Advanced Methods in Numerical Wave Propagation

- an Introduction -

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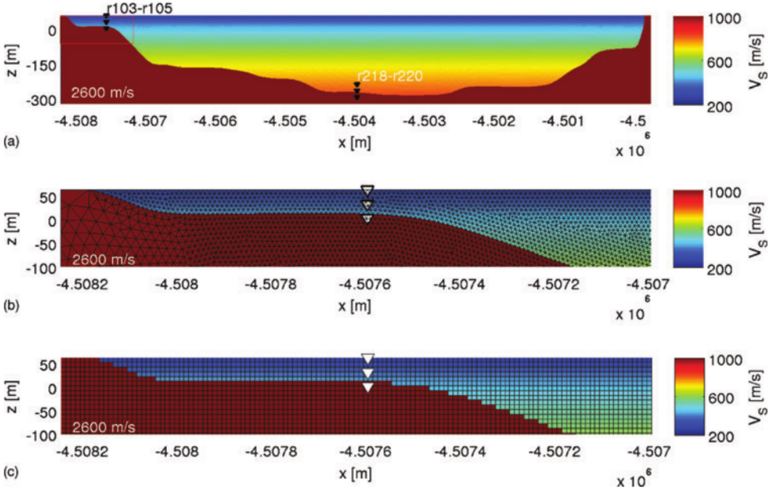
The Challenge

The Challenge?

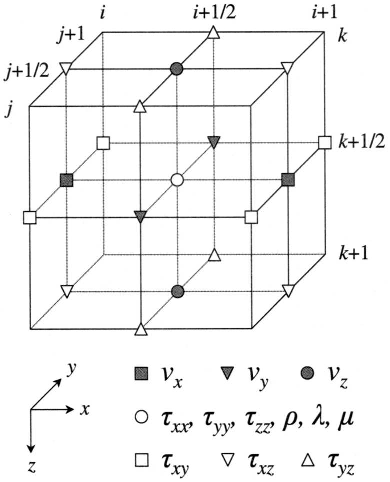
It is the year 2014, and we are still talking about the seismic forward problem. What makes it so hard?

Meshing / Gridding



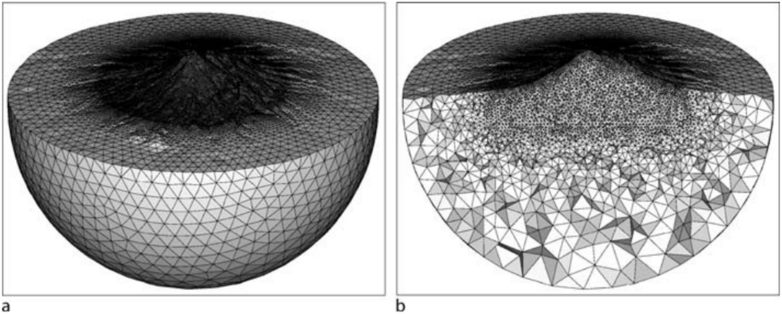
Pelties et al (2010)

Meshing / Gridding



Kang & Baag (2004)

Meshing / Gridding



Igel et al (2009)

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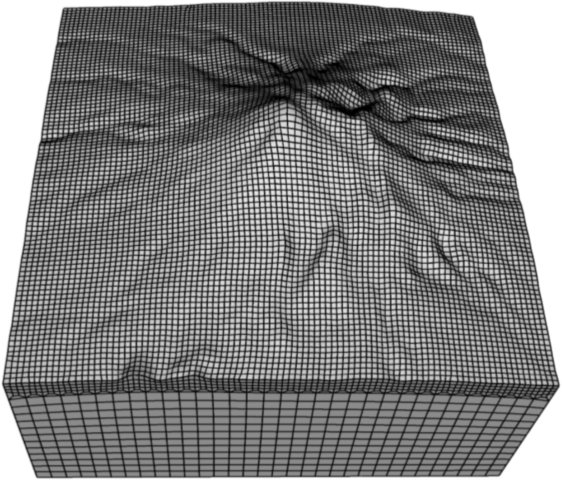

van Driel et al (submitted)

