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The Discontinuous Galerkin Method
Part 2

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Overview

1. Short **review** of the last presentation

2. From the advection to the **elastic wave equation**

3. A special time integration method: the **ADER-DG method**

The DG method in a nutshell (1D)

We looked at the linear advection equation:

$$\boxed{\partial_t u(x,t) + \mu \, \partial_x u(x,t) = 0}$$

Weak formulation of the equation

- Multiplying by a test function ϕ and integration over the domain Ω
- Applying the divergence theorem for the spatial derivative $\partial_x u$
- Discretization of the domain into K elements
  Approximation of the exact solution in the k-th element by

$$u_h^k(x,t) = \sum_{n=1}^{N_p} \hat{u}_n^k(t) \; \phi_n(x)$$

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The DG method in a nutshell (1D)

$$\partial_t u(x,t) + \mu \ \partial_x u(x,t) = 0$$

This leads to the weak formulation

Local matrices for every element!

$$M^k \cdot \partial_t u_h^k(t) - \mu \ (S^k)^T \cdot u_h^k(t) = (F_r^k(u_h^{k+1}(t), u_h^k(t)) - F_l^k(u_h^k(t), u_h^{k-1}(t)))$$

$$[M^k]_{ij} = \left[\int_{D_k} \phi_i^k(x) \ \phi_j^k(x) \ dx \right]_{ij}$$

Mass matrix of the k-th element

$$[S^k]_{ij} = \left[\int_{D_k} \phi_i^k(x) \partial_x \phi_j^k(x) \ dx \right]_{ij}$$

Stiffness matrix of the k-th element

$$u_h^k(t) = (u_1^k(t), \dots \ \ , u_{N_p}^k(t))^T$$

(time dependent) coefficients,
N+1 DOF for each element

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The DG method in a nutshell (1D)

$$\partial_t u(x,t) + \mu\ \partial_x u(x,t) = 0$$

This leads to the weak formulation

Local matrices for every element!

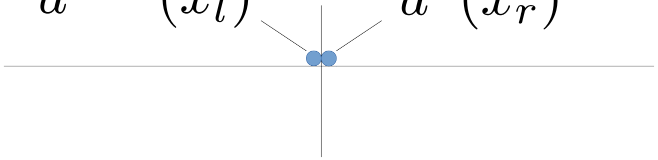
$$M^k \cdot \partial_t u_h^k(t) - \mu\ (S^k)^T \cdot u_h^k(t) = (F_r^k(u_h^{k+1}(t), u_h^k(t)) - F_l^k(u_h^k(t), u_h^{k-1}(t)))$$

$F_l^k(u_h^{k-1}(t), u_h^k(t))$ Left flux matrix

$F_r^k(u_h^k(t), u_h^{k+1}(t))$ Right flux matrix

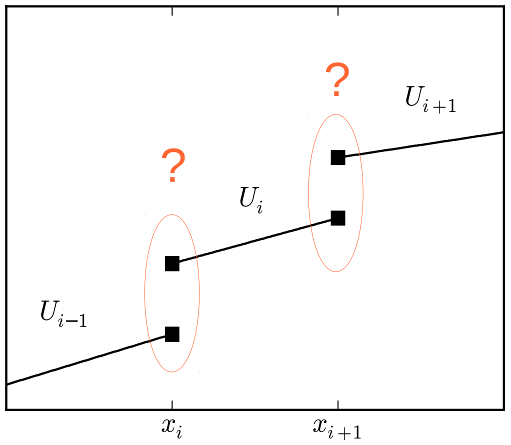
Dependent of the solution in the neighbouring element

$$u^{k-1}(x_l) \neq u^k(x_r)$$



Discontinous solution at the boundary of an element
→ Solution to a Riemann problem

The numerical flux



General concept:

We have a PDE together with a special initial condition boundary of the elements
→ **Riemann problem**

$$\begin{aligned} \partial_t u(x,t) + \mu \; \partial_x u(x,t) &= 0 \\ u(x,0) &= \begin{cases} u^{i-1}(x,t) & \text{if } x \in (x_{i-2}, x_{i-1}] \\ u^{i}(x,t) & \text{if } x \in (x_{i-1}, x_{i}] \end{cases} \end{aligned}$$

