Exercise 3

Momentum operator

Show that the normal ordered 4-momentum operator P_μ can be written as

$$P_\mu = \int \mathrm{d}^3x : T^0{}_\mu := \int \mathrm{d}\tilde{k}\, k_\mu \left[a^\dagger(\vec{k})\, a(\vec{k}) + b^\dagger(\vec{k})\, b(\vec{k})\right] ,$$

in terms of creation and annihilation operators.

We inspect the complex scalar field Lagrangian

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi \tag{1}$$

with the derivatives

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^\dagger \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} = \partial^\mu \phi. \tag{2}$$

The fields can be expressed as

$$\phi(x) = \int d\tilde{k} \left[a(\vec{k}) e^{-ikx} + b^\dagger(\vec{k}) e^{ikx}\right] \quad \text{and} \quad \phi^\dagger(x) = \int d\tilde{p} \left[b(\vec{p}) e^{-ipx} + a^\dagger(\vec{p}) e^{ipx}\right] \tag{3}$$

and we get the time derivatives

$$\Pi^\dagger = \partial^0 \phi(x) = \int d\tilde{k}\, (ik^0) \left[-a(\vec{k}) e^{-ikx} + b^\dagger(\vec{k}) e^{ikx}\right] \quad \text{and} \quad \Pi = \partial^0 \phi^\dagger(x) = \int d\tilde{p}\, (ip^0) \left[-b(\vec{p}) e^{-ipx} + a^\dagger(\vec{p}) e^{ipx}\right] \tag{4}$$

as well as the spacial derivatives

$$\partial_j \phi(x) = \int d\tilde{k}\, (ik^j) \left[-a(\vec{k}) e^{-ikx} + b^\dagger(\vec{k}) e^{ikx}\right] \quad \text{and} \quad \partial_j \phi^\dagger(x) = \int d\tilde{p}\, (ip^j) \left[-b(\vec{p}) e^{-ipx} + a^\dagger(\vec{p}) e^{ipx}\right] \tag{5}$$

because $e^{ikx} = e^{ik^0 x_0 - i\vec{k}\vec{x}}$.

Energy-Momentum-Tensor

For a complex field we have

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} \partial^\nu \phi^\dagger - g^{\mu\nu} \mathcal{L} \tag{6}$$

which results in

$$T^0{}_\mu = g_{\mu\lambda} T^{0\lambda} = \Pi\, (\partial_\mu \phi) + \Pi^\dagger \left(\partial_\mu \phi^\dagger\right) - g^0{}_\mu \mathcal{L}. \tag{7}$$

(Because of the metric tensor) we separate the cases

$$T^0{}_0 = \Pi\, (\partial_0 \phi) + \Pi \left(\partial_0 \phi^\dagger\right) - \mathcal{L} \tag{8}$$

(we don't express the other derivative as Π because of the lower indices) and

$$T^0{}_j = \Pi^\dagger\, (\partial_j \phi) + \Pi \left(\partial_j \phi^\dagger\right). \tag{9}$$

Space components

$$P_j = \int \mathrm{d}^3x : T^0{}_j : \tag{10}$$

$$
\begin{aligned}
P_j = \int \mathrm{d}^3x \int d\tilde{k} \int d\tilde{p} : (ip^0) \left[-b(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{ipx}\right] (ik_j) \left[-a(\vec{k})e^{-ikx} + b^\dagger(\vec{k})e^{ikx}\right] \quad (11)\\
+ (ik^0) \left[-a(\vec{k})e^{-ikx} + b^\dagger(\vec{k})e^{ikx}\right] (ip_j) \left[-b(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{ipx}\right] : \quad (12)\\
= -\int \mathrm{d}^3x \int d\tilde{k} \int d\tilde{p} : \quad (13)\\
p^0 k_j \left[b(\vec{p})a(\vec{k})e^{-i(k+p)x} - b(\vec{p})b^\dagger(\vec{k})e^{i(k-p)x} - a^\dagger(\vec{p})a(\vec{k})e^{-i(k-p)x} + a^\dagger(\vec{p})b^\dagger(\vec{k})e^{i(k+p)x}\right] \quad (14)\\
+k^0 p_j \left[a(\vec{k})b(\vec{p})e^{-i(k+p)x} - a(\vec{k})a^\dagger(\vec{p})e^{-i(k-p)x} - b^\dagger(\vec{k})b(\vec{p})e^{i(k-p)x} + b^\dagger(\vec{k})a^\dagger(\vec{p})e^{i(k+p)x}\right] : \quad (15)
\end{aligned}
$$