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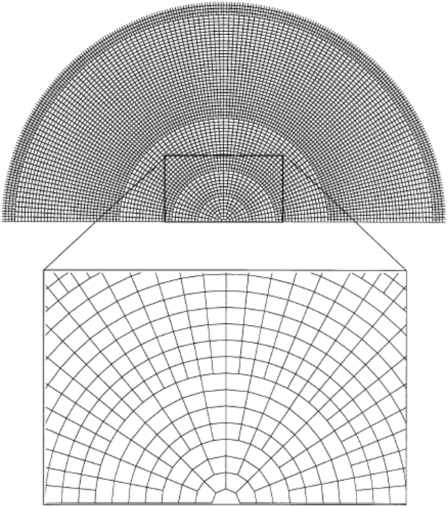
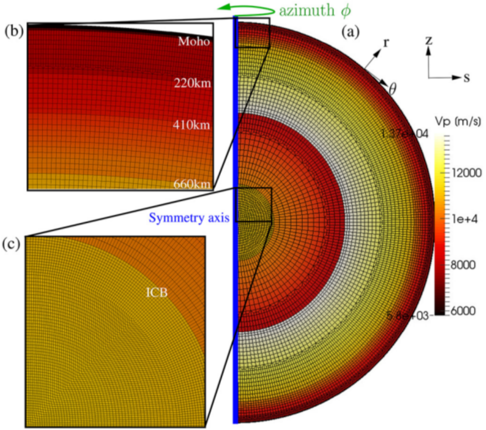


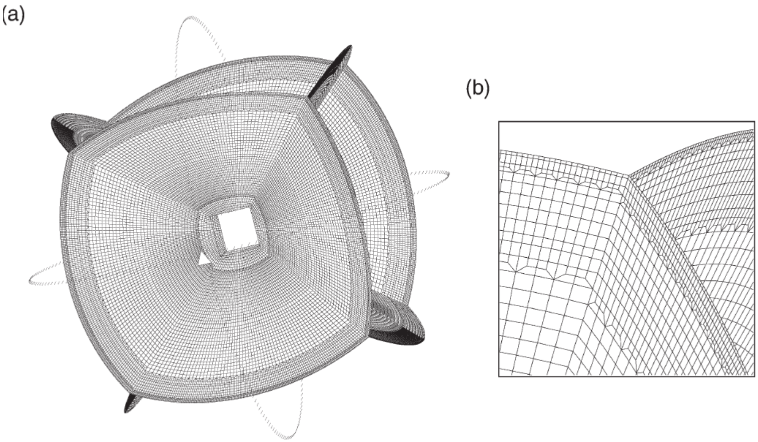
Thomas et al (2000)

Meshing / Gridding



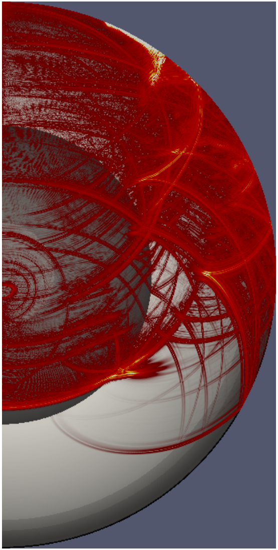
Nissen-Meyer et al (2014)

Meshing / Gridding



Tromp (2007)

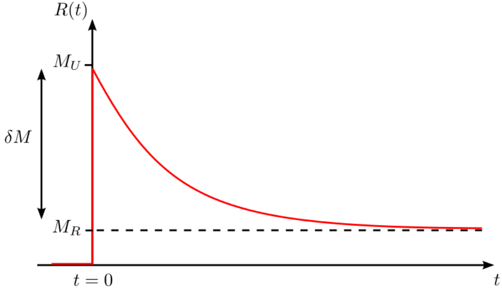
Surface / Diffracted Waves, Solid Fluid Boundaries

