(N) = -\varepsilon_0^b(N) / \sigma_0^b(N) & (9) \\ 2.3.2  \text{Recursions in order:} \\ \text{For } n = 1, \dots, n_{max} - 1 \\ \theta_{n+1}(N) = \begin{bmatrix} \theta_n(N) \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^f(N) \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} & (10) \\ \theta_{n+1}^b(N) = \begin{bmatrix} \theta_n(N) \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^f(N-1) \begin{bmatrix} 1 \\ \theta_n(N) \end{bmatrix} - \\ -\frac{(\varepsilon_n^b(N) + K_{n+1}^b(N-1) \varepsilon_n^f(N))}{\sigma_n^b(N)} & 0 \\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} & 0 \\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} & 0 \\ 1 \end{pmatrix} & n = 1  n \\ \hline (\star):  \text{If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N) \\ \hline \end{array}$		2.1 Initialize order recursions					
$ \begin{vmatrix} \gamma_0^0(N) = 1 \\ \text{Recursions in order:} \\ \text{ntru } n = 0, \dots, n_{max} - 1 \\ \alpha_n(N) = \alpha_n(N-1) + \varepsilon_n^f(N) \varepsilon_n^b(N) / \gamma_n^0(N) & (1) \\ K_{n+1}^f(N) = -\alpha_n(N) / \sigma_n^b(N) & (\star) & (2) \\ K_{n+1}^b(N) = -\alpha_n(N) / \sigma_n^b(N) & (\star) & (3) \\ \sigma_{n+1}(N) = \sigma_n(N) (1 - K_{n+1}^f(N) K_{n+1}^b(N)) & (4) \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^b(N-1) (1 - K_{n+1}^f(N-1) K_{n+1}^b(N-1)) & (5) \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^f(N) + K_{n+1}^f(N) \varepsilon_n^b(N) & (6) \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^b(N-1) + K_{n+1}^b(N-1) \varepsilon_n^f(N-1) & (7) \\ \gamma_{n+1}^0(N) = \varepsilon_n^b(N) - \varepsilon_n^b(N) / \varepsilon_n^b(N) & (8) \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ $							
$ \begin{array}{c} 2.2  \text{Recursions in order:} \\ & \text{ntru } n = 0, \dots, n_{max} - 1 \\ & \alpha_n(N) = \alpha_n(N-1) + \varepsilon_n^f(N) \varepsilon_n^b(N) / \gamma_n^0(N) & (1) & 1 & 1 & 1 \\ & K_{n+1}^f(N) = -\alpha_n(N) / \sigma_n^b(N) & (\star) & (2) & 0 & 1 & 0 \\ & K_{n+1}^b(N) = -\alpha_n(N) / \sigma_n^b(N) & (\star) & (3) & 0 & 1 & 0 \\ & \sigma_{n+1}(N) = \sigma_n(N) (1 - K_{n+1}^f(N) K_{n+1}^b(N)) & (4) & 2 & 0 & 1 \\ & \sigma_{n+1}^b(N) = \sigma_n^b(N-1) (1 - K_{n+1}^f(N-1) K_{n+1}^b(N-1)) & (5) & 1 & 0 & 0 \\ & \varepsilon_{n+1}^f(N) = \varepsilon_n^b(N) + K_{n+1}^f(N) \varepsilon_n^b(N) & (6) & 1 & 0 & 1 \\ & \varepsilon_{n+1}^b(N) = \varepsilon_n^b(N-1) + K_{n+1}^b(N-1) \varepsilon_n^b(N-1) & (7) & 1 & 0 & 1 \\ & \varepsilon_{n+1}^b(N) = \varepsilon_n^b(N-1) + K_{n+1}^b(N-1) \varepsilon_n^b(N-1) & (7) & 1 & 0 & 1 \\ & \gamma_{n+1}^b(N) = \gamma_n^b(N) - \varepsilon_n^b(N) / \sigma_n^b(N) & \star & (8) & 1 & 0 & 1 \\ & 2.3  \text{If the predictors are needed at time } N & & & \\ & & & & & & & & \\ & & & & & &$		$\sigma_0(N) = \sigma_0(N-1) + \varepsilon_0^{f2}(N);  \sigma_0^b(N) = \sigma_0(N-1)$	1	0	1		