Now use the Fourier transformation

$$
\int d^3x e^{\pm i(k+p)x} = (2\pi)^3 e^{\pm i\omega_k t} \delta(\vec{k} + \vec{p}) \quad (16)
$$

and $\omega_k = \omega_p := \omega$ if $\vec{k}^2 = \vec{p}^2$.

$$
\begin{aligned}
P_j = -(2\pi)^3 \int d\tilde{k} \int d\tilde{p} : \omega k_j \Big\{ \left[b(\vec{p})a(\vec{k})e^{-i2\omega t} + a^\dagger(\vec{p})b^\dagger(\vec{k})e^{i2\omega t}\right] \delta(\vec{k}+\vec{p}) - \left[b(\vec{p})b^\dagger(\vec{k}) + a^\dagger(\vec{p})a(\vec{k})\right] \delta(\vec{k}-\vec{p}) \Big\} \quad (17)\\
+\omega p_j \Big\{ \left[a(\vec{k})b(\vec{p})e^{-i2\omega t} + b^\dagger(\vec{k})a^\dagger(\vec{p})e^{i2\omega t}\right] \delta(\vec{k}+\vec{p}) - \left[a(\vec{k})a^\dagger(\vec{p}) + b^\dagger(\vec{k})b(\vec{p})\right] \delta(\vec{k}-\vec{p}) \Big\} : \quad (18)
\end{aligned}
$$

Evaluate $\int d\tilde{p} = \int \frac{d^3p}{(2\pi)^3 2\omega}$ and utilize that $[a, b] = \left[a, b^\dagger\right] = 0$.

$$
\begin{aligned}
P_j = -(2\pi)^3 \int d\tilde{k} : \omega k_j \Big\{ \cancel{\left[b(-\vec{k})a(\vec{k})e^{-i2\omega t} + a^\dagger(-\vec{k})b^\dagger(\vec{k})e^{i2\omega t}\right]} - \left[b(\vec{k})b^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right] \quad (19)\\
-\cancel{\left[a(\vec{k})b(-\vec{k})e^{-i2\omega t} + b^\dagger(\vec{k})a^\dagger(-\vec{k})e^{i2\omega t}\right]} + \left[a(\vec{k})a^\dagger(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right] \Big\} : \quad (20)
\end{aligned}
$$

Lastly we make use of the normal order.

$$
\begin{aligned}
P_j &= \int d\tilde{k} \frac{k_j}{2} : \left[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k}) + b(\vec{k})b^\dagger(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right] : \quad (21)\\
&= \int d\tilde{k} k_j \left[a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right] \quad (22)
\end{aligned}
$$

Time component

Recall that

$$
\begin{aligned}
T^0{}_0 &= \Pi\left(\partial_0\phi\right) + \Pi\left(\partial_0\phi^\dagger\right) - \mathcal{L} \quad (23)\\
&= \Pi\left(\partial_0\phi\right) + \Pi\left(\partial_0\phi^\dagger\right) - (\partial_\mu\phi^\dagger)(\partial^\mu\phi) + m^2\phi^\dagger\phi \quad (24)\\
&= \Pi\left(\partial_0\phi\right) + \left(\partial_j\phi^\dagger\right)\left(\partial^j\phi\right) + m^2\phi^\dagger\phi \quad (25)
\end{aligned}
$$

and

$$
H = P_0 = \int \mathrm{d}^3x : T^0{}_0 : . \quad (26)
$$

It is happening. Again...

$$
\begin{aligned}
P_0 = \int \mathrm{d}^3x \int d\tilde{k} \int d\tilde{p} : ip^0 \left[-b(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{ipx}\right](ik_0)\left[-a(\vec{k})e^{-ikx} + b^\dagger(\vec{k})e^{ikx}\right] && (27)\\
-ip^j \left[-b(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{ipx}\right](-ik^j)\left[-a(\vec{k})e^{-ikx} + b^\dagger(\vec{k})e^{ikx}\right] && (28)\\
+m^2 \left[b(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{ipx}\right]\left[a(\vec{k})e^{-ikx} + b^\dagger(\vec{k})e^{ikx}\right] : && (29)\\
= \int \mathrm{d}^3x \int d\tilde{k} \int d\tilde{p} : -p^0k_0 \left[b(\vec{p})a(\vec{k})e^{-i(k+p)x} - b(\vec{p})b^\dagger(\vec{k})e^{i(k-p)x} - a^\dagger(\vec{p})a(\vec{k})e^{-i(k-p)x} + a^\dagger(\vec{p})b^\dagger(\vec{k})e^{i(k+p)x}\right] && (30)\\
-p^jk_j \left[b(\vec{p})a(\vec{k})e^{-i(k+p)x} - b(\vec{p})b^\dagger(\vec{k})e^{i(k-p)x} - a^\dagger(\vec{p})a(\vec{k})e^{-i(k-p)x} + a^\dagger(\vec{p})b^\dagger(\vec{k})e^{i(k+p)x}\right] && (31)\\
+m^2 \left[b(\vec{p})a(\vec{k})e^{-i(+p)x} + b(\vec{p})b^\dagger(\vec{k})e^{i(k-p)x} + a^\dagger(\vec{p})a(\vec{k})e^{-i(k-p)x} + a^\dagger(\vec{p})b^\dagger(\vec{k})e^{i(k+p)x}\right] : && (32)
\end{aligned}
$$