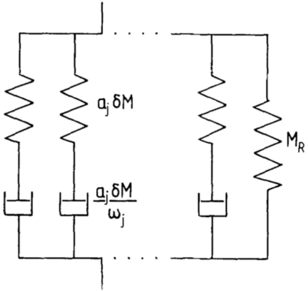


Hosseini et al (2013)

Attenuation

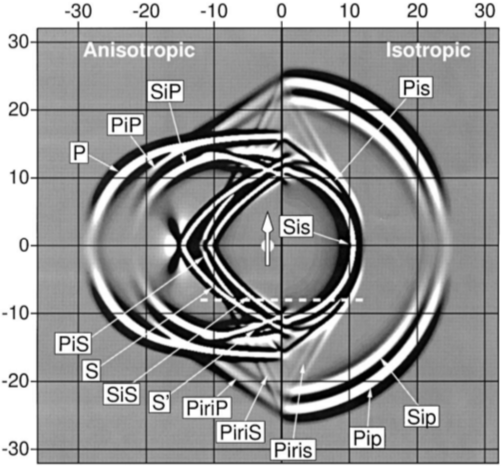
$$\sigma(t) = \int_{-\infty}^{\infty} M(t-\tau)\cdot\epsilon(\tau)\,\mathrm{d}\tau$$
$$= \int_{-\infty}^{\infty} R(t-\tau)\cdot\dot{\epsilon}(\tau)\,\mathrm{d}\tau.$$





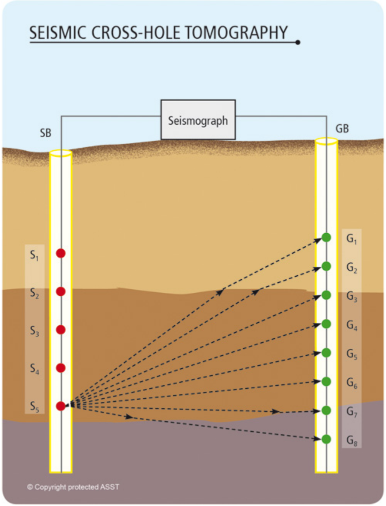
van Driel et al (2014b), Emmerich & Korn (1987)

Anisotropy



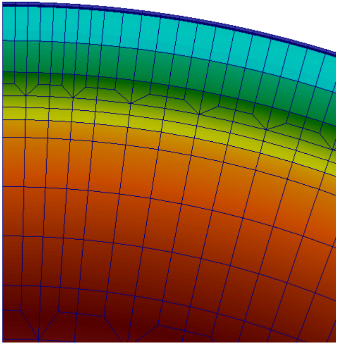
Komatitsch et al (2000)

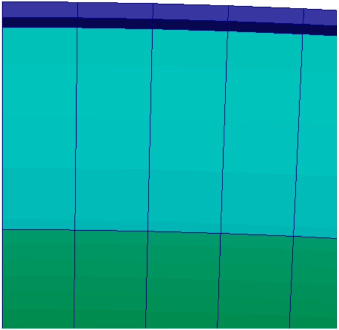
Scale Differences



whole domain (100m) vs borehole casing (1cm)

Scale Differences





Earth Diameter (13.462km) vs crust (10km) or ocean (few km)

Size

How many Gridpoints to mesh whole earth at 1Hz?