$ \begin{array}{c} \operatorname{ntru} \ n = 0, \dots, n_{max} - 1 \\ \alpha_n(N) = \alpha_n(N-1) + \varepsilon_n^f(N) \varepsilon_n^b(N) / \gamma_n^0(N) & (1) \\ K_{n+1}^f(N) = -\alpha_n(N) / \sigma_n^b(N) & (\star) & (2) \\ K_{n+1}^f(N) = -\alpha_n(N) / \sigma_n(N) & (\star) & (3) \\ K_{n+1}^f(N) = \sigma_n(N) / (1 - K_{n+1}^f(N) K_{n+1}^b(N)) & (4) \\ \sigma_{n+1}(N) = \sigma_n^b(N-1) (1 - K_{n+1}^f(N-1) K_{n+1}^b(N-1)) & (5) \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^f(N) + K_{n+1}^f(N) \varepsilon_n^b(N) & (6) \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^f(N) + K_{n+1}^f(N) \varepsilon_n^b(N) & (6) \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^b(N-1) + K_{n+1}^b(N-1) \varepsilon_n^f(N-1) & (7) \\ \gamma_{n+1}^0(N) = \gamma_n^0(N) - \varepsilon_n^b(N) / \sigma_n^b(N) & \star & (8) \\ 1 & 0 & 1 \\ \end{array} $ 2.3 If the predictors are needed at time $N$ Then: $ \begin{array}{c} 2.3.1 \text{ Initialize order recursions:} \\ \theta_1(N) = K_1^f(N); & \theta_1^b(N) = K_1^b(N-1) \\ K_1^*(N) = -\varepsilon_0^b(N) / \sigma_0^b(N) & (9) \\ 2.3.2 \text{ Recursions in order:} \\ \text{For } n = 1, \dots, n_{max} - 1 \\ \theta_{n+1}(N) = \begin{bmatrix} \theta_n(N) \\ \theta_n^b(N) \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^f(N) \begin{bmatrix} \theta_n^b(N) \\ 1 \\ 0 \\ \theta_n^b(N) \end{bmatrix} - \\ -\frac{(\varepsilon_n^b(N) + K_{n+1}^b(N-1) \varepsilon_n^f(N))}{\gamma_n^b(N)} \begin{bmatrix} 0 \\ K_n^*(N) \\ 0 \end{bmatrix} - \frac{(11)}{\varepsilon_n^b(N)} \\ K_{n+1}^*(N) = \begin{bmatrix} K_n^*(N) \\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\varepsilon_n^b(N)} \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} & (12) \\ n & 1 & n \\ \end{array} $ $ (\star): \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N) $		$\gamma_0^0(N) = 1$					
$\begin{array}{c} \alpha_n(N) = \alpha_n(N-1) + \varepsilon_n^f(N) \varepsilon_n^b(N) / \gamma_n^0(N) & (1) & 1 & 1 & 1 \\ K_{n+1}^f(N) = -\alpha_n(N) / \sigma_n^b(N) & (\star) & (2) & 0 & 1 & 0 \\ K_{n+1}^b(N) = -\alpha_n(N) / \sigma_n(N) & (\star) & (3) & 0 & 1 & 0 \\ \sigma_{n+1}(N) = \sigma_n(N) (1 - K_{n+1}^f(N) K_{n+1}^b(N)) & (4) & 2 & 0 & 1 \\ \sigma_{n+1}^b(N) = \sigma_n^b(N-1) (1 - K_{n+1}^f(N-1) K_{n+1}^b(N-1)) & (5) & 1 & 0 & 0 \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^b(N-1) (1 - K_{n+1}^f(N-1) K_{n+1}^b(N-1)) & (5) & 1 & 0 & 1 \\ \varepsilon_{n+1}^b(N) = \varepsilon_n^b(N-1) + K_{n+1}^b(N) - 1 \varepsilon_n^b(N-1) & (7) & 1 & 0 & 1 \\ \gamma_{n+1}^b(N) = \varepsilon_n^b(N-1) + K_{n+1}^b(N-1) \varepsilon_n^f(N-1) & (7) & 1 & 0 & 1 \\ \gamma_{n+1}^b(N) = \gamma_n^b(N) - \varepsilon_n^b(N) / \sigma_n^b(N) & \star & (8) & 1 & 0 & 1 \\ \end{array}$ $\begin{array}{c} 2.3.1 \text{ Initialize order recursions:} \\ \theta_1(N) = K_1^f(N); & \theta_1^b(N) = K_1^b(N-1) \\ K_1^*(N) = -\varepsilon_0^b(N) / \sigma_0^b(N) & (9) & 1 \\ 2.3.2 \text{ Recursions in order:} \\ \text{For } n = 1, \dots, n_{max} - 1 \\ \theta_{n+1}(N) = \begin{bmatrix} \theta_n(N) \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^f(N) \begin{bmatrix} \theta_n^b(N) \\ h(N) \end{bmatrix} & (10) & n & 0 & n \\ \theta_{n+1}^b(N) = \begin{bmatrix} \theta_n(N) \\ \theta_n^b(N) \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\gamma_n^b(N)} & \begin{pmatrix} \theta_n^b(N) \\ h(N) \end{bmatrix} & (11) \\ K_{n+1}^*(N) = \begin{bmatrix} K_n^*(N) \\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} & \begin{pmatrix} \theta_n^b(N) \\ 0 \end{bmatrix} & (12) & n & 1 & n \\ \end{pmatrix}$ $(\star): \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N) \end{array}$		2.2 Recursions in order:					
