Exercise 1

Dirac field momentum and charge

The solution of the Dirac equation can be expanded in the plane waves as follows

$$\psi(x) = \int \mathrm{d}\tilde{k} \sum_{\lambda=\pm} \left[a_\lambda(k) u(k,\lambda) e^{-ik\cdot x} + b_\lambda^\dagger(k) v(k,\lambda) e^{ik\cdot x} \right], \quad \text{with} \quad \mathrm{d}\tilde{k} = \frac{\mathrm{d}^3\tilde{k}}{(2\pi)^3 2\omega_k}$$

Therein $u(k,\lambda)$ and $v(k,\lambda)$ are Dirac spinors associated with positive and negative energy solutions, respectivly. They obey the relations

$$u^\dagger(k,\lambda)u(k,\lambda') = v^\dagger(k,\lambda)v(k,\lambda') = 2\omega_k \delta_{\lambda\lambda'}, \tag{1}$$

$$u^\dagger(\bar{k},\lambda)v(k,\lambda') = v^\dagger(\bar{k},\lambda)u(k,\lambda') = 0, \tag{2}$$

where $\vec{k} = (\omega_k, -\vec{k})^T$. At this stage, we leave open whether the $a^{(\dagger)}(k)$ and $b^{(\dagger)}(k)$ follow commutation or anti-commutation relations.

We start by showing $T^{0\mu} = \psi^\dagger i\partial^\mu\psi$

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}(\partial^\nu\psi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})}(\partial^\nu\overline{\psi}) - g^{\mu\nu}\mathcal{L} \tag{3}$$

for the Dirac-spinor Lagrangian $\mathcal{L} = \overline{\psi}(i\gamma^\alpha\partial_\alpha - m)\psi$.

$$\begin{aligned}
T^{0\mu} &= \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}(\partial^\mu\psi) + \frac{\partial\mathcal{L}}{\partial(\partial_0\overline{\psi})}(\partial^\mu\overline{\psi}) - g^{0\mu}\mathcal{L} && (4)\\
&= i\overline{\psi}\gamma^0(\partial^\mu\psi) + 0 - g^{0\mu}\overline{\psi}\,[\text{Dirac-Eq.}] && (5)\\
&= i\psi^\dagger\gamma^0\gamma^0\partial^\mu\psi && (6)\\
&= i\psi^\dagger\partial^\mu\psi && (7)
\end{aligned}$$

For the momentum operator, we integrate

$$\begin{aligned}
P^\mu = \int d^3x T^{0\mu} = \int d^3x \int d\tilde{k} \int d\tilde{p} \sum_{\lambda=\pm}\sum_{\kappa=\pm} \Big[& a_\lambda^\dagger(k)u^\dagger(k,\lambda)a_\kappa(p)u(p,\kappa)\cdot e^{i(k-p)x} && (8)\\
& - a_\lambda^\dagger(k)u^\dagger(k,\lambda)b_\kappa^\dagger(p)v(p,\kappa)\cdot e^{i(k+p)x} && (9)\\
& + b_\lambda(k)v^\dagger(k,\lambda)a_\kappa(p)u(p,\kappa)\cdot e^{-i(k+p)x} && (10)\\
& - b_\lambda(k)v^\dagger(k,\lambda)b_\kappa^\dagger(p)v(p,\kappa)\cdot e^{-i(k-p)x} \Big] && (11)
\end{aligned}$$

Using the Fourier-Transformation again:

$$\begin{aligned}
P^\mu = \int d^3x \int d\tilde{p} \sum_{\lambda=\pm}\sum_{\kappa=\pm} \Big[& a_\lambda^\dagger(p)u^\dagger(p,\lambda)a_\kappa(p)u(p,\kappa) && (12)\\
& - a_\lambda^\dagger(\overline{p})u^\dagger(\overline{p},\lambda)b_\kappa^\dagger(p)v(p,\kappa)e^{+i2\omega t} && (13)\\
& + b_\lambda(\overline{p})v^\dagger(\overline{p},\lambda)a_\kappa(p)u(p,\kappa)e^{-i2\omega t} && (14)\\
& - b_\lambda(p)v^\dagger(p,\lambda)b_\kappa^\dagger(p)v(p,\kappa) \Big] && (15)
\end{aligned}$$

Relations 1 cancel two lines and adds a $\delta_{\lambda\kappa}$ and thus

$$P^\mu = \int d\tilde{k} \sum_{\lambda=\pm} k_\mu \left[a_\lambda^\dagger(k)a_\lambda(k) - b_\lambda(k)b_\lambda^\dagger(k) \right]. \tag{16}$$

A similar calculation can be performed for

$$Q = \int d^3x\, \overline{\psi}(x)\gamma^0\psi(x) = \int d^3x\, \psi^\dagger(x)\psi(x). \tag{17}$$

Without the derivative we get

$$Qu = \int d\tilde{k} \sum_{\lambda=\pm} k_\mu \left[a_\lambda^\dagger(k) a_\lambda(k) + b_\lambda(k) b_\lambda^\dagger(k) \right]. \tag{18}$$

Both these cases make clear why an anti-commutation is needed for the normal order

$$: \left[b_\lambda(k) b_\lambda^\dagger(k) \right] := -b_\lambda^\dagger(k) b_\lambda(k). \tag{19}$$