- average s-wave speed: 5 km / s
- 6 points per wavelength
- 3 degrees of freedom per points
- $6 * 10^{12}$ DOF
- 20TB of memory in single precision just to store the wavefield at a single time

The 1D Wave equation

'strong form' - 'second order'

$$\rho \partial_t^2 u - \partial_x(\mu \partial_x u) = f$$

'strong form' - 'velocity stress' - 'first order'

$$\rho \partial_t^2 u - \partial_x T = f, \qquad T = \mu \partial_x u$$

'weak form' - 'variational form' - 'projection into subspace'

$$\int_x \left(v \cdot \rho \partial_t^2 u + (\partial_x v) \cdot \mu \partial_x u \right) \mathrm{d}x = \int_x v \cdot f \,\mathrm{d}x \qquad \forall v$$

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Boundary Conditions

'Dirichlet' = fixed

$$u(a) = 0$$

'Neumann' - 'free surface' - 'stress free' - 'loose end'

$$T = \mu \partial_x u = 0$$

'Absorbing' - 'outflow'

- PML = 'perfectly matched layer'
- ABC = 'absorbing boundary condition'

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Mass and Stiffness Matrix, time stepping

'semi discrete form' - 'ODE'

representation of the field variables in coefficients, e.g.:

$$\mathbf{u} = (u_1, \ldots, u_n)$$

semi discrete wave equation with mass matrix **M** and stiffness matrix **K**

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t)$$

'time stepping' - 'time integration'

advance the field variables in time, e.g. using Newmark scheme:

$$\mathbf{u}(t + \Delta t) = \mathbf{u}(t) + \Delta t \dot{\mathbf{u}}(t) + \frac{1}{2}\Delta t^2 \ddot{\mathbf{u}}(t),$$

$$\ddot{\mathbf{u}}(t + \Delta t) = \mathbf{M}^{-1}\left[\mathbf{f}(t + \Delta t) - \mathbf{K}\mathbf{u}(t + \Delta t)\right],$$

$$\dot{\mathbf{u}}(t + \Delta t) = \dot{\mathbf{u}}(t) + \frac{1}{2}\Delta t \left[\ddot{\mathbf{u}}(t) + \ddot{\mathbf{u}}(t + \Delta t)\right]$$

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\end{aligned}$$

Courant Number, CFL Condition

Courant - Friedrichs - Lewy Condition:

Necessary condition for time stepping schemes to be stable

$$C = \frac{v\Delta t}{\Delta x} < C_{\max}$$

$C_{\max}$ depends on the scheme and is determined empirically. Typical order of magnitude 0.1 to 1.

Reminder: Small Elements

The CFL condition is the reason, that small elements increase the computational cost: for global time stepping schemes, the smallest element dicates the timestep.

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Finite Differences (FD)

2nd order central differences

regular sampling of a function f

$$\partial_x f(n\Delta x) = (f_{n+1} - f_{n-1})/(2\Delta x), \quad n = 1, \ldots, N-2$$

4nd order central differences

regular sampling of a function f

$$\partial_x f(n\Delta x) = (-f_{n+2} + 8f_{n+1} - 8f_{n-1} + f_{n-2})/(12\Delta x), \quad n = 2, \ldots, N-3$$

Limitations

- simple domains
- Boundaries need special treatment
- accuracy of derivatives at the boundaries is less
- surface waves are very sensitive to that

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Pseudo Spectral Method (PS)

Compute derivatives using Fast Transforms (e.g. FFT)

$$F(l\Delta k) = \Delta x \sum_{n=0}^{N-1} f(n\Delta x)\exp(-2\pi i n l/N),$$

$$l = 0, \ldots, N-1$$

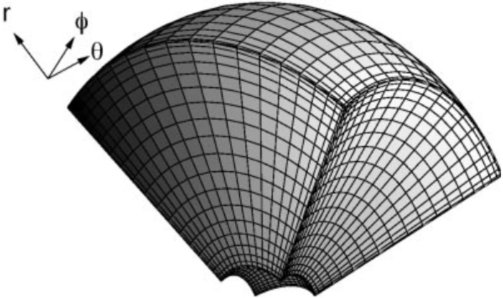
$$\frac{d}{dx} f(n\Delta x) = \frac{1}{N\Delta x} \sum_{l=0}^{N-1} i(l\Delta k) F(l\Delta k)\exp(2\pi i n l/N),$$

$$n = 0, \ldots, N-1$$

Pseudo Spectral

Limitations

- simple domains
- smooth media
- Natural boundary condition: periodic (due to global basis functions)
- Free surface easier to implement with Chebyshef Polynomials (local, higher sampling at the boundary)