$ \begin{cases} K_{n+1}^f(N) = -\alpha_n(N)/\sigma_n^b(N) \   (\star) & (2) \\ K_{n+1}^b(N) = -\alpha_n(N)/\sigma_n(N) \   (\star) & (3) \\ K_{n+1}^b(N) = -\alpha_n(N)/\sigma_n(N) \   (\star) & (3) \\ \sigma_{n+1}(N) = \sigma_n(N)(1 - K_{n+1}^f(N)K_{n+1}^b(N)) & (4) \\ \sigma_{n+1}^b(N) = \sigma_n^b(N - 1)(1 - K_{n+1}^f(N - 1)K_{n+1}^b(N - 1)) & (5) \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^b(N - 1)(1 - K_{n+1}^f(N - 1)K_{n+1}^b(N - 1)) & (5) \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^b(N - 1)(1 - K_{n+1}^f(N) - 1)\varepsilon_n^b(N) & (6) \\ \varepsilon_{n+1}^f(N) = \varepsilon_n^b(N - 1) + K_{n+1}^b(N - 1)\varepsilon_n^f(N - 1) & (7) \\ \gamma_{n+1}^0(N) = \gamma_n^b(N) - \varepsilon_n^{b2}(N)/\sigma_n^b(N) \   \star & (8) \\ \end{cases} $ 2.3 If the predictors are needed at time $N$ Then: $ \begin{cases} 2.3.1 \text{ Initialize order recursions:} \\ \theta_1(N) = K_1^f(N);  \theta_1^b(N) = K_1^b(N - 1) \\ K_1^*(N) = -\varepsilon_0^b(N)/\sigma_0^b(N) & (9) \\ 2.3.2 \text{ Recursions in order:} \end{cases} $ For $n = 1, \dots, n_{max} - 1$ $ \theta_{n+1}(N) = \begin{cases} \theta_n(N) \\ \theta_n^b(N) \\ \theta_n^b(N) \end{cases} + K_{n+1}^f(N - 1) \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix}  (10)  n = 0  n \end{cases} $ $ \theta_{n+1}^b(N) = \begin{bmatrix} \theta_n(N) \\ \theta_n^b(N) \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^f(N - 1) \begin{bmatrix} 1 \\ \theta_n(N) \\ 1 \end{bmatrix} - \frac{(\varepsilon_n^b(N) + K_{n+1}^b(N - 1)\varepsilon_n^f(N)}{\gamma_n^b(N)} \\ K_n^*(N) \end{bmatrix} = \begin{pmatrix} \kappa_n^*(N) \\ 0 \end{pmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix}  (12)  n = 1  n \end{cases} $ $ (\star): \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N) $		ntru $n = 0, \dots, n_{max} - 1$					
$K_{n+1}^{b}(N) = -\alpha_n(N)/\sigma_n(N)  (\star) \qquad (3) \qquad 0 \qquad 1 \qquad 0$ $\sigma_{n+1}(N) = \sigma_n(N)(1 - K_{n+1}^f(N)K_{n+1}^b(N)) \qquad (4) \qquad 2 \qquad 0 \qquad 1$ $\sigma_{n+1}^b(N) = \sigma_n^b(N - 1)(1 - K_{n+1}^f(N - 1)K_{n+1}^b(N - 1)) \qquad (5) \qquad 1 \qquad 0 \qquad 0$ $\varepsilon_{n+1}^f(N) = \varepsilon_n^f(N) + K_{n+1}^f(N)\varepsilon_n^b(N) \qquad (6) \qquad 1 \qquad 0 \qquad 1$ $\varepsilon_{n+1}^b(N) = \varepsilon_n^b(N - 1) + K_{n+1}^b(N - 1)\varepsilon_n^f(N - 1) \qquad (7) \qquad 1 \qquad 0 \qquad 1$ $\varepsilon_{n+1}^b(N) = \varepsilon_n^b(N - 1) + K_{n+1}^b(N - 1)\varepsilon_n^f(N - 1) \qquad (7) \qquad 1 \qquad 0 \qquad 1$ $\gamma_{n+1}^b(N) = \gamma_n^0(N) - \varepsilon_n^b(N)/\sigma_n^b(N) \qquad (8) \qquad 1 \qquad 0 \qquad 1$ 2.3 If the predictors are needed at time $N$ Then: $2.3.1 \text{ Initialize order recursions:}$ $\theta_1(N) = K_1^f(N);  \theta_1^b(N) = K_1^b(N - 