- Flux concepts as a (numerical) solution to Riemann problems
  For example: upwind flux (like in SeisSol)

Nodal vs. modal formulation

Modal

$$u_h^k(x,t) = \sum_{n=1}^{N_p} \hat{u}_n^k(t)\ P_n(x)$$

Legendre polynomials not depending on special grid points inside the element

→ DOF: coefficients for the approximation of the solution by Legendre polynomials

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Nodal

$$u_h^k(x,t) = \sum_{i=1}^{N_p} u_h^k(x_i,t)\ l_i^k(x)$$

Interpolating **Lagrange polynomials** at additional grid points inside the element

→ DOF: exact solution at the points x_i

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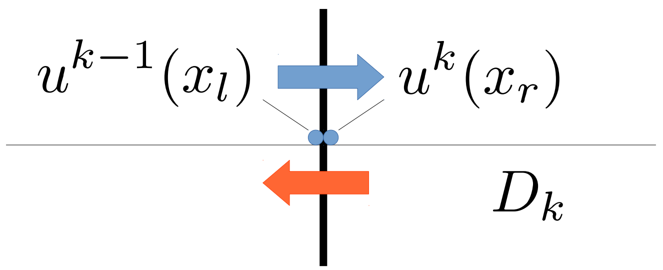
The numerical flux

Some simple choices for scalar equations



Central flux $$u^*(x^k_{BD}) = \frac{1}{2}(u_k^- + u_k^+)$$

-	interior information
+	exterior information

Upwind flux $$u^*(x^k_{BD}) = \frac{\mu}{2}(u_k^- + u_k^+) + \frac{|\mu|}{2}(n^- u_k^- + n^+ u_k^+)$$

(takes the boundary value from the direction where the wave is coming from)

Example **upwind flux** for the **left** boundary of an element

$$u^*(x_l^k) = \begin{cases} \mu \; u^k(x_l) & \text{if} \quad \mu \geq 0 \\ \mu \; u^{k-1}(x_r) & \text{if} \quad \mu < 0 \end{cases}$$

Advection velocity μ

Important aspects of the DG method

From the weak formulation

$$M^k \cdot \partial_t u_h^k(t) - \mu \ (S^k)^T \cdot u_h^k(t) = (F_r^k(u_h^{k+1}(t), u_h^k(t)) - F_l^k(u_h^k(t), u_h^{k-1}(t)))$$

we receive an **explicit scheme** in time since the mass matrix is diagonal!

$$\Rightarrow \quad \boxed{\partial_t u_h^k(t) = (M^k)^{-1}(\mu \ (S^k)^T u_h^k(t) + (F_r^k - F_l^k))}$$

Everything is defined **locally**

→ p-adaptivity (different choice of approximation order in every element)

→ easy to parallelize

→ possibility of local time stepping

From the 1D advection to the 1D elastic wave equation

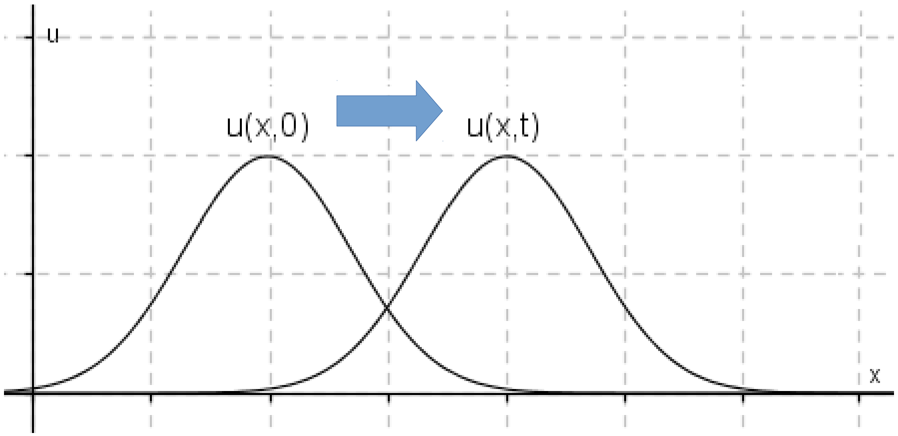
$$\rho\ \partial_t u(x,t) + \mu\ \partial_x u(x,t) = 0$$