Igel (1999)

Unstructured Meshes

Mapping between physical space (x) and reference coordinates ($\xi \in [-1, 1]$) in Elements Ω

using shape function N_a and control points x_a:

$$x(\xi) = \sum_a N_a(\xi) x_a$$

common shape functions are the Lagrange polynomials, so in 1D with two control points:

$$N_1(\xi) = 1 - \xi, \qquad N_2(\xi) = 1 + \xi, \qquad x_1 = -1, \qquad x_2 = 1$$

Representation of a function f and derivatives on these elements

$$f = f(x(\xi)) \qquad \Rightarrow \qquad \partial_x f = \partial_\xi f / \partial_\xi x = \partial_\xi f \cdot \partial_x \xi$$

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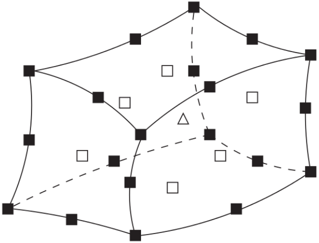
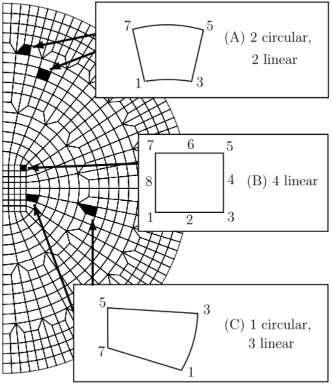




Finite Element Method

Shape functions as basis functions:

expansion coefficients f_a

$$f(x(\xi)) = \sum_a f_a N_a(\xi)$$

So the Gradient is

$$\partial_x f = \sum_a f_a N_a(\xi) \partial_x \xi$$

Quadrature (Integral from the weak form)

Commonly used: Gauss quadrature with Jacobian J

$$\int_\Omega f(x)\,\mathrm{d}x = \sum_\alpha \omega_\alpha f(\xi_\alpha) J(\xi_\alpha)$$

Finite Element Method

Limitations

- typically low order polynomials
- Non Diagonal Mass Matrix

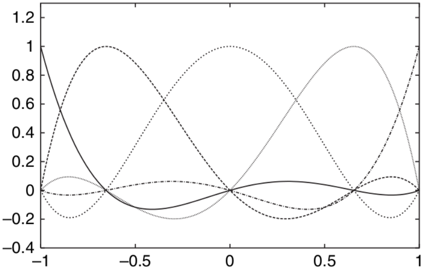
Spectral Element Method

Lagrangian Polynomials as basis functions:

expansion coefficients f_a

$$f(x(\xi)) = \sum_a f_a l_a(\xi)$$

Lagrangian Polynomials are defined on the Gauss-Lobatto-Legendre (GLL) Points, which are the same used in the quadrature:



Spectral Element Method

Quadrature

Gauss quadrature on the GLL points, which have the property

$$f(x(\xi_\alpha)) = f_\alpha$$

so

$$\int_\Omega f(x)\,\mathrm{d}x = \sum_\alpha \omega_\alpha f_\alpha J(\xi_\alpha)$$

and the mass matrix is diagonal with respect in this approximate quadrature:

$$\begin{aligned}\int_\Omega v\partial_t^2 u\,\mathrm{d}x &= \int_{-1}^{1} \rho v \partial_t^2 u \partial_\xi x\,\mathrm{d}\xi \\ &\approx \sum_\alpha \omega_\alpha \rho J_\alpha v_\alpha \ddot{u}_\alpha\end{aligned}$$

Spectral Element Method

Limitations

- typically uses hexahedrons
- hexahedral Meshing is very hard, no fully automatic algorithm exists.
- Medium constant (or at least smooth) inside each element.

More Schemes...

...we will look at this semester:

- Finite Volume
- Discontinuous Galerkin