1)$ $K_1^*(N) = -\varepsilon_0^b(N)/\sigma_0^b(N) \qquad (9) \qquad 1$ 2.3.2 Recursions in order: $For  n = 1, \dots, n_{max} - 1$ $\theta_{n+1}(N) = \begin{bmatrix} \theta_n(N) \\ \theta_n(N) \\ 0 \end{bmatrix} + K_{n+1}^f(N) \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} \qquad (10) \qquad n \qquad 0 \qquad n$ $\theta_{n+1}^b(N) = \begin{bmatrix} \theta_n(N) \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^f(N - 1) \begin{bmatrix} 1 \\ \theta_n(N) \end{bmatrix} \begin{bmatrix} -\frac{(\varepsilon_n^b(N) + K_{n+1}^b(N - 1)\varepsilon_n^f(N))}{\gamma_n^b(N)} \begin{bmatrix} 0 \\ K_n^*(N) \end{bmatrix} \qquad (11) \qquad 2n+1  1  2n+1$ $K_{n+1}^*(N) = \begin{bmatrix} K_n^*(N) \\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} \qquad (12) \qquad n \qquad 1  n$ $(\star):  \text{If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N)$		$\alpha_n(N) = \alpha_n(N-1) + \varepsilon_n^f(N)\varepsilon_n^b(N)/\gamma_n^0(N) \tag{1}$	1	1	1		
$\sigma_{n+1}(N) = \sigma_n(N)(1 - K_{n+1}^f(N)K_{n+1}^h(N)) \qquad (4) \qquad 2 \qquad 0 \qquad 1$ $\sigma_{n+1}^b(N) = \sigma_n^b(N-1)(1 - K_{n+1}^f(N-1)K_{n+1}^b(N-1)) \qquad (5) \qquad 1 \qquad 0 \qquad 0$ $\varepsilon_{n+1}^f(N) = \varepsilon_n^f(N) + K_{n+1}^f(N)\varepsilon_n^b(N) \qquad (6) \qquad 1 \qquad 0 \qquad 1$ $\varepsilon_{n+1}^b(N) = \varepsilon_n^b(N-1) + K_{n+1}^b(N-1)\varepsilon_n^f(N-1) \qquad (7) \qquad 1 \qquad 0 \qquad 1$ $\gamma_{n+1}^b(N) = \gamma_n^0(N) - \varepsilon_n^{b2}(N)/\sigma_n^b(N) \qquad \qquad (8) \qquad 1 \qquad 0 \qquad 1$ $\gamma_{n+1}^b(N) = \gamma_n^b(N) - \varepsilon_n^{b2}(N)/\sigma_n^b(N) \qquad \qquad (8) \qquad 1 \qquad 0 \qquad 1$ $2.3  \text{If the predictors are needed at time } N$ Then: $2.3.1  \text{Initialize order recursions:} \qquad \qquad$		$K_{n+1}^f(N) = -\alpha_n(N)/\sigma_n^b(N)  (\star) $	0	1	0		
$\sigma_{n+1}^{b}(N) = \sigma_{n}^{b}(N-1)(1-K_{n+1}^{f}(N-1)K_{n+1}^{b}(N-1)) \qquad (5) \qquad 1 \qquad 0 \qquad 0$ $\varepsilon_{n+1}^{f}(N) = \varepsilon_{n}^{f}(N) + K_{n+1}^{f}(N)\varepsilon_{n}^{b}(N) \qquad (6) \qquad 1 \qquad 0 \qquad 1$ $\varepsilon_{n+1}^{b}(N) = \varepsilon_{n}^{b}(N-1) + K_{n+1}^{b}(N-1)\varepsilon_{n}^{f}(N-1) \qquad (7) \qquad 1 \qquad 0 \qquad 1$ $\varepsilon_{n+1}^{b}(N) = \varepsilon_{n}^{b}(N-1) + K_{n+1}^{b}(N-1)\varepsilon_{n}^{f}(N-1) \qquad (7) \qquad 1 \qquad 0 \qquad 1$ $\varepsilon_{n+1}^{b}(N) = \varepsilon_{n}^{b}(N) - \varepsilon_{n}^{b}(N)/\sigma_{n}^{b}(N) \qquad \star \qquad (8) \qquad 1 \qquad 0 \qquad 1$ $2.3  \text{If the predictors are needed at time } N$ $\text{Then:}$ $2.3.1  \text{Initialize order recursions:}$ $\theta_{1}(N) = K_{1}^{f}(N);  \theta_{1}^{b}(N) = K_{1}^{b}(N-1) \qquad (9) \qquad 1$ $2.3.2  \text{Recursions in order:}$ $\text{For } n = 1, \dots, n_{max} - 1$ $\theta_{n+1}(N) = \begin{bmatrix} \theta_{n}(N) \\ \theta_{n}^{b}(N) \end{bmatrix} + K_{n+1}^{f}(N) \begin{bmatrix} \theta_{n}^{b}(N) \\ 1 \end{bmatrix} \qquad (10) \qquad n \qquad 0 \qquad n$ $\theta_{n+1}^{b}(N) = \begin{bmatrix} 0 \\ \theta_{n}^{b}(N) \end{bmatrix} + K_{n+1}^{b}(N-1) \begin{bmatrix} 