Advection equation

→ first order partial differential equation

→ scalar hyperbolic equation

→ nice description of the upwind flux concept



From the 1D advection to the 1D elastic wave equation

$$\boxed{\rho\ \partial_t u(x,t) + \mu\ \partial_x u(x,t) = 0}$$

Advection equation

$$\begin{aligned} \partial_t \sigma - \mu\ \partial_x v &= 0 \\ \partial_t v - \frac{1}{\rho}\ \partial_x \sigma &= 0 \end{aligned}$$

(coupled) linear elastic wave equation

hyperbolic first order PDE
(velocity-stress formulation)

$$\boxed{\partial_t Q + A\ \partial_x Q = 0}$$

Matrix-vector formulation with $Q = (\sigma, v)^T$ $\quad A = \begin{pmatrix} 0 & -\mu \\ -\frac{1}{\rho} & 0 \end{pmatrix}$

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From the 1D advection to the 1D elastic wave equation

$$\boxed{\partial_t q + \frac{\mu}{\rho}\ \partial_x q = 0} \quad \Longleftrightarrow \quad \boxed{\partial_t Q + A\ \partial_x Q = 0}$$

$$Q = (\sigma, v)^T$$

Advection equation

Elastic wave equation
(coupled) system of two PDEs

Question:

What is the relation between these equations?
How can we derive from the elastic wave equation to the advection equation?

→ apply all what we learned about the DG method to the elastic wave equation

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From the 1D advection to the 1D elastic wave equation

$$\boxed{\partial_t Q + A\ \partial_x Q = 0}$$

Elastic wave equation
(coupled) system of two PDEs

$$Q = (\sigma, v)^T \qquad A = \begin{pmatrix} 0 & -\mu \\ -\frac{1}{\rho} & 0 \end{pmatrix}$$

1. Decouple the system, i.e. find a diagonal matrix $\tilde{A}$ such that

$$\boxed{\partial_t \tilde{Q} + \tilde{A}\ \partial_x \tilde{Q} = 0}$$

This is possible because of the <u>definition of a hyperbolic equation:</u>

> A PDE is hyperbolic if the matrix A is diagonalizable
> and has only real eigenvalues.

From the 1D advection to the 1D elastic wave equation

→ eigenvalue decomposition of $A = \begin{pmatrix} 0 & -\mu \\ -\frac{1}{\rho} & 0 \end{pmatrix}$

$$\boxed{A = R^{-1} \cdot \Lambda \cdot R}$$

with $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\quad \lambda_1, \lambda_2$ eigenvalues

$R = \left[x_1 | x_2 \right]$ $\quad x_1, x_2$ corresponding eigenvectors

In our case:

$$\lambda_{1,2} = \pm \sqrt{\frac{\mu}{\rho}} = \pm\, c \qquad x_{1,2} = \begin{pmatrix} \pm\, \rho c \\ 1 \end{pmatrix}$$

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From the 1D advection to the 1D elastic wave equation

$$A = R^{-1} \cdot \Lambda \cdot R \qquad\qquad \Lambda = R \cdot A \cdot R^{-1}$$ **diagonal!**

We can rewrite

$$\boxed{\partial_t Q + A\, \partial_x Q = 0} \longrightarrow R^{-1}\partial_t Q + \underline{R^{-1}\ A\ R}\, R^{-1}\partial_x Q = 0$$

$$\longrightarrow \boxed{\partial_t W + \Lambda\, \partial_x W = 0}$$

with $W = R^{-1}\, Q$ Characteristic variable

➡ System of 2 advection equations travelling with speed $\pm\, c = \pm \sqrt{\dfrac{\mu}{\rho}}$

➡ Apply the DG method to these equations like in the last presentation

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The ADER-DG Scheme

- **ADER:** high-order time integration method for hyperbolic problems using **A**rbitrarily high-order **DER**ivatives
- first introduced by E.F. Toro, A.C. Millington, and L.A. Nejad in 2001 for high-order Finite Volume schemes.
- The first combination of the DG-scheme and the ADER time integration approach was presented by M. Dumbser in 2005.