We again use the Fourier transformation. For this case we can utilze that $\omega^2 = k_0k^0 = m^2 + \vec{k}^2$ to cancel the mixed operator brackets.

$$
\begin{aligned}
P_0 = (2\pi)^3 \int d\tilde{k} \int d\tilde{p} : -p^0k_0 \left\{\left[b(\vec{p})a(\vec{k})e^{-i2\omega t} + a^\dagger(\vec{p})b^\dagger(\vec{k})e^{i2\omega t}\right]\delta(\vec{k}+\vec{p}) - \left[b(\vec{p})b^\dagger(\vec{k}) + a^\dagger(\vec{p})a(\vec{k})\right]\delta(\vec{k}-\vec{p})\right\} && (33)\\
-p^jk_j \left\{\left[b(\vec{p})a(\vec{k})e^{-i2\omega t} + a^\dagger(\vec{p})b^\dagger(\vec{k})e^{i2\omega t}\right]\delta(\vec{k}+\vec{p}) - \left[b(\vec{p})b^\dagger(\vec{k}) + a^\dagger(\vec{p})a(\vec{k})\right]\delta(\vec{k}-\vec{p})\right\} && (34)\\
+m^2 \left\{\left[b(\vec{p})a(\vec{k})e^{-i2\omega t} + a^\dagger(\vec{p})b^\dagger(\vec{k})e^{i2\omega t}\right]\delta(\vec{k}+\vec{p}) + \left[b(\vec{p})b^\dagger(\vec{k}) + a^\dagger(\vec{p})a(\vec{k})\right]\delta(\vec{k}-\vec{p})\right\} : && (35)\\
= \frac{1}{2\omega}\int d\tilde{k} : -p^0k_0 \left\{\left[b(-\vec{k})a(\vec{k})e^{-i2\omega t} + a^\dagger(-\vec{k})b^\dagger(\vec{k})e^{i2\omega t}\right] - \left[b(\vec{k})b^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right]\right\} && (36)\\
+k^jk_j \left\{\left[b(-\vec{k})a(\vec{k})e^{-i2\omega t} - a^\dagger(-\vec{k})b^\dagger(\vec{k})e^{i2\omega t}\right] + \left[b(\vec{p})b^\dagger(\vec{k}) + a^\dagger(\vec{p})a(\vec{k})\right]\right\} && (37)\\
+m^2 \left\{\left[b(-\vec{k})a(\vec{k})e^{-i2\omega t} + a^\dagger(-\vec{k})b^\dagger(\vec{k})e^{i2\omega t}\right] + \left[b(\vec{k})b^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right]\right\} : && (38)
\end{aligned}
$$

$$
\begin{aligned}
P_0 = \int d\tilde{k} : \frac{\omega}{2}\left\{-\cancel{\left[b(-\vec{k})a(\vec{k})e^{-i2\omega t} - a^\dagger(-\vec{k})b^\dagger(\vec{k})e^{i2\omega t}\right]} + \left[b(\vec{k})b^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right]\right. && (39)\\
\left. +\cancel{\left[b(-\vec{k})a(\vec{k})e^{-i2\omega t} + a^\dagger(-\vec{k})b^\dagger(\vec{k})e^{i2\omega t}\right]} + \left[b(\vec{p})b^\dagger(\vec{k}) + a^\dagger(\vec{p})a(\vec{k})\right]\right\} : && (40)
\end{aligned}
$$

This yields

$$P_0 = \int d\tilde{k}\, k_0 \left[+a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right]. \tag{41}$$

Combined

Finally, if we combine equations 22 and 41 we get the expression

$$P_\mu = \int d\tilde{k}\, k_\mu \left[a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right]. \tag{42}$$