1 \\ \theta_{n}(N) \end{bmatrix} - \frac{\varepsilon_{n}^{b}(N)}{\gamma_{n}^{o}(N)} \qquad (11) \qquad 2n+1  1  2n+1$ $K_{n+1}^{*}(N) = \begin{bmatrix} K_{n}^{*}(N) \\ 0 \end{bmatrix} - \frac{\varepsilon_{n}^{b}(N)}{\sigma_{n}^{b}(N)} \begin{bmatrix} \theta_{n}^{b}(N) \\ 0 \end{bmatrix} \qquad (12) \qquad n \qquad 1  n$ $(\star) :  \text{If any of } \sigma_{n}(N), \sigma_{n}^{b}(N) \text{ sau } \gamma_{n}^{0}(N)$		$K_{n+1}^b(N) = -\alpha_n(N)/\sigma_n(N)  (\star) $ (3)	0	1	0		
$ \begin{cases} \varepsilon_{n+1}^f(N) = \varepsilon_n^f(N) + K_{n+1}^f(N) \varepsilon_n^b(N) & (6) \\ \varepsilon_{n+1}^b(N) = \varepsilon_n^b(N-1) + K_{n+1}^b(N-1) \varepsilon_n^f(N-1) & (7) \\ \gamma_{n+1}^0(N) = \gamma_n^0(N) - \varepsilon_n^{b2}(N) / \sigma_n^b(N) & \star & (8) \end{cases} $ 1 0 1 $ \gamma_{n+1}^0(N) = \gamma_n^0(N) - \varepsilon_n^{b2}(N) / \sigma_n^b(N) & \star & (8) \end{cases} $ 2.3 If the predictors are needed at time $N$ Then: $ \begin{cases} 2.3.1 \text{ Initialize order recursions:} \\ \theta_1(N) = K_1^f(N); & \theta_1^b(N) = K_1^b(N-1) \\ K_1^*(N) = -\varepsilon_0^b(N) / \sigma_0^b(N) & (9) \end{cases} $ 1 2.3.2 Recursions in order: $ \begin{cases} For \ n = 1, \dots, n_{max} - 1 \\ \theta_{n+1}(N) = \begin{bmatrix} \theta_n(N) \\ 0 \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^f(N) \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} & (10)  n = 0  n \end{cases} $ $ \begin{cases} \theta_n^b(N) \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^b(N-1) \begin{bmatrix} 1 \\ \theta_n(N) \end{bmatrix} - \frac{\varepsilon_n^b(N) + K_{n+1}^b(N-1)\varepsilon_n^f(N)}{\gamma_n^b(N)} \begin{pmatrix} 0 \\ K_n^*(N) \end{pmatrix} & (11) \\ K_n^*(N) \end{bmatrix} $ $ \begin{cases} K_n^*(N) \\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N) + K_{n+1}^b(N-1)\varepsilon_n^f(N)}{\gamma_n^b(N)} \begin{pmatrix} \theta_n^b(N) \\ 0 \end{pmatrix} & (12) \end{cases} $ $ \begin{cases} \lambda \end{cases} : \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N) \end{cases} $			2	0	1		
$ \begin{cases} \varepsilon_{n+1}^f(N) = \varepsilon_n^f(N) + K_{n+1}^f(N) \varepsilon_n^b(N) & (6) \\ \varepsilon_{n+1}^b(N) = \varepsilon_n^b(N-1) + K_{n+1}^b(N-1) \varepsilon_n^f(N-1) & (7) \\ \gamma_{n+1}^0(N) = \gamma_n^0(N) - \varepsilon_n^{b2}(N) / \sigma_n^b(N) & \star & (8) \end{cases} $ 1 0 1 $ \gamma_{n+1}^0(N) = \gamma_n^0(N) - \varepsilon_n^{b2}(N) / \sigma_n^b(N) & \star & (8) \end{cases} $ 2.3 If the predictors are needed at time $N$ Then: $ \begin{cases} 2.3.1 \text{ Initialize order recursions:} \\ \theta_1(N) = K_1^f(N); & \theta_1^b(N) = K_1^b(N-1) \\ K_1^*(N) = -\varepsilon_0^b(N) / \sigma_0^b(N) & (9) \end{cases} $ 1 2.3.2 Recursions in order: $ \begin{cases} For \ n = 1, \dots, n_{max} - 1 \\ \theta_{n+1}(N) = \begin{bmatrix} \theta_n(N) \\ 0 \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^f(N) \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} & (10)  n = 0  n \end{cases} $ $ \begin{cases} \theta_n^b(N) \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^b(N-1) \begin{bmatrix} 1 \\ \theta_n(N) \end{bmatrix} - \frac{\varepsilon_n^b(N) + K_{n+1}^b(N-1)\varepsilon_n^f(N)}{\gamma_n^b(N)} \begin{pmatrix} 0 \\ K_n^*(N) \end{pmatrix} & (11) \\ K_n^*(N) \end{bmatrix} $ $ \begin{cases} K_n^*(N) \\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N) + K_{n+1}^b(N-1)\varepsilon_n^f(N)}{\gamma_n^b(N)} \begin{pmatrix} \theta_n^b(N) \\ 0 \end{pmatrix} & (12) \end{cases} $ $ \begin{cases} \lambda \end{cases} : \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N) \end{cases} $		$\sigma_{n+1}^b(N) = \sigma_n^b(N-1)(1 - K_{n+1}^f(N-1)K_{n+1}^b(N-1)) $ (5)	1	0	0		
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$			1	0	1		
2.3 If the predictors are needed at time $N$ Then:		$\varepsilon_{n+1}^b(N) = \varepsilon_n^b(N-1) + K_{n+1}^b(N-1)\varepsilon_n^f(N-1) \tag{7}$	1	0	1		
Then:		$\gamma_{n+1}^0(N) = \gamma_n^0(N) - \varepsilon_n^{b2}(N) / \sigma_n^b(N)  \star \tag{8}$	1	0	1		
$\begin{array}{c} 2.3.1 \ \text{Initialize order recursions:} \\ \theta_1(N) = K_1^f(N);  \theta_1^b(N) = K_1^b(N-1) \\ K_1^*(N) = -\varepsilon_0^b(N)/\sigma_0^b(N) \\ 2.3.2 \ \text{Recursions in order:} \\ \text{For } n = 1, \ldots, n_{max} - 1 \\ \theta_{n+1}(N) = \begin{bmatrix} \theta_n(N) \\ 0 \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^f(N) \begin{bmatrix} \theta_n^b(N) \\ 1 \\ 0 \\ \theta_n^b(N) \end{bmatrix} - \\ -\frac{(\varepsilon_n^b(N) + K_{n+1}^b(N-1)\varepsilon_n^f(N))}{\gamma_n^b(N)} \begin{bmatrix} 0 \\ K_n^*(N) \\ 0 \\ 0 \end{bmatrix} - \frac{(10)}{\delta_n^b(N)} & (11) \\ K_n^*(N) \\ 1 \end{bmatrix} & (12) \\ n = 1  n \\ (\star) : \ \text{If any of } \sigma_n(N), \sigma_n^b(N) \ \text{sau } \gamma_n^0(N) \\ \end{array}$							
$\theta_{1}(N) = K_{1}^{f}(N);  \theta_{1}^{b}(N) = K_{1}^{b}(N-1)$ $K_{1}^{*}(N) = -\varepsilon_{0}^{b}(N)/\sigma_{0}^{b}(N) \qquad (9)$ 2.3.2 Recursions in order: For $n = 1, \dots, n_{max} - 1$ $\theta_{n+1}(N) = \begin{bmatrix} \theta_{n}(N) \\ 0 \\ \theta_{n}(N) \end{bmatrix} + K_{n+1}^{f}(N) \begin{bmatrix} \theta_{n}^{b}(N) \\ 1 \end{bmatrix} \qquad (10) \qquad n \qquad 0 \qquad n$ $\theta_{n+1}^{b}(N) = \begin{bmatrix} 0 \\ \theta_{n}^{b}(N) \end{bmatrix} + K_{n+1}^{b}(N-1) \begin{bmatrix} 1 \\ \theta_{n}(N) \end{bmatrix} - \frac{(\varepsilon_{n}^{b}(N) + K_{n+1}^{b}(N-1)\varepsilon_{n}^{f}(N)}{\gamma_{n}^{0}(N)} \begin{bmatrix} 0 \\ K_{n}^{*}(N) \end{bmatrix} \qquad (11)$ $K_{n+1}^{*}(N) = \begin{bmatrix} K_{n}^{*}(N) \\ 0 \end{bmatrix} - \frac{\varepsilon_{n}^{b}(N)}{\sigma_{n}^{b}(N)} \begin{bmatrix} \theta_{n}^{b}(N) \\ 1 \end{bmatrix} \qquad (12) \qquad n \qquad 1 \qquad n$ $(\star) : \text{ If any of } \sigma_{n}(N), \sigma_{n}^{b}(N) \text{ sau } \gamma_{n}^{0}(N)$		Then:					
$K_1^*(N) = -\varepsilon_0^b(N)/\sigma_0^b(N) \qquad (9) \qquad 1$ 2.3.2 Recursions in order: For $n = 1, \dots, n_{max} - 1$ $\theta_{n+1}(N) = \begin{bmatrix} \theta_n(N) \\ 0 \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^f(N) \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} \qquad (10) \qquad n \qquad 0 \qquad n$ $\theta_{n+1}^b(N) = \begin{bmatrix} 0 \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^b(N-1) \begin{bmatrix} 1 \\ \theta_n(N) \end{bmatrix} - \frac{(11)}{\sigma_n^0(N)} \begin{pmatrix} 0 \\ K_n^*(N) \end{pmatrix} \qquad (11) \qquad 2n+1 \qquad 1 \qquad 2n+1$ $K_{n+1}^*(N) = \begin{bmatrix} K_n^*(N) \\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} \qquad (12) \qquad n \qquad 1 \qquad n$ $(\star):  \text{If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N)$		2.3.1 Initialize order recursions:					