- Here we present the ADER scheme for the advection equation:

$$\partial_t q(x,t) + a\partial_x q(x,t) = 0$$

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The ADER-DG Scheme

- The basis of high-order time integration is the **Taylor Series Expansion** in time!

$$q(x,t^{n+1}) = q(x,t^n+\Delta t) \approx \sum_{j=0}^{N} \frac{\Delta t^j}{j!} \frac{\partial^j}{\partial t^j} q(x,t^n)$$

- Replace of the time derivative with the space derivative by using the governing PDE:

$$\frac{\partial}{\partial t} q(x,t) = -a \frac{\partial}{\partial x} q(x,t)$$

- Using of a recursion for the higher order derivative leads to (**Cauchy-Kowalewski Procedure)** :

$$\frac{\partial^{j+1}}{\partial t^{j+1}} q(x,t) = -a \frac{\partial}{\partial x} \left(\frac{\partial^j}{\partial t^j} q(x,t) \right)$$

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The ADER-DG Scheme

- In the **reference space** we get the analogue formulations:

$$q(\zeta, t^{n+1}) \approx \sum_{j=0}^{N} \frac{\Delta t^j}{j!} \frac{\partial^j}{\partial t^j} q(\zeta, t^n)$$

Taylor approximation

$$\frac{\partial^{j+1}}{\partial t^{j+1}} q(\zeta, t) = -\frac{a}{|I|} \frac{\partial}{\partial \zeta} \left(\frac{\partial^j}{\partial t^j} q(\zeta, t) \right)$$

(from the governing PDE)

The ADER-DG Scheme

- Replace q by $q(\zeta,t) \approx q_l(t)\phi_l(\zeta)$ and project onto the basis function $\phi_k(\zeta)$

$$\int_I \phi_k \frac{\partial^{j+1}}{\partial t^{j+1}} \phi_l \hat{q}_l d\zeta = \int_I \phi_k \left(-\frac{a}{|I|} \frac{\partial}{\partial \zeta} \frac{\partial^j}{\partial t^j} \phi_l \hat{q}_l \right) d\zeta$$

and after some reformulations we get

$$\frac{\partial^{j+1}}{\partial t^{j+1}} \hat{q}_l(t) = -\frac{a}{|I|} \frac{\int_I \phi_k \frac{\partial \phi_l}{\partial \zeta} d\zeta}{\int_I \phi_k \phi_l \partial \zeta} \frac{\partial^j}{\partial t^j} \hat{q}_l(t)$$

The ADER-DG Scheme

- Hence we can determine the required time integrals with the spatial order of accuracy

$$\mathrm{q}(\mathrm{t}^{n+1}) - q(t^n) =$$

$$-\left(aF^R_{kl}\boxed{\int_{t^n}^{t^{n+1}}\hat{q}_i^i(t_n)dt} - aF^L_{kl}\boxed{\int_{t^n}^{t^{n+1}}\hat{q}_l^{i-1}(t_n)dt} - aK_{kl}\boxed{\int_{t^n}^{t^{n+1}}\hat{q}_l^i(t_n)dt}\right)t/|I|M_{kl}$$

- with

$$\int_{t^n}^{n+1}\hat{q}(t)dt = \sum_{j=0}^{N}\frac{\Delta t^{j+1}}{(j+1)!}\frac{\partial^j}{\partial t^j}\hat{q}(t^n)$$

$$\frac{\partial^{j+1}}{\partial t^{j+1}}\hat{q}_l(t) = -\frac{a}{|I|}\frac{\int_I \phi_k \frac{\partial\phi_l}{\partial\zeta}d\zeta}{\int_I \phi_k\phi_l\partial\zeta}\frac{\partial^j}{\partial t^j}\hat{q}_l(t)$$