$2.3.2 \text{ Recursions in order:}$ For $n=1,\ldots,n_{max}-1$ $\theta_{n+1}(N)=\begin{bmatrix}\theta_n(N)\\0\\0\end{bmatrix}+K_{n+1}^f(N)\begin{bmatrix}\theta_n^b(N)\\1\end{bmatrix} \qquad (10) \qquad n=0 \qquad n$ $\theta_{n+1}^b(N)=\begin{bmatrix}0\\\theta_n^b(N)\end{bmatrix}+K_{n+1}^b(N-1)\begin{bmatrix}1\\\theta_n(N)\end{bmatrix}-\\-\frac{(\varepsilon_n^b(N)+K_{n+1}^b(N-1)\varepsilon_n^f(N))}{\gamma_n^0(N)}\begin{bmatrix}0\\K_n^*(N)\\N^0\end{bmatrix} \qquad (11) \qquad 2n+1 \qquad 1 \qquad 2n+1$ $K_{n+1}^*(N)=\begin{bmatrix}K_n^*(N)\\0\end{bmatrix}-\frac{\varepsilon_n^b(N)}{\sigma_n^b(N)}\begin{bmatrix}\theta_n^b(N)\\1\end{bmatrix} \qquad (12) \qquad n=1 \qquad n$ $(\star): \text{ If any of } \sigma_n(N),\sigma_n^b(N) \text{ sau } \gamma_n^0(N)$		$\theta_1(N) = K_1^f(N);  \theta_1^b(N) = K_1^b(N-1)$					
For $n=1,\ldots,n_{max}-1$ $\theta_{n+1}(N)=\begin{bmatrix}\theta_n(N)\\0\\\theta_n^k(N)\end{bmatrix}+K_{n+1}^f(N)\begin{bmatrix}\theta_n^b(N)\\1\end{bmatrix} \qquad (10) \qquad n \qquad 0 \qquad n$ $\theta_{n+1}^b(N)=\begin{bmatrix}0\\\theta_n^b(N)\end{bmatrix}+K_{n+1}^b(N-1)\begin{bmatrix}1\\\theta_n(N)\end{bmatrix}-\\-\frac{(\varepsilon_n^b(N)+K_{n+1}^b(N-1)\varepsilon_n^f(N))}{\gamma_n^0(N)}\begin{bmatrix}0\\K_n^*(N)\\d_n^*(N)\end{bmatrix} \qquad (11) \qquad 2n+1 \qquad 1 \qquad 2n+1$ $K_{n+1}^*(N)=\begin{bmatrix}K_n^*(N)\\0\end{bmatrix}-\frac{\varepsilon_n^b(N)}{\sigma_n^b(N)}\begin{bmatrix}\theta_n^b(N)\\1\end{bmatrix} \qquad (12) \qquad n \qquad 1 \qquad n$ $(\star): \text{ If any of } \sigma_n(N),\sigma_n^b(N) \text{ sau } \gamma_n^0(N)$		$K_1^*(N) = -\varepsilon_0^b(N)/\sigma_0^b(N) \tag{9}$		1			
$\theta_{n+1}(N) = \begin{bmatrix} \theta_n(N) \\ 0 \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^f(N) \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} \qquad (10) \qquad n \qquad 0 \qquad n$ $\theta_{n+1}^b(N) = \begin{bmatrix} 0 \\ \theta_n^b(N) \end{bmatrix} + K_{n+1}^b(N-1) \begin{bmatrix} 1 \\ \theta_n(N) \end{bmatrix} - \frac{(\epsilon_n^b(N) + K_{n+1}^b(N-1) \epsilon_n^f(N))}{\gamma_n^0(N)} \begin{bmatrix} 0 \\ K_n^*(N) \end{bmatrix} \qquad (11) \qquad 2n+1 \qquad 1 \qquad 2n+1$ $K_{n+1}^*(N) = \begin{bmatrix} K_n^*(N) \\ 0 \end{bmatrix} - \frac{\epsilon_n^b(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N) \\ 1 \end{bmatrix} \qquad (12) \qquad n \qquad 1 \qquad n$ $(\star) : \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N)$		2.3.2 Recursions in order:					
$-\frac{(\varepsilon_n^b(N)+K_{n+1}^b(N-1)\varepsilon_n^f(N))}{\gamma_n^0(N)} \begin{bmatrix} 0\\ K_n^*(N) \end{bmatrix} $ $K_{n+1}^*(N) = \begin{bmatrix} K_n^*(N)\\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N)\\ 1 \end{bmatrix} $ $(11)  2n+1  1  2n+1$ $(12)  n  1  n$ $(*): \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N)$		For $n = 1, \ldots, n_{max} - 1$					
$-\frac{(\varepsilon_n^b(N)+K_{n+1}^b(N-1)\varepsilon_n^f(N))}{\gamma_n^0(N)} \begin{bmatrix} 0\\ K_n^*(N) \end{bmatrix} $ $K_{n+1}^*(N) = \begin{bmatrix} K_n^*(N)\\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N)\\ 1 \end{bmatrix} $ $(11)  2n+1  1  2n+1$ $(12)  n  1  n$ $(*): \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N)$		$\left[\begin{array}{ccc} \theta_n(N) \end{array}\right] + \operatorname{ref}_{n}(N) \left[\begin{array}{ccc} \theta_n^b(N) \end{array}\right] \tag{10}$		0			
$-\frac{(\varepsilon_n^b(N)+K_{n+1}^b(N-1)\varepsilon_n^f(N))}{\gamma_n^0(N)} \begin{bmatrix} 0\\ K_n^*(N) \end{bmatrix} $ $K_{n+1}^*(N) = \begin{bmatrix} K_n^*(N)\\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N)\\ 1 \end{bmatrix} $ $(11)  2n+1  1  2n+1$ $(12)  n  1  n$ $(*): \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N)$		$\theta_{n+1}(N) = \begin{vmatrix} & & & \\ & 0 & \\ & & \end{vmatrix} + K_{n+1}(N) \begin{vmatrix} & & & \\ & 1 & \\ & & \end{vmatrix} $ (10)	n	Ü	n		
$-\frac{(\varepsilon_n^b(N)+K_{n+1}^b(N-1)\varepsilon_n^f(N))}{\gamma_n^0(N)} \begin{bmatrix} 0\\ K_n^*(N) \end{bmatrix} $ $K_{n+1}^*(N) = \begin{bmatrix} K_n^*(N)\\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N)\\ 1 \end{bmatrix} $ $(11)  2n+1  1  2n+1$ $(12)  n  1  n$ $(*): \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N)$							
$-\frac{(\varepsilon_n^b(N)+K_{n+1}^b(N-1)\varepsilon_n^f(N))}{\gamma_n^0(N)} \begin{bmatrix} 0\\ K_n^*(N) \end{bmatrix} $ $K_{n+1}^*(N) = \begin{bmatrix} K_n^*(N)\\ 0 \end{bmatrix} - \frac{\varepsilon_n^b(N)}{\sigma_n^b(N)} \begin{bmatrix} \theta_n^b(N)\\ 1 \end{bmatrix} $ $(11)  2n+1  1  2n+1$ $(12)  n  1  n$ $(*): \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N)$		$ \left  \begin{array}{c c} \theta_{n+1}^b(N) = \left  \begin{array}{c} \theta_n^b(N) \end{array} \right  + K_{n+1}^b(N-1) \left  \begin{array}{c} \theta_n(N) \end{array} \right  - $					
$(\star): \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N)$		$\begin{pmatrix} n & f \\ g^b(N) + K^b & f(N-1) + f(N) \end{pmatrix} = \begin{pmatrix} n & f(N) \\ 0 & 1 \end{pmatrix}$					
$(\star): \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N)$		$-\frac{(\varepsilon_n(N)+N_{n+1}(N-1)\varepsilon_n(N))}{\gamma_n^0(N)} \bigg _{K^*(N)} $ (11)	2n+1	1	2n+1		
$(\star): \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N)$		$\begin{bmatrix} K^*(N) \end{bmatrix} \xrightarrow{b \in \mathbb{N}} \begin{bmatrix} \theta^b(N) \end{bmatrix}$					
$(\star): \text{ If any of } \sigma_n(N), \sigma_n^b(N) \text{ sau } \gamma_n^0(N)$		$K_{n+1}^*(N) = \left  \begin{array}{c} \Gamma_n(V) \\ 0 \end{array} \right  - \frac{\varepsilon_n^*(N)}{\sigma_n^b(N)} \left  \begin{array}{c} \sigma_n(V) \\ 1 \end{array} \right  \tag{12}$	n	1	n		
		( ) TC ( (37) h(37) ()(27)					
are zero, set the corresponding division results to 0 0.							
		are zero, set the corresponding division results to 0 0.					